An Exact Solution of the Macroscopic Fluctuation Theory for SEP

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Outline

Introduction

1. Current and Tracer fluctuations in SEP: a microscopic approach
2. Fluctuating hydrodynamics: The Macroscopic Fluctuation Theory
3. Solving the MFT by Inverse Scattering

Conclusion

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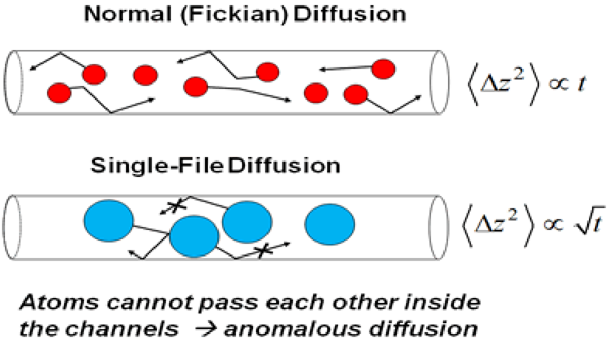
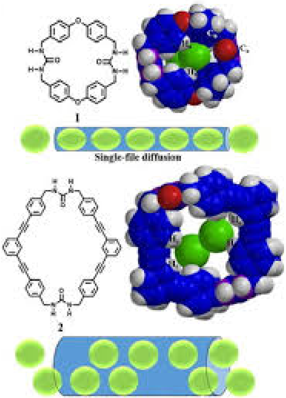
A. Grabsch, H. Moriya, KM, T. Sasamoto and O. Bénichou, Phys. Rev. Lett., **133**, *117102 (2024).*

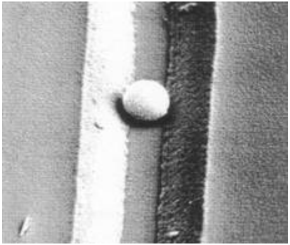
Single-file diffusion

Single-file diffusion is an important phenomena soft-condensed matter (for example, transport through cell membranes).



Experimental observations

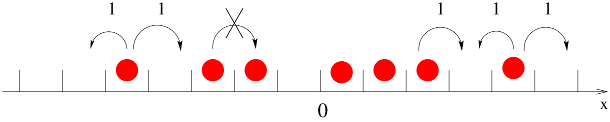




(C. Bechinger's group in Stuttgart)

The exclusion process

A pristine model for single-file diffusion is the Symmetric Exclusion Process, in which particles perform continuous-time random walks with hard-core (classical) exclusion interaction



This minimal model appears as a building block in many realistic studies of 1d transport and studied extensively in biophysics, condensed matter, polymer reptation, combinatorics, probability and even traffic flow.

The Symmetric Exclusion Process (SEP) on Z.

Consider the Symmetric Exclusion Process, $(p = q = 1)$ on $\mathbb{Z}$ with a uniform finite density ρ of particles.

Suppose that we tag and observe a particle that was initially located at site 0 and monitor its position X_t with time.

On the average $\langle X_t \rangle = 0$ but how large are its fluctuations?

- If the particles were non-interacting (no exclusion constraint), each particle would diffuse normally $\langle X_t^2 \rangle = Dt$.
- Because of the exclusion condition, a particle displays an anomalous diffusive behaviour: when $t \to \infty$, we have

$$\langle X_t^2 \rangle \simeq 2\frac{1-\rho}{\rho}\sqrt{\frac{Dt}{\pi}} \qquad \text{(Arratia, 1983)}$$

The exact probability distribution of X_t remained unknown for almost 40 years.

Microscopic Approach

The Exclusion process is an integrable model

One of the reasons that makes the exclusion process (and some variants) so attractive and popular is that it is **integrable**. It conceals hidden symmetries (combinatorial structures) that allows us to carry precise analysis and derive exact formulas and solutions.

A key observation was made Shlomo Alexander and, independently, by Deepak Dhar in the eighties. The Markov matrix of the exclusion process is **identical** to the Heisenberg Spin chain Hamiltonian:

$$M = \sum_{l=1}^{L} \left(\mathbf{S}_l^+ \mathbf{S}_{l+1}^- + q \mathbf{S}_l^- \mathbf{S}_{l+1}^+ + \frac{1+q}{4} \mathbf{S}_l^z \mathbf{S}_{l+1}^z - \frac{1+q}{4} \right)$$

where $\mathbf{S} = (S_x, S_y, S_z)$ are the Pauli matrices (and q represents the asymmetry of the jumps; $q = 1$ for symmetric walks).

