Rigorous Dean-Kawasaki Models for Reversible and Irreversible Diffusions

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New results based on Joint work with
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Ginzburg-Landau stochastic phase field model

$$\partial_t \phi + \nabla \cdot \left(-L(\phi) \nabla \frac{\delta H}{\delta \phi}(\phi) + \sqrt{T L(\phi)} \dot{W} \right) = 0,$$

L Onsager coefficient , H free energy.

Dean-Kawasaki equation

$$H = \beta \, \mathrm{Ent}, \; T = 1 \text{ and } L = \text{identity}$$

$$d\mu_t = \frac{\beta}{2} \Delta \mu_t dt + \nabla \cdot \left(\mu_t \nabla \frac{\delta F(\mu_t)}{\delta \mu_t} \right) dt + \nabla \cdot (\sqrt{\mu_t} dW_t)$$

G. Giacomin, J. Lebowitz, and E. Presutti. *Deterministic and stochastic hydrodynamic equations arising from simple microscopic model systems*, pages 107–152. American Mathematical Society, November 1999.

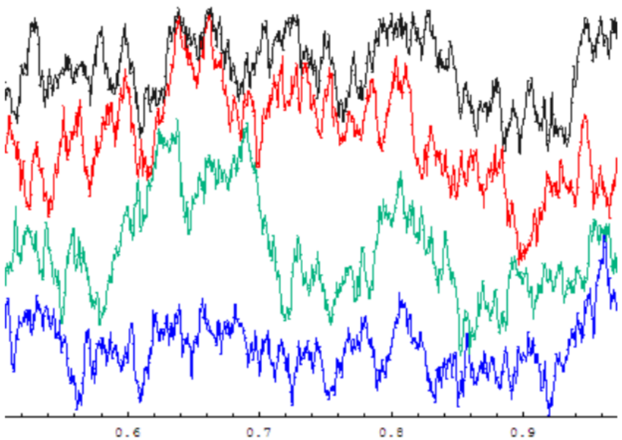
L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim. Macroscopic fluctuation theory. *Rev. Mod. Phys.*, 87(2):593–636, June 2015.

System of coupled two sided real Bessel processes

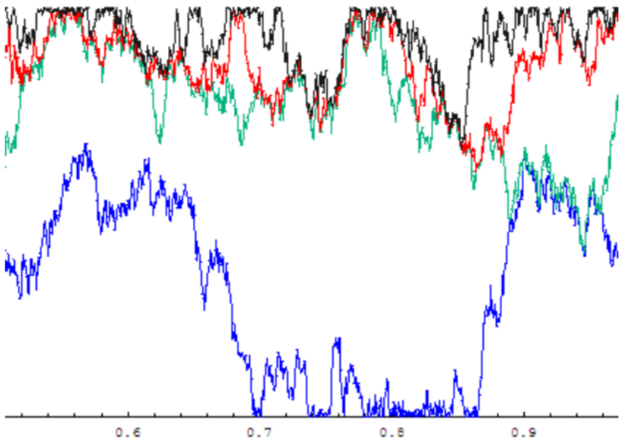
$$dx_t^i = (\frac{\beta}{N} - 1)\left(\frac{1}{x_t^i - x_t^{i-1}} - \frac{1}{x_t^{i+1} - x_t^i}\right)dt + \sqrt{2}dw_t^i + dl_t^{i-1} - dl_t^i, \quad i = 1, \cdots, N-1,$$

$\{w^i\}$ independent real Brownian motions , local times l^i satisfying

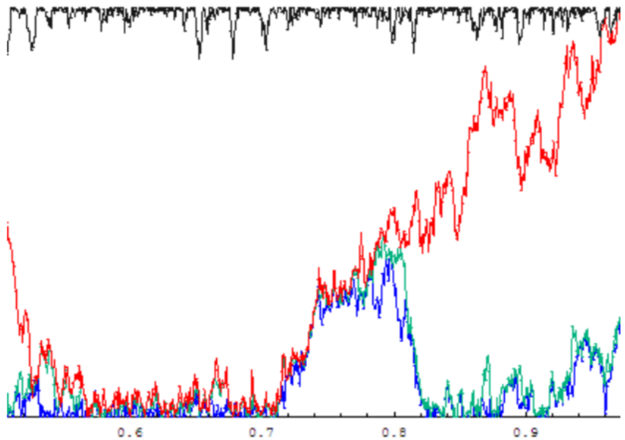
$$dl_t^i \geq 0, \quad l_t^i = \int_0^t \mathbb{1}_{\{x_s^i = x_s^{i+1}\}} dl_s^i.$$



