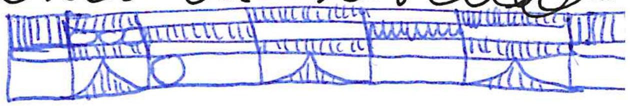
24/10/2021 I

Calcolare $f(x_0, y_0)$, Disegnare gli insiemi di livello



$\ell = -1$, $\ell = 0$, $\ell = 1$.

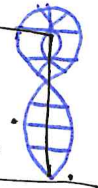
a) $f(x,y) = \sqrt{x+y}$. $\boxed{(x_0, y_0) = (1, 3)}$

b). $f(x,y) = \sqrt{e^y - x}$ $(x_0, y_0) = (\frac{1}{2}, 0)$.

c) $f(x,y) = \log\left(\dfrac{x^2 - y^2}{y - (1-x^2)^2}\right)$. $(x_0, y_0) = (1, \frac{1}{2})$

d) $f(x,y) = \cos(\sqrt{x^2+y^2})$ $(x_0, y_0) = (1, 3)$.

e). $f(x,y) = \dfrac{xy}{x^2+y^2}$. $(x_0, y_0) = (1, 3)$. $\ell = 0$, $\ell = \frac{1}{2}$, $\ell = -\frac{1}{2}$.



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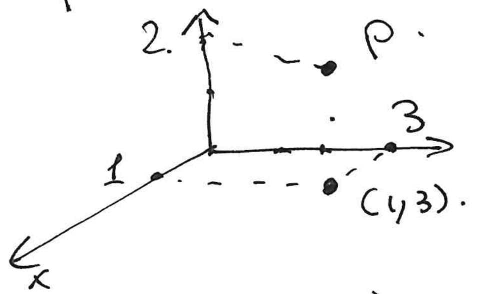
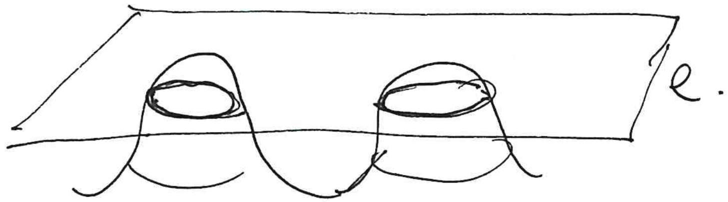
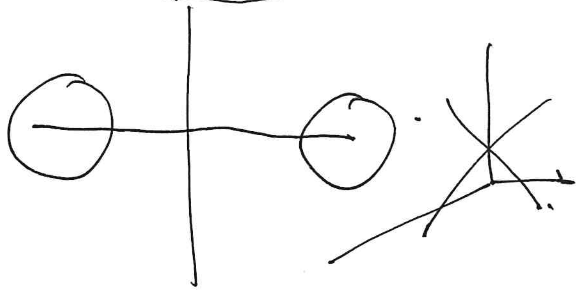
a). $f(1,3) = \sqrt{1+3} = \sqrt{4} = 2$. $(1,3,2) \in$ grafico di f.



Trovare l'insieme di livello significa trovare i punti (x,y) t.c. $\boxed{f(x,y) = \ell}$





