Conformally Invariant Random Geometry on Manifolds of Dimensions > 2

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Random Objects and Conformal Invariance

Basic observation: numerous beautiful, deep results on conformally invariant random objects in $n = 2$.

Question: does any of these have a counterpart in $n > 2$?

Surprising insights:

(A) Extending the groundbreaking results on

- conformally invariant Gaussian random fields,
- Liouville quantum gravity measures, and
- Polyakov-Liouville measure

to $n > 2$ relies on two properties

(i) conformally invariant energy/operator
(ii) logarithmic kernel for the operator

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(i) conformally invariant energy/operator
(ii) logarithmic kernel for the operator

(B) Both properties appear to be closely related. Indeed, (i) $\Rightarrow$ (ii).

(C) Also for $n > 2$, conformal invariance is a meaningful and powerful property which rules out all but one random objects.

Conformally Invariant Random Fields

Goal

Associate to each (M, g) a probability measure $\boldsymbol{\nu}_{M,g}$ on "fields" (continuous functions, distributions) on M such that

- $h \overset{(d)}{=} h' \circ \Phi$ if $\Phi : M \to M'$ is an isometry and h and h' are distributed according to $\boldsymbol{\nu}_{M,g}$ and $\boldsymbol{\nu}_{M',g'}$, resp.
- $\boldsymbol{\nu}_{M,g'} = \boldsymbol{\nu}_{M,g}$ if $g' = e^{2\varphi} g$ for some $\varphi \in \mathcal{C}(M)$

Assume that $\boldsymbol{\nu}_g$ is Gaussian, informally given as

$$d\boldsymbol{\nu}_g(h) = \frac{1}{Z_g} \exp\Big(-\frac{1}{2}\mathfrak{e}_g(h,h)\Big)\, dh$$

with bilinear form $\mathfrak{e}_g(u, v) = (u, Av)_{L^2}$.

Conformal Invariance Requirement

$$\mathfrak{e}_g(u,u) = \mathfrak{e}_{e^{2\varphi}g}(u,u) \qquad \forall \varphi, \ \forall u.$$

In case $n = 2$, celebrated property of the *Dirichlet energy*

$$\mathfrak{e}_g(u,u) := \int_M \left|\nabla_g u\right|^2 d\,\mathrm{vol}_g\,.$$

Conformally Invariant Random Fields

Gaussian measure $d\nu_g(h) = \frac{1}{Z_g}\exp\Big(-\frac{1}{2}\mathfrak{e}_g(h,h)\Big)\,dh$ with conformally invariant energy

$$\mathfrak{e}_g(u,u) = \mathfrak{e}_{e^{2\varphi}g}(u,u) \qquad \forall\varphi,\ \forall u.$$

In $n \neq 2$, Dirichlet energy no longer conformally invariant:

$$\mathfrak{e}_{e^{2\varphi}g}(u,u) = \int_M \left|\nabla_g u\right|^2 e^{(n-2)\varphi}\, d\,\mathrm{vol}_g\,.$$

In $n = 4$, more promising: bi-Laplacian energy

$$\tilde{\mathfrak{e}}_g(u,u) := \int_M \left(\Delta_g u\right)^2 d\,\mathrm{vol}_g\,.$$

Still not conformally invariant but close to:

$$\tilde{\mathfrak{e}}_{e^{2\varphi}g}(u,u) := \int_M \left(\Delta_g u + 2\nabla_g\varphi\,\nabla_g u\right)^2 d\,\mathrm{vol}_g = \tilde{\mathfrak{e}}_g(u,u) + \text{ low order terms.}$$

Paneitz:

$$\mathfrak{e}_g(u,u) = \int_M \left[(\Delta_g u)^2 - 2\,\mathrm{Ric}_g(\nabla_g u, \nabla_g u) + \frac{2}{3}\,\mathrm{scal}_g\cdot\left|\nabla_g u\right|^2\right] d\,\mathrm{vol}_g$$

is conformally invariant.

Energy, Operator, Kernel

Co-Polyharmonic Energy on n-Manifolds

Assume from now on that (M, g) is n-dimensional smooth, compact, connected Riemannian manifold without boundary, n even.

