Funzioni di più variabili

1°) Determinare e disegnare l'insieme di definizione.

$\log a \longrightarrow a > 0$

$\sqrt{a} \longrightarrow a \geq 0$

$\frac{1}{a} \longrightarrow a \neq 0$

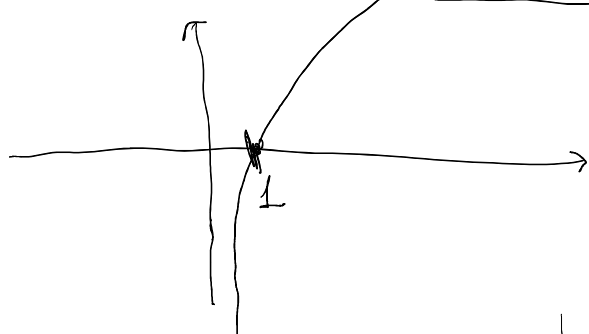
$\operatorname{tg} a \longrightarrow a \neq \frac{\pi}{2} + k\pi$

$$f(x,y) = \frac{\sqrt{9-(x-3)^2-(y-4)^2}}{\log(x^2-y^2)}$$

x^2-y^2 argomento del logaritmo $\rightarrow \boxed{x^2-y^2>0}$

argomento della radice $\boxed{9-(x-3)^2-(y-4)^2 \geq 0}$

argomento del rapporto: $\boxed{\log(x^2-y^2) \neq 0}$



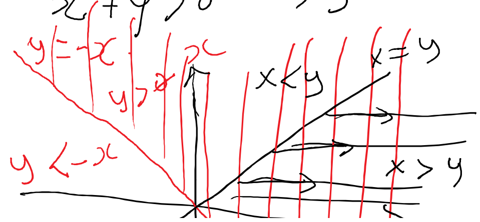
$$\begin{cases} x^2-y^2 > 0 & (1) \\ \log(x^2-y^2) \neq 0 & (2) \\ 9-(x-3)^2-(y-4)^2 \geq 0 & (3) \end{cases}$$

$x^2-y^2 = (x-y)(x+y)$

$(x-y)(x+y) > 0$

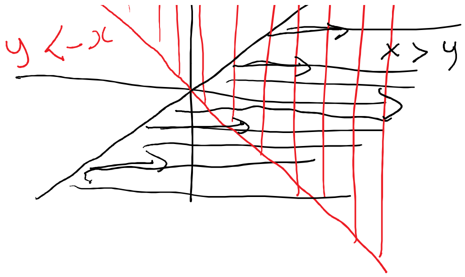
$x-y>0 \longrightarrow x>y$.

$x+y>0 \longrightarrow y>-x$.



$x>y$ dove? $x=y$

$x > y$ dove? $\quad x = y$



$y > -x$ dove? $\boxed{y = -x}$

Se [riquadro rosso a reticolo] $\begin{cases} x > y \\ y > -x \end{cases} \Longrightarrow \underset{+}{(x-y)}\underset{+}{(x+y)} > 0$

Se [riquadro tratteggiato] $\begin{cases} x > y \\ y < -x \end{cases} \Longrightarrow \underset{+}{(x-y)}\underset{-}{(x+y)} < 0$

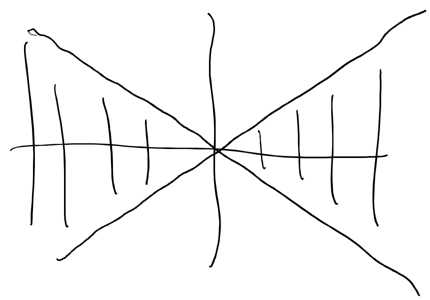
Se [riquadro rosso tratteggiato] $\begin{cases} x < y \\ y > -x \end{cases} \Longrightarrow \underset{-}{(x-y)}\underset{+}{(x+y)} < 0$

NON APPARTIENE

Se [riquadro vuoto] $\begin{cases} x < y \\ y < -x \end{cases} \Longrightarrow \underset{-}{(x-y)}\underset{-}{(x+y)} > 0$