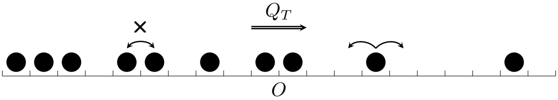
Thus, the exclusion process can be solved using (quantum) integrability methods (Bethe Ansatz).

The microscopic analysis of this interacting, non-equilibrium, N-body process, can be carried out to extreme precision (B. Derrida, M. Evans, J. Lebowitz, V. Hakim, D. Mukamel, G. Schütz, E. Speer, H. Spohn...).

Current fluctuations in the SEP

Consider the Symmetric Exclusion Process on $\mathbb{Z}$ with two-sided Bernoulli initial conditions ρ_- on the left, ρ_+ on the right at $t = 0$.

Time integrated current Q_T = total number of particles that have jumped from 0 to 1 *minus* the total number of particles that have jumped from 1 to 0 during the time interval $(0, T)$.

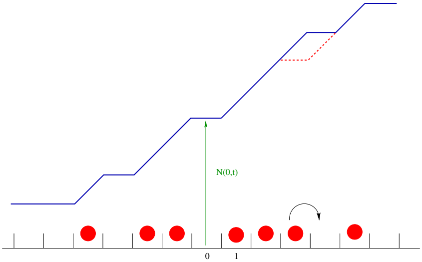


An exact microscopic combinatorial solution of the problem is possible: it yields the distribution of current and that of a tagged particle at any position and at any finite time. This involves the technology of Integrable Probabilities.

A macroscopic hydrodynamic picture, based on the MFT, is also exactly solvable and its results are totally consistent with the microscopic solution.

Mapping to an interface model

We represent the exclusion process by an interface model



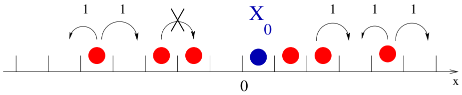
$N(0,t)$ represents the total current Q_t through $(0,1)$ in the duration t.

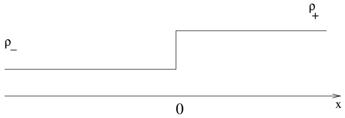
$$N(x,t) = N(0,t) + \begin{cases} \sum_{y=1}^{x} \eta_y(t)\,, & x > 0 \\ 0, & x = 0 \\ -\sum_{y=x+1}^{0} \eta_y(t)\,, & x < 0 \end{cases}$$

Note that $N(x,t)$ *is related to the KPZ height via* $h(x,t) = N(x,t) - \frac{x}{2}$

A Tagged Particle in the SEP with step profile

Consider SEP with a step-like Bernoulli initial condition with density ρ_- (resp. ρ_+) to the left (resp. right). The tagged particle (or tracer) is initially located at 0. Let the system evolve: X_t denotes the position of the tracer at time t.





What is the statistics of the position of the tracer X_t and its asymptotics in the long time limit?

Because of the non-crossing condition, the statistics of the current and that of a tagged particle are 'simply' related.

Tracer's position versus the height N(x,t)

Because the tracer is continuously moving, it is useful to relate its position X_t to the *local* observable $N(x,t)$, which is fixed at position x.

Using particle number conservation, one can show

$$\text{Prob}\left(X_t > x\right) = \text{Prob}\left(N(x,t) > 0\right)$$

Or, equivalently,

$$\text{Prob}\left(X_t \leq x\right) = \text{Prob}\left(N(x,t) \leq 0\right)$$

This relates the statistical properties of X_t and those of the height $N(x,t)$. In particular, one can deduce the large deviation function and the cumulants of X_t from the corresponding quantities for $N(x,t)$.

It is thus enough to focus on $N(x,t)$.

Exact expression of the generating function

It is possible to derive a formula for the characteristic function of the height $N(x,t)$, exact at any finite-time, in terms of a Fredholm determinant:

$$\langle e^{\lambda N(x,t)} \rangle = \det(1 + \omega K_{t,x}) W_0(\lambda)$$

where

$$\omega(\lambda) = \rho_+(e^{\lambda} - 1) + \rho_-(e^{-\lambda} - 1) + \rho_+\rho_-(e^{\lambda} - 1)(e^{-\lambda} - 1)$$

$$K_{t,x}(\xi_1, \xi_2) = \frac{\xi_1^{|x|} e^{\epsilon(\xi_1)t}}{\xi_1\xi_2 + 1 - 2\xi_2} \quad \text{with} \quad \epsilon(\xi) = \xi + \xi^{-1} - 2$$

$$W_0(\lambda) = \left(1 + \rho_\pm(e^{\pm\lambda} - 1)\right)^{|x|} \text{ with } \pm = \mathrm{sgn}(x)$$

From this result, information about the tracer can be deduced.

