Discrete Whittaker processes

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Stochastic Equations and Particle Systems

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Reverse plane partitions

$$\pi = (\pi_{ij} \, , \, (i,j) \in \lambda) \in \mathbb{Z}_+^{\lambda}$$

$$\pi_{ij} \geq \pi_{i,j-1} \vee \pi_{i-1,j}$$

Convention : $\pi_{i0} = \pi_{0j} = 0$

Reverse plane partition of shape 3211

0	1	1
2	4	
2		
3		

$\mathrm{RPP}(\lambda)$ = RPP's of shape λ

Markov chain on RPP(λ)

Subtract one from π_{ij} at rate

$$b_{ij}(\pi) = (\pi_{ij} - \pi_{i-1,j})(\pi_{ij} - \pi_{i,j-1})$$

conv. $\pi_{i0} = \pi_{0j} = 0$

Generator: $\quad G^\lambda = \sum_{(i,j)\in\pi} b_{ij}(\pi)\, D_{\pi_{ij}}$

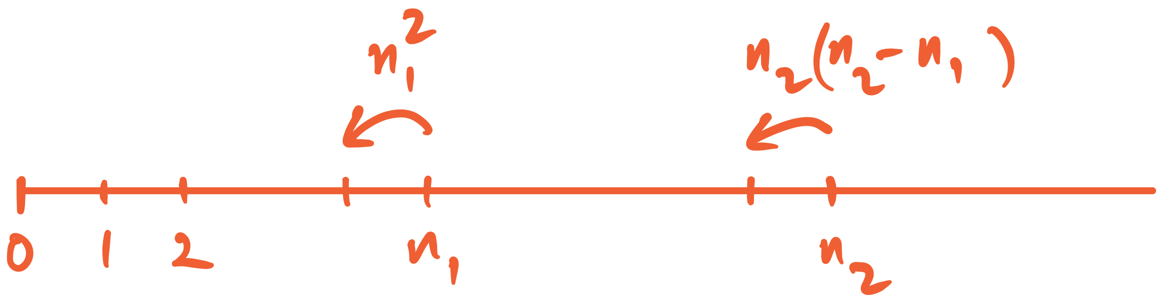
$$D_n f(n) = f(n-1) - f(n)$$

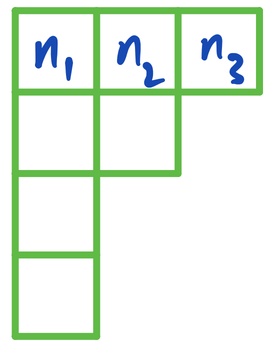
Some properties

1. Nested: if $\pi \sim G^\lambda$ and $\mu \subset \lambda$,

   then $\pi|_\mu \sim G^\mu$.

   In particular, first row $n_j = \pi_{1j}$:



