venerdì 13 novembre 2020 11:02

# Esercizi

1) $f(x,y) = \frac{x+y}{x^2+1}$ Calcolare la matrice Hessiana nei pti $(0,0)$ e $(1,-1)$

$f(x,y) = \frac{x}{x^2+1} + \frac{y}{x^2+1}$

$$\frac{\partial f}{\partial x} = \frac{(x^2+1) - (x+y)2x}{(x^2+1)^2} = \frac{-x^2 - 2xy + 1}{(x^2+1)^2} \leftarrow$$

$$\frac{\partial f}{\partial y} = \frac{1}{x^2+1}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{-x^2-2xy+1}{(x^2+1)^2}\right) =$$

$$= \frac{(-2x-2y)(x^2+1)^2 - (-x^2-2xy+1)2(x^2+1)\cdot 2x}{(x^2+1)^4}$$

messo in evidenza $(x^2+1)$

$$= \frac{[(-2x-2y)(x^2+1) - (-x^2-2xy+1)4x]}{(x^2+1)^3}$$

$$= \frac{2x^3 + 6x^2y - 6x - 2y}{(x^2+1)^3} = \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{1}{x^2+1}\right) = \frac{\partial}{\partial x}\left((x^2+1)^{-1}\right)$$

$$= -1\,(x^2+1)^{-2}\cdot 2x = \frac{-2x}{(1+x^2)^2} = \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{x^2+1}\right) = 0$$

Calcolare $D^2 f(0,0)$ e in $D^2 f(1,-1)$

$$D^2 f(0,0) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(0,0) & \frac{\partial^2 f}{\partial x \partial y}(0,0) \\ \frac{\partial^2 f}{\partial x \partial y}(0,0) & \frac{\partial^2 f}{\partial y^2}(0,0) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$D^2 f(1,-1) = \begin{pmatrix} \frac{2-6-6+2}{2^3} & \frac{\quad}{2^2} \\ -\frac{2}{2^2} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{8}{8} & -\frac{2}{4} \\ -\frac{2}{4} & 0 \end{pmatrix} = \begin{pmatrix} -1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$$

---

**Es 4:** $f(x,y) = x^2 y + 2x - 4y$

Trovare massimi e minimi assoluti in $D = \{(x,y) \text{ t.c. } |x| + |y| \leq 4\}$

1°) Trovare se esistono punti critici all'interno del Dominio $D$

2°) Trovare i massi e minimi sul bordo di $D$.

3°) Calcolare $f$ nei p.ti trovati, il valore massimo è il massimo assoluto, il valore minimo è il minimo assoluto.

1°)

$$\begin{cases} \frac{\partial f}{\partial x} = 2xy + 2 = 0 \\ \frac{\partial f}{\partial y} = x^2 - 4 = 0 \longrightarrow x^2 = 4 \end{cases}$$

$$\downarrow$$

$x = 2$

$x = -2$

Sostituiamo nella prima equazione

$x = 2 \longrightarrow 4y + 2 = 0 \longrightarrow y = -\frac{1}{2}$

$x = -2 \longrightarrow -4y + 2 = 0 \longrightarrow y = \frac{1}{2}$

$P_1 = (2, -\frac{1}{2})$

$P_2 = (-2, \frac{1}{2})$

Controllare se $P_1$ e $P_2$ appartengono all'interno di $D$

cioè se verificano $|x| + |y| < 4$

Per $P_1 \longrightarrow |2| + |-\frac{1}{2}| = 2 + \frac{1}{2} = \frac{5}{2} < 4$

Per $P_2 \longrightarrow |-2| + |\frac{1}{2}| = 2 + \frac{1}{2} = \frac{5}{2} < 4$