Long time asymptotics

In the long time limit, the asymptotics analysis of this Fredholm determinant, shows that the characteristic function behaves as

$$\langle e^{\lambda N(x,t)} \rangle \sim e^{-\sqrt{t}\mu(\xi,\lambda)}$$

where $\xi = -\frac{x}{\sqrt{4t}}$. The function $\mu(\xi, \lambda)$ is the cumulant generating function of $N(x, t)$:

$$\mu(\xi,\lambda) = \sum_{n=1}^{\infty} \frac{(-\omega)^n}{n^{3/2}} A(\sqrt{n}\,\xi) + \xi \log \frac{1+\rho_+(e^{\lambda}-1)}{1+\rho_-(e^{-\lambda}-1)}$$

with $A(u) = \xi + \int_{\xi}^{\infty} \mathrm{erfc}\,(u)\mathrm{d}u$ and

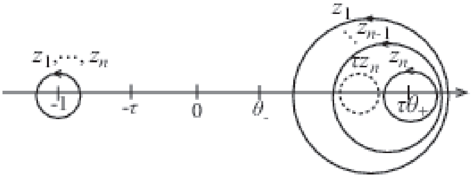
$$\omega(\lambda) = \rho_+(e^{\lambda}-1) + \rho_-(e^{-\lambda}-1) + \rho_+\rho_-(e^{\lambda}-1)(e^{-\lambda}-1)$$

Integrability: Schutz-Tracy-Widom Integral formulas

Inspired by the fact that ASEP is integrable by "Bethe Ansatz", the τ-correlation functions can be expressed as multiple contour integrals in the complex plane:

$$\langle \tau^{\sum_i N(x_i,t)} \rangle = \tau^{\sum_i i - \frac{x_i}{2}} \prod_{i=1}^{n} \left(1 - \frac{r_-}{\tau^i r_+}\right) \int \cdots \int \prod_{i<j} \frac{z_i - z_j}{z_i - \tau z_j} \prod_{i=1}^{n} \frac{e^{\Lambda_{x_i,t}(z_i)}}{(1 - \frac{z_i}{\tau\theta_+})(z_i - \theta_-)} \mathrm{d}z_i$$

with $r_\pm = \rho_\pm(1-\rho_\mp)$, $\theta_\pm = \rho_\pm/(1-\rho_\pm)$ and $e^{\Lambda_{x,t}(z)} = \left(\frac{1+z}{1+z/\tau}\right)^x e^{-\frac{q(1-\tau)^2 z}{(1+z)(\tau+z)} t}$



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T. Imamura, K.M, T. Sasamoto, CMP **384***:1409, (2021).*

From micro to Macro

• Exact solutions at the microscopic level require high-brow technology. However, at the level of large deviations, the cumulant generating function, $\mu(\xi, \lambda)$, is given by a rather simple expression.

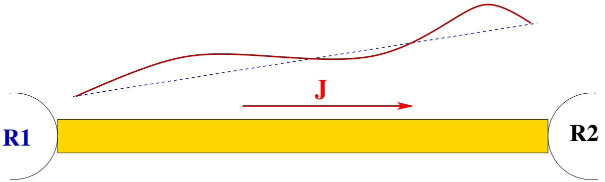
• We obtain the distribution of the height, current, tagged particle position in the long time limit. However, we have gained no knowledge on how large deviations (i.e. rare fluctuations) are dynamically generated.

• Time-dependent aspects seem to be out of reach of Bethe Ansatz (i.e. Integrable Probability) methods.

• A more physical picture, that would bypass combinatorics and asymptotics, and based on a more intuitive and direct approach, would be welcome.

Macroscopic Fluctuation Theory

The General Large Deviations Problem



The Probability to observe an atypical local current $j(x,t)$ and density profile $\rho(x,t)$ during $0 \le s \le L^2\,T$ (i.e. diffusive scaling, L is the size of the system) assumes a Large Deviation behaviour

$$\Pr\{j(x,t),\rho(x,t)\} \sim \mathrm{e}^{-L\,\mathcal{I}(j,\rho)}$$

Knowing $\mathcal{I}(j,\rho)$, one could deduce the large deviations of the current and of the density profile. For instance, $\Phi(j) = \min_\rho\{\mathcal{I}(j,\rho)\}$.

