10/12/21
(8 → Festa).

Integrali Doppi

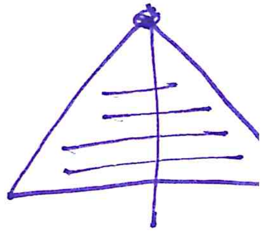
D è un dominio <u>normale</u>

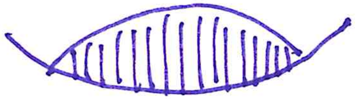
D è normale rispetto a "x" se

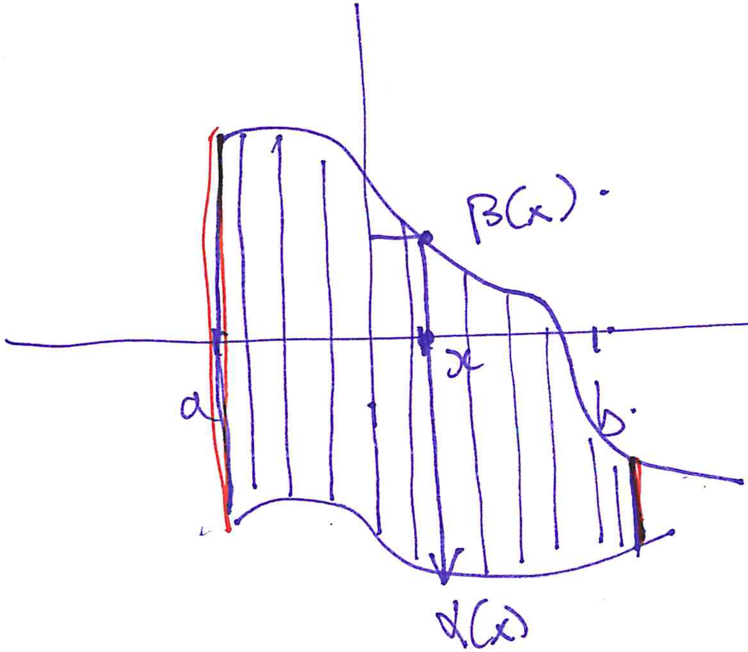
$$D_x = \{ (x,y) \text{ t.c. } a \le x \le b, \quad \alpha(x) \le y \le \beta(x) \}.$$

$a \in \mathbb{R}$, $b \in \mathbb{R}$, $\alpha : [a,b] \to \mathbb{R}$

$\beta : [a,b] \to \mathbb{R}$





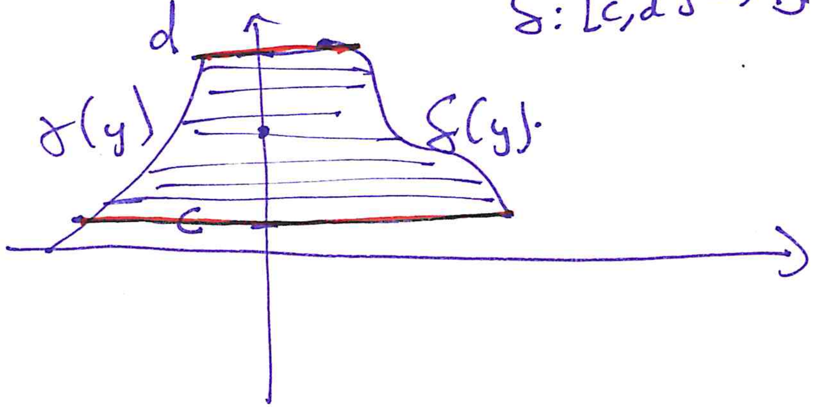


D è normale risp a "y"

$$D = \{ (x,y) \text{ t.c. } c \le y \le d, \quad \gamma(y) \le x \le \delta(y) \}.$$

$c \in \mathbb{R}$, $d \in \mathbb{R}$, $\gamma : [c,d] \to \mathbb{R}$

$\delta : [c,d] \to \mathbb{R}$



$$\iint_{D_x} f(x,y)\,dx\,dy = \int_a^b \underbrace{\left(\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy \right)}\,dx = \int_a^b \underbrace{F(x,\beta(x)) - F(x,\alpha(x))}\,dx$$

$D_x = \{a \le x \le b,\ \alpha(x) \le y \le \beta(x)\}$

si integra la f come una funzione della sola variabile "y", la "x" ha il ruolo di un parametro o di una costante. Possiamo usare il Teorema Fondamentale del Calcolo Integrale:

$\to F(x,y)$ t.c. $\dfrac{\partial F}{\partial y}(x,y) = f(x,y)$.

si applica il Teorema Fondamentale del Calcolo Integrale

IL RISULTATO È UN NUMERO, LE VARIABILI "x" e "y" sono sparite.