2°) $\partial D = \{(x,y) \text{ t.c. } |x| + |y| = 4\}$, $x > 0$, $y > 0$

$$\downarrow$$

$|x| + |y| = x + y = 4$ $\quad r_1$

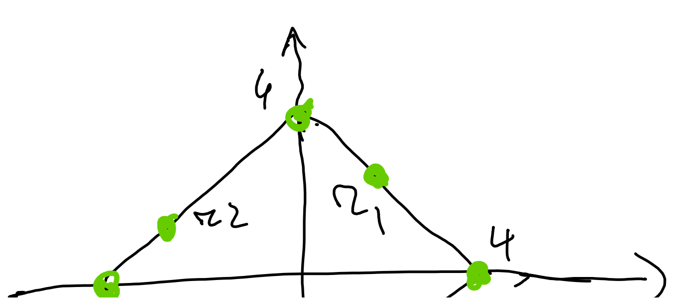
$y = 4 - x$

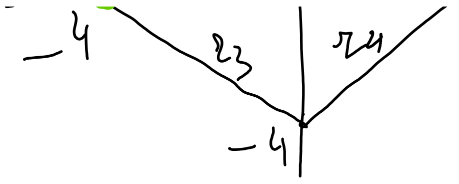
---

$x < 0$, $y > 0$

$|x| + |y| = -x + y = 4$

$y = 4 + x$






$x<0,\ y<0$

$|x|+|y| = -x-y = 4$ $r_3$

$y = -x-4$

$x>0,\ y<0$

$|x|+|y| = x-y = 4$

$y = x-4$

Restringere $f$ su ognuno dei segmenti $r_1, r_2, r_3, r_4$ per ognuno ci si riduce a una funzione di 1 variabile e si trova il max e il minimo

$r_1: y = 4-x$, con $\underline{x \in [0,4]}$

$f(x,y) = x^2y + 2x - 4y$

$f(x, 4-x) = x^2(4-x) + 2x - 4(4-x)$

$= -x^3 + 4x^2 + 6x - 16 = g(x) \nearrow \begin{matrix} g(0) \\ g(4) \end{matrix}$

$\sqrt{36} = 6$

$g'(x) = -3x^2 + 8x + 6 = 0$

$$x = \frac{-4 \pm \sqrt{16+18}}{-3} = \frac{-4 \pm \sqrt{34}}{-3}$$

$$x = \frac{4 \pm \sqrt{34}}{3}$$

$$0 < \frac{4+\sqrt{34}}{3} < \frac{4+\sqrt{36}}{3} = \frac{10}{3} < 4$$

NON È INCLUSO $\longrightarrow$ $\frac{4-\sqrt{34}}{3} < \frac{4-\sqrt{25}}{3} = -\frac{1}{3} < 0$

Calcolare

$g(0), g(4), g\left(\frac{4+\sqrt{34}}{3}\right)$

$f(0,4) = g(0) = -16$

$f(4,0) = g(4) = -x^3 + 4x^2 + 6x - 16 = 24 - 16 = 8$

$$g\left(\frac{4+\sqrt{34}}{3}\right) = -\left(\frac{4+\sqrt{34}}{3}\right)^3 + 4\left(\frac{4+\sqrt{34}}{3}\right)^2 + 6\left(\frac{4+\sqrt{34}}{3}\right) - 16 \simeq 11,4$$

In $r_2$: $y = 4+x \longrightarrow f(x, 4+x) = x^2(4+x) + 2x - 4(4+x) = g(x)$

$4 < x \le 0$

$g(x) = x^3 + 4x^2 - 2x - 16$

$g(0) = f(0,4) = -16$

$g(-4) = 8 - 16 = -8 = f(-4,0)$

$g'(x) = 3x^2 + 8x - 2 = 0$

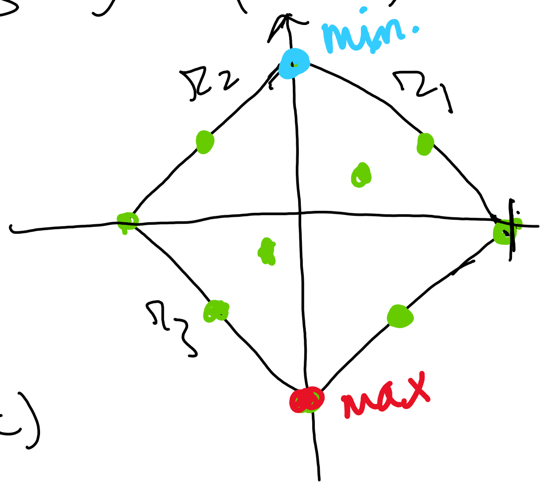
$$x = \frac{-4 \pm \sqrt{16+6}}{3} = \frac{-4 \pm \sqrt{22}}{3}$$