Is there a Principle which gives this large deviation functional for driven diffusive systems out of equilibrium?

The MFT action

For a weakly-driven diffusive system, the large deviation form of the probability to observe a current $j(x,t)$ and a density profile $\rho(x,t)$ during a time T, is given by

$$\Pr\{j(x,t),\rho(x,t)\} \sim \mathrm{e}^{-\,S_{MFT}(j,\rho)}\,,$$

with

$$S_{MFT}(j,\rho) = \int_0^T dt \int_{-\infty}^{+\infty} \frac{(j + D(\rho)\nabla\rho)^2\,dx}{2\sigma(\rho)}$$

under the constraint $\partial_t\rho = -\nabla.j$

(L. Bertini, D. Gabrielli, A. De Sole, G. Jona-Lasinio and C. Landim).

For a given problem, only the dominant paths will dominate the probability measure. They can be obtained by optimizing this action under constraints.

The Equations of Macroscopic Fluctuation Theory

The optimization of the action is a variational problem that leads to Euler-Lagrange equations. By a Legendre transform, a Hamiltonian structure is obtained by using a pair variables (ρ, H), conjugate to (ρ, j). Here, $\rho(x,t)$ is the density-field and $H(x,t)$ is a conjugate (momentum) field. The dynamics is given by

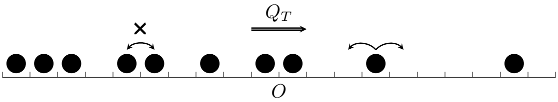
$$\partial_t \rho = \partial_x [D(\rho)\partial_x \rho] - \partial_x[\sigma(\rho)\partial_x H]$$

$$\partial_t H = -D(\rho)\partial_{xx} H - \frac{1}{2}\sigma'(\rho)(\partial_x H)^2$$

with Hamiltonian $\mathcal{H} = \sigma(\rho)(\partial_x H)^2/2 - D(\rho)\partial_x \rho \partial_x H.$

The information of the microscopic dynamics relevant at macroscopic scale is embodied in the transport coefficients D and σ other details are 'blurred' in this continuous hydrodynamic limit.

Height fluctuations at macroscopic scale



In the continuous limit:
$N(X,T) = \int_0^\infty [\rho(x,T) - \rho(x,0)]dx + \int_0^X [\rho(x,T)]dx$. And for large T, we have (LDP):

$$\langle e^{\lambda N(X,T)} \rangle \simeq e^{\sqrt{T}\mu(\xi,\lambda)}$$

What are the **profile** ρ and the conditioning **momentum field** H required to generate a given large fluctuation of the local height $N(X,T)$?

We want to extract the cumulant generating function (CGF) μ macroscopically.

We must average $e^{\lambda N(X,T)}$ under the MFT measure.

MFT equations for height fluctuations

Thus, we must solve the PDE's (for SEP, $D = 1,\ \sigma = 2\rho(1-\rho)$) :

$$\begin{aligned} \partial_t \rho &= \partial_x \left[\partial_x \rho - 2\rho(1-\rho)\partial_x H\right] \\ \partial_t H &= -\partial_{xx} H - (1-2\rho)(\partial_x H)^2 \end{aligned}$$

With non-local boundary conditions:

$$H(x,T) = \lambda\theta(x - X)$$

$$H(x,0) = \lambda\theta(x) + \log\frac{\rho(x,0)(1-\bar{\rho}(x))}{\bar{\rho}(x)(1-\rho(x,0))}$$

where $\bar{\rho}(x) = \rho_-\theta(-x) + \rho_+\theta(x)$ is the mean-initial step profile. The condition at $t = 0$ expresses the fact that the initial profile fluctuates with two-sided Bernoulli measure.

Knowing the optimal profile ρ^* solving this system, the CGF will be obtained from

$$\sqrt{T}\frac{d\mu}{d\lambda} = N(X,T) = \int_X^\infty \rho^*(x,T) - \int_0^\infty \rho^*(x,0)dx$$

• The MFT equations describe the non-equilibrium behaviour of many diffusive interacting particle systems (dynamical transitions, shocks...).

• Mathematical/Numerical difficulties : well-posedness; non-local boundary conditions.

• Time-dependent equations were solved only in noninteracting case and for years no analytic time-dependent solutions of these coupled PDE's were known.