$\beta = 10$



$\beta = 1$



$\beta = 0.3$

Sebastian Andres and Max-K. von Renesse, *Particle approximation of the Wasserstein diffusion*, J. Funct. Anal. **258** (2010), no. 11, 3879–3905.

$$\mu_t^N = \frac{1}{N-1} \sum_{i=1}^{N-1} \delta_{x^i_{N\cdot t}} \in \mathcal{P}([0,1])$$

Theorem $(\mu_\cdot^N) \overset{N\to\infty}{\Longrightarrow} (\mu_\cdot)$ *in* $C_{\mathbb{R}_+}\big((\mathcal{P}([0,1]), \tau_w)\big)$

$$d\mu_t = \beta \Delta \mu_t dt + \ \Gamma(\mu_t) dt + \operatorname{div}(\sqrt{\mu}_t dW_t),$$

$$\Gamma(\mu_t) = \frac{1}{2} \sum_{z \in \operatorname{supp}(\mu_t)} (\delta_z)''$$

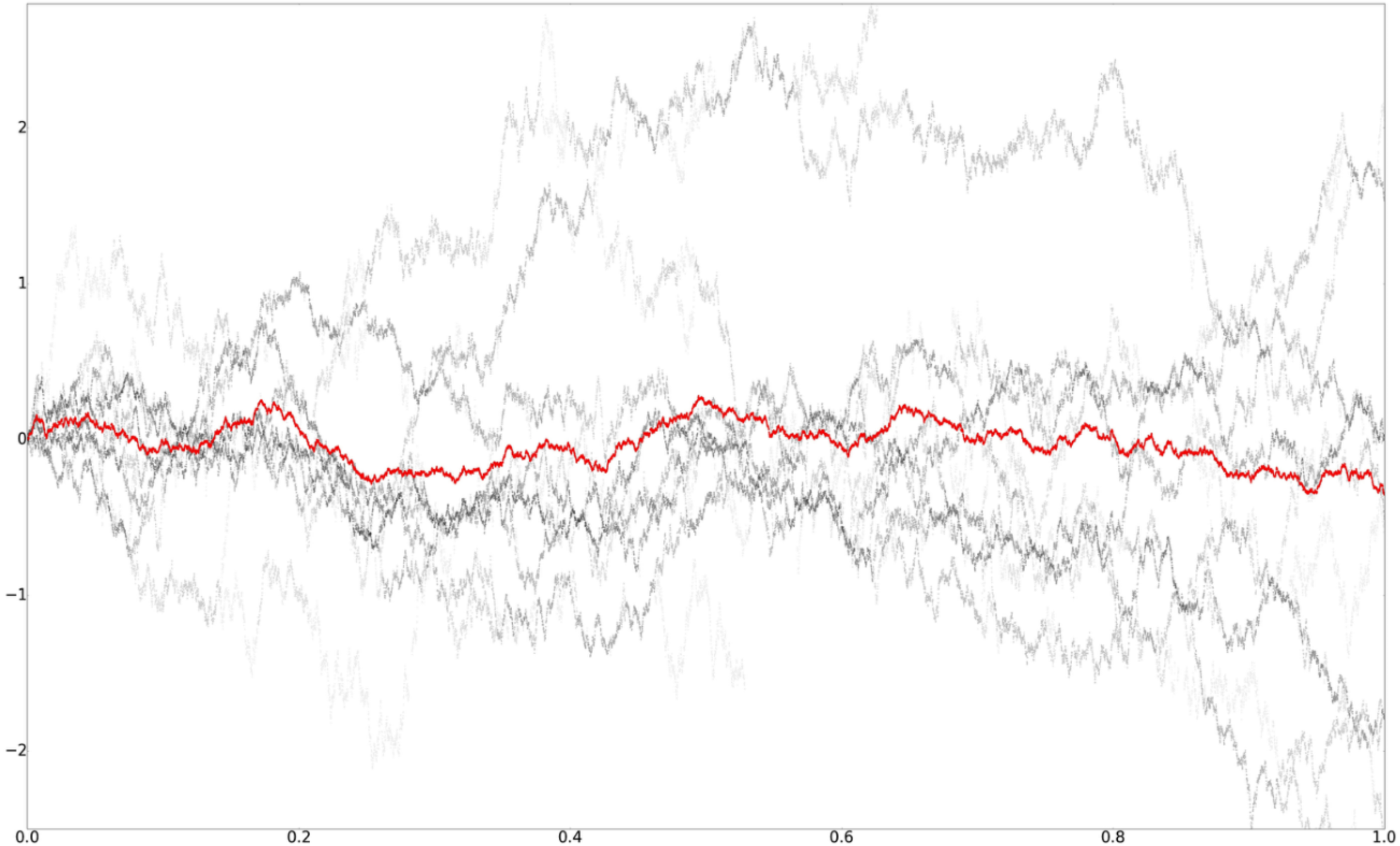
Wasserstein Diffusion

$$\lim_{t\to 0} t \log p_t(A,B) = -\frac{d_W(A,B)^2}{2}.$$

M.-K. von Renesse and K.-T. Sturm. Entropic measure and Wasserstein diffusion. *Ann. Probab.*, 37(3):1114–1191, 2009.

$\beta = 0$

$$d\mu_t = \Gamma(\mu_t)dt + \mathrm{div}(\sqrt{\mu}_t dW_t)$$

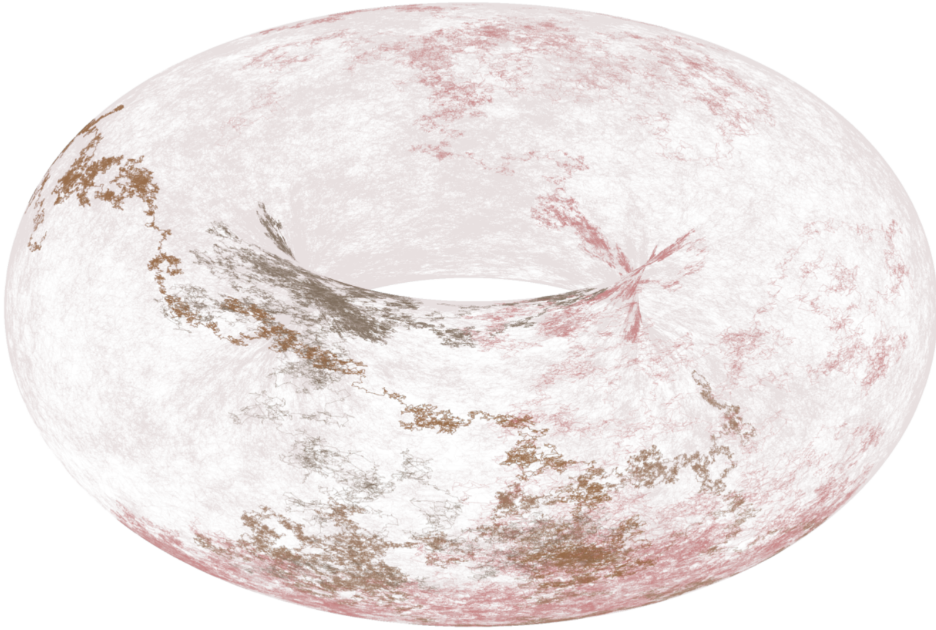