$x = \frac{-4+\sqrt{22}}{3} > 0$ non incluso

$\sqrt{22} < \sqrt{25}$

$$-4 < -3 = \frac{-4-5}{3} < x = \frac{-4-\sqrt{22}}{3} < 0$$

$$-0{,}95 \simeq g\left(\frac{-4-\sqrt{22}}{3}\right) = \left(\frac{-4-\sqrt{22}}{3}\right)^3 + 4\left(\frac{-4-\sqrt{22}}{3}\right)^2 - 2\left(\frac{-4-\sqrt{22}}{3}\right) - 16$$




$r_3: \ y = -x-4$ con $x \in [-4,0]$

$$f(x,-x-4) = x^2 \cdot \overbrace{(-x-4)}^{y} + 2x - 4\overbrace{(-x-4)}^{y}$$

$$= -x^3 - 4x^2 + 6x + 16 = g(x)$$

Calcolare $g(-4)$, $g(0)$ e trovare se $g'(x) = 0$

$g'(x) = -3x^2 - 8x + 6 = 0$

$$x = \frac{4 \pm \sqrt{16+18}}{-3} = \frac{4 \pm \sqrt{34}}{-3}$$

Determinare quali di questi due punti appartengono a $[-4,0]$. $x = \frac{4-\sqrt{34}}{-3} > 0$ non appartiene

$$x = \frac{4+\sqrt{34}}{-3}$$

$g(-4) = f(-4,0) = -8$

$g(0) = f(0,-4) = 16$

$$4\cancel{4},63 \simeq g\left(\frac{4+\sqrt{34}}{-3}\right) = -\left(\frac{4+\sqrt{34}}{-3}\right)^3 - 4\left(\frac{4+\sqrt{34}}{-3}\right)^2 + 6\left(\frac{4+\sqrt{34}}{-3}\right) + 16$$

In $r_4$, $x \in [0,4]$

$y = x-4$ , $x \in [0, 4]$

$$f(x, x-4) = x^2(x-4) + 2x - 4(x-4)$$
$$= x^3 - 4x^2 - 2x + 16 = g(x)$$

Calcolare $g(0), g(4)$ e i pt. in $[0,4]$ che verificano $g'(x)=0$

$$g'(x) = 3x^2 - 8x - 2 = 0$$

$$x = \frac{4 \pm \sqrt{16+6}}{3} = \frac{4 \pm \sqrt{22}}{3}$$

$$x = \frac{4-\sqrt{22}}{3} < 0 \quad \notin [0,4]$$

$$0 < \frac{4+\sqrt{22}}{3} < \frac{4+\sqrt{25}}{3} = \frac{9}{3} = 3 < 4$$

$$g(0) = f(0,-4) = 16$$
$$g(4) = f(4,0) = 8$$
$$0{,}94 = g\left(\frac{4+\sqrt{22}}{3}\right) = \left(\frac{4+\sqrt{22}}{3}\right)^3 - 4\left(\frac{4+\sqrt{22}}{3}\right)^2 - 2\left(\frac{4+\sqrt{22}}{3}\right) + 16$$

Punti Interni critici

$$f\left(2, -\frac{1}{2}\right) = 4 \cdot \left(-\frac{1}{2}\right) + 2 \cdot 2 - 4\left(-\frac{1}{2}\right) = 4$$

$$f\left(-2, \frac{1}{2}\right) = 4 \cdot \left(\frac{1}{2}\right) + 2(-2) - 4\left(\frac{1}{2}\right) = -4$$

$$f(4, 0) = 8$$
$$f(0, 4) = -16$$
$$f(-4, 0) = -8$$
$$f(0, -4) = 16$$

$$f\left(\frac{4+\sqrt{34}}{3}, 4 - \frac{4+\sqrt{34}}{3}\right) \simeq 1{,}4$$
$$f\left(\frac{-4-\sqrt{22}}{3}, \frac{-4-\sqrt{22}}{3} + 4\right) \simeq 0{,}95$$
$$f\left(\frac{4+\sqrt{34}}{-3}, \ldots\right) \simeq 1{,}63$$
$$f\left(\frac{4+\sqrt{22}}{3}, \frac{4+\sqrt{22}}{3} - 4\right) = 0{,}94$$

