Integrali doppi

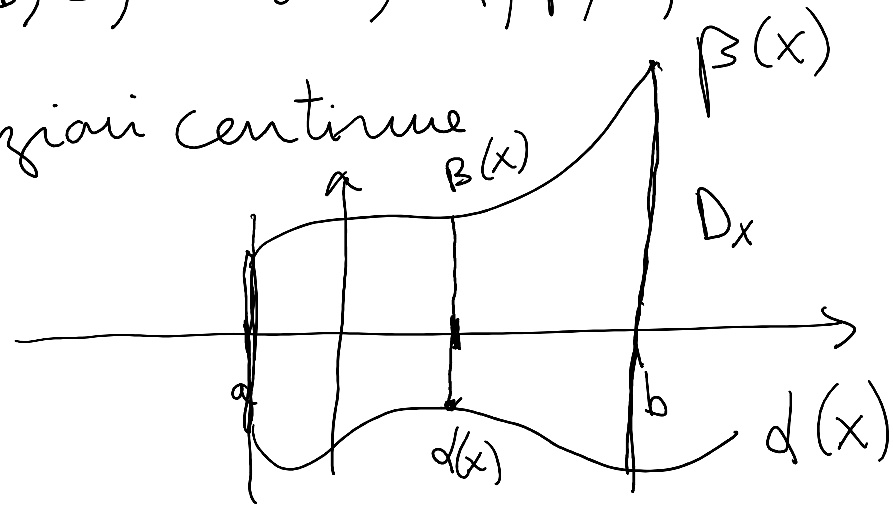
$f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$

Formula di riduzione: e

$D_x = \{ (x,y) \in \mathbb{R}^2 \ a \leq x \leq b, \ \alpha(x) \leq y \leq \beta(x) \}$

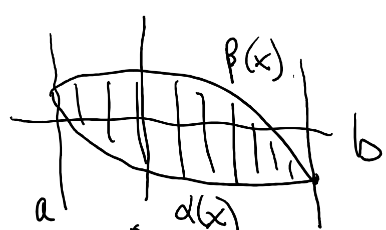
$D_y = \{ (x,y) \in \mathbb{R}^2, \ c \leq y \leq d, \ \gamma(y) \leq x \leq \delta(y) \}$

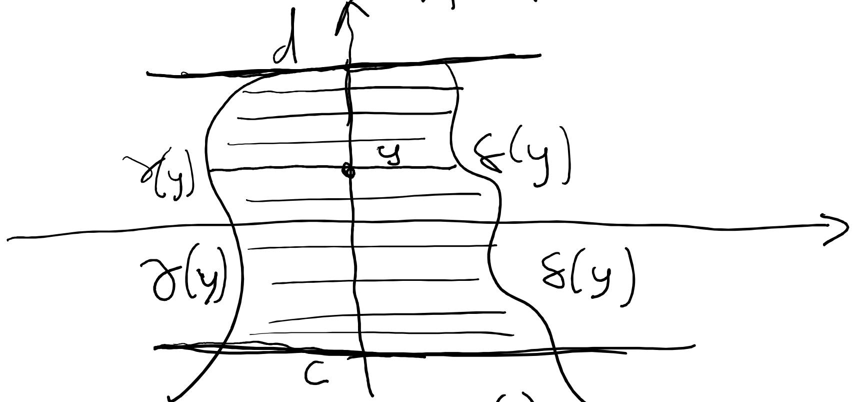
$a, b, c, d \in \mathbb{R}$, $\alpha, \beta, \gamma, \delta$ sono delle funzioni continue



$x = a$

$x = b$





$h \quad \beta(x)$

$$\iint_{D_x} f(x,y)\,dx\,dy = \int_a^b \underbrace{\left(\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy\right)}_{g(x)} dx$$

$D_x = \{a \le x \le b,\ \alpha(x) \le y \le \beta(x)\}$

trovare la primitiva considerando x costante come funzione della sola y sostituiste a y le funzione $\beta(x)$ e $\alpha(x)$