V. V. Konarovskyi and M.-K. von Renesse. Modified Massive Arratia flow and Wasserstein diffusion. *Comm. Pure Appl. Math.*, 72(4):764–800, 2019.

V. V. Konarovskyi and M.-K. von Renesse. Reversible coalescing-fragmentating Wasserstein dynamics on the real line. *J. Funct. Anal.*, 286(8):110342, April 2024.

Massive Particle Systems and Wasserstein Brownian Motions

$$\mathrm{d}\mu_t = \nabla\cdot(\sqrt{\mu_t}\,\xi) + \sum_{x\in\mu_t} \Delta\delta_x \,\mathrm{d}t + \frac{\beta}{2}\nabla\cdot\left(\mu_t \int \frac{(\cdot - y)}{|\cdot - y|_{\mathbb{T}^d}^{p+2}}\,\mathrm{d}\mu_t(y)\right)\mathrm{d}t$$



$$\mathrm{d}X_t^i = \mathrm{d}\mathsf{W}^i_{t/s_i} - \frac{\beta}{2}\sum_j^\infty s_i\, s_j \frac{X_t^i - X_t^j}{\left|X_t^i - X_t^j\right|_{\mathbb{T}^d}^{p+2}}\,\mathrm{d}t\,,$$

L. Dello Schiavo. The Dirichlet–Ferguson Diffusion on the Space of Probability Measures over a Closed Riemannian Manifold. *Ann. Probab.*, 50(2):591–648, 2022.