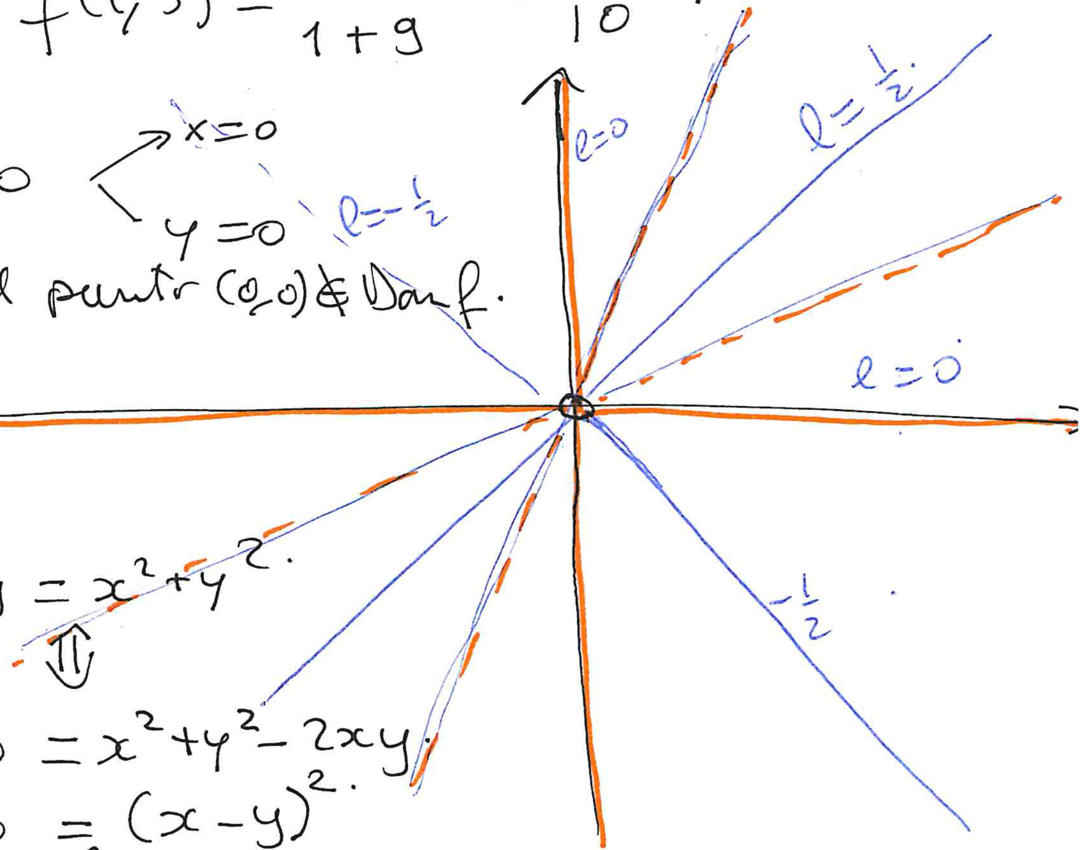
$l=1$, $f(x,y)=1 \iff \sqrt{e^y - x} = 1 \iff e^y - x = 1$.

$\boxed{e^y = x+1.} \iff y = \log(x+1)$.

$$f(x,y)=\frac{xy}{x^2+y^2} \qquad f(1,3)=\frac{3}{1+9}=\frac{3}{10}.$$

$l=0$ $\quad \frac{xy}{x^2+y^2}=0 \iff xy=0$ → $x=0$, $y=0$

escluso il punto $(0,0) \notin \mathrm{Dom}\, f$.



$l=\frac{1}{2}$. $\quad \frac{xy}{x^2+y^2}=\frac{1}{2} \iff 2xy = x^2+y^2$.

$\Updownarrow$

$0 = x^2+y^2-2xy$.

$0 = (x-y)^2$.

$\Updownarrow$

$0 = x-y$

$\Updownarrow$

$\boxed{x=y.}$

$\ell = -\frac{1}{2}$ $\qquad \frac{xy}{x^2+y^2} = -\frac{1}{2} \iff 2xy = -(x^2+y^2) \iff x^2+y^2+2xy = 0$

V.

$$\Updownarrow$$

$$(x+y)^2 = 0 \iff x+y=0$$

$$\boxed{y=-x}$$

$\ell = \frac{1}{4}$?

$$\frac{xy}{x^2+y^2} = \frac{1}{4} \iff 4xy = x^2+y^2$$

$$\boxed{x^2+y^2-4xy=0}$$

$\ell = \frac{1}{4}$

$y \neq 0$ dividiamo per y^2 $\rightarrow$ $\left(\frac{x}{y}\right)^2 + 1 - 4\left(\frac{x}{y}\right) = 0$