Integrable functions (or distributions) u on M will be called grounded if $\langle u \rangle_g := \frac{1}{vol_g(M)} \int_M u \, d\,\mathrm{vol}_g = 0$

Grounded Sobolev spaces $\mathring{H}^s(M,g) = (-\Delta_g)^{-s/2} \mathring{L}^2(M, \mathrm{vol}_g)$ for $s \in \mathbb{R}$,
usual Sobolev spaces $H^s(M,g) = (1-\Delta)^{-s/2} L^2(M, \mathrm{vol}_g) = \mathring{H}^s(M,g) \oplus \mathbb{R} \cdot \mathbf{1}$

Laplacian $-\Delta : H^s \to \mathring{H}^{s-2}$; grounded Green operator $\mathring{\mathrm{G}}_g : \mathring{H}^s \to \mathring{H}^{s+2}$.

Graham/Jenne/Mason/Sparling.

The co-polyharmonic energy

$$\mathfrak{e}_g(u,v) = c \int_M \big(-\Delta_g \big)^{n/4} u \cdot \big(-\Delta_g \big)^{n/4} v \; d\,\mathrm{vol}_g + \text{low order terms}$$

is conformally invariant. We choose $c = a_n := \frac{2}{\Gamma(n/2)\,(4\pi)^{n/2}}$.

$\mathfrak{e}_g(u,v) = \int_M p_g u \cdot v \;\, d\,\mathrm{vol}_g$ with co-polyharmonic operator

$$p_g u := c\,(-\Delta)^{n/2} u + \text{ low order terms}$$

Co-Polyharmonic Energy on n-Manifolds

Definition

The n-manifold (M, g) is called admissible if $\mathfrak{e}_g > 0$ on $\mathring{H}^{n/2}(M)$.

Large classes of n-manifolds are admissible. For instance in $n = 4$:

- all compact Einstein 4-manifolds with $\mathrm{Ric} \geq 0$ are admissible.
- all compact hyperbolic 4-manifolds with spectral gap $\lambda_1 > 2$ are admissible.

For the sequel, we always assume that (M, g) is admissible.

Two Key Properties of the Co-Polyharmonic Green Kernel

Define co-polyharmonic Green operator

$$\mathsf{k}_g := \mathsf{p}_g^{-1} : H^{-n}(M) \to \mathring{L}^2(M)$$

and associated bilinear form with domain $H^{-n/2}(M)$ by

$$\mathcal{K}_g(u,v) := \langle u, \mathsf{k}_g v\rangle_{L^2}.$$

Theorem

k_g *is an integral operator with an integral kernel* k_g *which is grounded, symmetric, and satisfies*

$$\left| k_g(x,y) - \log \frac{1}{d_g(x,y)} \right| \le C_0.$$

Theorem

Assume that $g' := e^{2\varphi} g$ *for some* $\varphi \in \mathcal{C}^\infty(M)$*. Then the co-polyharmonic Green kernel* $k_{g'}$ *for the metric* g' *is given by*

$$k_{g'}(x,y) = \ k_g(x,y) - \bar{\phi}(x) - \bar{\phi}(y) + c$$

with $\phi(x) := \langle k_g(x,.)\rangle_{g'}$ *and* $c := \langle \phi\rangle_{g'}$

Gaussian Fields

Co-Polyharmonic Gaussian Field – Definition, Construction

Definition

A co-polyharmonic Gaussian field on (M, g) is a centered Gaussian random variable on $\mathring{H}^{-\epsilon}(M)$ for some $\epsilon > 0$ with covariance

$$\mathbf{E}\big[\langle h, u\rangle \langle h, v\rangle\big] = \mathcal{K}_g(u, v) \qquad \forall u, v \in \mathring{H}^{\epsilon}(M).$$

Existence and uniqueness follows from theory of abstract Wiener spaces.

Interpretation: $\mathbf{E}\big[h(x)\big] = 0, \qquad \mathbf{E}\big[h(x)\, h(y)\big] = k_g(x, y) \qquad (\forall x, y)$

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Let a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ be given and an i.i.d. sequence $(\xi_j)_{j\in\mathbb{N}}$ of $\mathcal{N}(0, 1)$ random variables. Furthermore, let $(\psi_j)_{j\in\mathbb{N}_0}$ and $(\nu_j)_{j\in\mathbb{N}_0}$ denote the sequences of eigenfunctions and eigenvalues for p_g (counted with multiplicities).

