ESERCIZIO 1 π

F(x,y)=(3x^2y+cos x, x^3+e^y)

a) $F(0,0) = (1, 1)$. $\quad Def = \mathbb{R}^2$.

b). $\dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x}$. $\quad \dfrac{\partial F_1}{\partial y} = 3x^2 = \dfrac{\partial F_2}{\partial x} = 3x^2$

F è irrotazionale e Ins di Def è semplicemente connesso

$\Longrightarrow$ F è conservativo

c) $L(F, \gamma) = ?$ $\quad \gamma(0) = (2,0) = \gamma(2\pi) = (2,0)$.

γ è chiusa e F è ~~[illegible]~~ conservativo $\Longrightarrow L(F,\gamma) = 0$.

d) $\gamma_1 = (2t^2, 3t^3)$ $\quad \gamma_1(0) = (0,0) \neq \gamma_1(1) = (2,3)$.

Trovare il potenziale f t.c. $\nabla f = F$

$$\begin{cases} \dfrac{\partial f}{\partial x} = 3x^2y + \cos x \longrightarrow f(x,y) = \int 3x^2y + \cos x \, dx = \boxed{x^3y + \sin x + g(y)} \, * \\ \dfrac{\partial f}{\partial y} = x^3 + e^y \longrightarrow \dfrac{\partial f}{\partial y} = x^3 + g'(y) = x^3 + e^y \rightarrow \quad g'(y) = e^y \rightarrow g(y) = e^y \end{cases}$$

Calcolare usando $\rightarrow$ $f(x,y) = x^3y + \sin x + e^y$.

$L(F,\gamma_1) = f(2,3) - f(0,0) = 24 + \sin 2 + e^3 - 1$.

$= 23 + \sin 2 + e^3$

nel punto finale $\quad$ pto iniziale

e) Trovare una curva γ_2 t.c. $L(F,\gamma_2)=3$. $\quad a=0,\ b=1 \qquad \gamma_2(0)=(x_0,y_0)$ $\qquad$ III

$\gamma_2(1)=(x_1,y_1)$.

$$L(F,\gamma_2)=f(\gamma_2(b))-f(\gamma_2(a))$$

$$=f(x_1,y_1)-f(x_0,y_0)=3.$$

scelgo $\boxed{(x_0,y_0)=(0,0).}$

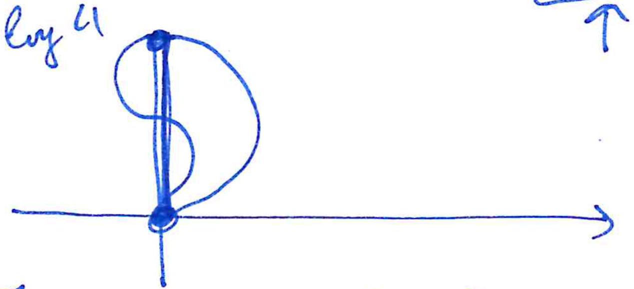
$$f(x_1,y_1)-1=3$$

$$f(x_1,y_1)=4 \Longrightarrow x_1^3y_1+\sin x_1+e^{y_1}=4.$$

$\boxed{x_1=0} \qquad e^{y_1}=4$

$y_1=\log 4.$

$(x_1,y_1)=(0,\log 4)$



$\gamma_2(t)$

$$\boxed{\begin{cases} x(t)=0 \\ y(t)=t \end{cases} \quad t\in[0,\log 4]}$$

$$\boxed{\begin{cases} x(t)=0 \\ y(t)=t\log 4. \end{cases} \quad t\in[0,1]}$$

$x=0 \Longrightarrow \boxed{y(t)=t} \to t\in[0,\log 4]$

$y(0)=0$

$y(b)=\log 4.$

$y(t)=\log(t+1). \to t\in[0,3].$

- $y_0=\log 4 \longrightarrow x_0. \qquad f(x_0,y_0)=f(0,\log 4)=4. \qquad f(x_1,y_1)$