$$\left(\frac{x}{y}\right)^2 - 4\left(\frac{x}{y}\right) + 1 = 0$$

$$t^2 - 4t + 1 = 0$$

$$t = 2 \pm \sqrt{4-1} = 2 \pm \sqrt{3}$$

$$\frac{x}{y} = 2+\sqrt{3}$$

$$\frac{x}{y} = 2-\sqrt{3}$$

$$y = \left(\frac{1}{2+\sqrt{3}}\right)x$$

$$y = \left(\frac{1}{2-\sqrt{3}}\right)x$$

OSS: $\ell > 0$ $\quad \sqrt{x+y} = \ell$

$x + y = \ell^2.$

$y = -x + \ell^2.$

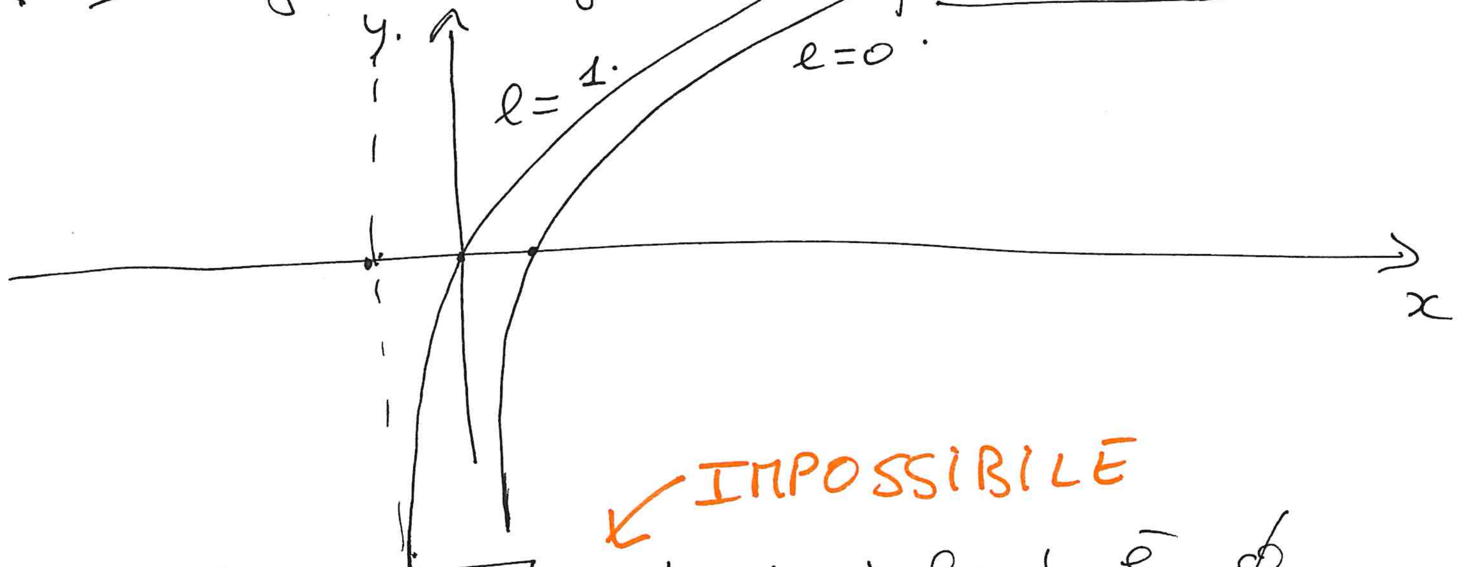
Tutti gli insiemi di livello non nulli sono rette parallele alla retta $y = -x$.

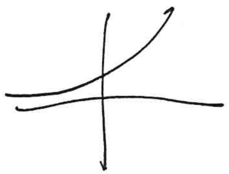
b). $f(\frac{1}{2}, 0) = \sqrt{e^0 - \frac{1}{2}} = \sqrt{1 - \frac{1}{2}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$

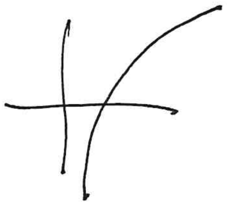
$\ell = 0 \quad f(x,y) = 0 \iff \sqrt{e^y - x} = 0 \iff \boxed{e^y - x = 0}$