Theorem

A co-polyharmonic field is given by

$$h := \sum_{j\in\mathbb{N}} \nu_j^{-1/2}\, \xi_j\, \psi_j.$$

Co-Polyharmonic Gaussian Field – Smooth Approximation

Theorem

A co-polyharmonic field is given by

$$h := \sum_{j\in\mathbb{N}} \xi_j \cdot \sqrt{\mathsf{k}_g}\, \psi_j = \sum_{j\in\mathbb{N}} \nu_j^{-1/2}\, \xi_j\, \psi_j.$$

More precisely,

1. *For each $\ell \in \mathbb{N}$, a centered Gaussian random variable h_ℓ with values in $\mathcal{C}^\infty(M)$ is given by*

$$h_\ell := \sum_{j=1}^{\ell} \nu_j^{-1/2}\, \xi_j\, \psi_j.$$

2. *The convergence $h_\ell \to h$ holds in $L^2(\mathbf{P}) \times H^{-\epsilon}(M)$ for every $\epsilon > 0$. In particular, for a.e. ω and every $\epsilon > 0$,*

$$h^\omega \in H^{-\epsilon}(M),$$

3. *For every $u \in H^{-n/2}(M)$, the family $(\langle u, h_\ell\rangle)_{\ell\in\mathbb{N}}$ is a centered $L^2(\mathbf{P})$-bounded martingale and*

$$\langle u, h_\ell\rangle \to \langle u, h\rangle \quad \text{in } L^2(\mathbf{P}) \text{ as } \ell \to \infty.$$

The Ungrounded Co-Polyharmonic Gaussian Field

The law $\boldsymbol{\nu}_g$ of the **ungrounded** co-polyharmonic Gaussian field is defined as

$d\boldsymbol{\nu}_g :=$ image of $d\mathring{\boldsymbol{\nu}}_g(h) \otimes d\mathcal{L}^1(t)$ under the map $(h,t) \mapsto h+t$

where $\mathring{\boldsymbol{\nu}}_g =$ denotes the law of the ("grounded") co-polyharmonic Gaussian field as defined before.

Theorem

The ungrounded co-polyharmonic Gaussian field is conformally invariant:

- *If $g' = e^{2\varphi} g$ on M then $h' \overset{(d)}{=} h$.*
- *If $\Phi : (M,g) \to (M',g')$ is an isometry then $h' \circ \Phi \overset{(d)}{=} h$.*

The ("grounded") co-polyharmonic Gaussian field is conformally invariant modulo re-grounding:
If $g' = e^{2\varphi} g$ on M then $h' \overset{(d)}{=} h - \big\langle h \big\rangle_{\mathrm{vol}_{g'}}$.

Co-Polyharmonic Gaussian Field – Discrete Approximation

Let M be the continuous torus $\mathbb{T}^n \cong [0,1)^n$ and consider its discrete approximations $\mathbb{T}^n_L \cong \{0, \frac{1}{L}, \ldots, \frac{L-1}{L}\}^n$ for $L \in \mathbb{N}$.

Co-polyharmonic Gaussian Field on the discrete torus $\mathbb{T}^n_L$

Centered Gaussian field $(h_L(v))_{v\in\mathbb{T}^n_L}$ with covariance function

$$k_L(u,v) = \frac{1}{a_n}\,\mathring{G}^{n/2}_L(u,v) = \frac{1}{a_n}\sum_{z\in\mathbb{Z}^n_L\setminus\{0\}} \frac{1}{\lambda^{n/2}_{L,z}}\cdot\cos\Big(2\pi\, z\cdot(v-u)\Big)$$

where $\lambda_{L,z} = 4L^2\sum_{k=1}^n \sin^2\big(\pi z_k/L\big)$ and $\mathbb{Z}^n_L = \{z\in\mathbb{Z}^n : 0 < \|z\|_\infty < L/2\}$.

Note that $\lambda_{L,z}$ are eigenvalues of the discrete Laplacian whereas $\lambda_z := 4\pi^2|z|^2$ are the corresponding eigenvalues of the continuous Laplacian.
Also note that $\sum_{z\in\mathbb{Z}^n\setminus\{0\}}\frac{1}{\lambda_z^{n/2}} = c\sum_{z\in\mathbb{Z}^n\setminus\{0\}}\frac{1}{|z|^n} = \infty$.