IV

$$G(x,y) = F(x,y) + (y,0).$$

$$L(G,\gamma) = \underbrace{L(F,\gamma)}_{\substack{\| \\ 0 \\ \text{(punto c)}}} + L((y,0),\gamma) = L((y,0),\gamma) = -6\pi.$$

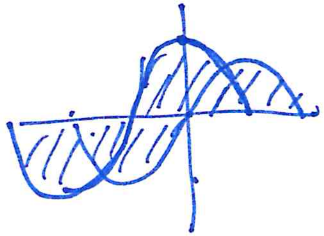
$\cancel{\nabla g = (y,0)}$

$$L((y,0),\gamma) = \int_0^{2\pi} 3\sin t\,(-2\sin t)\,dt = -6\int_0^{2\pi} \sin^2 t\,dt = -6\pi.$$

$$\boxed{\int_0^{2\pi} \sin^2 t\,dt = \pi.}$$

$$\cos^2 t + \sin^2 t = 1.$$

$$\int_0^{2\pi} \sin^2 t\,dt = \int_0^{2\pi} \cos^2 t\,dt$$



$$\int_0^{2\pi} \sin^2 t\,dt = \frac{1}{2}\int_0^{2\pi} dt = \pi.$$

$$\|$$

$$\frac{1}{2}\int_0^{2\pi} \sin^2 t\,dt + \frac{1}{2}\int_0^{2\pi} \sin^2 t\,dt = \frac{1}{2}\int_0^{2\pi} \cos^2 t + \sin^2 t\,dt$$

a) $F(0,0) = (0, 1)$.

$x^2+2x+1+y^2 \neq 0$.

$(x+1)^2+y^2 \neq 0 \iff$

$(x+1)^2+y^2=0 \iff \begin{cases} x+1=0 \\ y=0 \end{cases}$ $\quad x=-1$, $y=0$

$Def = \mathbb{R}^2 \setminus \{(-1,0)\}$.

(OSS: il dominio non è semplicemente connesso).

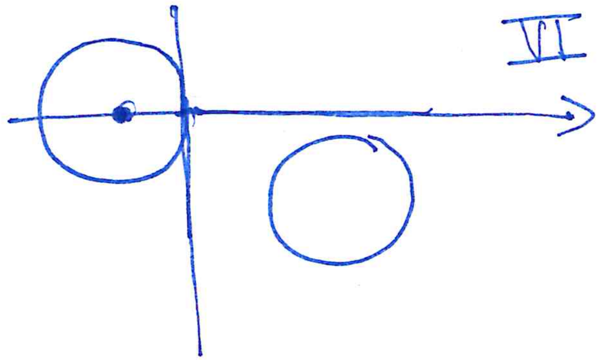
b). $\dfrac{\partial F_1}{\partial y} \overset{?}{=} \dfrac{\partial F_2}{\partial x}$.

$$\frac{\partial F_1}{\partial y} = \frac{-1(x^2+2x+1+y^2)+y\cdot 2y}{(x^2+2x+1+y^2)^2} = \frac{y^2-x^2-2x-1}{(x^2+2x+1+y^2)^2}$$

$$\frac{\partial F_2}{\partial x} = \frac{1(x^2+2x+1+y^2)-(x+1)(2x+2)}{(x^2+2x+1+y^2)^2} = \frac{y^2+(x+1)^2-2(x+1)^2}{(x^2+2x+1+y^2)^2}$$

$$= \frac{y^2-(x+1)^2}{(x^2+2x+1+y^2)^2} = \frac{\partial F_1}{\partial y}.$$

F è irrotazionale.



VI

$L(F,\gamma) =$

$\gamma'(t) = (-\sin t, \cos t).$

$$F(x,y) = \left(\frac{-y}{(x+1)^2 + y^2}, \frac{x+1}{(x+1)^2+y^2} \right).$$

$$\int_0^{2\pi} \underbrace{\left[\frac{-\sin t}{(\cos t - 1 + 1)^2 + (\sin t)^2} \right]}_{F_1(\gamma(t))} \cdot \underset{x'(t)}{(-\sin t)} + \underbrace{\frac{\cos t - 1 + 1}{(\cos t - 1 + 1)^2 + (\sin t)^2}}_{F_2(\gamma(t))} \cdot \underset{y'(t)}{\cos t} \, dt \doteq$$

$$= \int_0^{2\pi} \frac{\sin^2 t}{1} + \frac{\cos^2 t}{1} \, dt = \int_0^{2\pi} 1 \, dt = 2\pi \neq 0$$

$$x^2 + 2x + 1 + y^2 = (\cos t - 1)^2 + 2(\cos t - 1) + 1 + \sin^2 t$$
$$= \cos^2 t \cancel{- 2\cos t} + 1 + \cancel{2\cos t} - 2 + 1 + \sin^2 t$$
$$= 1.$$