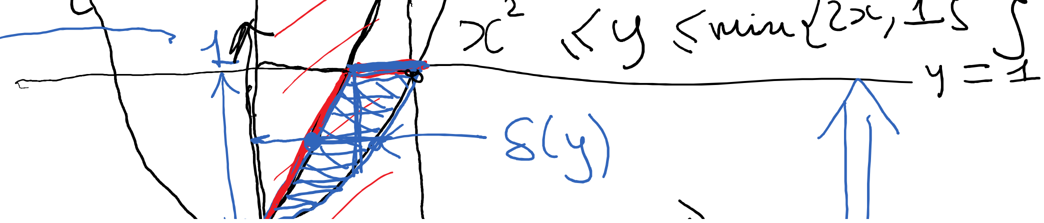
$$\iint_{D_y} f(x,y)\,dx\,dy = \int_c^d \underbrace{\left(\int_{\gamma(y)}^{\delta(y)} f(x,y)\,dx\right)}_{g(y)} dy$$

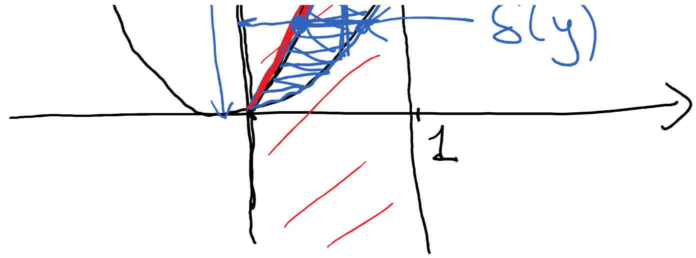
$D_y = \{c \le y \le d,\ \gamma(y) \le x \le \delta(y)\}$

Esercizio

$$\iint_D \frac{2x}{4-y}\,dx\,dy$$

$$\boxed{D = \{(x,y) \in \mathbb{R}^2 :\ 0 \le x \le 1,\ x^2 \le y \le \min\{2x, 1\}\}}$$





$$\boxed{D_y = \left\{ (x,y) \in \mathbb{R}^2 \quad 0 \leq y \leq 1;\ \frac{1}{2}y \leq x \leq \sqrt{y} \right\}}$$

$$y = x^2 \longrightarrow x = \sqrt{y} = \delta(y)$$

$$y = 2x \longrightarrow x = \frac{1}{2}y = \gamma(y)$$

$$\iint_{D_y} \frac{2x}{4-y}\,dx\,dy = \int_0^1 \left(\int_{\frac{1}{2}y}^{\sqrt{y}} \frac{2x}{4-y}\,d\underline{\underline{x}} \right) dy$$

$$= \int_0^1 \frac{1}{4-y} \cdot x^2 \Big|_{\frac{y}{2}}^{\sqrt{y}} dy = \int_0^1 \frac{1}{4-y}\left(y - \frac{y^2}{4}\right) dy$$

NON CI SONO PIÙ "X".

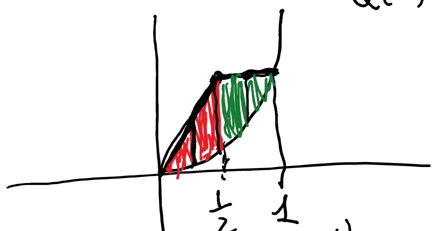
$$= \frac{1}{4} \int_0^1 \frac{4y - y^2}{(4-y)}\,dy = \frac{1}{4} \int_0^1 \frac{y(4-y)}{(4-y)}\,dy = \frac{1}{4}\int_0^1 y\,dy$$

$$= \frac{1}{4} \frac{y^2}{2}\Big|_0^1 = \frac{1}{8} \longrightarrow \boxed{\text{È UNA COSTANTE}}$$

$$\iint_D \frac{2x}{4-y}\,dx\,dy$$

D

$$D=\{(x,y) \text{ t.c. } \underline{\underline{0}} \le x \le \underline{\underline{1}},\ \underbrace{x^2}_{\alpha(x)} \le y \le \underbrace{\min\{2x,1\}}_{\beta(x)}\}$$