L. Dello Schiavo. Massive Particle Systems, Wasserstein Brownian Motions, and the Dean–Kawasaki Equation. *arXiv:2411.14936*, pages 1–103, 2024.

Theorem (Existence and uniqueness of solutions to the Dean-Kawasaki equation) *Let* $\nu \in \mathcal{P}(\mathbb{R}^d)$ *and* $F \in \mathrm{C}_b^{2,2}(\mathcal{M}_F(\mathbb{R}^d))$.
Then the Dean-Kawasaki equation

$$d\mu_t = \frac{\beta}{2}\Delta\,\mu_t dt + \nabla\cdot\left(\mu_t\nabla\frac{\delta F(\mu_t)}{\delta\mu_t}\right)dt + \nabla\cdot(\sqrt{\mu_t}dW_t)$$

has a (unique in law) solution μ_t, $t \geq 0$, *starting from* ν, *i.e.* $\mu_0 = \nu$. *if and only if* $\beta \in \mathbb{N}$ *and* $\nu = \frac{1}{n}\sum_{i=1}^n \delta_{x^i}$ *for some* $x^i \in \mathbb{R}^d$, $i \in [n] := \{1,\ldots,n\}$.
Moreover,

$$\mu_t = \frac{1}{n}\sum_{i=1}^n \delta_{X^i(t)}, \quad t \geq 0,$$

where $X(t) = (X^1(t),\ldots,X^n(t))$, $t \geq 0$, *is a (unique) solution to the equation*

$$dX^i(t) = -\nabla\frac{\delta F(\mu_t)}{\delta\mu_t}(X^i(t))dt + \sqrt{\beta}\,dw^i(t), \quad i \in [n],$$

with $X(0) = (x^1,\ldots,x^n)$, *and* $w^i(t)$, $t \geq 0$, $i \in [n]$, *are independent standard Wiener processes on* $\mathbb{R}^d$.

V. V. Konarovskyi, T. Lehmann, and M.-K. von Renesse. On Dean-Kawasaki Dynamics with Smooth Drift Potential. *J. Statist. Phys.*, 178(3):666–681, nov 2019.

Federico Cornalba, Tony Shardlow, and Johannes Zimmer, *A regularized Dean-Kawasaki model: derivation and analysis*, SIAM J. Math. Anal. **51** (2019), no. 2, 1137–1187.

F. Cornalba and J. Fischer. Multilevel Monte Carlo methods for the Dean–Kawasaki equation from Fluctuating Hydrodynamics. *arXiv:2311.08872*, 2023.

F. Cornalba and J. Fischer. The Dean–Kawasaki Equation and the Structure of Density Fluctuations in Systems of Diffusing Particles. *Arch. Rational Mech. Anal.*, 247(5) 2023.

B. Fehrman and B. Gess. Well-Posedness of the Dean–Kawasaki and the Nonlinear Dawson–Watanabe Equation with Correlated Noise. *Arch. Rational Mech. Anal.*, 248(2), March 2024.

Benjamin Gess, Rishabh S. Gvalani, and Vitalii Konarovskyi, *Conservative SPDEs as fluctuating mean field limits of stochastic gradient descent*, arXiv:2207.05705 (2022).

A. Martini and A. Mayorcas. An Additive Noise Approximation to Keller–Segel–Dean–Kawasaki Dynamics Part I: Local Well-Posedness of Paracontrolled Solutions. *arXiv:2207.10711v2*. 2022.

N. Dirr, B. Fehrman, and B. Gess. Conservative stochastic pde and fluctuations of the symmetric simple exclusion process. *arXiv:2012.02126*, 2021.

A. Djurdjevac, H. Kremp, and N. Perkowski. Weak error analysis for a nonlinear spde approximation of the dean-kawasaki equation, 2022.

Well-Posedness for Dean–Kawasaki Models of

Vlasov-Fokker-Planck Type.

James F. Lutsko. A dynamical theory of nucleation for colloids and macromolecules. *J. Chem. Phys.*, 136:034509, 2012.

$$
\begin{aligned}
\mathrm{d}x_t^i &= v_t^i \mathrm{d}t, \\
\mathrm{d}v_t^i &= -\frac{1}{n}\sum_{j=1}^{n} \nabla H(v_t^i - v_t^j)\mathrm{d}t + \mathrm{d}W_t^i
\end{aligned}
$$

Michael E. Cates and Julien Tailleur. Motility-induced phase separation. *Annu. Rev. Condens. Matter Phys.*, 6(1):219–244, 2015.