Conclusione

$x^2 - y^2 > 0 \longrightarrow$



(Attenzione: $\boxed{x^2 - y^2 > 0} \Longrightarrow \boxed{x^2 > y^2 \quad \cancel{\Downarrow} \quad \cancel{x > y}}$ $\cancel{x > \pm y}$

$x^2 - y^2 = 0$

$x^2 = y^2$

$\Downarrow$

$x = y$

$x = -y$

$x = -2 \to x^2 = 4$

$y = 1 \to y^2 = 1$

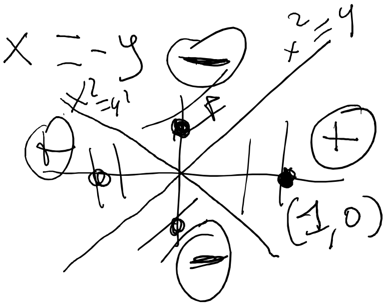
$4 > 1$

$\Downarrow$

$-2 > 1$

$(0,1)$

$0^2 - (1)^2 = -1 < 0$



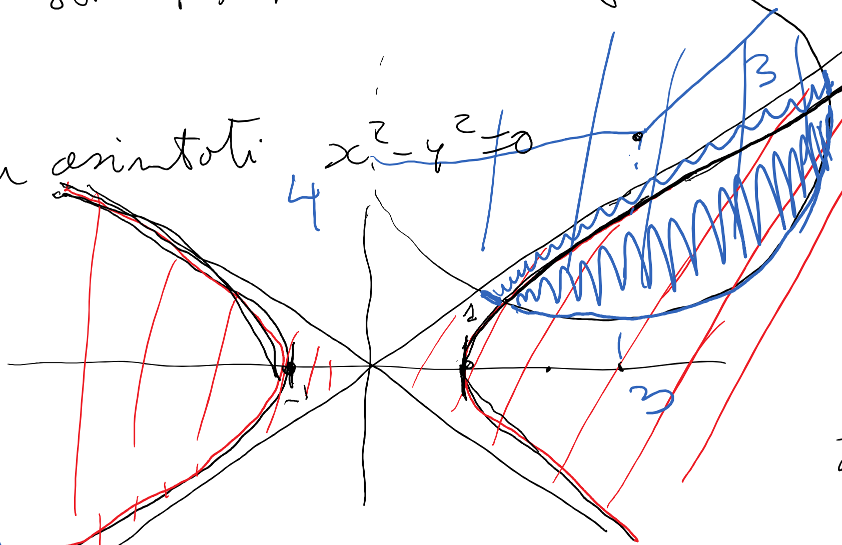
$1^2 - 0^2 = 1 > 0 \Rightarrow x^2 - y^2 > 0$)

... $(x^2 - y^2) = 0 \Longrightarrow \boxed{x^2 - y^2 = 1}$

② $\log(x^2-y^2) \neq 0 \longmapsto \log(x^2-y^2)=0 \longmapsto \boxed{x^2-y^2=1}$

$\uparrow$ iperbole

Con asintoti $x^2-y^2=0$

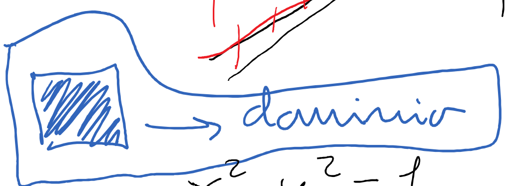


$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

$\downarrow$

asintoti

$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=0$



$x^2-y^2=1$

$y^2=x^2-1$

$\nearrow y=\sqrt{x^2-1} \longrightarrow$

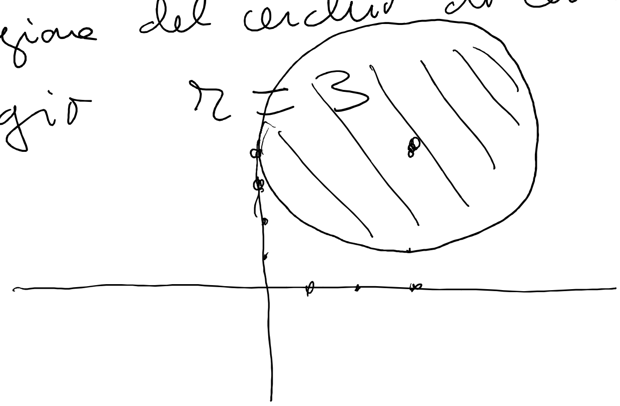
$\searrow y=-\sqrt{x^2-1} \longrightarrow$

