Exercises on continuous functions and limits
Calculus Unit 1, Birindelli

1. Find the point of discontinuity of the functions defined by

$$f(x) = [x^2]$$

$$g(x) = x[x-1]$$

2. Without looking for the result, prove that there exists a solution of

$$e^x = 2x$$

and find an interval of size $1/4$ that contains the solution.

3. Prove that the polinome $P(x) = x^4 + 7x^3 - 9$ has at least two zeroes in $\mathbb{R}$.

4. Prove the following fact: If f is a functions continuous, strictly montone increasing in $[a, b]$ then the inverse function is continuous and strictly monotone increasing.

5. Let $f(x)$ and $g(x)$ be two continuous in (a, b) prove that

$$h(x) = \max\{f(x), g(x)\} \text{ and } k(x) = \min\{f(x), g(x)\}$$

are continuous functions

6. Consider $f : [0, 3] \to \mathbb{R}$ continuous in $[0, 3]$ such that $\text{Im}f = f([0, 3]) = [0, 4]$ Prove that there exists an $x \in [0, 3]$ such that $f(x) = \frac{4}{3}x$

7. For $x_o > 0$ and for $x_1 \in \mathbb{R}$ determine the following limits, using the special limits:

$$a)\ \lim_{x\to x_o} \frac{\log(x)-\log(x_o)}{x-x_o}, \quad b)\ \lim_{x\to x_1} \frac{e^x-e^{x_1}}{x-x_1}, \quad c)\ \lim_{x\to x_1} \frac{x^2-x_1^2}{x-x_1}$$

$$d)\ \lim_{x\to x_1} \frac{x^\alpha-x_1^\alpha}{x-x_1}, \quad e)\ \lim_{x\to x_1} \frac{\sin x-\sin x_1}{x-x_1}$$

(For the point limti "e" use the fact that $\sin(a-b) = 2\cos(\frac{a+b}{2})\sin(\frac{a-b}{2})$)