Eq. dell'insieme di livello

$\boxed{e^y = x} \iff \log(e^y) = \log x \iff \boxed{y = \log x}$







IMPOSSIBILE

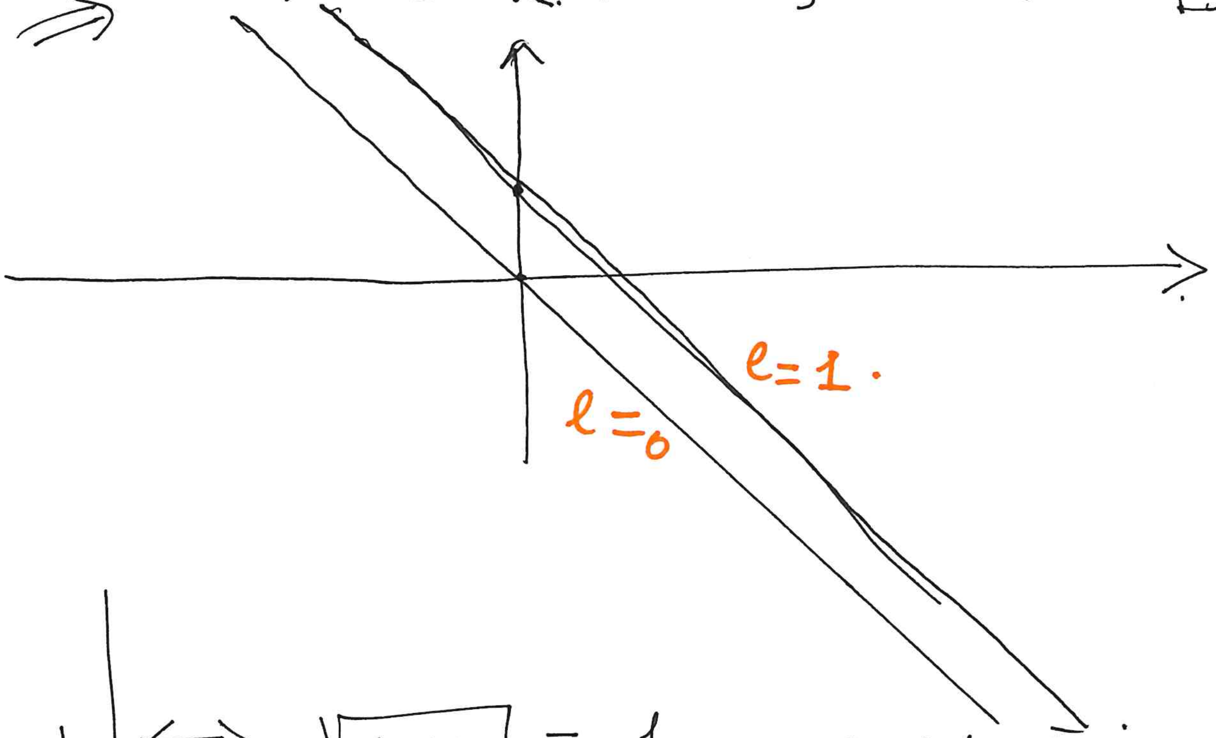
$\ell = -1 \quad f(x,y) = -1 \iff \sqrt{e^y - x} = -1 \iff \ell = -1$ è ϕ.

II

$\ell = 0$

$f(x,y)=0 \implies \sqrt{x+y} = 0 \iff x+y=0 \iff \boxed{y=-x}$ → Equazione che determina i punti dell'insieme di livello.

Si tratta di una retta



$\ell = -1$

$f(x,y) = -1 \iff \sqrt{x+y} = -1$. questo è impossibile, per la definizione di $\sqrt{\ }$ quindi Insieme di livello -1 è l'insieme ~~vuoto~~ vuoto $\implies \emptyset$.

$\ell = 1$

$f(x,y)=1 \iff \sqrt{x+y} = 1 \iff x+y=1$

$y = -x+1$

$a = b \iff a^2 = b^2$

$$f(x,y) : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$ se esiste finito.

$$\frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}.$$ se esiste finita

Per calcolare le derivate parziale basta usare le regole di derivazione per le funzioni reali considerando volta per volta una delle variabili costante e l'altra variabile.