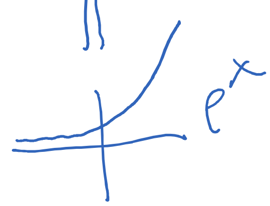
studio di funzione

③ $9-(x-3)^2-(y-4)^2 \geq 0 \longrightarrow =$

$\boxed{3^2=9 \geq (x-3)^2+(y-4)^2}$

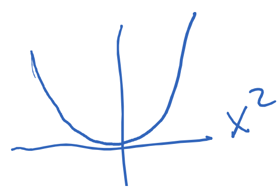
Equazione del cerchio di centro $(3,4)$ e raggio $r=3$



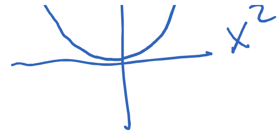


$x^2+y^2=r^2$ cerchio

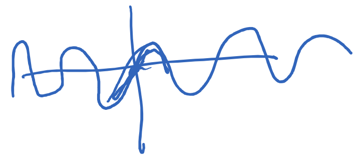
$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ Iperbole



$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ Iperbole

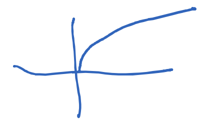


$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellisse



$x^2 \to (x-x_0)^2$ $y^2 \to (y-y_0)^2$

$(0,0) \to (x_0, y_0)$

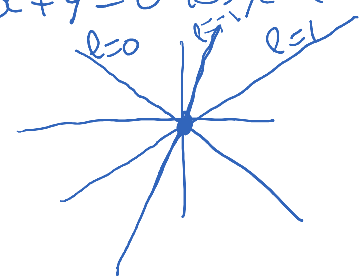


Insieme di livello "l" di una funzione

$$f = \{(x,y) \text{ t.c. } f(x,y) = l\}$$

$$\boxed{f(x,y) = \frac{x+y}{2x-3y}} \to l=0,\ l=1,\ l=-1,\ l=2$$

$f(x,y)=0 \iff \frac{x+y}{2x-3y} = 0 \iff x+y=0 \iff y=-x$



$f(x,y)=1 \iff \frac{x+y}{2x-3y} = 1$

$\Downarrow$

$x+y = 2x-3y$

$4y = x \Rightarrow y = \frac{1}{4}x$

$f(x,y) = -1 \iff \frac{x+y}{2x-3y} = -1 \iff x+y = -2x+3y$

$3x = 2y$

$y = \frac{3}{2}x$

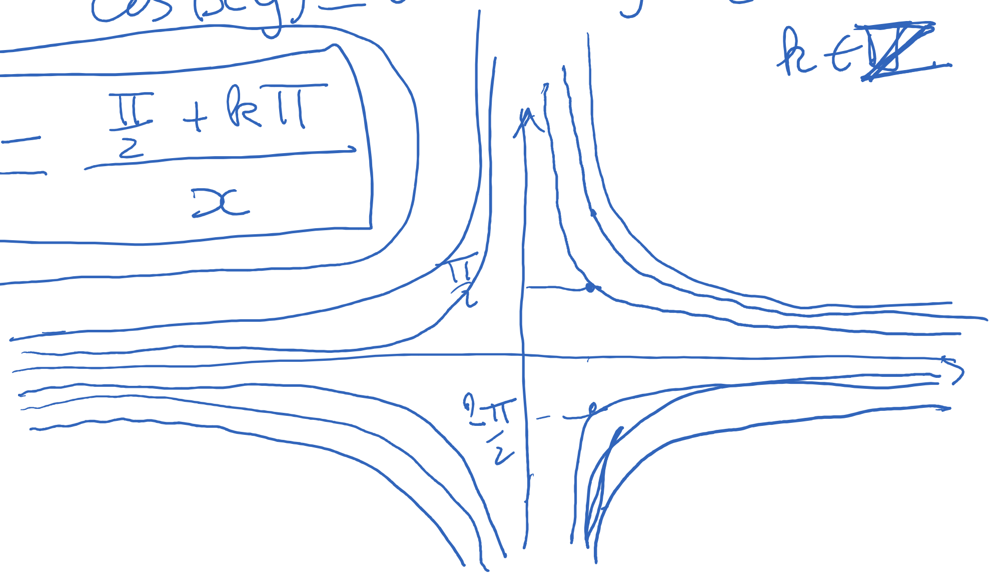
OSS: Se insiemi di livelli diversi si "toccano" in un punto, il limite della funzione in quel punto non esiste.