$$\iint_D \frac{2x}{4-y}\,dx\,dy = \int_0^1 \left(\int_{x^2}^{\min\{2x,1\}} \frac{2x}{4-y}\,dy \right) dx$$

$$= \int_0^{\frac{1}{2}} \left(\int_{x^2}^{2x} \frac{2x}{4-y}\,dy \right) dy + \int_{\frac{1}{2}}^{1} \left(\int_{x^2}^{1} \frac{2x}{4-y}\,dy \right) dx$$

$$\int_{x^2}^{2x} \frac{1}{4-y}\,dy = -\log(4-y)\Big|_{x^2}^{2x} = -\log(4-2x)+\log(4-x^2)$$

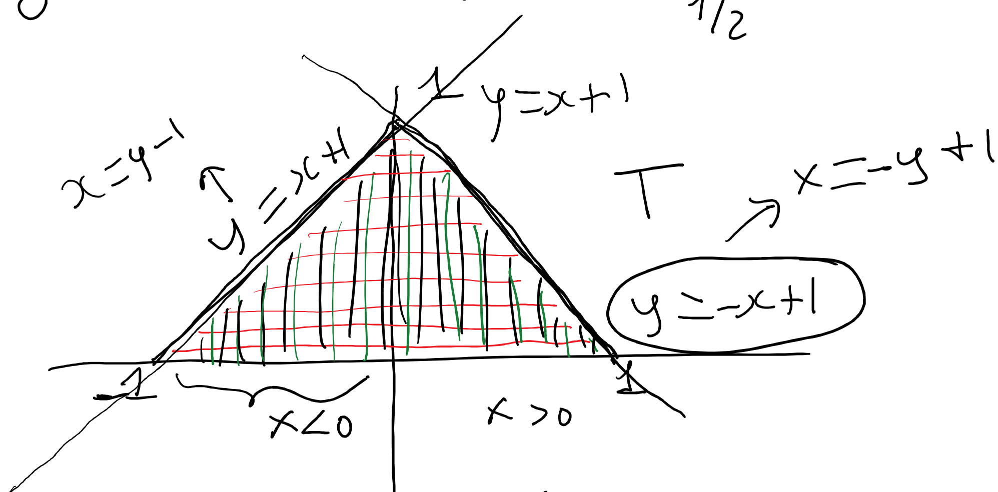
$$= \log(4-x^2)-\log(4-2x)$$

$$\int_{x^2}^{1} \frac{1}{4-y}\,dy = -\log(4-y)\Big|_{x^2}^{1} = -\log 3+\log(4-x^2)$$

$$= \int_0^{1/2} 2x\left[\log(4-x^2)-\log(4-2x)\right]dx + \int_{1/2}^{1} 2x\left[\log(4-x^2)-\log 3\right]dx$$

$$\int_0^1\left(\int_{x^2}^{\min\{2x,1\}} \frac{2x}{4-y}\,dy\right)dx = \int_0^1 2x\left(-\log(4-y)\right)\Big|_{x^2}^{\min(2x,1)} dx$$

$$= \int_0^{1/2} 2x\left(-\log(4-y)\right)\Big|_{x^2}^{2x} dx + \int_{1/2}^{1} 2x\left(-\log(4-y)\right)\Big|_{x^2}^{1} dx$$



$$\iint_T f(x,y)\,dx\,dy = \iint_T y+x^2\,dx\,dy$$

$$T=\{(x,y) \text{ t.c. } -1\le x\le 1,\ 0\le y\le \min\{1+x,\ 1-x\}\}$$

$$=\int_{-1}^{1}\left(\int_0^{\min\{1+x,\,1-x\}} y+x^2\,dy\right)dx =$$

$$=\int^{1} y^2 + x^2 y\,\Big|^{\min\{1+x,\,1-x\}} dx$$

$$= \int_{-1}^{1} \frac{y^2}{2} + x^2 y \Big|_{0} \quad dx$$

$$= \int_{-1}^{0} \frac{y^2}{2}+x^2y \Big|_0^{1+x} dx + \int_0^1 \frac{y^2}{2}+x^2y\Big|_0^{1-x} dx$$