III

$$\iint_{D_y} f(x,y)\,dx\,dy = \int_c^d \left(\underbrace{\int_{\gamma(y)}^{\delta(y)} f(x,y) \quad d\underline{\underline{x}}} \right) dy = \int_c^d \underbrace{G(\delta(y),y) - G(\gamma(y),y)}_{\text{solo la variabile } y.}\, dy$$

$$D_y = \{ c \le y \le d,\ \gamma(y) \le x \le \delta(y) \}$$

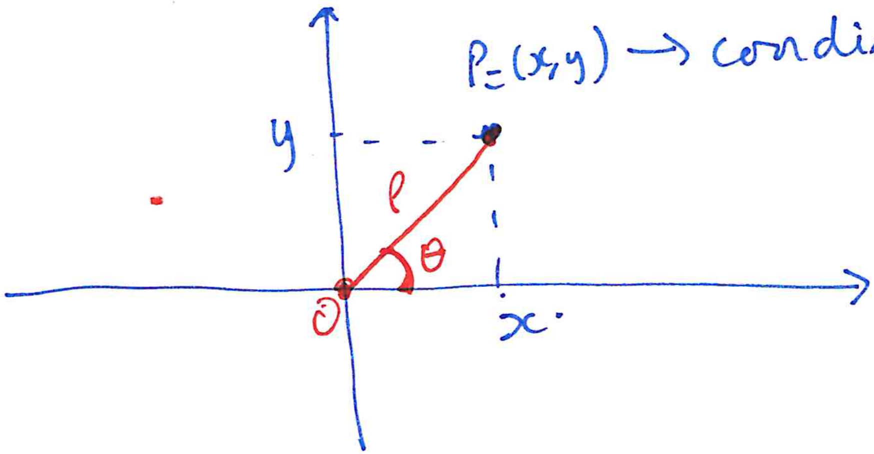
c, d: costanti — $\gamma(y)$, $\delta(y)$: funzioni

si integra la funzione f come una funzione della variabile "x" e la "y" è un parametro. Usando Teorema Fondamentale del Calcolo Integrale $G(x,y)$ t.c. $\frac{\partial G}{\partial x}(x,y) = f(x,y)$.

IL RISULTATO È UN NUMERO.

Cambiamento di Variabili per gli integrali doppi

Coordinate polari



$P=(x,y)$ → coordinate cartesiane

$$\boxed{\rho = \sqrt{x^2+y^2}}$$

θ è l'angolo compreso tra l'asse x e la retta OP

$$\operatorname{tg}\theta = \frac{y}{x}, \quad \boxed{\theta = \operatorname{arctg}\frac{y}{x}}$$

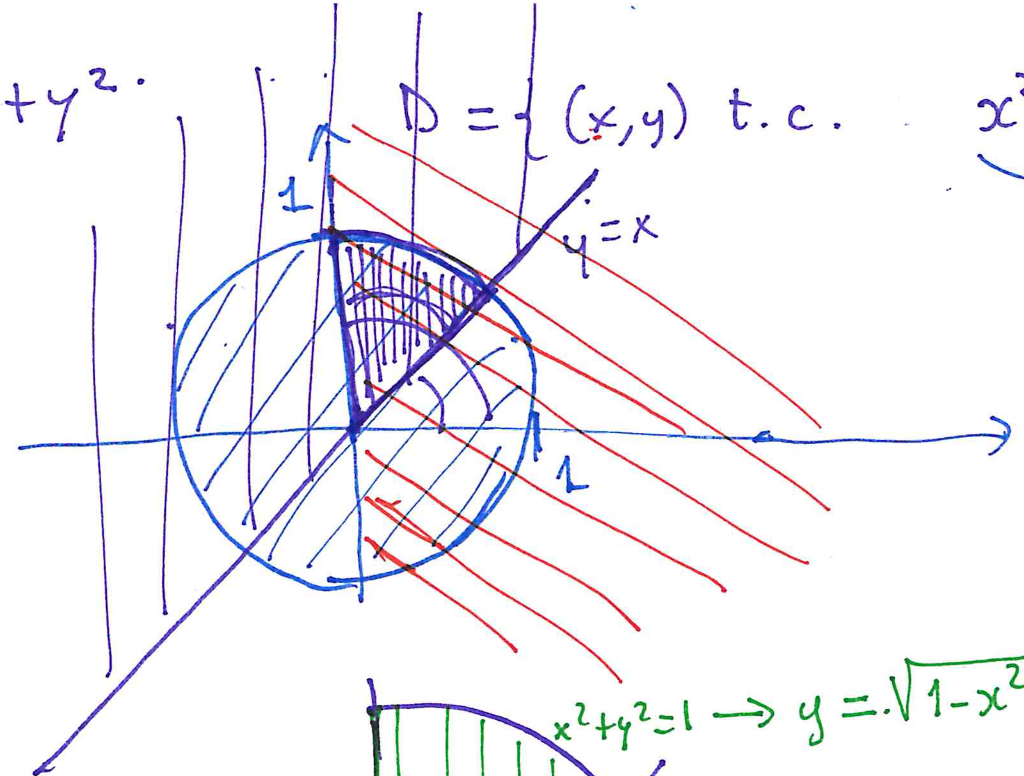
$$\begin{cases} \rho = \sqrt{x^2+y^2} \\ \theta = \operatorname{arctg}\frac{y}{x} \end{cases} \longleftrightarrow \begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

