About some notions of irregularity with application to the enhanced dissipation of rough shear flows

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Introduction

Consider the equation

$$\begin{cases} \dot{x}_t = b(x_t) + \dot{\omega}_t, \\ x_0 = x. \end{cases} \quad \rightsquigarrow \quad \theta_t = x + \int_0^t b(\theta_s + \omega_s)\, ds,$$

with the position $\theta_t = x_t - \omega_t$. Use the **averaging operator**

$$T^{\omega}_{st} b(x) = \int_s^t b(x + \omega_r)\, dr = b \star \mu^{\omega}_{0t}(x)$$

where μ^{ω}_{0t} is the occupation measure

$$\mu^{\omega}_{st} = \int_s^t \delta_{\omega_r}\, dr$$

Davie: if b is bounded and ω is Bm, then

$$x \mapsto T^{\omega}_{0t} b$$

is (almost) Lipschitz.

[Geman, Horowitz, 1980] [Davie, 2007] [Flandoli, 2015] [Catellier, Gubinelli, 2016] [Galeati, Gubinelli, 2020]

[Galeati, Gubinelli, 2022]

A path $\omega : [0, T] \to \mathbb{R}^d$ is (ρ, γ)-**irregular** if

$$\|\mu^\omega\|_{\rho,\gamma,T} := \sup_{a \in \mathbb{R}^d} \sup_{s<t} (1 + |a|^\rho) \frac{|\widehat{\mu^\omega_{st}}|}{|t-s|^\gamma}$$

For instance, for the fractional Brownian motion of index H, and $2H\rho < 1$,

$$\|T^\omega_{st}\|_{\mathcal{L}(\mathscr{F}L^{\alpha,1}, \mathscr{F}L^{\alpha+\rho,1})} \lesssim |t-s|^\gamma$$

on the Fourier-Lebesgue spaces

$$\mathscr{F}L^{\alpha,p} = \left\{ f : \int |\widehat{f}(\xi)|^p (1 + |\xi|)^{\alpha p} \right\}$$

[Catellier, Gubinelli, 2016]

Heuristic idea:

$$\int_0^t b(\theta_s + \omega_s)\, ds \approx \sum_{i=0}^{n-1} T^{\omega}_{s_i s_{i+1}} b(\theta_{s_i}) \approx \int_0^t T^{\omega}_{ds}(\theta_s).$$

Theorem (sewing lemma)

Let $\chi : [0,T]^2 \to \mathbb{R}$ *be such that*

$$|\xi_{st} - \xi_{su} - \xi_{ut}| \lesssim |t-s|^{1+\epsilon},$$

for $s \leqslant u \leqslant t$*, then the following limit exits*

$$\lim \sum_{i=0}^{n-1} \xi_{s_i s_{i+1}}.$$

For instance, for Hölder functions f, g, set $\xi_{st} = f(s)(g(t) - g(s))$,

$$\xi_{st} - \xi_{su} - \xi_{ut} = (f(s) - f(u))(g(t) - g(u)) \lesssim |s-u|^{\alpha}|t-u|^{\beta} \lesssim |t-s|^{\alpha+\beta}$$

[Gubinelli, 2004]

Small balls notion of irregularity

For a measure μ, define

$$F_\mu(r, y) = \mu(\{x : |x - y| \leqslant r\})$$

and for a function f, $\tau_y f(x) = f(x - y)$.

Definition

For $\alpha > 0$, $1 \leqslant p, q \leqslant \infty$, and a compactly supported measure μ,

$$\|\mu\|_{\mathrm{SBE}_q^{\alpha,p}} := \|r^{-\alpha-d}\Delta_k F_\mu(r, y)\|_{L_y^q L_r^p(dy\otimes\frac{dr}{r})}$$

Here Δ_k are multiscale differences,

$$\begin{cases} \Delta_0 f(r) = f(r) - f(r/2), \\ \Delta_{k+1} f(r) = \Delta_k f(r) - 2^{k+1}\Delta_k f(r/2) \end{cases}$$

