Calculus-Unit 1

Applied Computer Science for AI

Blank examination

Voto finale

Postazione:

Cognome:

Nome:

Matricola:

Canale:

Esercizio	Punteggio
1	
2	
3	
4	
Risp. Mult.	
Totale	

Es. 1 [1+2+1 Points] Let $a_n = \frac{n^2+2}{2n^2+1}$ for $n \in \mathbb{N}$

1. Prove that the sequence is bounded by giving an upper bound and a lower for the sequence. (Justify your answer)
   Observe that $n^2 + 2 = \frac{1}{2}(2n^2+1) + \frac{3}{2}$ *so that* $\frac{n^2+2}{2n^2+1} = \frac{1}{2} + \frac{3}{2(2n^2+1)} \geq \frac{1}{2}$. *So that* $\frac{1}{2}$ *is a lower bound and*
   $$\frac{n^2+2}{2n^2+1} \leq \frac{2n^2+2}{2n^2+1} \leq \frac{2n^2+2}{n^2+1} = 2$$
   so that 2 *is an upper bound.*

2. Find $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \frac{1+\frac{2}{n^2}}{2+\frac{1}{n^2}} = \frac{1}{2}$

3. Prove that the sequence is monotone.
   Since $2(2n^2+1) \leq 2(2(n+1)^2+1)$ *then*
   $$a_n = \frac{n^2+2}{2n^2+1} = \frac{1}{2} + \frac{3}{2(2n^2+1)} \geq \frac{1}{2} + \frac{3}{2(2(n+1)^2+1)} = a_{n+1}$$
   And the sequence is monotone decreasing.

Es 2 [3 Points] Prove using induction that for any $n \in \mathbb{N}$ and any $x \in \mathbb{R}$

$$(\sum_{k=0}^{n} x^k)(1-x) = (1-x^{n+1})$$

For $n = 1$, *the equality becomes* $(1+x)(1-x) = 1-x^2$ *which is true.*
Suppose that it is true for n, *let's prove it for* $n+1$.

$$(\sum_{k=0}^{n+1} x^k)(1-x) = (\sum_{k=0}^{n} x^k)(1-x) + x^{n+1}(1-x) = (1-x^{n+1}) + x^{n+1}(1-x) = 1-x^{n+1}+x^{n+1}-x^{n+2} = (1-x^{n+2})$$

Es 3 [4 points] Compute the following limit (justify your answer)

$\lim_{x\to 0^+} \frac{\ln(1+\sqrt[3]{x^2})}{\sqrt{2x}\cdot \sin x}$ *We want to use the special limits*

$$\lim_{x\to 0^+} \frac{\ln(1+\sqrt[3]{x^2})}{\sqrt{2x}\cdot \sin x} = \lim_{x\to 0^+} \frac{\ln(1+\sqrt[3]{x^2})}{\sqrt[3]{x^2}} \cdot \frac{\sqrt[3]{x^2}}{\sqrt{2x}\cdot \sin x} = \lim_{x\to 0^+} \frac{\ln(1+\sqrt[3]{x^2})}{\sqrt[3]{x^2}} \cdot \frac{x^{\frac{2}{3}}}{x^{\frac{3}{2}}\sqrt{2}} \frac{x}{\sin x} = +\infty$$

Es 4 [1+2+1+2+1 points] Given the function $f(x) = \frac{x^2}{x-3}$. Determine:
a) Domain: *The function exists if* $x-3\neq 0$ *su the domain is* $D = \mathbb{R}\setminus\{3\} = (-\infty,3)\cup(3,+\infty)$
b) Asymptotes.

We need to compute the limits at the boundaries of the domain: $\lim_{x\to 3^+} \frac{x^2}{x-3} = +\infty$, $\lim_{x\to 3^-} \frac{x^2}{x-3} = -\infty$,
$\lim_{x\to +\infty} \frac{x^2}{x-3} = +\infty$ *and* $\lim_{x\to -\infty} \frac{x^2}{x-3} = -\infty$. *So* $x=3$ *is vertical asymptote. While at infinity we look for non horizontal asymptotes. Observe that* $f(x) = \frac{x^2}{x-3} = \frac{x^2-9+9}{x-3} = x+3+\frac{9}{x-3}$ *Hence*

$$\lim_{x\to +\infty} f(x) - (x+3) = \lim_{x\to \pm\infty} \frac{9}{x-3} = 0$$

and $y = x+3$ *is an asymptote both at* $+\infty$ *and at* $-\infty$*:*
c) Derivative: *Using* $f(x) = x+3+\frac{9}{x-3}$*, we get that* $f'(x) = 1 - \frac{9}{(x-3)^2}$
d) Interval of monotonicity *In order to determine the interval of monotonicity, we just need to determine the sign of* f' *in* $\mathbb{R}\setminus\{3\}$ $1-\frac{9}{(x-3)^2} = 0$ *for* $x-3 = \pm 3$ *i.e.* $x=6$ *and* $x=0$. *So that*
f *is increasing in* $(-\infty,0)$
f *is decreasing in* $(0,3)$
f *is decreasing in* $(3,6)$
f *is increasing in* $(6,\infty)$
e) Graph

Es 5 [2 o -1 points] The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^{-x^2}$

(A) Has a minimum and a maximum (**XB**) Has a maximum but no minimum

(C) Has a minimum but no maximum (D) Its minimum is at infinity

Es 6 [2 o -1 punti] The derivative of $f(x) = \sin x\, e^{\cos x}$ is:

(A) $\cos x e^{\cos x}$ (B) $\cos x e^{-\sin x}$ (C) $-\sin^2 x e^{\cos x}$

(**XD**) $e^{\cos x}(\cos^2 x + \cos x - 1)$ (E) None of the previous answers is correct

Es 7 Let $f : [1, 2] \to \mathbb{R}$ differentiable such that $f(1) = 1$, $f(2) = \pi$. Then

(A)[1/2] f is increasing $(1, 2)$ **T** **XF** (B)[1/2] $\exists x_o \in (1, 2)$ such that $f'(x_o) = \pi$ **T** **XF**

(C)[1/2] f has a maximum and a minimum **XT** **F** (D)[1/2] $\exists x_o \in (1, 2)$ su that $f(x_o) = 2$ **XT** **F**

Es 8 [2 o -1 punti] $(1 + i)^3$ equals:

(A) $2 + 2i$ (B) $2 - 2i$ (C) $-2i$

(**XD**) $-2 + 2i$ (E) 2

Es 9 [3 o -1 punti] The $\lim_{n\to+\infty} \dfrac{-n^3 + 2n + \ln n^5}{(-1)^n n + 2n^3 + \sqrt{n}}$ equals

(A) 1 (**XB**) $\frac{-1}{2}$ (C) $+\infty$

(D) $-\infty$ (E) The limit does not exist (F) None of the previous answers is correct

Es 10 The function $f : [a, b] \to \mathbb{R}$ is continuous. Say which of the following holds true

(A)[1/2] If $f(a) = f(b)$ then the maximum of f is in (a, b) **T** **XF**

(B)[1/2] If $f(\frac{a+b}{2}) = \frac{f(a)+f(b)}{2}$ then f is constant **T** **XF**

(C)[1/2] If $f(b) > f(a)$, then f is increasing in (a, b). **T** **XF**

(D)[1/2] If $f(x) = 2f(a) + b(x - a)$, then $f(a) = 0$ **XT** **F**

(E)[1/2] There exists an x such that $f(x) = \frac{f(a)+f(b)}{2}$ **XT** **F**