cambio di coordinate da coordinate cartesiane (x,y) e coordinate polari (ρ, θ).

$f(x,y) = x^2 + y^2.$ $\qquad D = \{(x,y) \text{ t.c. } x^2+y^2 \leq 1, \; x \geq 0, \; y \geq x\}$

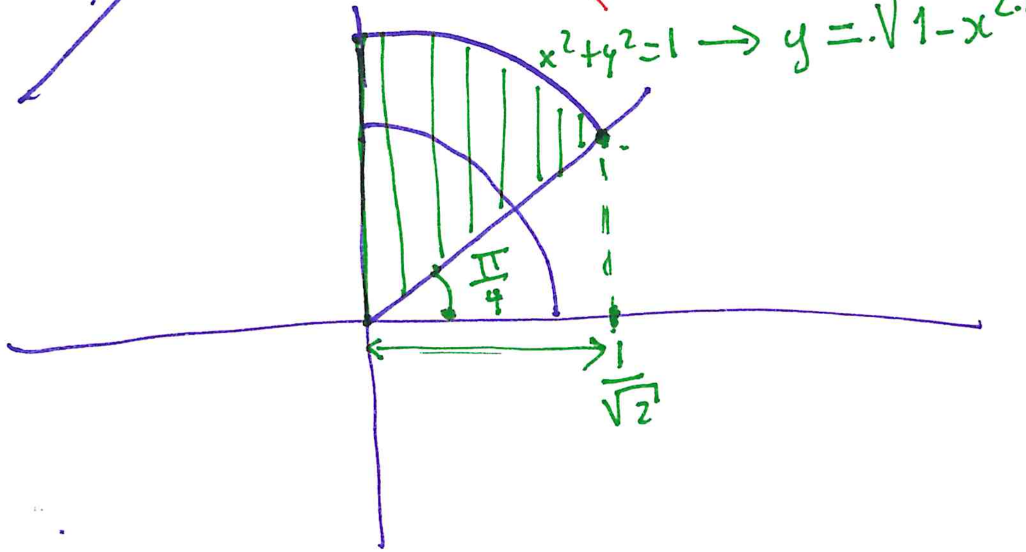
$x^2+y^2 = 1$

$y = x.$



$x^2+y^2=1 \rightarrow y = \sqrt{1-x^2}$

$$\left.\begin{array}{l} x^2+y^2=1 \\ y = x. \end{array}\right\} \quad \begin{array}{c} x^2+x^2=1 \\ 2x^2=1 \\ x^2 = \frac{1}{2} \\ \downarrow \\ x = \frac{1}{\sqrt{2}} \end{array}$$



$$D = \left\{(x,y) \cdot \; 0 \leq x \leq \frac{1}{\sqrt{2}}, \; x \leq y \leq \sqrt{1-x^2}\right\}.$$

VI

$$\iint_D x^2+y^2\,dx\,dy = \int_0^{\frac{1}{\sqrt{2}}}\left(\int_x^{\sqrt{1-x^2}} x^2+y^2\,dy\right)dx = \int_0^{\frac{1}{\sqrt{2}}}\left(\frac{1}{3}+\frac{2}{3}x^2\right)\sqrt{1-x^2}-\frac{4}{3}x^3\,dx.$$