• Recently, several exact results for closely related problems of optimal fluctuation paths have appeared: Krajenbrink and Le Doussal (weak-noise KPZ); Bettelheim, Smith and Meerson (KMP); Grabsch, Poncet, Rizkallah, Illien and Bénichou (Single Files) and Moriya-M-Sasamoto (SEP).

SOLVING THE MFT BY

INVERSE SCATTERING

The key observation is that the MFT equations for SEP are a *classically integrable Hamiltonian system, in the sense of Liouville (i.e. they have a Lax pair in modern setting).*

Reflecting Brownian Motions: Cole-Hopf mapping

In the limiting case of very low density, the simple exclusion process reduces to a system of Brownian Motions with specular reflection (RBM) and the MFT equations read:

$$\begin{aligned} \partial_t \rho &= \partial_x \left[\partial_x \rho - 2\rho \partial_x H\right] \\ \partial_t H &= -\partial_{xx} H - (\partial_x H)^2 \end{aligned}$$

These equations are solved by mapping them to two decoupled heat equations thanks to the Cole-Hopf transformation:

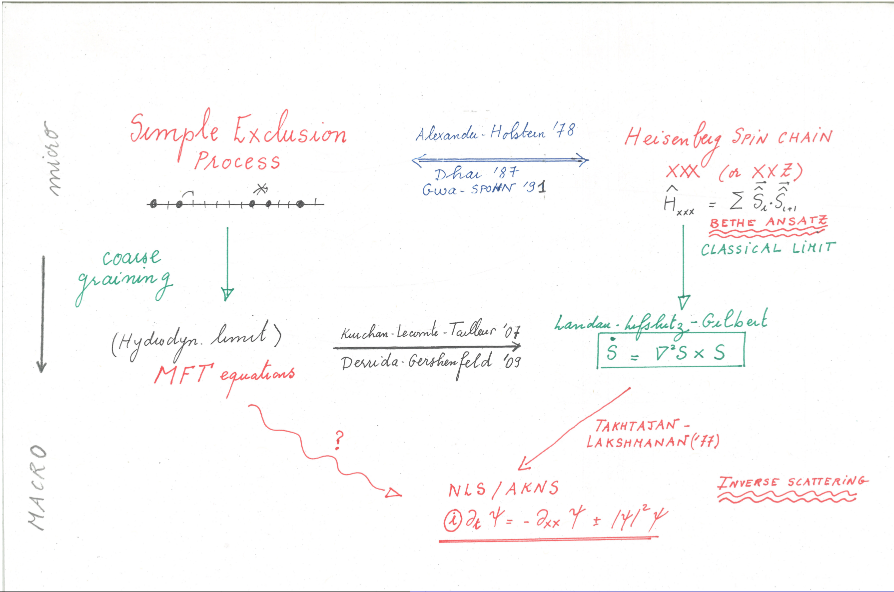
$$\begin{aligned} u(x,t) &= \rho \mathrm{e}^{-H} \\ v(x,t) &= \mathrm{e}^{H} \end{aligned}$$

In these new variables, the above equations become

$$\begin{aligned} \partial_t u &= \partial_{xx} u \\ \partial_t v &= -\partial_{xx} v \end{aligned}$$

The particles are in fact non-interacting: this is a “free” model.

SOLVING MFT: A chart of models



A generalization of the Cole-Hopf mapping

The following novel non-local transformation

$$u(x,t) = \left(\frac{\partial \rho}{\partial x} - \rho(1-\rho)\frac{\partial H}{\partial x}\right) \exp\left[-\int_{-\infty}^{x} dy (1-2\rho)\partial_y H\right],$$
$$v(x,t) = -\frac{\partial H}{\partial x} \exp\left[\int_{-\infty}^{x} dy (1-2\rho)\partial_y H\right]$$

maps the MFT to the *Ablowitz-Kaup-Newell-Segur (AKNS) system:*

$$\partial_t u(x,t) = \partial_{xx} u(x,t) - 2u(x,t)^2 v(x,t)$$
$$\partial_t v(x,t) = -\partial_{xx} v(x,t) + 2u(x,t) v(x,t)^2$$

The boundary conditions transform also well (still non-local in time):

$u(x,0) = \omega\delta(x)$ and $v(x,T) = \delta(x)$

with $\omega = (e^{\lambda} - 1)\rho_-(1-\rho_+) + (e^{-\lambda} - 1)\rho_+(1-\rho_-)$

Classical Integrability

The AKNS equations can be viewed as an 'imaginary time' analog to the Non-Linear Schrödinger (NLS) equation ($t \to it$, $\psi \to u$ and $\psi^* \to v$).

$$i\partial_t \psi = -\partial_{xx}\psi + 2|\psi|^2\psi$$

It is known that NLS is an integrable PDE.