Theorem

We have that

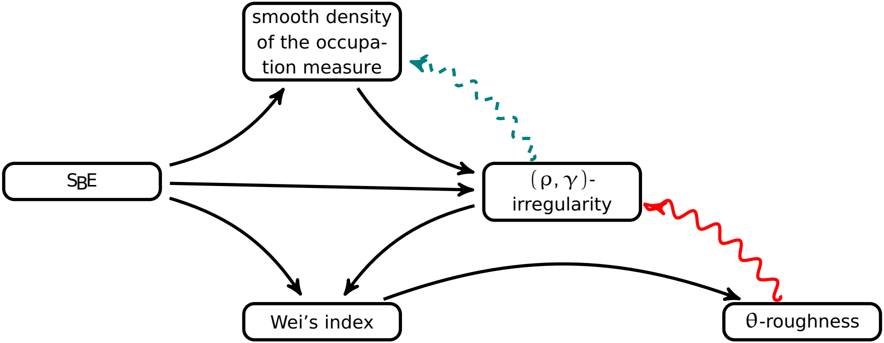
$$\|\mu\|_{B^{\alpha}_{q,\infty}} \lesssim \|\mu\|_{\mathsf{SBE}^{\alpha,p}_{q}} \lesssim \|\mu\|_{B^{\alpha}_{q,1}}$$

Moreover, for occupation measures of paths,

- stability by reparametrisation with bi-Lipschitz maps,
- stability by r-variation perturbation of the path (for suitable r),
- if the occupation measure is γ-Hölder with values in $\mathsf{SBE}^{\alpha,p}_{q}$, then ω is (α,γ)-irregular.

- if ω is a real Gaussian process with $Cov(\omega_{ts}) \gtrsim |t-s|^{2H}$, then $\mu^\omega \in$ SBE.
- more generally, if ω is Gaussian and **locally non deterministic**,
- solution of
$$dx_t = b(t, x_t)\,dt + \sigma(t, x_t)\,dB_t,$$
- fractional Brownian motion of index H
- equation driven by fractional Brownian motion,

[Berman, 1983]



- teal dashed arrow: conjecture.
- wiggled red line: the implication does not hold.

Implied arrows are not drawn.

An example: uniqueness of a 1D continuity equation

Consider

$$\begin{cases} \partial_t u + \left[b(\omega_t, x) + \dot{\omega}_t) u(t,x) \right]_x = 0, \\ u(0,\cdot) = u_0 \end{cases} \qquad x \in \mathbb{R}, t \geqslant 0,$$

with (heavy technical) assumptions on ω, on its occupation measure, and on $b \in C^{\beta}(\mathbb{R}, B^{\alpha}_{p,q})$.

Theorem

The problem

$$x_t = x_0 - \omega_t + \int_0^t b(\omega_s, x_s)\, ds$$

generates a flow ϕ of diffeomorphisms.
For every $u_0 \in L^1 \cap L^{p'}_{loc}$, there is a unique solution of the continuity equation with initial condition u_0, given by

$$u_0(\phi_t^{-1}(x)) D_x \phi_t^{-1}(x).$$

The key point is to solve the equation for characteristics $\theta_t = x_t + \omega_t$,

$$\theta_t = x_0 + \int_0^t b(\omega_s, \theta_s - \omega_s)\, ds$$

which in turns amounts to understand

$$\int_0^t f(s, \theta_s - \omega_s)\, ds$$

for a generic f. This is done through a suitable **sewing lemma**,

$$\chi_{st} = \int_s^t f(s, \theta_s - \omega_r)\, dr = f(s, \cdot) \star \mu^{\omega}_{st}(\theta_s),$$

where μ^{ω}_{st} is the occupation measure of ω on $[s, t]$.

Total enhanced dissipation for very rough shear flows

Consider the following advection-diffusion equation on the 2D torus of a **passive scalar**

$$\begin{cases} \partial_t f + u \cdot \nabla f = \nu \Delta f, \\ f(0, \cdot) = f_0, \end{cases}$$

driven by a transport velocity u with

$$\operatorname{div} u = 0.$$

Dissipation: multiply by f and integrate by parts

$$\frac{d}{dt} \|f\|_{L^2}^2 + \nu \|\nabla f\|_{L^2}^2 = 0,$$

thus

$$\|f\|_{L^2}^2 \lesssim \|f\|_{L^2}^2 \, e^{-c\nu t}$$

Enhanced dissipation with rate r,

$$\|f_t\|_{L^2}^2 \lesssim \|f_0\|_{L^2}^2 e^{-cr(\nu)t}.$$

- [Constantin et al] relaxation enhancing flows.