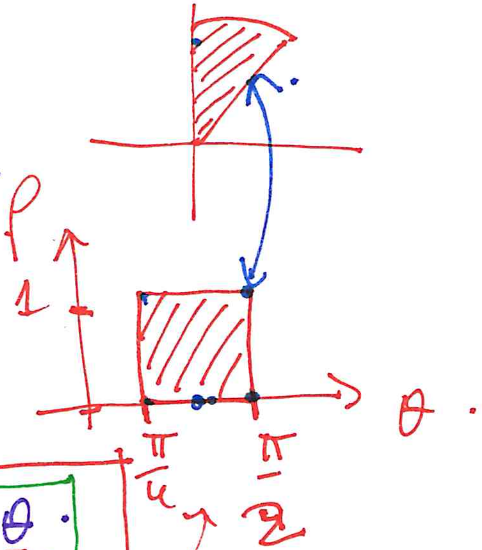
$$\int_x^{\sqrt{1-x^2}} x^2+y^2\,dy = x^2y+\frac{y^3}{3}\Bigg|_x^{\sqrt{1-x^2}} = x^2\sqrt{1-x^2}+\frac{1}{3}\left(\sqrt{1-x^2}\right)^3-\left(x^3+\frac{x^3}{3}\right).$$

$$= x^2\sqrt{1-x^2}+\frac{1}{3}(1-x^2)\sqrt{1-x^2}-\frac{4}{3}x^3.$$

$$= \left(\frac{1}{3}+\frac{2}{3}x^2\right)\sqrt{1-x^2}-\frac{4}{3}x^3.$$

$= \int \ldots$ si risolve l'integrale.

$$D = \{(x,y) \text{, proprietà di } x \text{ e } y\} = \{(\rho,\theta) \text{, proprietà di } \rho \text{ e } \theta\}.$$



Esempio

$$D = \{(x,y) \quad (x^2+y^2) \le 1, \quad x \ge 0, \quad y \ge x\}.$$

$$= \{(\rho,\theta) \quad 0 \le \rho \le 1, \quad \frac{\pi}{4} \le \theta \le \frac{\pi}{2}\}.$$

$$\iint_{D_{x,y}} f(x,y)\, \boxed{dx\,dy} = \iint_{D_{\rho,\theta}} f(\rho\cos\theta, \rho\sin\theta)\, \boxed{\rho\, d\rho\, d\theta}.$$

integrale nel rettangolo.

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta. \end{cases}$$

$dx\,dy = \rho\, d\rho\, d\theta.$

→ Valgono le regole degl'integrali su domini normali risp. ρ o risp. θ.

VIII

$$\iint_D x^2+y^2\,dx\,dy = \iint \underbrace{[(\rho\cos\theta)^2+(\rho\sin\theta)^2]}_{=\rho^2}\rho\,d\rho\,d\theta = \boxed{\iint_D \rho^3\,d\rho\,d\theta}$$

$x^2+y^2 = \rho^2$

$D = \{x^2+y^2\leq 1,\ 0\leq x,\ y\geq x\}$

$D = \{(\rho,\theta),\ 0\leq\rho\leq 1,\ \frac{\pi}{4}\leq\theta\leq\frac{\pi}{2}\}$

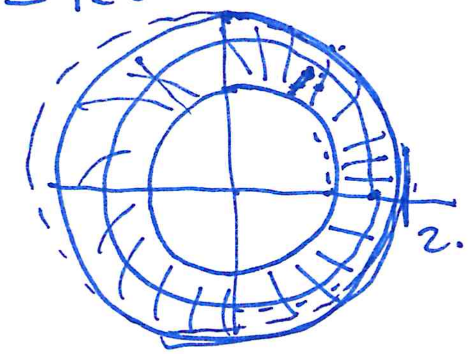
in termini di ρ e θ.

$$= \int_0^1\left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\rho^3\,d\theta\right)d\rho = \int_0^1 \rho^3\theta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}\,d\rho = \int_0^1\frac{\pi}{4}\rho^3\,d\rho = \frac{\pi}{16}\rho^4\Big|_0^1 = \frac{\pi}{16}.$$

$$\overset{\to \text{ coord. cartesiane}}{D_{x,y}} = \{(x,y) \text{ t.c. } 1 \le x^2+y^2 \le 4\}.$$

$x^2+y^2=4=2^2.$

$x^2+y^2=1.$



$$\iint_D x+y \, dx\,dy$$

$$\begin{cases} \rho = \sqrt{x^2+y^2} \\ \theta = \operatorname{arctg} \frac{y}{x}. \end{cases}$$