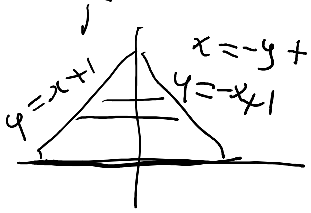
$$= \int_{-1}^{0} \frac{(1+x)^2}{2} + x^2(1+x)\,dx + \int_0^1 \frac{(1-x)^2}{2}+x^2(1-x)\,dx$$

$$= \frac{(1+x)^3}{6} + \frac{x^3}{3}+\frac{x^4}{4}\Big|_{-1}^{0} + \left(-\frac{(1-x)^3}{6}+\frac{x^3}{3}-\frac{x^4}{4}\right|_0^1$$

$$= \frac{1}{6} - \left[-\frac{1}{3}+\frac{1}{4}\right] + \left[\frac{1}{3}-\frac{1}{4}+\frac{1}{6}\right] = \frac{2}{6}+\frac{2}{3}-\frac{2}{4}$$

$$= 1-\frac{1}{2}=\frac{1}{2}$$

$$\iint_T y+x^2\,dx\,dy = \int_0^1 \int_{y-1}^{-y+1} y+x^2\,dx \quad dy$$



$$= \{(x,y);\ 0 \le y \le 1,\ y-1 \le x \le -y+1\}$$

$$= \int_0^1 yx+\frac{x^3}{3}\Big|_{y-1}^{-y+1} dy = \int_0^1 y(-y+1)+\frac{(-y+1)^3}{3} -$$

$$\left(y(y-1) + \frac{(y-1)^3}{3}\right) dy$$

$$\left(y(y-1) + \frac{(y-1)^3}{3}\right) dy$$

$$= \int_0^1 2y^2 + 2y + \frac{(-y+1)^3}{3} - \frac{(y-1)^3}{3}\, dy$$

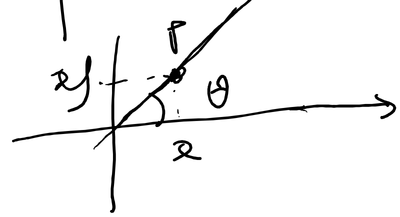
$$= \frac{2y^3}{3} + y^2 - \frac{(-y+1)^4}{12} - \frac{(y-1)^4}{12} \Big|_0^1$$

$$= -\frac{2}{3} + 1 + \frac{1}{12} + \frac{1}{12} = -\frac{2}{3} + 1 + \frac{1}{6} = 1 - \frac{1}{2} = \frac{1}{2}$$

Coordinate polari

$\rho = \sqrt{x^2+y^2}$

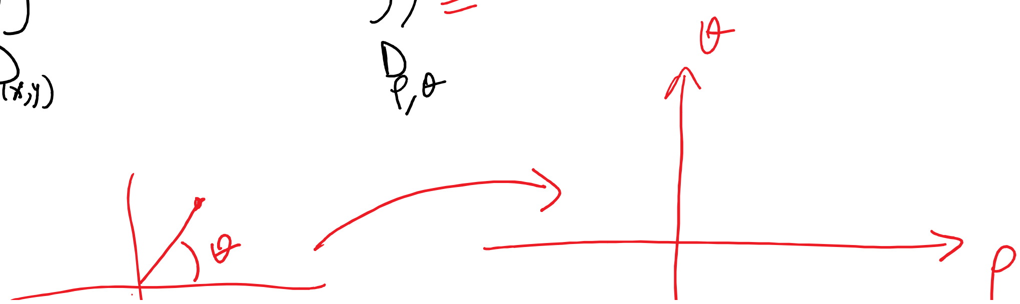
$x \qquad \boxed{\theta = \mathrm{arctg}\, \frac{y}{x}}$



$$\begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \end{cases} \qquad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \theta = \mathrm{arctg}\, \frac{y}{x} \end{cases}$$