Given iid standard normals $(\xi_z)_{z\in\mathbb{Z}^n_L}$ and Fourier basis functions $\varphi_z(x) = \frac{1}{\sqrt{2}}\cos(2\pi xz)$ and $\varphi_{-z}(x) = \frac{1}{\sqrt{2}}\sin(2\pi xz)$, a co-polyharmonic Gaussian field is given as

$$h_L = \frac{1}{\sqrt{a_n}}\sum_{z\in\mathbb{Z}^n_L\setminus\{0\}}\frac{1}{\lambda^{n/4}_{L,z}}\cdot\xi_z\,\varphi_z.$$

Co-Polyharmonic Gaussian Field – Discrete Approximation

The law of the ungrounded polyharmonic Gaussian field is given explicitly as

$$c_n \exp\left(-\frac{a_n}{2N}\left\|(-\Delta_L)^{n/4}h\right\|^2\right) d\mathcal{L}^N(h)$$

on $\mathbb{R}^N \cong \mathbb{R}^{\mathbb{T}_L^n}$ where $N = L^n$.

Theorem

- *Convergence of fields $h_L \to h$ as $L \to \infty$: tested against $f \in \bigcup_{s>n/2} H^s(\mathbb{T}^n)$*
- *Convergence of Fourier extension of h_L to h: in each $H^{-\epsilon}(\mathbb{T}^n)$ and also tested against $f \in H^{-n/2}(\mathbb{T}^n)$*

Liouville Geometry

Liouville Quantum Gravity Measure

Let M as before be a closed manifold of even dimension and h the (grounded) co-polyharmonic Gaussian field. For $\ell \in \mathbb{N}$ define a random measure

$$d\mu^{\gamma h_\ell}(x) := \exp\Big(\gamma h_\ell(x) - \frac{\gamma^2}{2} k_\ell(x,x)\Big)\, d\,\mathrm{vol}_g(x)$$

on M where $h_\ell(x) := \langle q_\ell(x,.), h\rangle$ for suitable family of kernels $q_\ell(x,y)$ and $k_\ell(x,y) := \mathbf{E}\big[h_\ell(x)h_\ell(y)\big] = \iint k(x',y')q_\ell(x,x')q_\ell(y,y')dx'dy'$. Based on Kahane 1986, Shamov 2016, Berestycki 2017,

Theorem

If $|\gamma| < \sqrt{2n}$, then there exists a random measure $\mu^{\gamma h}$ on M with $\mu^{\gamma h_\ell} \to \mu^{\gamma h}$. More precisely, for every $u \in \mathcal{C}(M)$,

$$\int_M u\, d\mu^{\gamma h_\ell} \longrightarrow \int_M u\, d\mu^{\gamma h} \quad \textit{in } L^1(\mathbf{P}) \textit{ and } \mathbf{P}\textit{-a.s. as } \ell \to \infty.$$

The random measure $\mu^{\gamma h} := \lim_{\ell\to\infty} \mu^{\gamma h_\ell}$ is called *plain Liouville Quantum Gravity measure*.

Theorem

The previous convergence $\mu^{\gamma h_\ell} \to \mu^{\gamma h}$ also holds true if we put $h_\ell := \sum_{j=1}^{\ell} \nu_j^{-1/2}\, \xi_j\, \psi_j$.

Liouville Quantum Gravity Measure

We define the plain LQG measure for the ungrounded Gaussian field $h = h_0 + t$ with $(h_0, t) \sim \mathring{\boldsymbol{\nu}}_g \otimes \mathcal{L}^1$ by

$$\mu^{\gamma h} := e^{\gamma t} \mu^{\gamma h_0}.$$

The adjusted LQG measure is given by

$$\bar{\mu}^{\gamma h} := e^{\frac{\gamma^2}{2} r_g} \mu^{\gamma h}$$

with

$$r_g(x) := \limsup_{y \to x} \left[k_g(x, y) - \log \frac{1}{d_g(x, y)} \right].$$

Equivalently, $\bar{\mu}^{\gamma h} = \mu^{\gamma \tilde{h}}$ where the 'refined' field $\tilde{h}$ is associated with the covariance kernel

$$\tilde{k}_g(x, y) := k_g(x, y) - \frac{1}{2} r_g(x) - \frac{1}{2} r_g(y) + c$$

where $c = \langle r_g \rangle + \frac{1}{4} \mathfrak{p}_g(r_g, r_g)$.