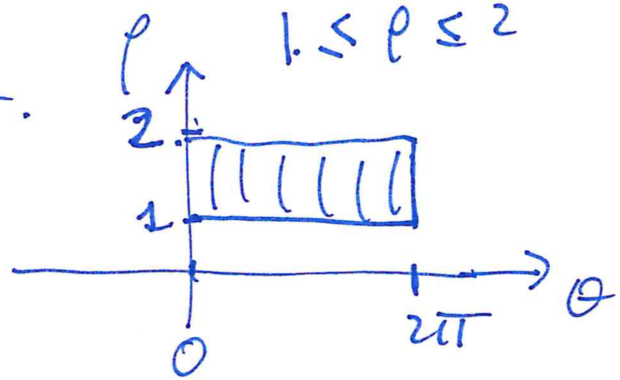
$1 \le x^2+y^2 \le 4$

$1 \le \rho^2 \le 4$

$1 \le \rho \le 2$

$$D_{\rho,\theta} = \{(\rho,\theta) \text{ t.c. } 1 \le \rho \le 2, \; 0 \le \theta \le 2\pi\}.$$

Coordinate polari $\longrightarrow$ D è un rettangolo.



$$\iint_{D_{x,y}} x+y \, dx\,dy = \iint_{D_{\rho,\theta}} (\rho\cos\theta + \rho\sin\theta)\rho \, d\rho\, d\theta = .$$

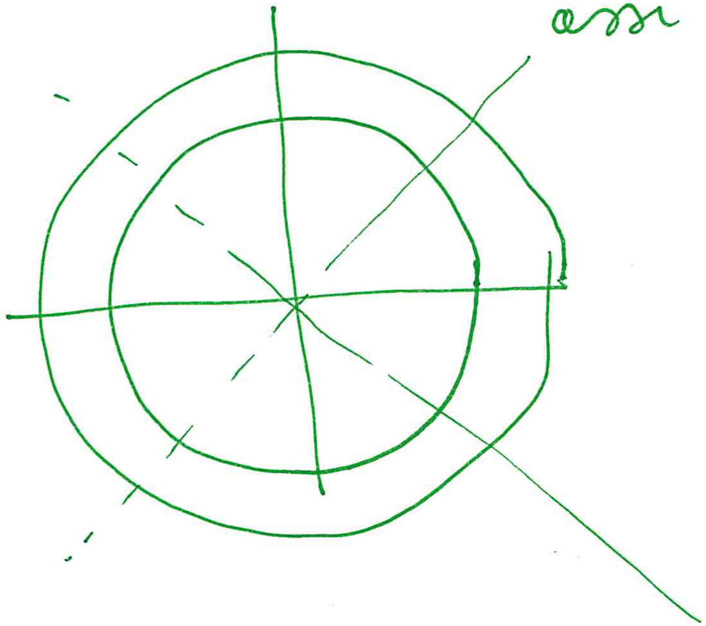
$$= \iint_D \rho^2(\cos\theta+\sin\theta)\, d\rho\, d\theta = \int_0^{2\pi}\left(\int_1^2 \rho^2(\cos\theta+\sin\theta)\,d\rho\right) d\theta = \int_0^{2\pi}\left(\frac{8}{3}-\frac{1}{3}\right)(\cos\theta+\sin\theta)d$$

$$= \frac{7}{3}\int_0^{2\pi} \cos\theta + \sin\theta \, d\theta = \frac{7}{3}\left(\sin\theta - \cos\theta\right)\Big|_0^{2\pi} = \frac{7}{3}\left(0 - 1 - (0 - 1)\right) = 0.$$

X

$$\iint_D x+y \, dx\,dy = 0.$$

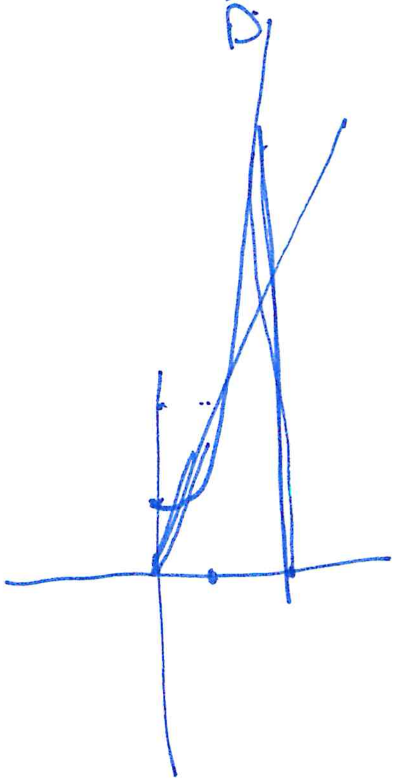
(La funzione è dispari in x e in y e il dominio è simmetrico rispetto agli assi cartesiani.)