$$
\mu_t(x, v) = \frac{1}{n}\sum_{i=1}^{n} \delta_{(x_t^i, v_t^i)},
$$

$$
\begin{aligned}
\mathrm{d}\langle f,\mu_t\rangle &= \frac{1}{n}\sum_{i=1}^n \mathrm{d}f(x_t^i,v_t^i)\\
&= \frac{1}{n}\sum_{i=1}^n \nabla_x f(x_t^i,v_t^i)\cdot v_i\mathrm{d}t - \frac{1}{n^2}\sum_{i=1}^n \nabla_v f(x_t^i,v_t^i)\cdot\sum_{j=1}^n \nabla H(v_t^i-v_t^j)\mathrm{d}t\\
&\quad + \frac{1}{n}\sum_{i=1}^n \nabla_v f(x_t^i,v_t^i)\mathrm{d}B_t^i + \frac{1}{2n}\sum_{i=1}^n \Delta_v f(x_t^i,v_t^i)\mathrm{d}t\\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_x f(x,v)\cdot v\mu_t(\mathrm{d}x\mathrm{d}v)\mathrm{d}t - \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_v f(x,v)\cdot(\nabla H*\nu_t)(v)\mu_t(\mathrm{d}x\mathrm{d}v)\mathrm{d}t\\
&\quad + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} \Delta_v f(x,v)\mu_t(\mathrm{d}x\mathrm{d}v)\mathrm{d}t + M_t^f,
\end{aligned}
$$

$$
\mathrm{d}\left[M_\cdot^f, M_\cdot^f\right] = \frac{1}{n}\int_{\mathbb{R}^d\times\mathbb{R}^d} \left|\nabla_v f(x,v)\right|^2 \mu_t(\mathrm{d}x\mathrm{d}v)\mathrm{d}t.
$$

$$
\nu_t(V) = \mu_t(\mathbb{R}^d, V)
$$

$$\langle \mu_t, f\rangle - \langle \mu_0, f\rangle - \int_0^t \langle \nabla_x f(x,v)\cdot v\rangle_{\mu_s} \,\mathrm{d}s$$
$$+ \int_0^t \langle \nabla_v f(x,v)\cdot (\nabla H * \nu_t)(v)\rangle_{\mu_s} \,\mathrm{d}s - \frac{1}{2}\int_0^t \langle \Delta_v f\rangle_{\mu_s} \mathrm{d}s$$

is a local martingale with quadratic variation

$$\frac{1}{n}\int_0^t \left\langle |\nabla_v f|^2 \right\rangle_{\mu_s} \mathrm{d}s.$$

$$\mathrm{d}\mu_t = -\nabla_x(v\mu_t)\mathrm{d}t + \nabla_v(\nabla H * \nu(x)\mu)\mathrm{d}t + \frac{1}{2}\Delta_v \mu_t \mathrm{d}t + \alpha^{-\frac{1}{2}}\nabla_v \cdot (\sqrt{\mu_t}\mathrm{d}W_t)$$

$$\alpha = n.$$

$$\partial_t \mu_t = \alpha L^* \mu_t + \nabla \cdot (\mu_t F_{\mu_t}) + \nabla \cdot (\sqrt{\mu_t} \sigma \dot{W}_t)$$

$$L = b \cdot \nabla + \frac{1}{2} \sigma \sigma^T : \nabla^2$$

Definition 1.1 (Martingale solutions). *A continuous $\mathcal{M}_F(\mathbb{R}^k)$-valued process $(\mu_t)_{t \geq 0}$ is a* solution *to Equation* (6) *if for each $\varphi \in C_b^2(\mathbb{R}^k)$*

$$M_t(\varphi) := \langle \mu_t, \varphi \rangle - \int_0^t \langle \mu_s, \alpha L \varphi + F_{\mu_s} \cdot \nabla \varphi \rangle \, ds$$

is a martingale with quadratic variation

$$[M_\cdot(\varphi)]_t = \int_0^t \left\langle \mu_s, |\sigma^T \nabla \varphi|^2 \right\rangle ds.$$

Assumption 1.2. *(L.1) The operator L acting on $\mathcal{C}_b^2(\mathbb{R}^k)$ admits a unique in law Markovian family of diffusion processes $X = (X_t^z)_{t\geq 0}^{z\in\mathbb{R}^k}$ solving the associated $(L, \mathcal{C}_b^2(\mathbb{R}^k), \delta_z)_{z\in\mathbb{R}^k}$-martingale problems.*

(L.2) There is a set $\mathcal{D} \subset \mathcal{C}_b^2(\mathbb{R}^k)$ which is dense with respect to the topology of locally uniform convergence and is stable under composition with functions $\psi \in \mathcal{C}^\infty(\mathbb{R})$ satisfying $\psi(0) = 0$ such that for all $\varphi \in \mathcal{D}$ and $t \geq 0$, it holds that $P_t\varphi \in \mathcal{C}_b^2(\mathbb{R}^k)$ and $P_t L\varphi = LP_t\varphi \in \mathcal{C}_b(\mathbb{R}^k)$, where $P_t f(z) = \mathbb{E}f(X_t^z)$.