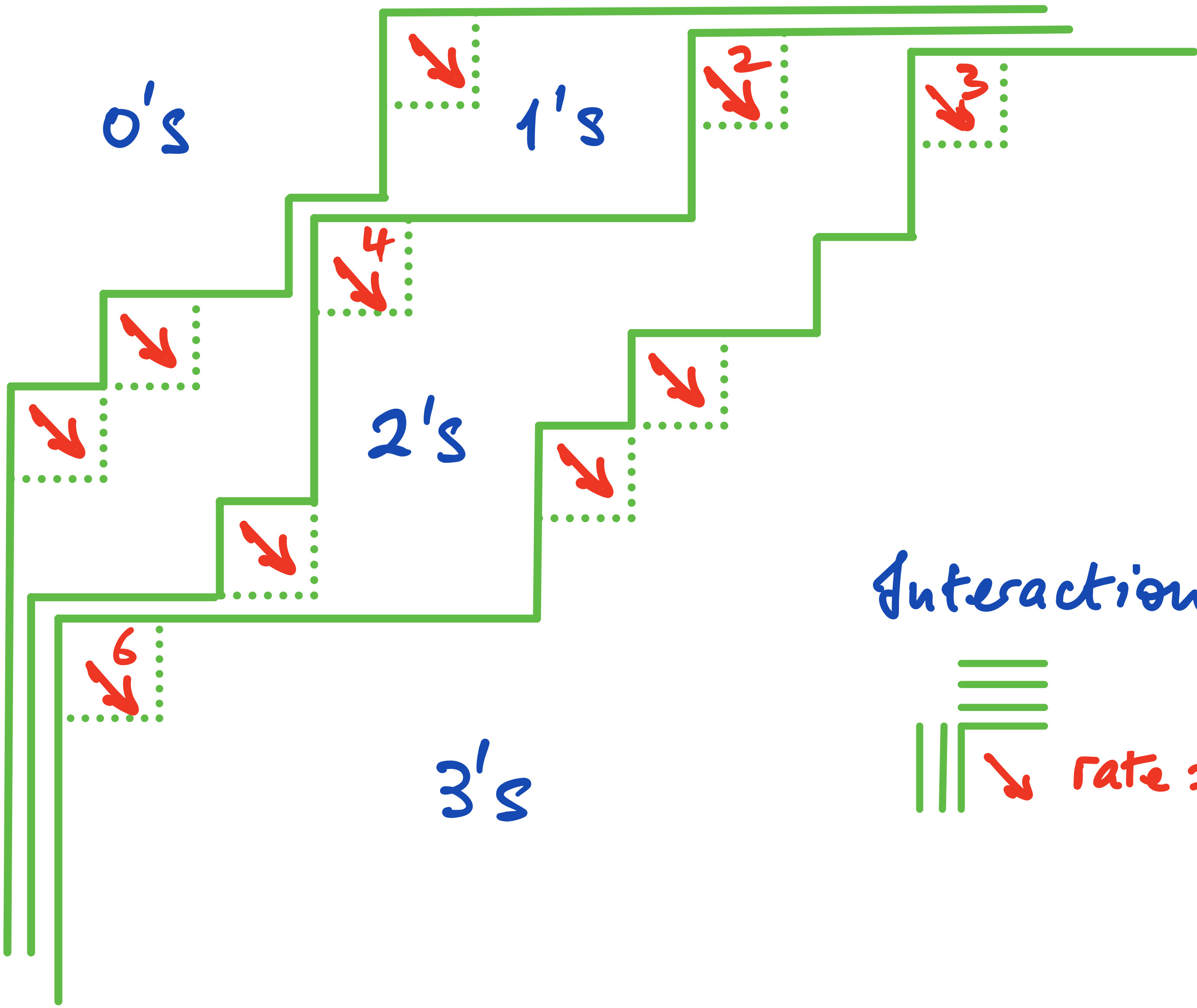


2. Theorem: Unique entrance law from

   $\pi_{ij} = +\infty \quad \forall\, i,j \in \lambda$

3. Can take $\lambda = \mathbb{N}^2$

As a system of interacting corner growth processes



Shape $\lambda = \mathbb{N}^2$

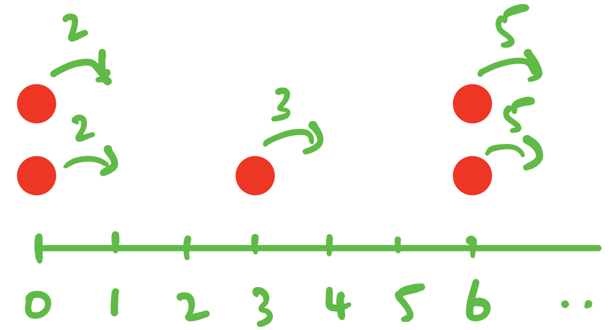
Interaction :

↘ rate $= 3 \times 4 = 12$

Discrete (repulsive) δ-Bose gas

First row (or column):