The AKNS equations have an infinite number of conserved quantities in involution. They are classically integrable in the sense of Liouville.

The AKNS equations can be solved by using the Inverse Scattering Theory.

Classical Integrability I: Lax Pair

Consider the following auxiliary linear problem:

$$\begin{cases} \frac{\partial}{\partial x}\Psi(x,t) & = U(x,t;k)\Psi(x,t) \\ \frac{\partial}{\partial t}\Psi(x,t) & = V(x,t;k)\Psi(x,t) \end{cases}$$

with $\Psi^T(x,t) = (\psi_1(x,t), \psi_2(x,t))$; $U(x,t)$ and $V(x,t)$ are the matrices:

$$U = \begin{pmatrix} -ik & v(x,t) \\ u(x,t) & ik \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 2k^2 + uv & 2ik\, v - \partial_x v \\ 2ik\, u + \partial_x u & -2k^2 - uv \end{pmatrix}$$

The compatibility of these equations, $\partial_t \partial_x \Psi = \partial_x \partial_t \Psi$, is ensured by the zero curvature condition:

$$\boxed{\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U,V] = 0}$$

This condition is ensured if the functions u and v satisfy the AKNS system.

Classical Integrability II: Scattering

The first equation of the pair reads, in components:

$$\begin{cases} \frac{\partial}{\partial x}\psi_1(x,t) & = -ik\psi_1 + v(x,t)\psi_2 \\ \frac{\partial}{\partial x}\psi_2(x,t) & = u(x,t)\psi_1 + ik\psi_2 \end{cases}$$

This is a linear scattering problem on $\mathbb{R}$, for any given value of the time t, in which $u(x,t)$ and $v(x,t)$ that solve AKNS appear as potentials.

Because these potentials vanish at infinity, asymptotic states are well-defined: ψ_1 and ψ_2 behave as plane waves at $x = \pm\infty$.

Therefore, incoming/outgoing plane waves from $x \to -\infty$

$$\phi(x;k) \sim \begin{pmatrix} e^{-ikx} \\ 0 \end{pmatrix} \qquad \text{and} \qquad \bar{\phi}(x;k) \sim -\begin{pmatrix} 0 \\ e^{ikx} \end{pmatrix}$$

will scatter at $x \to +\infty$ as follows

$$\phi(x;k) \sim \begin{pmatrix} a(k,t)e^{-ikx} \\ b(k,t)e^{ikx} \end{pmatrix} \quad \text{and} \quad \bar{\phi}(x;k) \sim \begin{pmatrix} \bar{b}(k,t)e^{-ikx} \\ -\bar{a}(k,t)e^{ikx} \end{pmatrix}$$

The functions $a, \bar{a}, b, \bar{b}$ are the scattering amplitudes associated to this (Dirac) scattering process.

Evolution of the scattering data

$x = -\infty$ $\qquad\qquad\qquad\qquad$ $x = +\infty$

$t = 0$ $\qquad \phi \sim \begin{pmatrix} e^{-ikx} \\ 0 \end{pmatrix}$ $\quad \xrightarrow[\text{at } t=0]{\text{SCATTERING}} \quad$ $\begin{pmatrix} a(k,0)\, e^{-ikx} \\ b(k,0)\, e^{+ikx} \end{pmatrix}$

time EVOLUTION $\quad V_{-\infty} \simeq \begin{pmatrix} 2k^2 & 0 \\ 0 & -2k^2 \end{pmatrix}$ $\downarrow$ $\qquad\qquad$ $\downarrow$ time evolution with $V_{+\infty}$

$$\begin{pmatrix} e^{2k^2 t}\, a(k,0)\, e^{-ikx} \\ e^{-2k^2 t}\, b(k,0)\, e^{+ikx} \end{pmatrix}$$

