

Exercise on limits and continuity

1. Prove using the definition of limit that:

$$\lim_{x \rightarrow 1} 2x^2 - 3x = -1, \lim_{x \rightarrow 1} \log(x) = 0, \lim_{x \rightarrow 0} \frac{1}{1 + 2x} = 1, \lim_{x \rightarrow 1} 3 = 3$$

2. Prove, using the fact that  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$  that  $\lim_{x \rightarrow 0} \sin x = 0$
3. Using Prostaferesi's formula:  $(\cos a - \cos b = -2 \sin(\frac{a+b}{2}) \sin(\frac{a-b}{2}))$  and the limit in the previous exercise, prove that the function  $f(x) = \cos x$  is continuous.
4. Using the previous exercise, prove that  $f(x) = (\cos x)^2$  is continuous.
5. Prove that the function  $f(x) = \begin{cases} x^2 + 1 & \text{if } x > 0 \\ -x^2 - 1 & \text{if } x \leq 0 \end{cases}$  is not continuous at zero
6. Let  $f(x)$  be the Dirichlet function i.e.

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Prove that  $f$  is not continuous in any  $x$ .