Independent Poisson processes with rank dependent rates



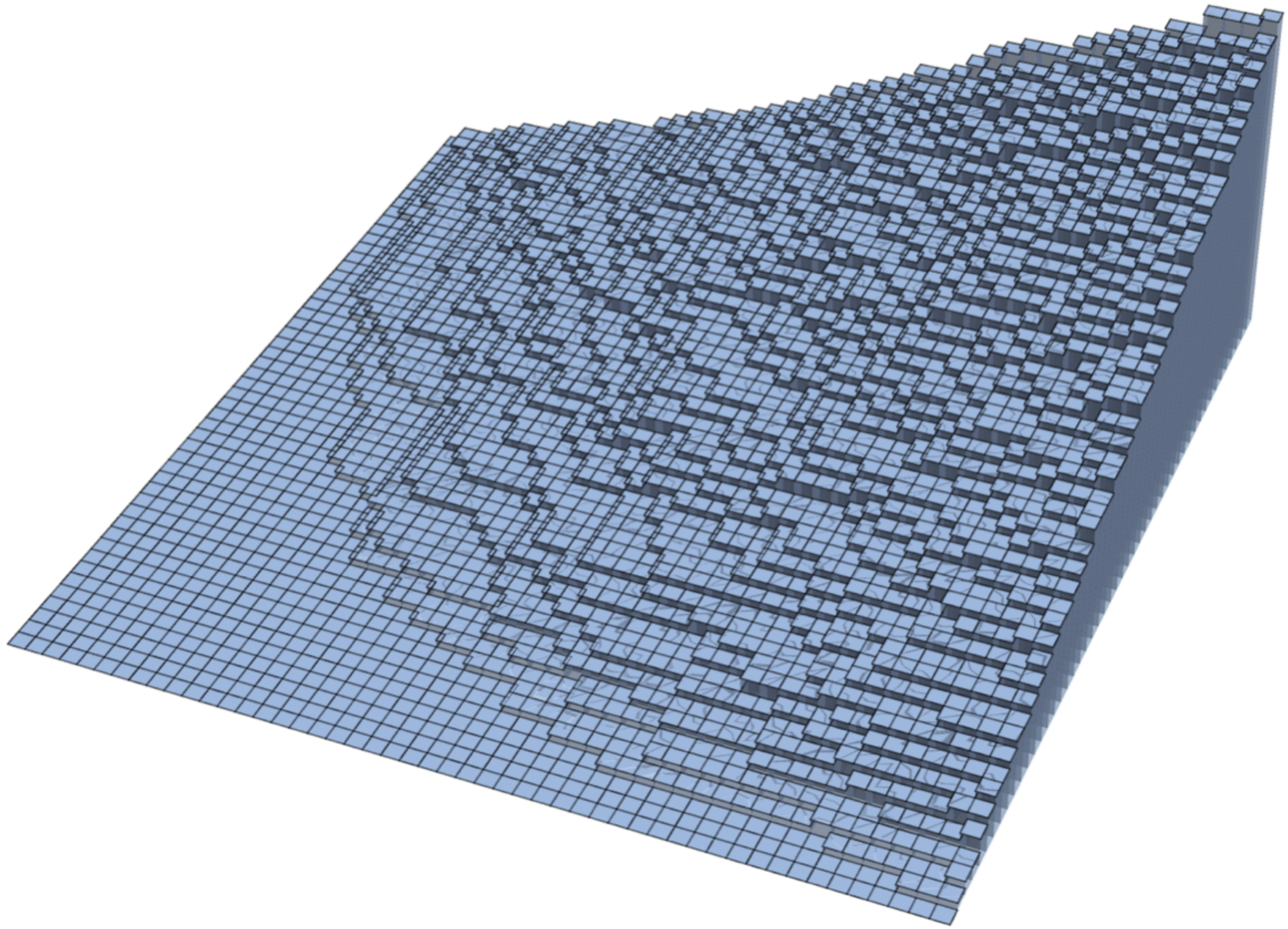
Fwd diff. op.

Doob transform of $\sum B_{x_i} - \sum_{i \neq j} \mathbb{1}_{x_i = x_j}$

via $\varphi(x) = \prod_i i^{x_{(i)}}$

$x_{(1)} \leq x_{(2)} \leq \cdots$

Simulation

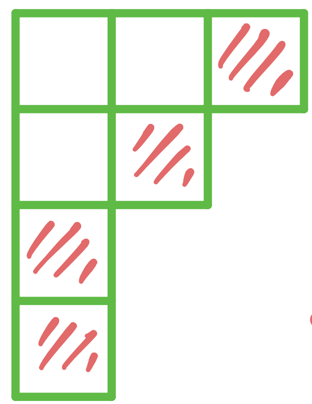


Restrictions to skew-shapes

Let $\mu \subset \lambda^0 = \{(i,j)\in\lambda : (i+1,j),(i,j+1)\in\lambda\}$

e.g. $\lambda = 3211$,

$\mu = \lambda^0 = 21$



λ/μ

Thm If initial law chosen correctly and $\pi \sim G^\lambda$ then $\pi|_{\lambda/\mu}$ is Markov.