$f(x,y) = \cos(xy)$ livello 0.

Trovare (x,y) t.c. $\cos(xy) = 0$ $\quad xy = \frac{\pi}{2} + k\pi$

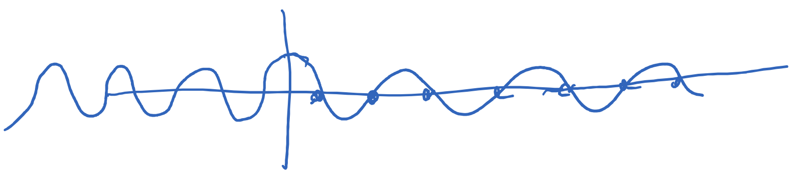
$k \in \mathbb{Z}$

$$\boxed{y = \frac{\frac{\pi}{2} + k\pi}{x}}$$



$k=0$

$k=1$



Esercizio 7: $f(x,y) = 4x^2 - y^2$.

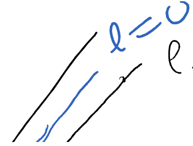
Disegnare il grafico di questa funzione usando: segno della funzione, curve di livello e piano tangente $(0,0)$.

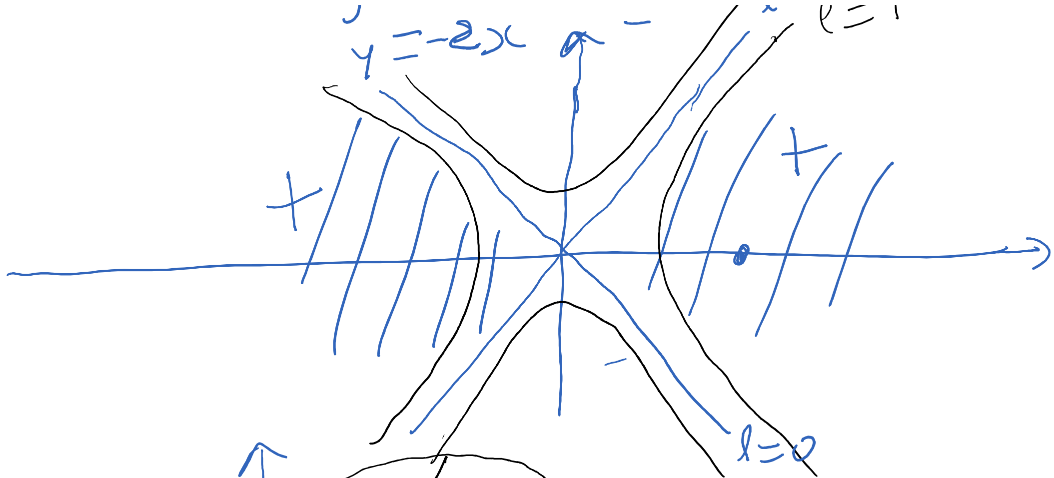
$$4x^2 - y^2 > 0$$

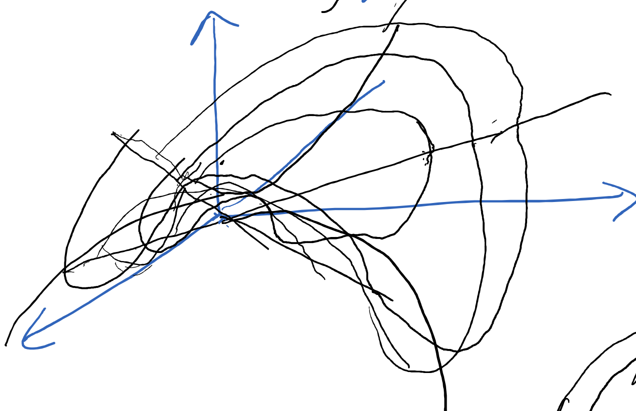
$$(2x-y)(2x+y) > 0 \quad (=0)$$

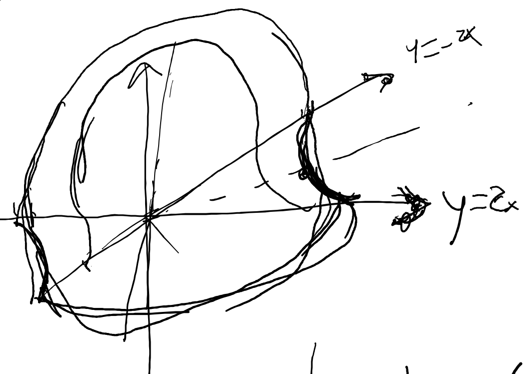
$y = 2x$

$y = -2x$







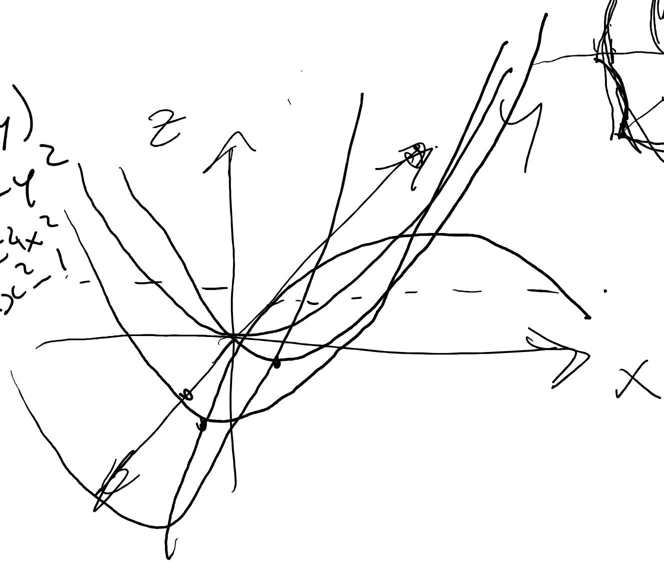


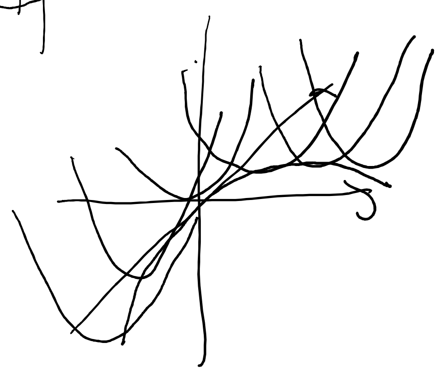
$z = f(x,y)$

$z = 4x^2 - y^2$

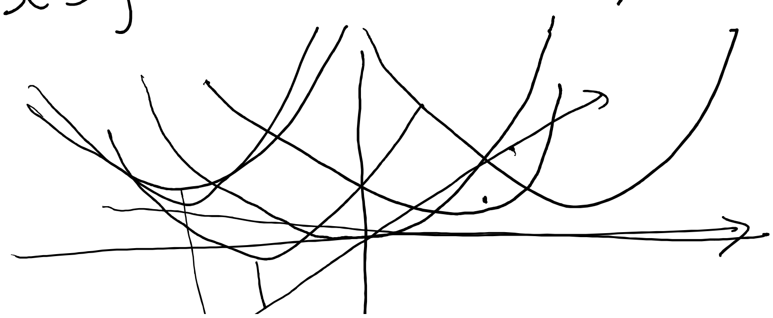
$y=0 \rightarrow z = 4x^2$

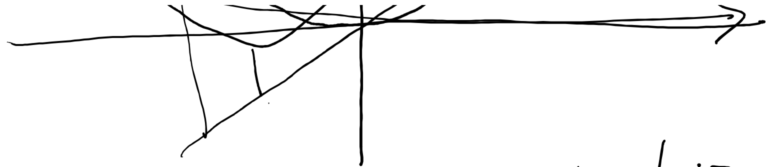