---

$$f(x,y) = yx^2 + 2xy - y^2 - y$$

Trovare i pti critici e determinarne la natura cioè det. se sono pt di max

o minimo locale
ptr. di sella o nessuno
dei tre.

$$\begin{cases} \frac{\partial f}{\partial x} = 2xy + 2y = 0 \\ \frac{\partial f}{\partial y} = x^2 + 2x - 2y - 1 = 0 \end{cases}$$

$$\boxed{\begin{cases} 2y(x+1) = 0 \\ x^2+2x-2y-1=0 \end{cases}} \longrightarrow y = 0 \text{ o } x = -1$$

$y=0 \Downarrow$

$x^2+2x-1=0$

$x = -1 \pm \sqrt{1+1}$

$x = -1+\sqrt{2}$

$x = -1-\sqrt{2}$

$x=-1 \Downarrow$

$1-2-2y-1=0$

$2y=-2$

$y=-1$

$P_1 = (-1+\sqrt{2}, 0)$, $P_2 = (-1-\sqrt{2}, 0)$, $P_3 = (-1,-1)$

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(2xy+2y) = 2y$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}(2xy+2y) = 2x+2$

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(x^2+2x-2y-1) = -2$

$$D^2 f(-1+\sqrt{2}, 0) = \begin{pmatrix} 0 & +2\sqrt{2} \\ +2\sqrt{2} & -2 \end{pmatrix}$$

$\det D^2 f = -(+2\sqrt{2})^2$
$= -8 < 0$

$(-1+\sqrt{2}, 0)$ è ptr di Sella

$$D^2 f(-1-\sqrt{2}, 0) = \begin{pmatrix} 0 & -2\sqrt{2} \\ -2\sqrt{2} & -2 \end{pmatrix} \rightarrow \det D^2 = -(-2\sqrt{2})^2 = -8 < 0$$

$\Downarrow$

$(-1-\sqrt{2}, 0)$ è ptr di Sella

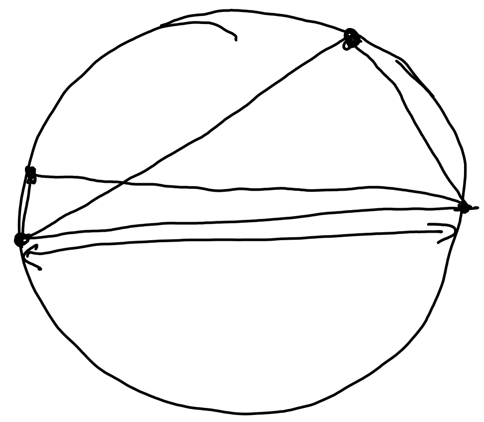
$$D^2 f(-1,-1) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

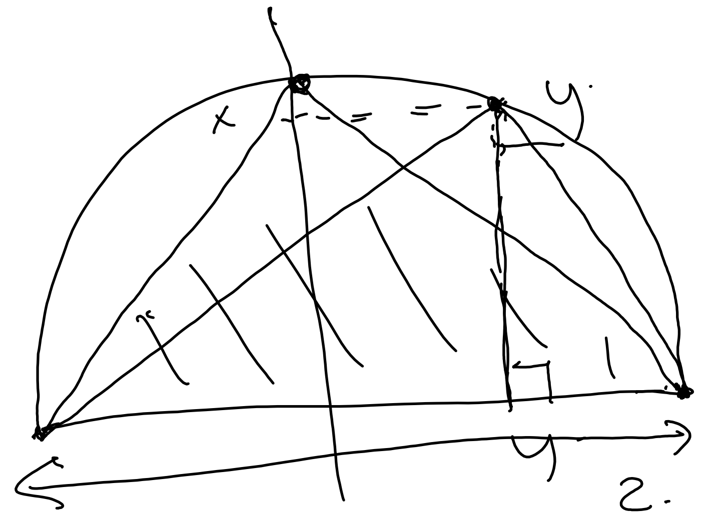
$\det D^2 f = -2 \cdot -2 = 4 > 0$

... tr di

$\frac{\partial^2 f}{\partial x^2} < 0 \implies (-1,-1)$ è un p.to

massimo locale






$(x, y)$

$x^2 + y^2$

$A = \frac{2 \cdot y}{2} = y$