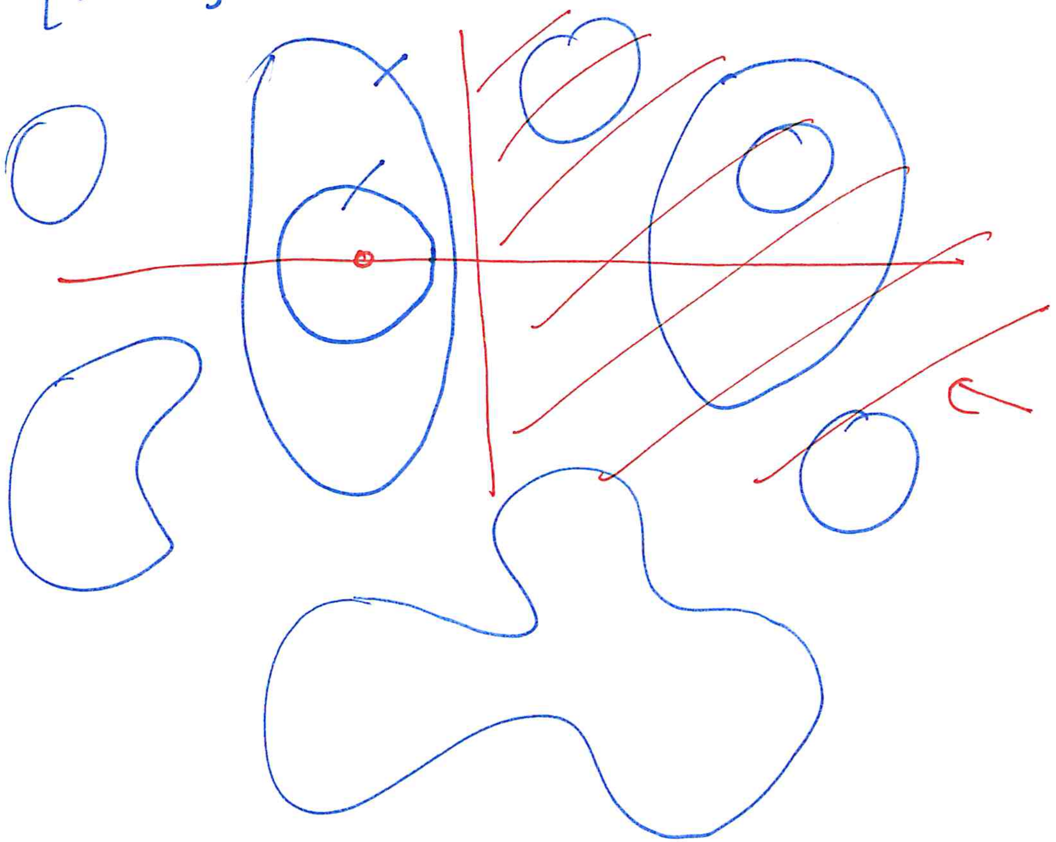
d) siccome γ è chiuso e $L(F,\gamma) \neq 0 \implies F$ non è conservativo.

e) Se γ è una curva chiusa contenuta nel semispazio $x>0$.