Example

$\lambda = 21, \quad \mu = 1$

$$\begin{array}{|c|c|}\hline \pi_{11} & \pi_{12} \\ \hline \pi_{21} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline k & n \\ \hline m & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

$$G = k^2 D_k + n(n-k) D_n + m(m-k) D_m$$

Suppose that, initially, $\pi_{12} = n$, $\pi_{21} = m$

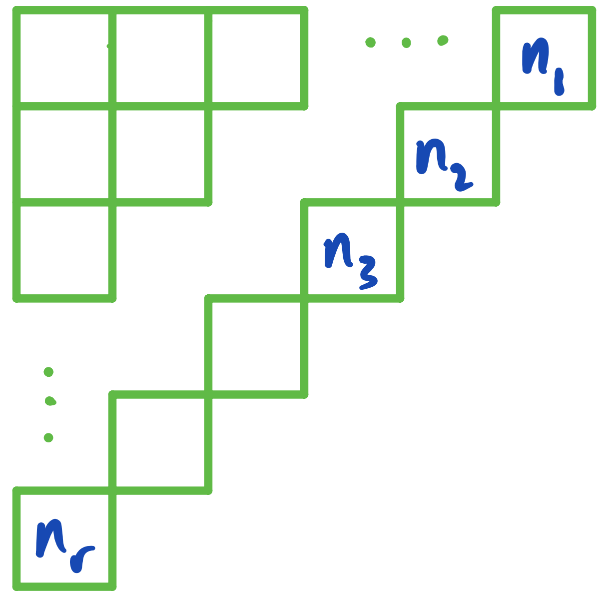
and $\pi_{11} \sim \binom{n+m}{n}^{-1} \binom{n}{k} \binom{m}{k}$.

Then, if $\pi \sim G$, $\pi|_{\lambda/\mu} \equiv (\pi_{12}, \pi_{21})$ is Markov w/ gen. $L = \frac{n^3}{n+m} D_n + \frac{m^3}{n+m} D_m$.

Staircase shapes

$\lambda = \delta_{r+1} = (r, r-1, \dots, 1)$

$\mu = \delta_r \qquad \pi|_{\lambda/\mu} \in \mathbb{Z}_+^r .$



$$A_r(n) = \sum_{\substack{\pi \in RPP(\lambda) \\ \pi|_{\lambda/\mu} = n}}{}' \; \underbrace{\prod_{(i,j)\in\delta_r} \binom{\pi_{i+1,j}}{\pi_{ij}} \binom{\pi_{i,j+1}}{\pi_{ij}}}_{W_r(\pi)}$$

Theorem Suppose that, initially, either

$$\pi|_{\partial/\mu} = n \quad \text{and} \quad \pi \sim A_r(n)^{-1} W_r(\pi)$$

$$\text{or} \quad \pi_{ij} = +\infty \quad \forall\, i,j \in \partial .$$

Then, if $\pi \sim G^{\partial}$, $\pi|_{\partial/\mu}$ is a Markov chain on $\mathbb{Z}_+^r$ with generator

$$L^r = \sum_{i=1}^{r}{}' \frac{A_r(n-e_i)}{A_r(n)} n_i^2 D_{n_i}$$

$$\equiv A_r(n)^{-1} \circ H^r \circ A_r(n), \quad \text{where} \ldots$$

Fundamental Whittaker Functions

$$\phi_r(x) = \sum_{n \in \mathbb{Z}_+^r} \frac{A_r(n)}{\prod_{i=1}^r n_i!^2} \prod_{i=1}^r e^{n_i(x_i - x_{i+1})}$$

satisfies $\mathcal{H}^r \phi_r = 0$ where

$$\mathcal{H}^r = -\frac{1}{2} \sum_{i=1}^{r+1} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^r e^{x_i - x_{i+1}}$$