$t = t$ $\qquad e^{2k^2 t} \begin{pmatrix} e^{-ikx} \\ 0 \end{pmatrix}$ $\quad \xrightarrow[\text{at } t=t]{\text{Scattering}} \quad$ $\equiv e^{2k^2 t} \begin{pmatrix} a(k,t)\, e^{-ikx} \\ b(k,t)\, e^{+ikx} \end{pmatrix}$

$$\begin{cases} a(k,0) = a(k,t) \\ b(k,t) = b(k,0)\, e^{-4k^2 t} \end{cases}$$

action / angle variables

Classical Integrability III: Diagonalization

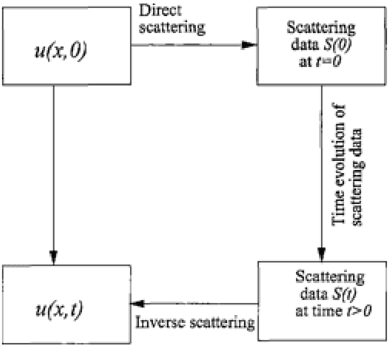
Using the second equation of the Lax pair, which describes the time dynamics of Ψ and the asymptotic plane-wave expressions, the time evolution of the scattering amplitudes is obtained explicitly:

$$a(k,t) = a(k,0), \quad b(k,t) = b(k,0)e^{-4k^2 t}$$
$$\bar{a}(k,t) = \bar{a}(k,0), \quad \bar{b}(k,t) = \bar{b}(k,0)e^{4k^2 t}$$

Key feature: **The dynamics drastically simplifies in terms of the scattering amplitudes.** (The scattering amplitudes are the action-angle variables of the dynamics.)

If we know the scattering amplitudes at initial time, they are determined at all times. Then, the potentials $u(x,t)$ and $v(x,t)$ can be reconstructed at any time by the inverse-scattering procedure (Gelfand-Levitan-Marchenko).

ISM as a non-linear Fourier Transform



The Inverse Scattering Method can be viewed as a Non-Linear Fourier Transform that diagonalizes the evolution of integrable non-linear equations.

We shall explain in the following that the MFT equations for SEP can be analyzed by the ISM scheme.

Solving MFT by Inverse Scattering

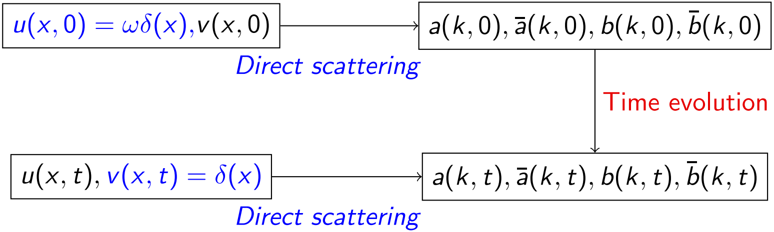
We wish to apply ISM to Simple Exclusion (MFT). However, we have non-local boundary conditions (*not* Cauchy initial conditions):

$u(x,0) = \omega\delta(x)$ but $v(x,0)$ is unknown

$v(x,T) = \delta(x)$ but $u(x,T)$ is unknown

1. Calculate implicitly the scattering amplitudes both at $t = 0$ and $t = T$ in terms of the unknown potentials $v(x,0)$ and $u(x,T)$, knowing that the other potential is a Dirac function.

2. Match the scattering data at initial and final times using the 'trivial' action-angle dynamics (integrability).



The scattering amplitudes

The Dirac scattering problem at initial time with $u(x,0) = \omega\delta(x)$ and $v(x,0)$ unspecified is elementary to solve. One finds (in terms of the half Fourier-transforms of v):

$$a(k,0) = 1 + \omega\hat{v}_+(k), \quad b(k,0) = \omega$$
$$\bar{a}(k,0) = 1 + \omega\hat{v}_-(k), \quad \bar{b}(k,0) = -\left[\hat{v}(k) + \omega\hat{v}_+(k)\hat{v}_-(k)\right]$$

Similarly, the Dirac scattering problem at final time with $u(x,T)$ unknown and with $v(x,T) = \delta(x)$ gives

$$a(k,T) = 1 + \hat{u}_+(k), \quad b(k,T) = (\hat{u}(k) + \hat{u}_+(k)\hat{u}_-(k))e^{-2ikX}$$
$$\bar{a}(k,T) = 1 + \hat{u}_-(k), \quad \bar{b}(k,T) = -e^{-2ikX}$$

From the simple evolution of the scattering data, we deduce that $\hat{u}_\pm = \omega\hat{v}_\pm$ and

$$\hat{u}(k) + \hat{u}_+(k)\hat{u}_-(k) = \omega e^{-4k^2T+2ikX}$$

Equation for the density profile

Hence, we have shown that the half Fourier transform of the final profile

$$\hat{u}_{\pm}(k) = \int_{\mathbb{R}_{\mp}} u(x+X, T) e^{-2ikx} dx$$

satisfies a scalar Riemann–Hilbert factorization problem:

$$(\hat{u}_{+}(k) + 1)(\hat{u}_{-}(k) + 1) = 1 + \omega e^{-4k^2 T + 2ikX}$$

where $1 + \hat{u}_{\pm}$ is analytic on the upper (respectively lower) complex plane, with a given product along $\mathbb{R}$.