$y=1 \quad z = 4x^2 - 1$

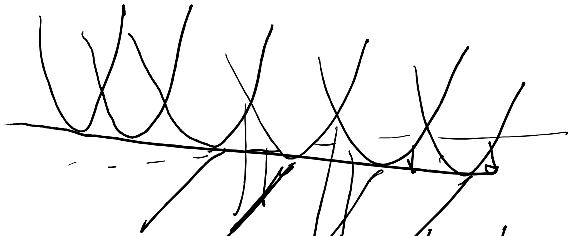


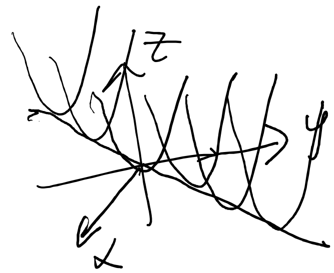


$f(x,y) = x^2 - y$ $\qquad$ $z = x^2 - y$









I) Trovare Max, Min relativi, assoluti per le seguenti:

1.) $f(x,y) = 25x^3 + 12y^3 - 75x^2 - 36y$

2.) $f(x,y) = x^2 - xy^2 + x^2 y$

3.) $f(x,y) = (x-y)^2 - 2(x-y)$

4.) $f(x,y) = x^2 \log(y-1) - 2y + y^2$ $(y>1)$

5.) $f(x,y) = -\sin x \sin 2y$

T. di Fermat

(x_0,y_0) è un punto estremale

(x_0, y_0) ...

locale di una funzione f derivabile allora $\nabla f(x_0, y_0) = (0, 0)$

Se (x_0, y_0) è un p.to critico $\nabla f(x_0, y_0) = (0,0)$

e $D^2 f(x_0, y_0) = \begin{pmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{xy}(x_0, y_0) & f_{yy}(x_0, y_0) \end{pmatrix}$ t.c.

$\det D^2 f > 0$ ~~e~~ e $f_{xx}(x_0, y_0) > 0 \rightarrow$ <u>minimo</u>