(L.3) For all $T > 0$ and every function $\varphi \in \mathcal{D}$, the function $(z,t) \to \left|\sigma^T \nabla(P_t\varphi)\right|(z)$ is uniformly bounded on $\mathbb{R}^k \times [0,T]$.

(L.4) There exists an exhaustion $(A_n)_{n\in\mathbb{N}} \subset \mathcal{B}(\mathbb{R}^k)$ with $A_n \nearrow \mathbb{R}^k$ such that for any $t > 0$ there exists a sequence $(c_n)_{n\in\mathbb{N}} \subset [0,1)$ such that $P_t 1_{A_n}(z) \leq c_n$ for all $z \in \mathbb{R}^k$ and $n \in \mathbb{N}$.

Assumption 1.4. *(L.5) The fields σ respectively b are continuous and bounded respectively of at most linear growth. That is, there is some $K > 0$ such that $|\sigma(z)| \leq K$ and $|b(z)| \leq K(1 + |z|)$ for $z \in \mathbb{R}^k$.*

(F.1) For some $G \in \mathcal{C}^2_{loc}(\mathcal{M}^2)$, we have

$$F_\mu(\cdot) = \left(\sigma\sigma^T \nabla \frac{\delta G}{\delta \mu}\right)(\cdot).$$

(F.2) It holds that $\sigma^T \nabla \frac{\delta G}{\delta \mu}(\mu, .)$ is bounded and $\nabla^l \frac{\delta^m G}{\delta \mu^m}(\mu, .)$ is for $m, l \in \{1, 2\}$ of at most linear growth, locally uniformly with respect to μ on $\mathcal{M}_F$ and $\mathcal{M}^2$ respectively.

Theorem 1.5. *Let Assumption 1.2 hold, and additionally Assumption 1.4 be satisfied if the interaction F is nonvanishing, $F \neq 0$. Let $\mu_0 \in \mathcal{M}_F$ have a finite second moment. Then the initial value problem*

$$\partial_t \mu_t = \alpha L^* \mu_t + \nabla \cdot (\mu_t F_{\mu_t}) + \nabla \cdot (\sqrt{\mu_t} \sigma \dot{W}_t)$$

with initial condition μ_0 has a solution if and only if $\alpha\, \mu_0(\mathbb{R}^k) =: n \in \mathbb{N}$.

In this case, the solution is given by

$$\mu_t = \frac{1}{\alpha} \sum_{i=1}^{n} \delta_{z_t^i},$$

where the system $\{z_t^i\}_{t\geq 0}^{i=1\cdots,n}$ is the unique in law solution to

$$dz_t^i = \alpha\, b(z_t^i)dt + \sqrt{\alpha}\, \sigma(z_t^i) dW_t^i + F\left(\frac{1}{\alpha}\sum_{j=1}^{n} \delta_{z_t^j}, z_t^i\right) dt, \quad i = 1, \cdots, n,$$

driven by n independent $\mathbb{R}^l$-valued Brownian motions $\{(W_t^i)_{t\geq 0}\}^{i=1,\cdots,n}$.

Examples

1 Inertial Langevin dynamics without interaction

$$\begin{aligned} \mathrm{d}x_t &= v_t \mathrm{d}t, \\ \mathrm{d}v_t &= -\gamma v_t \mathrm{d}t - \nabla U(x_t)\mathrm{d}t + \sqrt{2}\mathrm{d}W_t, \end{aligned}$$

To fit Equation (7) into the framework of Equation (6), we set $k = 2d$ and write $z \in \mathbb{R}^k$ as a combined vector of position and velocity, $z = (x, v) \in \mathbb{R}^d \times \mathbb{R}^d$. Then the drift vector b and the multiplicative noise σ in (6) are given by

$$b(x, v) = \begin{pmatrix} v \\ -\gamma v - \nabla U(x) \end{pmatrix} \quad \text{and} \quad \sigma = \sqrt{2} \begin{pmatrix} 0_{d\times d} \\ 1_{d\times d} \end{pmatrix},$$

where $0_{d\times d}$ is a $d \times d$ matrix of zeros and $1_{d\times d}$ is the identity matrix in d dimensions. This defines the operator L of (1).