Siccome F è irrotazionale in $D=\{x>0\}$ e $\{x>0\}$ è semplicemente connesso $\Longrightarrow$ F è conservativo in D

$$\Downarrow$$

$$L(F,\gamma)=0.$$



è semplicemente connesso

Si ricordano le formule di riduzione per domini normali.

$$D_x = \{(x,y) \quad a \le x \le b, \ \alpha(x) \le y \le \beta(x)\}$$

$$D_y = \{(x,y) \quad c \le y \le d \ , \ \gamma(y) \le x \le \delta(y)\}$$

$$\iint_{D_x} f(x,y)\,dx\,dy = \int_a^b \left(\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy \right) dx$$

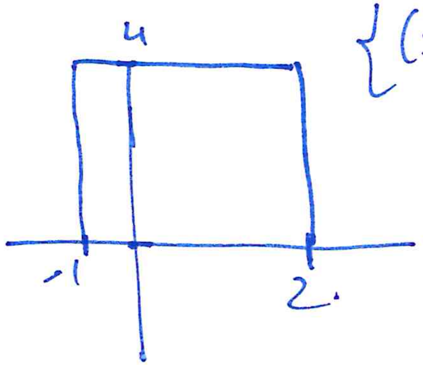
$$\iint_{D_y} f(x,y)\,dx\,dy = \int_c^d \left(\int_{\gamma(y)}^{\delta(y)} f(x,y)\,dx \right) dy$$

Esercizio 4, il secondo dei due

a) $$\iint_D x+2y\,dx\,dy = \int_{-1}^{2} \left(\int_0^4 x+2y\,dy \right) dx = \int_{-1}^{2} \left(xy+y^2 \right)\Big|_0^4 dx = \int_{-1}^{2} 4x+16\,dx =$$

$$D = [-1,2] \times [0,4] = \{(x,y) \quad -1 \le x \le 2, \quad 0 \le y \le 4\}.$$

$\uparrow$ x. $\qquad \uparrow$ y.



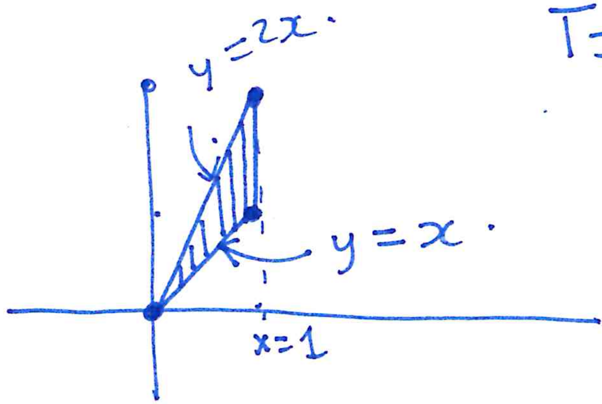
$$= 2x^2+16x \Big|_{-1}^{2} = 8+32-(2-16) = 40+14 = 54$$

IX

b) $\iint_T y\,dx\,dy$

$T=\{(x,y)\quad 0\le x\le 1\quad x\le y\le 2x\}$

T triangolo
$(0,0),\ (1,1),\ (1,2)$



La retta per due punti (x_0, y_0), (x_1, y_1).

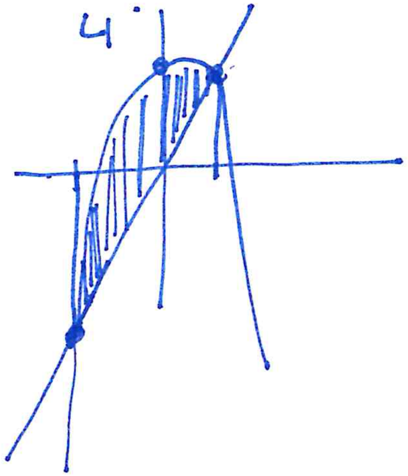
$$\boxed{\frac{x-x_0}{x_1-x_0}=\frac{y-y_0}{y_1-y_0}}$$

$$\iint_T y\,dx\,dy = \int_0^1\left(\int_x^{2x} y\ dy\right)dx.$$

$$=\int_0^1\left(\frac{y^2}{2}\right)_x^{2x} dx = \int_0^1 \frac{4x^2}{2}-\frac{x^2}{2}\,dx = \int_0^1 \frac{3}{2}x^2\,dx = \frac{1}{2}x^3\Big|_0^1 = \frac{1}{2}.$$

$$\iint_D xy\,dx\,dy \qquad D = \{(x,y) \quad 3x \leq y \leq 4 - x^2\}. \qquad \begin{cases} y = 4 - x^2 \\ y = 3x \end{cases}$$