$\det D^2 f > 0$ e $f_{xx}(x_0, y_0) < 0 \rightarrow$ max

$\det D^2 f < 0 \Longrightarrow$ p.to di <u>Sella</u>

Se $\det D^2 f = 0$ non abbiamo informazioni

1) $f(x,y) = 25x^3 + 12y^3 - 75x^2 - 36y$

Cercare i punti critici:

$$\begin{cases} \dfrac{\partial f}{\partial x} = 75x^2 - 2 \cdot 75x = 75x(x-2) = 0 \\ \dfrac{\partial f}{\partial y} = 36y^2 - 36 = 36(y^2 - 1) = 0 \end{cases}$$

$$\begin{cases} x = 0 \quad \text{o} \quad x = 2 \leftarrow \\ y^2 = 1 \rightarrow y = 1 \quad \text{o} \quad y = -1 \end{cases}$$

I punti critici: $(0,1)$; $(0,-1)$; $(2,1)$; $(2,-1)$

$\Big(150(x-1) \qquad 0 \Big)$

$(0,1)$,

$$D^2f(x,y) = \begin{pmatrix} 150(x-1) & 0 \\ 0 & 72y \end{pmatrix}$$

$$D^2f(0,1) = \begin{pmatrix} -150 & 0 \\ 0 & 72 \end{pmatrix} \rightarrow \det D^2 f = -150 \cdot 72 < 0$$

$\Downarrow$

p.to di sella

né max, né min el.