In this case the corresponding degenerate Dean-Kawasaki Equation (Vlasov-Fokker-Planck equation) reads (with $\alpha = 1$ and $F = 0$)

$$\partial_t \mu_t = (\Delta_v \mu_t - \nabla_x \cdot (v\mu_t)) + \nabla_v \cdot ((\gamma v + \nabla U(x))\mu_t) + \nabla_v \cdot (\sqrt{\mu_t}\dot{W}_t).$$

2 Active matter

We now consider a class of active matter models, where individual agents at position $x_t \in \mathbb{R}^d$ are driven by a propulsion mechanism $g\colon \mathbb{R}^l \to \mathbb{R}^d$, for some $l \leq d$,

$$\begin{aligned} \mathrm{d}x_t &= g(\theta_t)\mathrm{d}t, \\ \mathrm{d}\theta_t &= \mathrm{d}B_t. \end{aligned}$$

In this case, the corresponding degenerate Dean-Kawasaki equation (Vlasov-Fokker-Planck equation) reads

$$\partial_t \mu_t = \left(\frac{1}{2} \Delta_\theta \mu_t - \nabla_x \cdot (g(\theta)\mu_t) \right) + \nabla_\theta \cdot (\sqrt{\mu_t} \dot{W}_t).$$

The classical example is $g(\theta) = v \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$ for some $v \in \mathbb{R}$. Then the angle θ can be interpreted as the direction the active particle is propelled into. See, for example, [35, 2] and references therein. We assume that $g \in \mathcal{C}_b^2\left(\mathbb{R}^l, \mathbb{R}^d\right)$. Then, a strong solution to the SDE (8) exist; since the coefficients are Lipschitz continuous, the solution is unique. Hence $(x_t, \theta_t)_{t \geq 0}$ solves the $\left(L, \mathcal{C}_b^2\left(\mathbb{R}^{d+l}\right)\right)$ martingale problem for the generator

$$L\varphi(x,\theta) = g(\theta) \cdot \nabla_x \varphi(x,\theta) + \frac{1}{2} \Delta_\theta \varphi(x,\theta).$$

3 Fluctuating hydrodynamics for Vlasov-Fokker-Planck equation with interaction

$$\rho(.) = \mu(.\times\mathbb{R}^d)$$

$$F_\mu(x) = \int_{\mathbb{R}^d} f_{\mathrm{int}(x,x')}\mathrm{d}\rho(x') = f_{\mathrm{int}} * \rho(x)$$

$$\partial_t\mu_t = \left(\frac{1}{2}\Delta_v\mu_t - \nabla_x\cdot(v\mu_t)\right) + \nabla_v\cdot((\gamma v + \nabla U(x))\mu_t + \mu_t(f*\rho_t)) + \nabla_v\cdot(\sqrt{\mu_t}\dot{W}_t),$$

corresponding particle system $(x_t^i, v_t^i)_{i=1}^n$ solves for $i = 1, \dots, n$

$$\begin{aligned} \mathrm{d}x_t^i &= v_t^i\mathrm{d}t, \\ \mathrm{d}v_t^i &= -\gamma v_t^i\mathrm{d}t - \nabla U(x_t^i)\mathrm{d}t - \sum_{i=1}^n f(x_t^i, x_t^j) + \sqrt{2}\mathrm{d}W_t^i. \end{aligned}$$

One checks that condition (F.1) of Assumption 1.4 is verified by choosing

$$G(\mu) = \langle \mu, v\cdot F_\mu\rangle\,,$$

so that indeed

$$\nabla_v\frac{\delta G}{\delta\mu}(x,v) = \nabla_v\left(v\cdot F_\mu(x) + \int_{\mathbb{R}^{2d}} v'\cdot f_{\mathrm{int}}(x,x')\mathrm{d}\mu(x',v')\right) = F_\mu(x).$$

