A bridge between Schramm-Loewner Evolutions and Random Matrix Theory

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STOCHASTIC EQUATIONS AND PARTICLE SYSTEMS

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Planar Statistical Physics. What is the scaling limit?

- A sample of a planar Loop-Erased Random Walk:

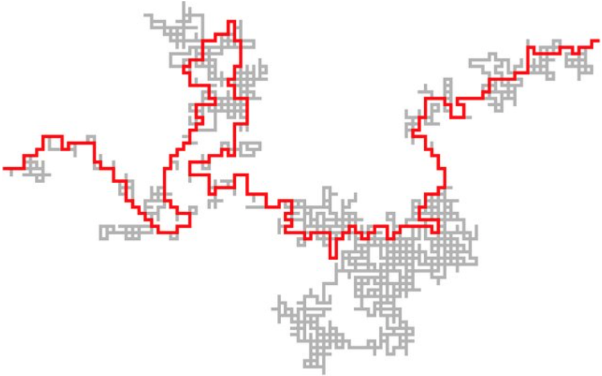
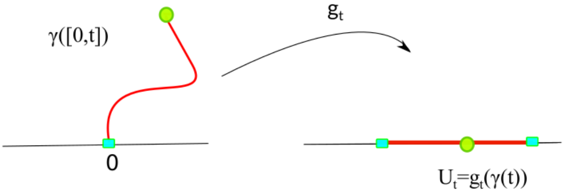


Figure: Credit to E. Peltola, A. Karrila, K. Kytola.

Conformal maps and the Loewner equation

- In general, for a non-self crossing curve $\gamma(t) : [0, \infty) \to \bar{\mathbb{H}}$ with $\gamma(0) = 0$ and $\gamma(\infty) = \infty$, we consider the simply connected domain $\mathbb{H} \setminus \gamma([0, t])$.



- Using the Riemann Mapping Theorem for the simply connected domain $\mathbb{H} \setminus \gamma([0, t])$, we have a three parameter family of conformal maps $g_t : \mathbb{H} \setminus \gamma([0, t]) \to \mathbb{H}$.
- Loewner Equation encodes the dependence between the evolution of the maps g_t when the curve $\gamma([0, t])$ grows.

Description of the conformal maps

- For $g_t(z)$, as $|z| \to \infty$ we use the normalization:

$$g_t(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots$$

- We take $b_0 = 0$.
- The coefficient $b_1 = b_1(\gamma([0,t]))$ is called the *half-plane capacity* of $\gamma(t)$ and is proved to be an additive, continous and increasing function. Hence, by reparametrizing the curve $\gamma(t)$ such that $b_1(\gamma([0,t])) = 2t$, we obtain

$$g_t(z) = z + \frac{2t}{z} + \dots$$

- The maps satisfy the Loewner Differential Equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Definition of SLE

Definition

Let B_t be a standard real Brownian motion starting from 0. The chordal SLE(κ) is defined as the law on curves induced by the solution to the following ordinary differential equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}\,, \qquad g_0(z) = z\,.$$

- SLE trace: $\gamma(t) = \lim_{y \to 0+} g_t^{-1}\left(\sqrt{\kappa} B_t + iy\right)$.
  [Rohde-Schramm ($\kappa \neq 8$), Lawler-Schramm-Werner($\kappa = 8$)]