$$D^2f(0,-1) = \begin{pmatrix} -150 & 0 \\ 0 & -72 \end{pmatrix} \rightarrow \det D^2 f > 0$$

$$\frac{\partial^2 f}{\partial x^2}(0,-1) < 0$$

$(0,-1)$ pto di max locale

$$D^2f(2,1) = \begin{pmatrix} 150 & 0 \\ 0 & 72 \end{pmatrix} \rightarrow \det D^2 f > 0$$

$$\frac{\partial^2 f}{\partial x^2}(2,1) > 0$$

$(2,1)$ pto di min. locale

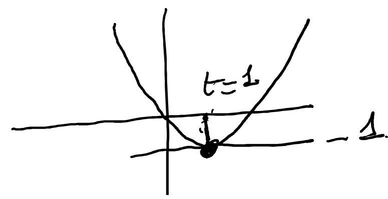
$$D^2f(2,-1) = \begin{pmatrix} 150 & 0 \\ 0 & -72 \end{pmatrix} \rightarrow \det D^2 f < 0$$

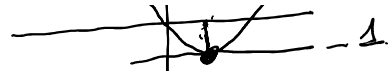
$\Downarrow$

pto di sella.

3) $f(x,y) = (\underbrace{x-y}) ^2 - 2(\underbrace{x-y})$

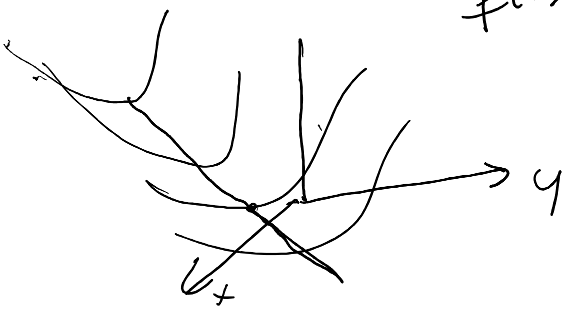
$$g(t) = \underbrace{t^2 - 2t} = (t-1)^2 - 1$$





$\boxed{x-y=1}$ $f(x,y)=1^2-2\cdot 1=-1$

$f(x,y)\geq -1$



$y=x-1$

$\nabla f(x,y)$

$$\begin{cases} \frac{\partial f}{\partial x}=2(x-y)-2=0 \implies x-y=1 \\ \frac{\partial f}{\partial y}=-2(x-y)+2=0 \end{cases}$$

$(x,x-1)$ sono tutti i pti critici di f.

$\boxed{f(x,x-1)=-1} \leq f(x,y)$

sono tutti punti di minimo assoluto.

$$f(x,y)=(x-y)^2-2(x-y)$$
$$=\left[(x-y)-1\right]^2-1\geq -1=f(x,x-1)$$

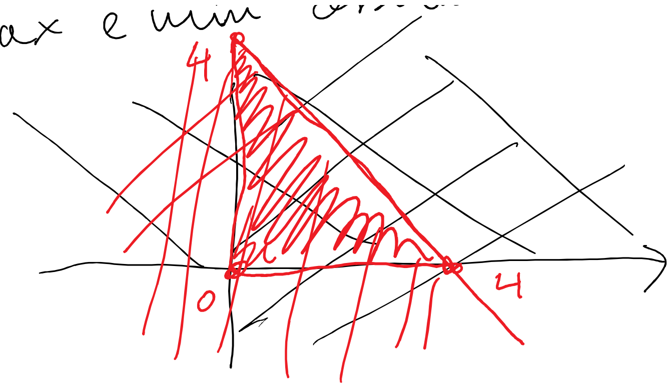
$$D^2f=\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \longrightarrow \det D^2 f=0$$

$$f(x,y)=x^2y+xy^2+x^3-x$$

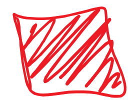
max e min assoluti in $D=\{(x,y);\ x\geq 0,\ y\geq 0,\ x+y\leq 4\}$

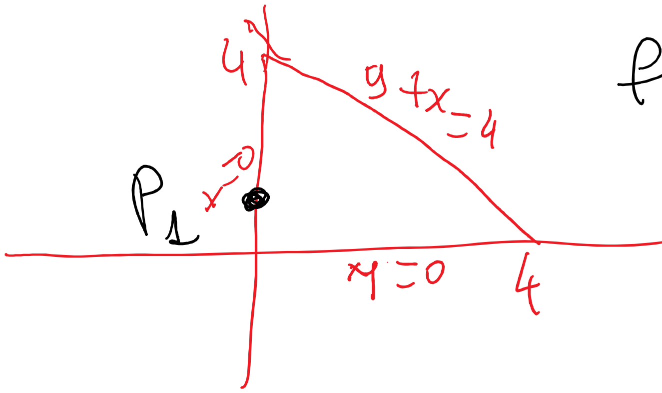
max e min ...