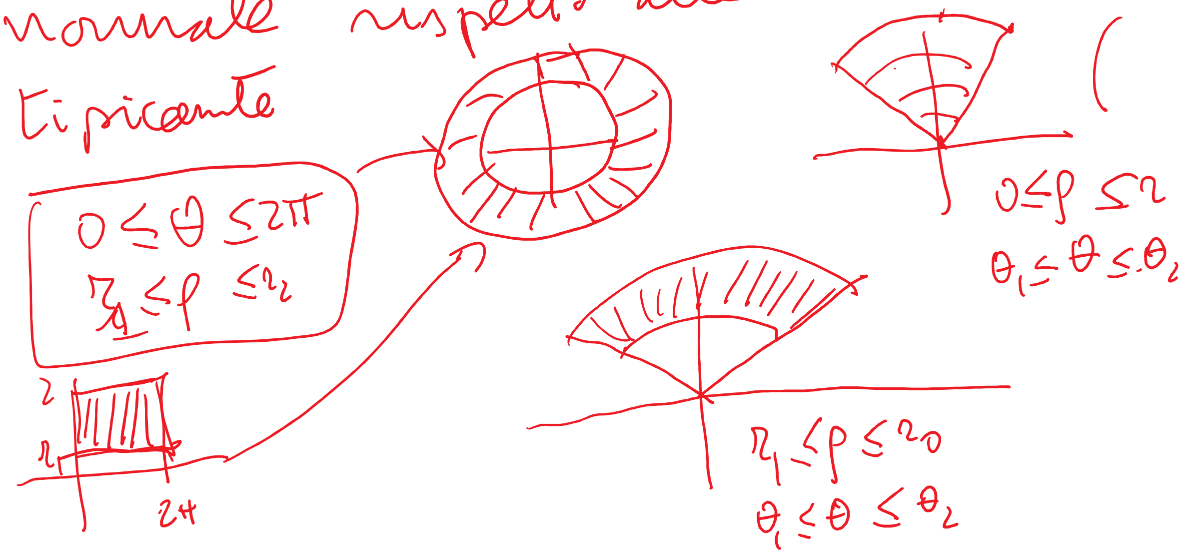
$$D_{(x,y)} = \{(x,y) \ldots\ldots\} = D_{\rho,\theta} = \{(\rho,\theta) \ldots\}$$

$$\iint_{D_{(x,y)}} f(x,y)\,dx\,dy = \iint_{D_{\rho,\theta}} f(\rho\cos\theta, \rho\sin\theta)\cdot \rho \; d\rho\, d\theta$$



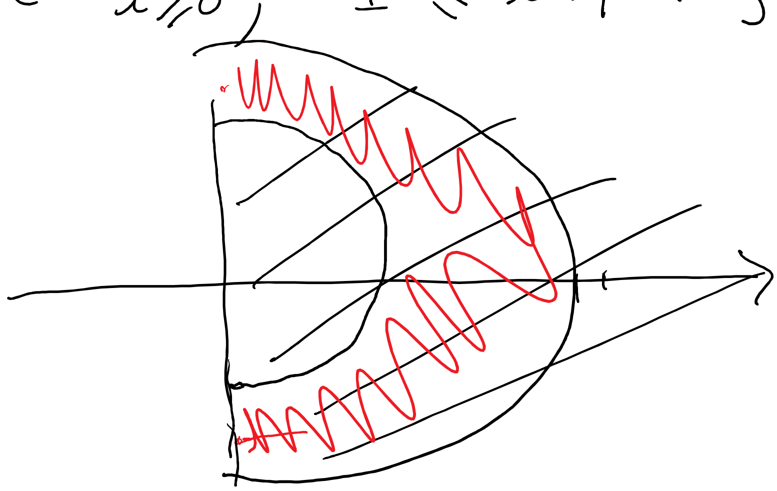
Si fa questa sostituzione se l'insieme D espresso in coordinate polari è normale rispetto alle coordinate polari

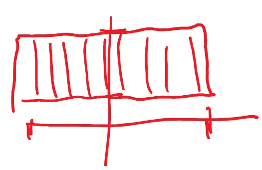
tipicamente



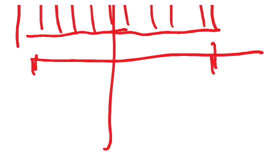
Esercizio $\displaystyle\iint_D \frac{x}{1+x^2+y^2}\,dx\,dy$