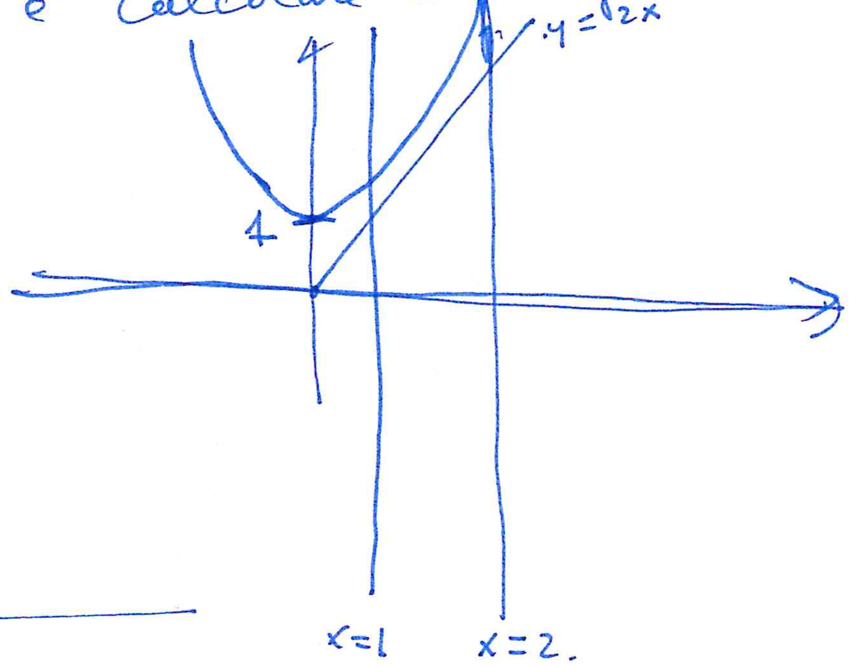


$$\iint_D xy\,dx\,dy.$$

$D = \{(x,y)$ t.c. $1 \le x \le 2,\ 2x \le y \le x^2+1\}$

Disegnare D e calcolare l'integrale.





$y = 2x.$

$y = x^2+1.$

$$\iint_D xy\,dx\,dy = \int_1^2 \left(\int_{2x}^{x^2+1} xy \;\; dy \right) dx = \int_1^2 \left(x \left(\frac{y^2}{2}\right) \Bigg|_{2x}^{x^2+1} \right) dx = .$$

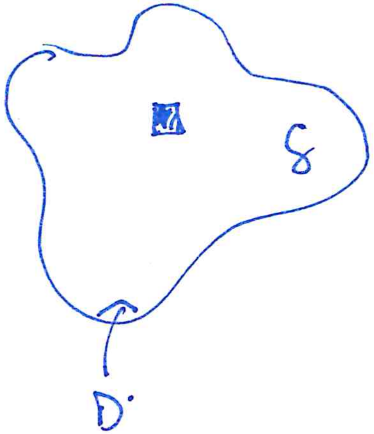
$$= \int_1^2 \frac{x}{2}\left((x^2+1)^2 - (2x)^2\right) dx = \frac{1}{2}\int_1^2 x(x^2+1)^2 - 4x^3\, dx.$$

$$= \frac{1}{2}\left[\frac{(x^2+1)^3}{6} - x^4\right]_1^2 = \frac{1}{2}\left[\frac{5^3}{6} - 2^4 - \left(\frac{2^3}{6} - 1\right)\right]$$

$$= \frac{1}{2}\left(\frac{125}{6} - 16 - \frac{8}{6} + 1\right) = \frac{1}{2}\left(\frac{117}{6} - 15\right).$$

Il Baricentro

$$\frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = x_B$$



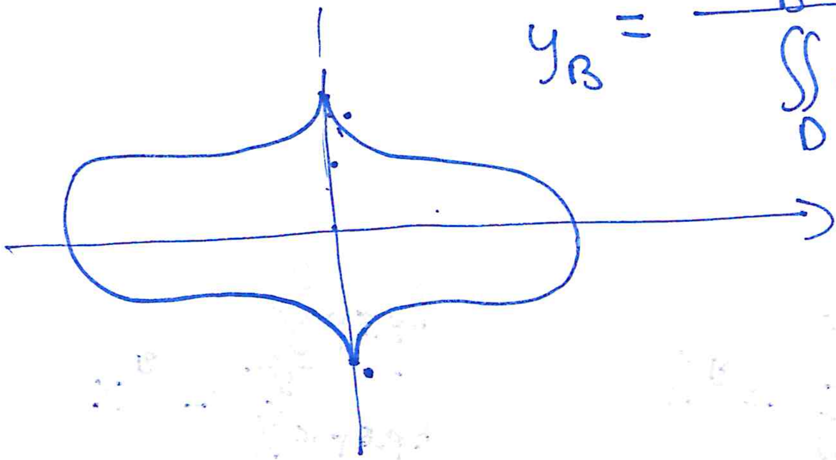
δ densità

$$M = \iint_D \delta(x,y)\,dx\,dy$$

Baricentro (x_B, y_B). Se $\delta(x,y) \equiv 1$.