Simulations of the SLE traces obtained with our algorithm

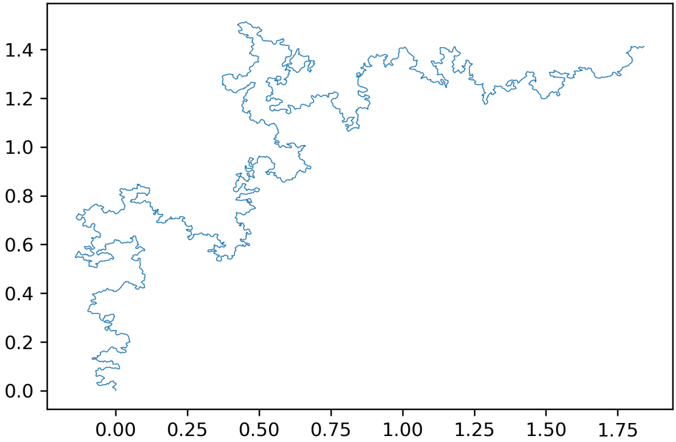
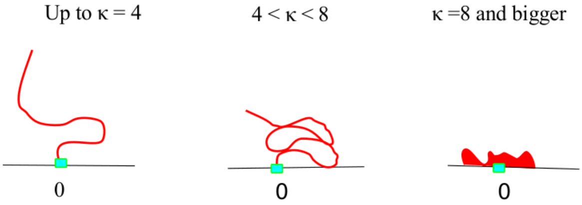


Figure: A sample of SLE_κ trace for $\kappa = 8/3$ obtained with our code.

SLE phase transitions

- It is proved that there are two phase transitions when κ varies between 0 and ∞.



Multiple SLE model

- **Multiple SLE:** Alberts, Ang, Bauer, Beffara, Bernard, Binder, Byun, Cardy, Del Monaco, Dubédat, Duplantier, Hotta, Izyurov, Kang, Karrila, Katori, Koshida, Kozdron, Kytölä, Lawler, Lenells, Makarov, Miller, Olsiewski Healey, Peltola, Schleissinger, Sun, Viklund, Wang, Wu, Yu, Zhan, ...
- **Brownian Motion** $\rightarrow$ **Dyson Brownian Motion**(DBM), $\beta = 8/\kappa$.

$$d\lambda_t^i = \frac{\sqrt{2}}{\sqrt{N\beta}} dB_t^i + \frac{1}{N} \sum_{j \neq i} \frac{dt}{\lambda_t^j - \lambda_t^i}, i = 1, \cdots, N.$$

- **Multiple SLE** simultaneous growth

$$\partial_t g_t^N(z) = \frac{1}{N} \sum_{j=1}^{N} \frac{2}{g_t^N(z) - \lambda_t^j}.$$

RMT and SLE

Theorem (Campbell-Luh-M, RMTA 2025.)

Let $\beta = 1$ or $\beta = 2$, and let K_T^N be the multiple SLE hull with N drivers at time $T > 0$. Then, for any $\epsilon > 0$ and compact $G \subset \mathbb{H} \setminus \cup_N K_T^N$, for the multiple SLE maps with N-drivers, we have that

$$\sup_{t \in [0,T],\ z \in G} |g_t^N(z) - g_t^\infty(z)| = O_{G,T}\left(\frac{1}{N^{1/3-\epsilon}}\right),$$

with overwhelming probability.

- E holds with o.p. if, for every $p > 0, \mathbb{P}(E) \geq 1 - O\left(N^{-p}\right)$.
- Uses modern RMT techniques such as Stieltjes transforms, self-consistent equations, etc.

Idea of the proof of the result

First, Del Monaco and Schleissinger:

$$\frac{\partial}{\partial t} g_t^{\infty}(z) = M_t^{\infty}(g_t(z)), \quad g_0(z) = z,$$

where M_t^{∞} is a solution to the complex Burgers equation

$$\begin{cases} \frac{\partial M_t^{\infty}(z)}{\partial t} = -2M_t^{\infty}(z)\frac{\partial M_t^{\infty}(z)}{\partial z}, t > 0 \\ M_0^{\infty}(z) = \int_{\mathbb{R}} \frac{2}{z-x} d\mu_0(x) \end{cases}$$

- Let $M_t^N(\bullet) = \frac{1}{N}\sum_{j=1}^{N} \frac{2}{\bullet - \lambda_t^j}$.
- Let us consider the time interval $[0, 1]$ and a uniform partition with $t_k = \frac{k}{n}, k = 0, 1, \dots, n$. Let $t \in (t_1, t_2)$.
- The proof is based on controlling $\sup_{\bullet \in G} \left| M_t^{\infty}(\bullet) - M_t^N(\bullet) \right| \leq \sup_{\bullet \in G} \left| M_t^{\infty}(\bullet) - M_{t_1}^{\infty}(\bullet) \right| + \sup_{\bullet \in G} \left| M_{t_1}^{\infty}(\bullet) - M_{t_1}^N(\bullet) \right| + \sup_{\bullet \in G} \left| M_{t_1}^N(\bullet) - M_t^N(\bullet) \right|$, for $G \subset \mathbb{H}$.