4 Active swimmers and flocking

$$F_\mu(x,\theta) = \int \chi_R(|x-x'|)\nabla H(\theta-\theta')\mathrm{d}\mu(x',\theta'),$$

$$\partial_t \mu_t = \alpha\left(\frac{1}{2}\Delta_\theta \mu_t - \nabla_x \cdot (g(\theta)\mu_t)\right) + \nabla_\theta \cdot (\mu_t F_{\mu_t}) + \nabla_\theta \cdot (\sqrt{\mu_t}\dot{W}_t),$$

corresponding particle system

$$\begin{aligned} \mathrm{d}x_t^i &= g(\theta_t^i)\mathrm{d}t, \\ \mathrm{d}\theta_t^i &= -\sum_{j\sim i} \chi_R(|x-x'|)\nabla H(\theta_t^i - \theta_t^j)\mathrm{d}t + \mathrm{d}B_t^i, \end{aligned}$$

$$G(\mu) = \frac{1}{2}\int\int \chi_R(|x-x'|)H(\theta-\theta')\mathrm{d}\mu(x,\theta)\mathrm{d}\mu(x',\theta').$$

Proof of the main theorem

Interaction-free Case $F = 0$:

Proposition 3.2 (Laplace duality). *Assume $\alpha > 0$ and $\mu_0 \in \mathcal{M}_F$. Then for all non-negative functions $\varphi \in \mathcal{C}_b(\mathbb{R}^k)$*

$$\mathbb{E}\left(e^{-\langle \mu_t, \varphi \rangle}\right) = e^{-\langle \mu_0, V_t \varphi \rangle}$$

with Hamilton-Jacobi equation $V_t \varphi = \psi_t$

$$\partial_t \psi_t = \alpha L \psi_t - \Gamma(\psi_t), \; \psi_0 = \varphi \in \mathcal{D}$$

$$\Gamma(f, g) = (\sigma^T \nabla f) \cdot \sigma^T \nabla g \;, \Gamma(f) = \Gamma(f, f).$$

□

Moment generating function for $\alpha \mu_t(A)$

$$g(s) = \mathbb{E}\left(s^{\alpha \mu_t(A)} \right) = \mathbb{E}\left(e^{-|\ln(s)| \alpha \mu_t(A)}\right)$$

$$= \exp\left(\left\langle \mu_0, \alpha \ln\left(P_t e^{-|\ln(s)| 1_A}\right)\right\rangle\right).$$

Interacting case $F \neq 0$:

Lemma 3.6. *Let b be of at most linear growth and $F\colon \mathcal{M}_F \times \mathbb{R}^k \to \mathbb{R}^k$ be continuous and bounded. Assume that $(\mu_t)_{t\geq 0}$ is a solution to Equation (6) with initial condition $\mu_0 \in \mathcal{M}^2$. Then $\mu_t \in \mathcal{M}^2$ almost surely for all times $t \geq 0$.*

Proposition 3.7. *For a solution $(\mu_t)_{t\geq 0}$ of equation (6) with initial condition $\mu_0 \in \mathcal{M}^2$ and a function $E \in \mathcal{C}_b^2(\mathcal{M}_F)$, the following process is a local martingale*

$$M_t^E = E(\mu_t) - E(\mu_0) - \int_0^t \left\langle \mu_s, \alpha L \frac{\delta E}{\delta \mu}(\mu_s, \cdot) + F_{\mu_s} \cdot \nabla \frac{\delta E}{\delta \mu}(\mu_s, \cdot) \right\rangle ds$$
$$- \frac{1}{2} \int_0^t \left\langle \mu_s, \sigma\sigma^T : D \frac{\delta^2 E}{\delta \mu^2}(\mu_s, \cdot, \cdot) \right\rangle ds,$$

where above $(D\psi)(z) := \partial_i \partial_{k+j} \psi(z, z)$ for $\psi \in \mathcal{C}^1(\mathbb{R}^k \times \mathbb{R}^k)$, with quadratic variation

$$[M_\cdot^E]_t = \int_0^t \left\langle \mu_s, \Gamma\left(\frac{\delta E}{\delta \mu}\right) \right\rangle ds.$$

Proposition 3.10. *Let the interaction F satisfy Assumption 1.4 and let $(\mu_t)_{t\geq 0}$ be a solution to Equation (6) with initial condition $\mu_0 \in \mathcal{M}_c^2$. Define $\mathbb{Q}$ by $d\mathbb{Q} = d\mathbb{Q} = e^{-M_t^G - \frac{1}{2}[M^G]_t} d\mathbb{P}$ on $\mathcal{F}_t$, for $t \geq 0$. Then, μ_t solves*

$$d\mu_t = L^* \mu_t \, dt + \nabla \cdot (\sqrt{\mu_t} \sigma \, dW_t)$$

on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$, where L^* denotes the dual of L. □