Liouville Quantum Gravity Measure

A key property of the adjusted Liouville Quantum Gravity measure is its quasi-invariance under conformal transformations.

Theorem

Assume that $h \sim \boldsymbol{\nu}_g$ and $h' \sim \boldsymbol{\nu}_{g'}$ where $g' = e^{2\varphi} g$, then

$$\bar{\mu}_{g'}^{\gamma h'} \overset{(d)}{=} e^{(n+\frac{\gamma^2}{2})\varphi} \bar{\mu}_g^{\gamma h}$$

or, in other words,

$$\bar{\mu}_{g'}^{\gamma h'} \overset{(d)}{=} \bar{\mu}_g^{\gamma\, T(h)}$$

with the shift $T : h \mapsto h + \left(\frac{n}{\gamma} + \frac{\gamma}{2}\right) \varphi$.

Liouville Brownian Motion, Random Paneitz Operator

If $\gamma < 2$ then a.s. the LQG measure $\mu^{\gamma h}$ does not charge sets of vanishing H^1-capacity

$\longrightarrow$ Dirichlet form $\int_M |\nabla u|^2 d\,\mathrm{vol}_g$ on $L^2(M, \mu^{\gamma h})$
$\longrightarrow$ Liouville Brownian motion (random time change of BM)

If $\gamma < \sqrt{2n}$ then a.s. the LQG measure $\mu^{\gamma h}$ does not charge sets of vanishing $H^{n/2}$-capacity

$\longrightarrow$ energy form $\int u((-\Delta)^{n/2} + l.o.t.)u\, d\,\mathrm{vol}_g$ on $L^2(M, \mu)$
$\longrightarrow$ random Paneitz operators, conformally invariant

LQG Measure – Discrete Approximation

- Let M be the continuous torus $\mathbb{T}^n \cong [0,1)^n$
- Consider its discrete approximations $\mathbb{T}^n_L \cong \{0, \frac{1}{L}, \ldots, \frac{L-1}{L}\}^n$ for $L \in \mathbb{N}$.
- Recall the co-polyharmonic Gaussian field on the discrete torus $\mathbb{T}^n_L$

$$h_L = \frac{1}{\sqrt{a_n}} \sum_{z \in \mathbb{Z}^n_L \setminus \{0\}} \frac{1}{\lambda_{L,z}^{n/4}} \cdot \xi_z \, \varphi_z.$$

For the “spectrally reduced field” replace here the discrete eigenvalues $\lambda_{L,z}$ by the corresponding continuous ones λ_z (which are larger).

For given $\gamma \in \mathbb{R}$, the **discrete LQG measure** μ_L is the random measure on $\mathbb{T}^n_L$ defined by

$$d\mu_L(v) = \exp\left(\gamma h_L(v) - \frac{\gamma^2}{2} k_L(v,v)\right) dm_L(v),$$

where m_L denotes the normalized counting measure $\frac{1}{L^n} \sum_{u \in \mathbb{T}^n_L} \delta_u$.

In accordance to the approximation of the Co-polyharmonic fields, we have convergence of μ_L to the LQG measure μ on $\mathbb{T}^n$.

LQG Measure – Discrete Approximation

Theorem

(i) *For* $\gamma < \sqrt{n}$ *and* $L = a^{\ell}$, $a \in \mathbb{N}_{\geq 2}$,

$$\mu_{a^{\ell}} \to \mu \qquad \text{in law in } L^1(\mathbf{P}) \text{ as } \ell \to \infty.$$

(ii) *An analogous convergence result holds for the LQG measure associated to the Fourier extension of the spectrally reduced field in the range* $\gamma < \sqrt{2n}$.

The range of γ in (i) differs from the Gaussian multiplicative chaos construction since this construction uses the eigenvalues of the discrete Laplacian instead of the Laplacian.