$$\left.\begin{array}{l} y \geq 0 \\ x+y \leq 4 \end{array}\right\}$$



$y \leq -x+4$

 $= D$



$$f(x,y) = x^2y + xy^2 + x^3 - x$$

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2xy + y^2 + 3x^2 - 1 = 0 \\[2mm] \dfrac{\partial f}{\partial y} = x^2 + 2xy = 0 \end{cases}$$

$x(x+2y)=0$

$x=0 \qquad x=-2y$

$1^\circ eq \to y^2-1=0$

$y=1$ o $y=-1$

$\boxed{(0,1)}$

$(0,-1)$

$1^\circ eq.$

$$-4y^2 + y^2 + 12y^2 - 1 = 0$$

$$9y^2 - 1 = 0 \implies y^2 = \frac{1}{9} \begin{cases} y = \frac{1}{3} \\ y = -\frac{1}{3} \end{cases}$$

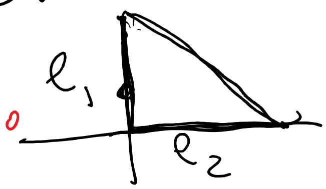
$y=\frac{1}{3} \to x = -\frac{2}{3} \qquad \left(-\frac{2}{3}, \frac{1}{3}\right)$

$y=-\frac{1}{3} \to x = \frac{2}{3} \qquad \left(\frac{2}{3}, -\frac{1}{3}\right)$

Osservazione: I punti $(0,-1)$, $\left(-\frac{2}{3}, \frac{1}{3}\right)$ e $\left(\frac{2}{3}, -\frac{1}{3}\right)$ non appartengono a D.

Osservazione: I punti $(0, \ldots)$ $\ldots$ e $\left(\frac{2}{3}, -\frac{1}{3}\right)$ non appartengono a D.

Rimane solo $\boxed{(0,1) = P_1}$ $\to f(P_1) = 0$



$f(x,y)$ $\quad \ell_1 : \{ x = 0 \,, \; 0 \le y \le 4 \}$

$\boxed{f(0,y) = 0}$ $\quad \forall y$

$f(x,y) \to \ell_2 : \{ y = 0 \,, \; 0 \le x \le 4 \}$

$f(x,0) = x^3 - x = g(x) \to g'(x) = 3x^2 - 1$

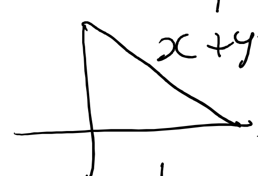
$g'(x) = 0 \to x = \frac{1}{\sqrt{3}}$

$\cancel{x = -\frac{1}{\sqrt{3}}}$

$g(0) = 0$

$g(4) = 4^3 - 4 = 4(16-1) = 60$

$g\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}\left(\frac{1}{3} - 1\right) = -\frac{2}{3\sqrt{3}}$



$f(x,y) \longrightarrow \ell_3 = \{ y = 4 - x \,, \; 0 \le x \le 4 \}$

$f(x,y) = x^2 y + y^2 x + x^3 - x$

$f(x, 4-x) = x^2(4-x) + (4-x)^2 x + x^3 - x$

$= 4x^2 - x^3 + 16x - 8x^2 + x^3 + x^3 - x$

$= x^3 - 4x^2 + 15x = g(x)$

$g'(x) = 3x^2 - 8x + 15 = 0$ Imp.

$\Delta = 16 - 45 < 0$

$f(0,4) = g(0) = 0 \qquad g(4) = 60 = f(4,0)$

$$\text{min} \; -\frac{2}{3\sqrt{3}} < 0 < 60 \longrightarrow \text{Max}$$

$$\max_D f(x,y) = 60 = f(4,0)$$

$$\min_D f(x,y) = -\frac{2}{3\sqrt{3}} = f\left(\frac{1}{\sqrt{3}}, 0\right).$$