$g(x) = f(x,y)$ y costante, un parametro

$$\frac{\partial f}{\partial x}(x,y) = g'(x).$$

$h(y) = f(x,y)$ x costanta, un parametro

$$\frac{\partial f}{\partial y}(x,y) = h'(y)$$

Def: $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right).$

"gradiente di f"

Def. $P=(\bar{x},\bar{y})$ è un punto critico se $\nabla f(\bar{x},\bar{y}) = (0,0)$. $\Longleftrightarrow$

$$\begin{cases} \frac{\partial f}{\partial x}(\bar{x},\bar{y}) = 0 \\ \frac{\partial f}{\partial y}(\bar{x},\bar{y}) = 0 \end{cases}$$

Trovare i punti critici delle funzioni

a) $\boxed{f(x,y) = xy - 3x^2y}$

$\boxed{\text{b) } f(x,y) = x\,e^{xy^2 + y^2}.}$

$\boxed{\text{c) } f(x,y) = (1-x^2)^2(y-1)(y-2).}$

Per trovare i punti critici è necessario determinare le derivate parziali $\dfrac{\partial f}{\partial x}$ e $\dfrac{\partial f}{\partial y}$.

a) $\dfrac{\partial f}{\partial x}$.

$f(x,y) = xy - 3x^2y = h(y). \longrightarrow h'(y) = x - 3x^2$

$f(x,y) = xy - 3x^2y = g(x) \longrightarrow g'(x) = y - 6xy.$

$$\begin{cases} \dfrac{\partial f}{\partial x} = y - 6xy = 0 \\[2ex] \dfrac{\partial f}{\partial y} = x - 3x^2 = 0 \end{cases} \qquad \begin{cases} y(1-6x) = 0 \longrightarrow y = 0 \text{ oppure } 1-6x=0 \longrightarrow \boxed{y=0} \text{ oppure } \boxed{x = \tfrac{1}{6}} \\[2ex] x(1-3x) = 0 \longrightarrow x = 0 \text{ oppure } x = \tfrac{1}{3}. \end{cases}$$

SONO I PUNTI CRITICI

$y = 0 \longrightarrow \boxed{(0,0)}$ e $\boxed{\left(\tfrac{1}{3}, 0\right)}$

$x = 0$ $\qquad$ $x = \tfrac{1}{3}$

Se scelgo $x = \frac{1}{6}$ si annulla la 1° eq. ma non la seconda. $x = \frac{1}{6}$ non può essere scelto.

c) $f(x,y) = (1-x^2)^2 (y-1)(y-2) = (1-x^2)^2 (y^2 - 3y + 2)$

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = (y-1)(y-2) \cdot 2(1-x^2)(-2x) = -4x(1-x^2)(y-1)(y-2) = 0 \\ \dfrac{\partial f}{\partial y}(x,y) = (1-x^2)^2 \cdot [1(y-2) + 1(y-1)] = (1-x^2)^2 \cdot (2y-3) = 0 \end{cases}$$

$x = 0$ opp. $1-x^2 = 0$ opp $(y-1) = 0$ oppure $(y-2) = 0$.

$$\begin{cases} x = 0 \\ 1-x^2 = 0 \text{ opp. } 2y-3 = 0 \end{cases}$$

$$\begin{cases} \boxed{x=0} \;/\; \boxed{x=1} \;/\; x=-1 \;/\; y=1 \;/\; y=2 \\ x=1, \; x=-1 \;/\; \boxed{y=\frac{3}{2}} \end{cases}$$

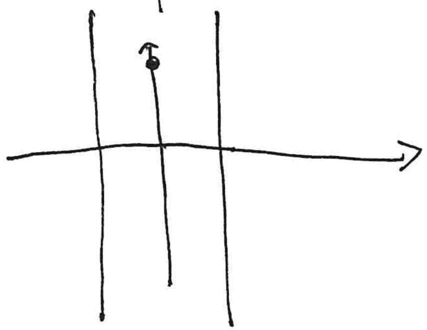
questi valori annullano $\frac{\partial f}{\partial x}$

$\longrightarrow$ questi valori annullano $\frac{\partial f}{\partial y}$.