$D = \{(x,y) \text{ t.c. } x \geq 0,\ 1 \leq x^2+y^2 \leq 4\}$





$$D_{\rho,\theta} = \left\{(\rho,\theta),\ 1 < \rho \le 2,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right\}$$



$\rho = \sqrt{x^2+y^2}$

$x = \rho\cos\theta$

$x \ge 0 \iff \cos\theta \ge 0$

$$\iint_D \frac{x}{1+x^2+y^2}\,dx\,dy = \iint_{D_{\rho,\theta}} \frac{\rho\cos\theta}{1+\rho^2}\cdot\rho\,d\rho\,d\theta$$

$$= \int_1^2 \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\rho^2}{1+\rho^2}\cdot\cos\theta\,d\theta\right) d\rho = \int_1^2 \frac{\rho^2}{1+\rho^2}\,\sin\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\rho$$

$$= 2\int_1^2 \frac{\rho^2}{1+\rho^2}\,d\rho = 2\int_1^2 \frac{\rho^2+1-1}{\rho^2+1}\,d\rho = 2\int_1^2 1-\frac{1}{\rho^2+1}\,d\rho$$

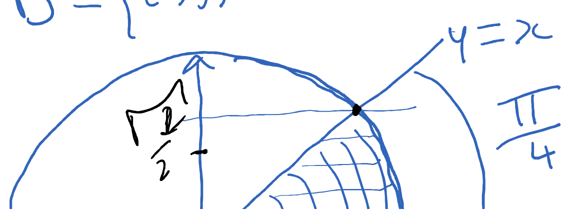
$$= 2\left(\rho - \operatorname{arctg}\rho\right)\Big|_1^2 = 2\left(2-\operatorname{arctg}2-1+\frac{\pi}{4}\right)$$

$$= 2\left(1-\frac{\pi}{4}-\operatorname{arctg}2\right)$$

$$\iint_D \frac{y}{x(1+x^2+y^2)}\,dx\,dy$$

$D = \{(x,y)\ \text{t.c.}\ x \ge y \ge 0,\ x^2+y^2 \le 1\}$

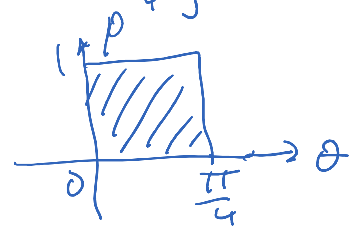
$\Downarrow$



$D = \left\{(x,y),\ 0 < y \le \frac{\sqrt{2}}{2},\ y < x \le \sqrt{1-y^2}\right\}$

$y \le x \le \sqrt{1-y^2}\}$

$$D_{\rho,\theta} = \{(\rho,\theta)\ \ 0 \le \rho \le 1,\ \ 0 \le \theta \le \frac{\pi}{4}\}$$



$$\iint_D \frac{y}{x(1+x^2+y^2)}\,dx\,dy =$$

$$= \iint_{D_{\rho,\theta}} \frac{\rho\sin\theta}{\rho\cos\theta(1+\rho^2)} \cdot \rho \cdot d\rho\, d\theta = \iint_{D_{\rho,\theta}} \frac{\sin\theta}{\cos\theta} \cdot \frac{\rho}{1+\rho^2}\, d\rho\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(\int_0^1 \frac{\sin\theta}{\cos\theta} \frac{\rho}{1+\rho^2}\, d\rho \right) d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos\theta} \log(1+\rho^2)\Big|_0^1$$

$$\int \frac{\rho}{1+\rho^2}\, d\rho = \frac{1}{2}\int \frac{2\rho}{1+\rho^2}\, d\rho = \frac{1}{2}\log(1+\rho^2)$$

$$= -\frac{\log 2}{2} \int_0^{\frac{\pi}{4}} \frac{-\sin\theta}{\cos\theta}\, d\theta = -\frac{\log 2}{2} \log(\cos\theta)\Big|_0^{\frac{\pi}{4}}$$

$$= -\frac{\log 2}{2}\left(\log\left(\frac{\sqrt{2}}{2}\right)\right) = \frac{\log 2}{2} \cdot \log\left(\frac{2}{\sqrt{2}}\right)$$

$$\iint \frac{y}{x(1+x^2+y^2)}\,dx\,dy = \int^{\frac{\sqrt{2}}{2}} \left(\int^{\sqrt{1-y^2}} \frac{y}{x(1+x^2+y^2)}\, dx \right) dy$$