$$x_B = \frac{\iint_D x\,\delta(x,y)\,dx\,dy}{\iint_D \delta(x,y)\,dx\,dy} \qquad x_B = \frac{\iint_D x\,dx\,dy}{\iint_D dx\,dy}$$

$$y_B = \frac{\iint_D y\,\delta(x,y)\,dx\,dy}{\iint_D \delta(x,y)\,dx\,dy} \qquad y_B = \frac{\iint_D y\,dx\,dy}{\iint_D dx\,dy}$$

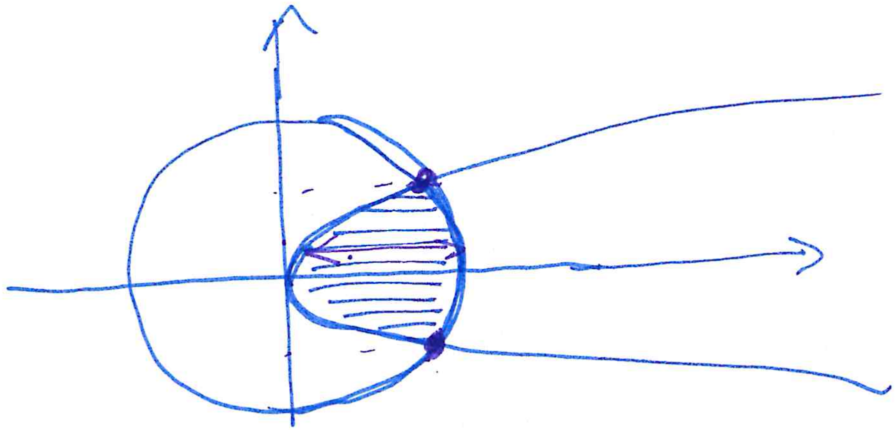


Es. XIV

$A = \{(x,y)$ t.c. $x^2+y^2 \le 1, \quad x \ge 2y^2\}.$

$d = 2x$. a) Disegnare A, b) Calcolare la massa di A

d è la densità



$$\begin{cases} x^2+y^2=1 \\ x = 2y^2 \end{cases}$$

$x^2 = 4y^4$

$4y^4+y^2-1=0$

$y^2 = \dfrac{-1 \pm \sqrt{1+4}}{8}$

$y^2 = \dfrac{-1+\sqrt{5}}{8}$

$y = \sqrt{\dfrac{-1+\sqrt{5}}{8}}$

$y = -\sqrt{\dfrac{-1+\sqrt{5}}{8}}$

$A = \left\{(x,y) \text{ t.c. } -\sqrt{\dfrac{-1+\sqrt{5}}{8}} \le y \le \sqrt{\dfrac{-1+\sqrt{5}}{8}}, \quad 2y^2 \le x \le \sqrt{1-y^2}\right\}.$

b) $M = \iint_A 2x\,dx\,dy = \int_{-\sqrt{\ }}^{\sqrt{\ }} \left(\int_{2y^2}^{\sqrt{1-y^2}} 2x\,dx \right) dy.$

$= \int x^2 \Big|^{\sqrt{1-y^2}} dy = \int^{\sqrt{\ }} (1-y^2) - 4y^4\,dy = y - \dfrac{y^3}{3} - \dfrac{4y^5}{5} \Big|_{\sqrt{\ }}^{\sqrt{\ }}$

$$2\sqrt{\frac{-1+\sqrt{5}}{8}} - \frac{2}{3}\left(\frac{-1+\sqrt{5}}{8}\right)\sqrt{\frac{-1+\sqrt{5}}{8}} - \frac{8}{5}\left(\frac{-1+\sqrt{5}}{8}\right)^2\sqrt{\frac{-1+\sqrt{5}}{2}} =$$

$$= \sqrt{\frac{-1+\sqrt{5}}{8}}\left(2 + \frac{2-2\sqrt{5}}{24} - \frac{\cancel{8}}{5}\left(\frac{1+5-2\sqrt{5}}{8^2}\right)\right)$$

$$= \sqrt{\frac{-1+\sqrt{5}}{8}}\left(2 + \frac{1-\sqrt{5}}{12} - \frac{6-2\sqrt{5}}{40}\right).$$