This Riemann–Hilbert problem is solved by using the Cauchy Formula (after taking logarithms) and we obtain:

$$\hat{u}_{\pm}(k) + 1 = \exp\left[-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-\omega e^{-4k^2 T + 2ikX})^n}{n}\mathrm{erfc}(\mp i\sqrt{4nT}(k - \frac{iX}{4T})\right]$$

Cumulant Generating Function of the current

Calculating the height $N(X, T)$ from the optimal profiles at $t = 0$ and $t = T$ yields its Cumulant Generating Function (CGF).
In the long time limit, $\langle e^{\lambda N(X,T)} \rangle \simeq e^{\sqrt{T}\mu(\xi,\lambda)}$, with

$$\mu(\xi, \lambda) = \sum_{n=1}^{\infty} \frac{(-\omega)^n}{n^{3/2}} A(\sqrt{n}\,\xi) + \xi \log \frac{1 + \rho_+(e^{\lambda} - 1)}{1 + \rho_-(e^{-\lambda} - 1)}$$

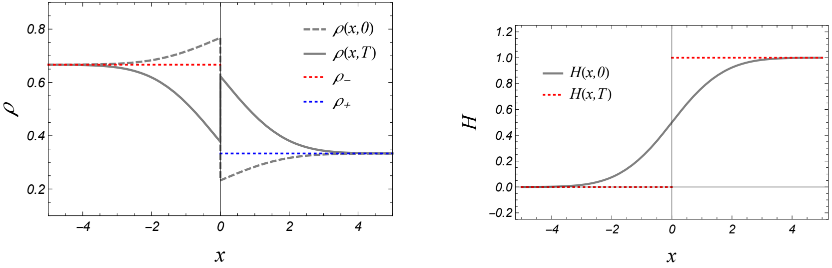
with $\mathrm{A(u)} = \xi + \int_{\xi}^{\infty} \mathrm{erfc}\,(\mathrm{u})\mathrm{du}$ and $\omega = (\mathrm{e}^{\lambda} - 1)\rho_-(1 - \rho_+) + (\mathrm{e}^{-\lambda} - 1)\rho_+(1 - \rho_-)$

This is the same formula that was obtained by previously using Integrable Probabilities, *it is now derived directly at the macroscopic level*.

Optimal profiles describing the dynamical evolution that generates a given fluctuation (rare event) that were out of reach by the microscopic techniques are now found by solving the MFT at hydrodynamic scale.

Optimal Profiles and Control Fields

Gathering all the pieces and going back to the variables of the original MFT equations, explicit formulas for the optimal fields (ρ^*, H^*) are obtained.



Optimal profiles for the total current at $X = 0$ of ρ (left) and H (right) at $t = 0$ and at $t = T$, with $\rho_+ = 1/3$, $\rho_- = 2/3$, $\lambda = 1$ and $T = 1$.

Conclusions

A major challenge in non-equilibrium physics is to determine the large deviations, considered to be the relevant generalizations of the thermodynamic potentials (Free Energy) far from equilibrium.

Interacting particle processes (such as the exclusion process) are ideal toy-models to investigate these questions with a large variety of methods:

• Microscopic scale: Combinatorics, Matrix representation, Bethe Ansatz, Integrable Probabilities...

• Coarse-grained level: hydrodynamic limits, fluctuating hydrodynamics (SPDE), Macroscopic Fluctuation Theory for optimal paths (PDE)...

Finding explicit time-dependent solutions of the MFT has been a challenge since this theory was proposed (2001). Very recently, several exact results appeared: Krajenbrink-Le Doussal (weak-noise KPZ); Bettelheim-Smith-Meerson (KMP); Grabsch et al. and MMS (SEP).

These exact results are based on the Inverse Scattering Method, originally developed to study non-linear dispersive hydrodynamics.

Applications of the ISM to non-equilibrium statistical mechanics seems very promising. The relation between microscopic and macroscopic integrability is very intriguing.