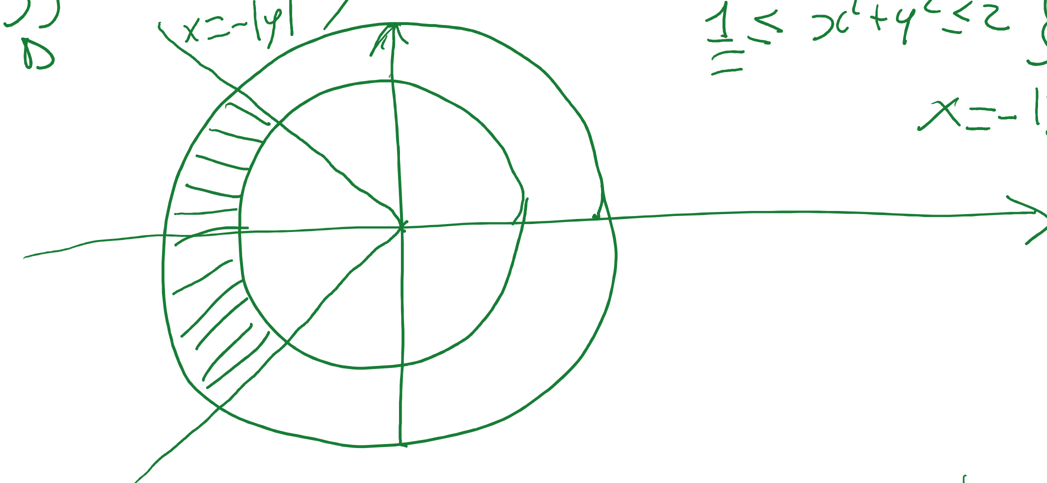
$$\iint_{D_{21}} \frac{y}{x(1+x^2+y^2)}\,dx\,dy = \int_0 \left(\int_y \frac{\ldots}{x(1+x^2+y^2)} \right) dy$$

$$\left\{ (x,y) \quad 0 \le y \le \frac{\sqrt{2}}{2} \; ; \; y \le x \le \sqrt{1-y^2} \right\}$$

Es.

$$\iint_D x\,d\rho\,dy \qquad D = \left\{ (x,y) \text{ t.c. } x \le -|y| ,\ \underline{1} \le x^2+y^2 \le 2 \right\}$$

$x = -|y|$



$$D = D_{\rho,\theta} = \left\{ (\rho,\theta) \text{ t.c. } 1 \le \rho \le \sqrt{2} ,\ \frac{3\pi}{4} \le \theta \le \frac{5\pi}{4} \right\}$$

$$\iint_D x\,dx\,dy = \iint_{D_{\rho,\theta}} \rho\cos\theta \cdot \rho\,d\rho\,d\theta = \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \left(\int_1^{\sqrt{2}} \rho^2 \cos\theta\,d\rho \right) d\theta$$

$$= \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \cos\theta \left. \frac{\rho^3}{3} \right|_1^{\sqrt{2}} d\theta = \left(\frac{(\sqrt{2})^3}{3} - \frac{1}{3} \right) \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \cos\theta\,d\theta$$

$$= \left(\frac{(\sqrt{2})^3}{3} - \frac{1}{3} \right) \left(+ \sin\theta \right) \Big|_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} = -\boxed{\sqrt{2}} \left(\frac{(\sqrt{2})^3}{3} - 1 \right)$$

Suggerimento $\qquad x = g(y)$ è una $\sqrt{\quad}$

Suggerimento $x = g(y)$ è un

riuscite a disegnarla $\longrightarrow$ $\boxed{y = g(x)}$