Polyakov-Liouville Measure

Polyakov-Liouville Measure

In 20016-2019 Rhodes–Vargas with David, Garban, and Kupiainen provided a rigorous definition to the Polyakov–Liouville measure $\boldsymbol{\pi}_g$, informally given as

$$d\boldsymbol{\pi}_g(h) = \exp\Big(- S_g(h)\Big)\, dh$$

with (non-existing) uniform distribution dh on the set of fields
Classical Liouville theory is a two-dimensional conformal field theory.

For n=2: The Liouville action functional is defined as

$$S_g(h) := \int_M \Big(\frac{1}{4\pi} \left|\nabla h\right|^2 + \frac{\Theta}{2} R_g h + m e^{\gamma h} \Big)\, d\,\mathrm{vol}_g,$$

where $m, \Theta, \gamma > 0$ are parameters and R_g denotes the Gauss curvature.

With appropriate choice of constants,

- minimizers h of the action functional satisfy the Liouville equation

  $$R_{e^{\gamma h} g} = -\frac{1}{2} m\gamma^2$$

  which produces metrics with constant negative curvature.
- semiclassical limit ($\gamma \to 0$): Polyakov–Liouville measure concentrates on surfaces of constant curvature (Lacoin/Rhodes/Vargas 2019+).

Polyakov-Liouville Measure, $n = 2$

Arbitrary even $n \geq 2$: Ansatz for Polyakov–Liouville action

$$S_g(h) := \int_M \left(\frac{1}{2} \left| \sqrt{\mathsf{p}_g}\, h \right|^2 + \Theta\, Q_g h + m e^{\gamma h} \right) d\,\mathrm{vol}_g\,.$$

Here

- p_g is the co-polyharmonic operator,
- Q_g denotes Branson's Q-curvature,
- m, Θ, γ are parameters.

In the case

- $n = 2,\ Q_g = \frac{1}{2} R_g,$
- $n = 4,\ Q_g = -\frac{1}{6}\Delta_g \mathrm{scal}_g - \frac{1}{2}|\mathrm{Ric}_g|^2 + \frac{1}{6}\mathrm{scal}_g^2.$

In general, total Q-curvature is conformally invariant, and if $g' = e^{2\varphi} g$ then

$$e^{n\varphi} Q_{g'} = Q_g + \frac{1}{a_n}\, \mathsf{p}_g \varphi.$$

Polyakov-Liouville Measure, $n \geq 2$

Minimizers of S_g satisfy

$$\mathsf{p}_g h + \Theta Q_g + m\gamma e^{\gamma h} = 0.$$

Choose $\Theta = \frac{na_n}{\gamma}$, $m = -\frac{na_n}{\gamma^2}\bar{Q}$ for some $\bar{Q} \in \mathbb{R}$ and put $\varphi = \frac{\gamma}{n}h$.
Then this reads as

$$\frac{1}{a_n}\mathsf{p}_g\varphi + Q_g = e^{n\varphi}\bar{Q}.$$

In other words, $g' = e^{2\varphi}g$ is a metric of constant Branson curvature $Q_{g'} = \bar{Q}$.

Polyakov-Liouville Measure

Informal ansatz

$$d\boldsymbol{\pi}_g(h) = \exp\left(- \int_M \left(\frac{1}{2}\left|\sqrt{\mathsf{p}_g}\, h\right|^2 + \Theta\, Q_g h + m e^{\gamma h}\right) d\,\mathrm{vol}_g \right) dh$$

Rigorous

$$d\boldsymbol{\nu}_g^*(h) := \exp\left(- \Theta\langle h, Q_g\rangle - m\,\bar{\mu}_g^{\gamma h}(M)\right) d\boldsymbol{\nu}_g(h)$$

$$d\boldsymbol{\pi}_g(h) := \sqrt{\frac{\mathrm{vol}_g(M)}{\det'(\frac{1}{2\pi}\mathsf{p}_g)}} \cdot d\boldsymbol{\nu}_g^*(h)$$

where

- $\boldsymbol{\nu}_g$ is the law of ungrounded co-polyharmonic Gaussian field on (M, g),
- $\bar{\mu}_g^{\gamma h}$ is the adjusted LQG measure
- $\det'(\ldots)$ denotes the regularized determinant.