In particular for shear flows $u = \begin{pmatrix} u(y) \\ 0 \end{pmatrix}$

- [Bedrossian et al] $r(\nu) \sim \nu^{\frac{n}{n+2}}$ for $u \in C^{n+1}$
- [Wei] there are Weierstrass-type functions such that $r(\nu) = \nu^{\frac{\alpha}{\alpha+2}}$
- [Colombo et al] for $\alpha \in (0,1)$, **there is** $u \in C^\alpha$ such that enhanced dissipation holds with $r(\nu) = \frac{\alpha}{\alpha+2}$.
- [Coti Zelati et al] The above rates are sharp.
- [Gubinelli et al] For **generic** shear flows $u \in B^\alpha_{1,\infty}$, enhanced dissipation holds with $r(\nu) \approx \nu^{\frac{\alpha}{\alpha+2}}$.

[Constantin, Kiselev, Ryzhik, Zlatoš, 2008] [Bedrossian, Coti Zelati, 2017] [Wei, 2021]

[Colombo, Coti Zelati, Widmayer 2021] [Coti Zelati, Drivas, 2022] [Galeati, Gubinelli, 2023]

Theorem (MR, L. Roveri)

Let $\alpha \in (-\frac{1}{2}, 0)$, $u \in B^{\alpha}_{1,\infty}$ *with* $\Lambda(\alpha, 0, 2, u) > 0$ *(almost), then*

$$\|f\|^2_{L^2} \lesssim \|f\|^2_{L^2} e^{-c r(\nu) t},$$

with

$$\nu^{\frac{\alpha}{\alpha+2}} \lesssim r(\nu) \lesssim \nu^{-\frac{1/2-\alpha}{5/2-\alpha}}.$$

In particular,

- about $\alpha > -\frac{1}{2}$,
- unmatched lower and upper bound,
- instantaneous total enhanced dissipation,
- triviality of the advected (inviscid) passive scalar,

For $p \in [1, \infty)$, $\alpha \in \mathbb{R}$, $k \in \mathbb{N}$, $f \in L^p(0,1)$.

$$\Lambda(\alpha, k, p, f) = \inf_{|J| \leqslant 1, P \in \mathcal{P}_k} |J|^{-\alpha} \|f - P\|_{L^p,\text{norm}}$$

Some properties.

- $\Lambda(\alpha, k, p, f) = 0$ if $f \in B^{\alpha}_{p,q}$, $q < \infty$, or $f \in B^{\alpha'}_{p,\infty}$, $\alpha' > \alpha$,
- $\Lambda(\alpha', k, p, f) = 0$ for $\alpha' > \alpha$, for **generic** $f \in B^{\alpha}_{1,\infty}$,
- $\Lambda(\alpha, k, p, f + \varphi) \sim \Lambda(\alpha, k, f)$ for φ "smooth".
- if $\Lambda(\alpha, k, p, f) > 0$, then f is α-Hölder rough,
- if f is ρ, γ-irregular, then $\Lambda(\alpha, 0, p, f) > 0$,

Conjectures:

- $\Lambda(\alpha, k, 1, f) \sim \Lambda(\alpha, k, p, f)$ ($\lesssim$ holds),
- $\Lambda(\alpha, k, p, f) \sim \Lambda(\alpha - 1, k - 1, p, f_x)$ ($\lesssim$ holds)

Upper bound (on the L^2-norm): spectral estimates on a suitable primitive $\partial_y U$ of u, where

$$\begin{cases} -\partial_y^2 U = u - \bar{u}, \\ U \text{ periodic and mean zero,} \end{cases}$$

with $\Lambda(\alpha + 1, 1, 2, \partial_y U) > 0$.

Lower bound (on the L^2-norm) A Feynman-Kac formula depending on

$$\int_0^t u(y + \sqrt{2\nu} B_s)\, ds$$

estimated via Itō's formula (Itō's trick).