X



$$4 - x^2 = 3x.$$

$$x^2 + 3x - 4 = 0$$

$$x = 1$$

$$x = -4.$$

$$\iint_D xy\,dx\,dy = \int_{-4}^{1}\left(\int_{3x}^{4-x^2} xy\,dy\right)dx = \int_{-4}^{1} x\frac{y^2}{2}\Bigg|_{3x}^{4-x^2} dx = \int_{-4}^{1} \frac{x}{2}(4-x^2)^2 - \frac{x}{2}9x^2\,dx$$

$$= \frac{1}{2}\int_{-4}^{1} x(16 - 8x^2 + x^4) - 9x^3\,dx = \frac{1}{2}\int_{-4}^{1} 16x - 17x^3 + x^5\,dx.$$

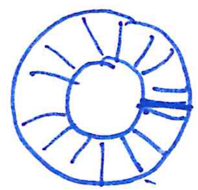
$$= \frac{1}{2}\left(8x^2 - \frac{17}{4}x^4 + \frac{1}{6}x^6\right)_{-4}^{1} = \frac{1}{2}\left(8 - \frac{17}{4} + \frac{1}{6}\right) - \frac{1}{2}\left(8(-4)^2 \dots = \dots\right)$$

XI

$$\iint_D e^{x^2+y^2}\,dx\,dy \qquad D=\{(x,y) \text{ t.c. } 1\le x^2+y^2\le 3\}.$$

$x^2+y^2=1$ cerchio unitario

$x^2+y^2=3$ —— di raggio $\sqrt{3}$.

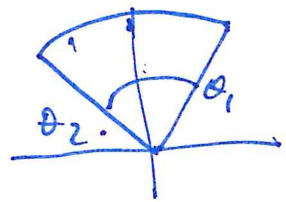


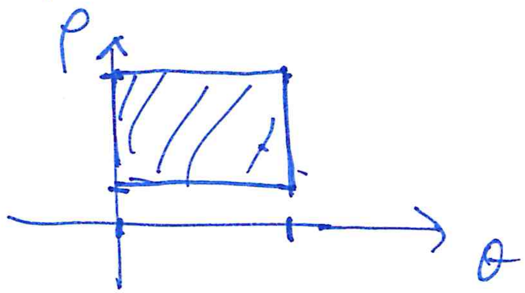
$$\begin{cases}\rho=\sqrt{x^2+y^2}\\ \theta=\operatorname{arctg}\frac{y}{x}\end{cases}$$

$$\begin{cases}x=\rho\cos\theta\\ y=\rho\sin\theta\end{cases}$$

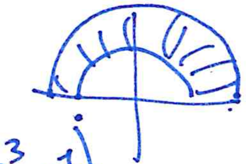
$$\iint_{D_{(x,y)}} f(x,y)\,dx\,dy=\iint_{D_{\rho,\theta}} f(\rho\cos\theta,\rho\sin\theta)\,\rho\,d\rho\,d\theta$$

$$D_{\rho,\theta}=\{(\rho,\theta)\quad 1\le\rho\le\sqrt{3},\ 0\le\theta\le 2\pi\}$$





$$\iint_D e^{x^2+y^2}\,dx\,dy=\iint_{D_{\rho,\theta}} e^{\rho^2}\rho\,d\rho\,d\theta. \qquad s=\rho^2\quad ds=2\rho\,d\rho$$



$$=\int_0^{2\pi}\left(\int_1^{\sqrt{3}} e^{\rho^2}\rho\,d\rho\right)d\theta=\int_0^{2\pi}\frac{1}{2}\left(\int_\sim^\sim e^s\,ds\right)d\theta=\int_0^{2\pi}\frac{1}{2}\,e^{\rho^2}\Big|_1^{\sqrt{3}}\,d\theta=\frac{2\pi}{2}(e^3-e^1)$$

$$=\pi(e^3-e).$$