Idea of the proof of the result

- Let $\beta = 1$. We have that $M_t^N(\bullet) = \frac{-2}{N} \operatorname{tr} \left(A_t - \bullet \cdot I\right)^{-1}$, where $A_t = \sqrt{t}A$, with A a GOE.
- For M_t^∞, we have $M_t^\infty(\bullet) - S^N\left(z - 2tM_t^\infty(\bullet)\right) = 0$, with $S^N(\bullet) = -\frac{2}{N} \operatorname{tr} Q(\bullet)$, where $Q(\bullet)$ is a certain resolvent matrix.
- By a union bound, we have that, for any $\epsilon > 0$

$$
\begin{aligned}
&\mathbb{P}\left(\bigcup_{t_k} \left| M_{t_k}^\infty(\bullet) - M_{t_k}^N(\bullet) \right| = \Omega\left(\frac{C}{N^{1/3-\epsilon}}\right)\right) \\
&\leq \sum_{k=1}^{n} \mathbb{P}\left(\left| M_{t_k}^\infty(\bullet) - M_{t_k}^N(\bullet) \right| = \Omega\left(\frac{C}{N^{1/3-\epsilon}}\right)\right) \\
&\leq ne^{-CN}.
\end{aligned}
$$

- Note $g = \Omega(f) \leftrightarrow f = O(g)$.

Summary of the toolbox between RMT and SLE

$$\partial_t g_t(z) = \frac{1}{N}\sum_{j=1}^{N}\frac{2}{g_t(z)-\lambda_t^j}.$$

- Rewrite RHS of Multiple Loewner equation as $M_t^N(\bullet) = \frac{1}{N}\sum_{j=1}^{N}\frac{2}{\bullet-\lambda_t^j}$.
- Recognize Stieltjes transform for a given choice of measure.
- **Let** $\beta = 1$. **We have that** $M_t^N(\bullet) = \frac{-2}{N}\,\mathrm{tr}\left(A_t - \bullet\cdot I\right)^{-1}$, **where** $A_t = \sqrt{t}A$, **with** A **a GOE**
- RHS of Loewner can be recovered as a Random Matrix Theory object.
- **The critical parameters** $\kappa = 4$ **and** $\kappa = 8$ **in the SLE theory correspond to the 'nice'** $\beta = 2$ **and** $\beta = 1$ **parameters in the Random Matrix Theory.**

The result presented today is a first application of this toolbox.

Future directions

Medium term goals: Analysis and Geometry of the Multiple SLE curves; Study of the fixed N ($N=2$ is nicer), asymptotic case, etc. Approximations schemes (**working on this over the summer REU/G**).

- Interplay between **RMT/DBM** (gaps, etc.) and **Loewner theory**.

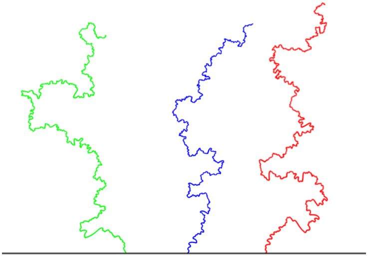


Figure: Multiple SLE for N=3. Credits to K. Luh

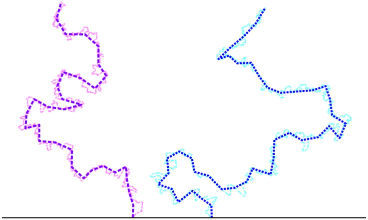


Figure: Approximation Scheme. Credits to K. Luh

Thank you very much for your attention!