$\left(0, \frac{3}{2}\right)$

$(1, y)$ sono tutti punti critici $\forall y \in \mathbb{R}$

$(-1, y)$ sono tutti punti critici $\forall y \in \mathbb{R}$



$f(x,y) = \arctan \frac{x}{y}$

$$\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot x \cdot \frac{\partial}{\partial y}\left(\frac{1}{y}\right) = \frac{1}{1+\left(\frac{x}{y}\right)^2} x \cdot \left(-\frac{1}{y^2}\right)$$

<u>oss</u> $\frac{\partial}{\partial y}\left(\frac{1}{y}\right) = \frac{\partial}{\partial y}(y^{-1}) = -1\, y^{-1-1} = \frac{-1}{y^2}.$

$$\frac{\partial f}{\partial y} = \frac{-x}{x^2+y^2}.$$

$f(x,y) = x^3 e^{x^2-y^3}$ Calcolare le $\frac{\partial f}{\partial x}$ e $\frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial x}(x,y) = 3x^2 e^{x^2-y^3} + x^3 \cdot \frac{\partial}{\partial x}\left(e^{x^2-y^3}\right)$$

$$= 3x^2 e^{x^2-y^3} + x^3 \cdot e^{x^2-y^3} \cdot 2x = x^2 e^{x^2-y^3}\left[3+2x^2\right]$$

$$\frac{\partial f}{\partial y}(x,y) = x^3 \cdot \frac{\partial}{\partial y} e^{x^2-y^3} = x^3 e^{x^2-y^3} \cdot (-3y^2) = -3x^3y^2 e^{x^2-y^3}.$$

Det. l'eq. del piano tangente.

$f(x,y) = x^2 + 3xy^2$ $\qquad P = (1,-1) = (x_0, y_0)$

$$z = \frac{\partial f}{\partial x}(P)(x-x_0) + \frac{\partial f}{\partial y}(P)(y-y_0) + f(P).$$

Calcolare.

$$f(1,-1) = 1^2 + 3 \cdot 1 \cdot (-1)^2 = 1 + 3 = 4.$$

$$\frac{\partial f}{\partial x} = 2x + 3y^2 \longrightarrow \frac{\partial f}{\partial x}(1,-1) = 2 \cdot 1 + 3(-1)^2 = 2 + 3 = 5.$$

$$\frac{\partial f}{\partial y} = 3x \cdot 2y = 6xy \longrightarrow \frac{\partial f}{\partial y}(1,-1) = 6 \cdot 1 \cdot (-1) = -6.$$

$$z = \underbrace{5}_{\frac{\partial f}{\partial x}(1,-1)}(x - \underbrace{1}_{x=1}) \underbrace{-6}_{\frac{\partial f}{\partial y}(1,-1)}(y + \underbrace{1}_{y=-1}) + \underbrace{4}_{f(1,-1)}.$$

$$\boxed{z = 5x - 6y - 7}$$ equazione del piano tangente nel punto del grafico $(1, -1, 4)$.

A) $f(x,y) = \arctan \frac{x}{y}$

B) $f(x,y) = x^3 e^{x^2 - y^3}$

$(\arctan x)' = \frac{1}{1+x^2}$

$f(x,y) = \cancel{y\, \frac{1}{y} \arctan x}$.

$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$

A) $\frac{\partial f}{\partial x} = \frac{1}{1 + \left(\frac{x}{y}\right)^2} \cdot \frac{1}{y} = \frac{y}{x^2+y^2}$.

$\frac{\partial f}{\partial y} = \frac{1}{1 + \left(\frac{x}{y}\right)^2} \cdot \left(-\frac{x}{y^2}\right) = \frac{-x}{x^2+y^2}$.

~~arctan y~~

$\left(\frac{1}{y}\right)' = \left(-\frac{1}{y^2}\right)$

$\left(\frac{c}{y}\right)' = -\frac{c}{y^2}$.