In case $M = \mathbb{S}^n$, cf. Levy–Oz (2018), Cerclé (2019).

Polyakov-Liouville Measure

Recall

$$d\boldsymbol{\nu}_g^*(h) := \exp\Big(-\Theta\langle h, Q_g\rangle - m\,\bar{\mu}_g^{\gamma h}(M)\Big)\, d\boldsymbol{\nu}_g(h)$$

Theorem

Assume that $0 < \gamma < \sqrt{2n}$ and $\Theta\, Q(M) < 0$. Then $\boldsymbol{\nu}_g^*$ is a finite measure.

Theorem

If $\Theta = a_n \left(\frac{n}{\gamma} + \frac{\gamma}{2}\right)$, then $\boldsymbol{\nu}_g^*$ is conformally quasi-invariant under the shift $T: h \mapsto h - \Theta\varphi$ with A-type conformal anomaly

$$\frac{d\boldsymbol{\nu}_{g'}^*}{dT_*\boldsymbol{\nu}_g^*} = \exp\left(\left(\frac{n}{\gamma} + \frac{\gamma}{2}\right)^2 \left[\frac{1}{2}\mathfrak{p}_g(\varphi,\varphi) + a_n \int \varphi\, Q_g\, d\,\mathrm{vol}_g\right]\right).$$

Polyakov-Liouville Measure

Now consider

$$d\boldsymbol{\pi}_g(h) := \sqrt{\frac{\mathrm{vol}_g(M)}{\det'(\frac{1}{2\pi}\mathsf{p}_g)}} \cdot \exp\Big(- \Theta\langle h, Q_g\rangle - m\,\bar{\mu}_g^{\gamma h}(M)\Big)\, d\boldsymbol{\nu}_g(h),$$

again with $\Theta = a_n\left(\frac{n}{\gamma} + \frac{\gamma}{2}\right)$.

Theorem

Assume n = 2. *Then $\boldsymbol{\pi}_g$ is conformally quasi-invariant under the shift $T: h \mapsto h - \Theta\varphi$ with conformal anomaly*

$$\frac{d\boldsymbol{\pi}_{g'}}{dT_*\boldsymbol{\pi}_g} = \exp\left(\left[\frac{1}{6} + \left(\frac{2}{\gamma} + \frac{\gamma}{2}\right)^2\right]\left[\frac{1}{2}\mathfrak{p}_g(\varphi,\varphi) + a_n \int \varphi\, Q_g\, d\,\mathrm{vol}_g\right]\right).$$

David–Kupiainen–Rhodes–Vargas '16, David–Rhodes–Vargas '16, Huang–Rhodes–Vargas '18, Guillarmou–Rhodes–Vargas '19.

Polyakov-Liouville Measure

Now consider

$$d\boldsymbol{\pi}_g(h) := \sqrt{\frac{\mathrm{vol}_g(M)}{\det'(\frac{1}{2\pi}\mathsf{p}_g)}} \cdot \exp\Big(-\Theta\langle h, Q_g\rangle - m\,\bar{\mu}_g^{\gamma h}(M)\Big)\, d\boldsymbol{\nu}_g(h),$$

again with $\Theta = a_n\left(\frac{n}{\gamma} + \frac{\gamma}{2}\right)$.

Theorem

Assume n = 4 *and for simplicity that (M, g) is conformally flat, i.e. it is conformally equivalent to a flat manifold.*
Then $\boldsymbol{\pi}_g$ is conformally quasi-invariant under the shift $T: h \mapsto h - \Theta\varphi$ with conformal anomaly

$$\frac{d\boldsymbol{\pi}_{g'}}{dT_*\boldsymbol{\pi}_g} = \exp\left(\left[\frac{7}{45} + \left(\frac{4}{\gamma} + \frac{\gamma}{2}\right)^2\right] \cdot \left[\frac{1}{2}\mathfrak{p}_g(\varphi,\varphi) + a_n \int \varphi\, Q_g\, d\,\mathrm{vol}_g\right]\right)$$
$$\cdot \exp\left(\frac{1}{45\pi^2}\left[-\int \mathrm{scal}_{g'}^2\, d\,\mathrm{vol}_{g'} + \int \mathrm{scal}_g^2\, d\,\mathrm{vol}_g\right]\right).$$