Equivalently, $H^r A_r = 0$ where

$$H^r = \sum_{i=1}^r n_i^2 D_{n_i} + \sum_{i=1}^{r-1} n_i n_{i+1} .$$

Hashizume (1982), Ishii-Stade (2007)

Givental (1997), Batyrev et al (2000)

Absorption times $r = 1, 2$

$r = 1$, $L = n^2 D_n$

Absorption time from n: $\tau_n = \sum_{k=1}^{n} e_{k^2}$

$r = 2$, $L = \frac{n^3}{n+m} D_n + \frac{m^3}{n+m} D_m$

Absorption time from (n, m):

$$\tau_{n,m} \overset{d}{=} \tau_n + \tau'_m$$

indep. copy of τ_m

Hitting probabilities ($r = 2$)

primitive cube root of unity

$$S_p = -\sum_{n=1}^{\infty}{}' \frac{n^p q^n}{(1-q^n)^2} \qquad q = e^{2\pi i \omega} = -e^{-\pi\sqrt{3}}$$

Ramanujan : $S_2 = \frac{1}{24\pi^2}$, $S_4 = \frac{3\,\Gamma(1/3)^{18}}{40960\,\pi^{12}}$, ...

$$\mathbb{P}_\infty(T_{ij} < \infty) = \text{lin. comb. of } S_p\text{'s.}$$

$$\mathbb{P}_\infty(T_{01} < \infty) = 12\pi^2 S_2 = \frac{1}{2}$$

$$\mathbb{P}_\infty(T_{11} < \infty) = 24\pi^2 S_5 \simeq 0.878..$$

Imaginary exponential Fctls

$B_1, B_2, \ldots, B_{r+1}$ indep. BM's

$$Y_k(t) = e^{i(B_k(t) - B_{k+1}(t))}, \quad Z_k(t) = \int_0^t Y_k(s)\,ds \qquad k = 1, \ldots, r.$$

Transition probabilities for L^r process:

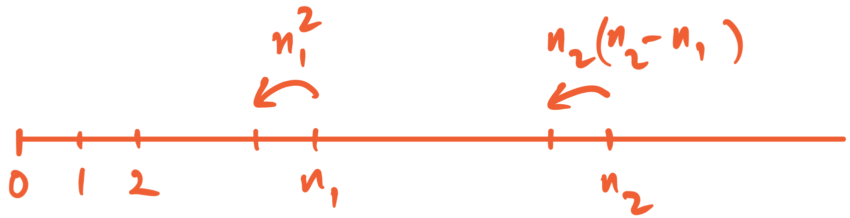
$$p_t(n,m) = \frac{n!^2}{m!^2 (n-m)!} \frac{A_r(m)}{A_r(n)} \mathbb{E}\, Y(t)^{-n} Z(t)^{n-m}$$

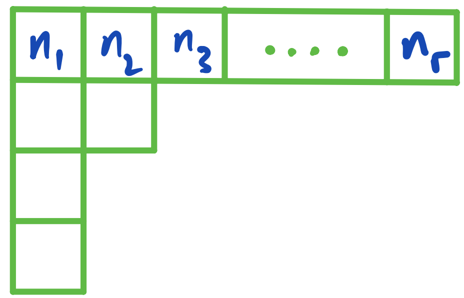
$$\mathbb{P}_n(T_0 \le t) = p_t(n,0) = \frac{n!}{A_r(n)} \mathbb{E}\, Z(t)^n.$$

$n! = n_1! \cdots n_r!$ $\quad y^n = y_1^{n_1} \cdots y_r^{n_r}$ $\quad$ etc.

Semidiscrete polymer with imag. disorder

Evolution of first row :





Transition kernel $\quad q_t(n,m) \quad n,m \in \mathbb{Z}_+^r$

$$q_t\Big(\underbrace{(n,n,\dots,n)}_{r},(0,0,\dots,0)\Big) = \frac{1}{n!^r}\, E\, u_r(t)^n$$

$$u_r(t) = \int_{0<s_1<s_2<\dots<s_r<t} y_1(s_1)\dots y_r(s_r)\, ds_1\dots ds_r$$