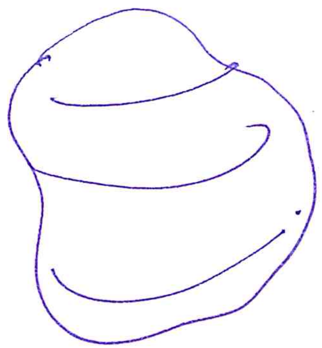
$$\cancel{A = \{(\rho,\theta) \quad \rho \le 1, \quad \rho\cos\theta \ge 2\rho^2\sin^2\theta\}}.$$

$$\cancel{\frac{\cos\theta}{2\sin^2\theta} \ge}$$

$$\iiint_V f(x,y,z)\,dx\,dy\,dz$$

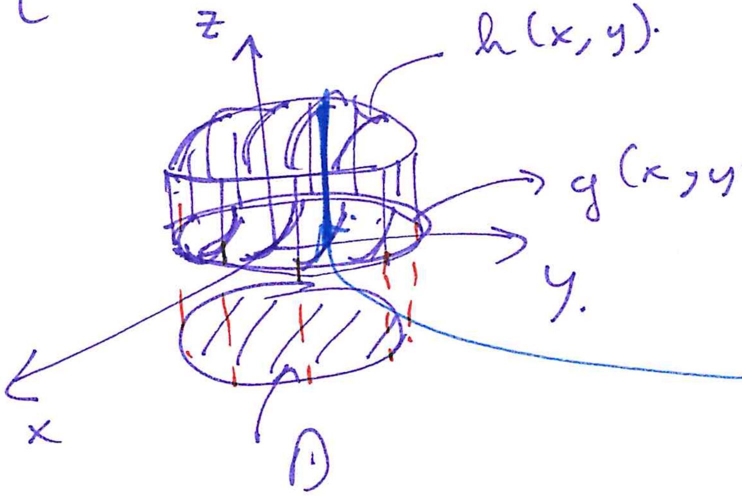
Formula di riduzione.

Formula di Riduzione:



$f(x,y,z)$ densità $\longrightarrow$ Massa $= \iiint_V f(x,y,z)\,dx\,dy\,dz$

$$V = \left\{ (x,y,z) \text{ t.c. } (x,y) \in D,\ g(x,y) \leq z \leq h(x,y) \right\}.$$

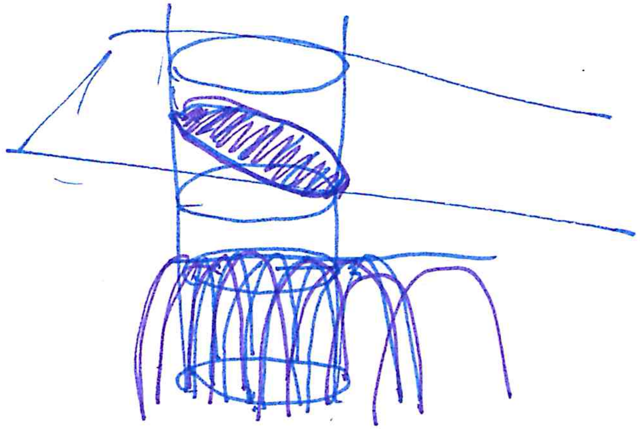


$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iint_D \Bigg(\underbrace{\int_{g(x,y)}^{h(x,y)} f(x,y,z)\,dz}_{1\,\text{VAR}} \Bigg) \underbrace{dx\,dy}_{2\,\text{VAR}}$$

Funzione di due variabili
"Integrare in D"

Determinare il volume.

$$V = \{(x,y,z) \text{ t.c. } \quad x^2+y^2 \le 1 \quad , \quad -x^2 \le z \le 2x+3y+6 \}$$



$$\text{Volume} = \iiint_V 1 \, dx\,dy\,dz = \iint_D \left(\int_{-x^2}^{2x+3y+6} 1 \, dz \right) dx\,dy$$

$D = \{x^2+y^2 \le 1\}$

$\{\rho^2 \le 1, \ 0 \le \theta \le 2\pi\}$

$$= \iint_D 2x+3y+6+x^2 \, dx\,dy = \iint_D 6+x^2 \, dx\,dy = 6\pi + \iint_D x^2 \, dx\,dy.$$

$D = \{x^2+y^2 \le 1\}$

$$= 6\pi + \int_0^1 \left(\int_0^{2\pi} \rho^2 \cos^2\theta \ \rho \, d\theta \right) d\rho = 6\pi + \frac{1}{3} \cdot \pi = \frac{19}{3}\pi.$$