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3 authors:

Lorenzo Carlucci
Sapienza University of Rome

John William Case
University of Delaware

Sanjay Jain
National University of Singapore

Learning Correction Grammars

Lorenzo Carlucci[1], John Case[2], and Sanjay Jain *[3]

[1] Via Elea 8, 00183, Roma, Italy. carlucci5@unisi.it
[2] Department of Computer and Information Sciences, University of Delaware, Newark, DE 19716-2586, U.S.A. case@cis.udel.edu
[3] Department of Computer Science, National University of Singapore, Singapore 117543, Republic of Singapore. sanjay@comp.nus.edu.sg

Abstract. We investigate a new paradigm in the context of learning in the limit, namely, learning *correction grammars* for classes of *r.e.* languages. Knowing a language may feature a representation of the target language in terms of *two* sets of rules (two grammars). The second grammar is used to make corrections to the first grammar. Such a pair of grammars can be seen as a single description of (or grammar for) the language. We call such grammars *correction grammars*. Correction grammars capture the observable fact that people *do* correct their linguistic utterances during their usual linguistic activities.

Is the need for self-corrections implied by using correction grammars instead of normal grammars compensated by a *learning* advantage? We show that learning correction grammars for classes of r.e. languages in the **TxtEx**-model (i.e., converging to a single correct correction grammar in the limit) is sometimes more powerful than learning ordinary grammars even in the **TxtBc**-model (where the learner is allowed to converge to infinitely many syntactically distinct but correct conjectures in the limit). For each $n \geq 0$, there is a similar learning advantage, again in learning correction grammars for classes of r.e. languages, but where we compare learning correction grammars that make $n+1$ corrections to those that make n corrections.

The concept of a correction grammar can be extended into the constructive transfinite, using the idea of counting-down from notations for transfinite constructive ordinals. This transfinite extension can also be conceptualized as being about learning Ershov-descriptions for r.e. langauges. For u a notation in Kleene's general system $(O, <_o)$ of ordinal notations for constructive ordinals, we introduce the concept of an u-correction grammar, where u is used to bound the number of corrections that the grammar is allowed to make. We prove a general hierarchy result: if u and v are notations for constructive ordinals such that $u <_o v$, then there are classes of r.e. languages that can be **TxtEx**-learned by conjecturing v-correction grammars but not by conjecturing u-correction grammars.

Surprisingly, we show that — above "ω-many" corrections — it is not possible to strengthen the hierarchy: **TxtEx**-learning u-correction grammars of classes of r.e. lan-

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guages, where u is a notation in O for *any* ordinal, can be simulated by **TxtBc**-learning w-correction grammars, where w is any notation for the smallest infinite ordinal ω.

1 Introduction and Motivation

We investigate a new model in the context of Gold-style computability-theoretic learning theory (see [23, 25]): learning "correction grammars". Burgin [8] suggested that knowing a language may feature a representation of the language in terms of *two* sets of rules, i.e., two grammars, say g_1 and g_2: g_2 is used to "edit" errors of (make corrections to) g_1. In set-theoretic terms, the language L is represented as the difference $(L_1 - L_2)$, where L_i is the language generated by the grammar g_i. The pair $\langle g_1, g_2\rangle$ can thus be seen as a single description of (or "grammar" for) the language L. Burgin called these grammars *grammars with prohibition*. We prefer to call them *correction grammars*. The idea is readily formalizable in a recursion-theoretic context: p is a *correction grammar* for L if and only if $p = \langle i, j\rangle$ and $L = (W_i - W_j)$.[4] It is natural to consider extensions of the correction grammars paradigm. The concept is readily generalizable to descriptions of r.e. languages as finite differences of r.e. languages. This idea formalizes the concept of a finite, fixed number of successive editing for errors. For $k > 0$, a k-correction grammar for an r.e. language L is a number p such that $p = \langle i_1, \ldots, i_k\rangle$ and $L = W_{i_1} - (W_{i_2} - \cdots (W_{i_{k-1}} - W_{i_k}) \cdots)$.

Correction grammars — besides being mathematically interesting — can be seen as capturing the observable fact that people *do* correct their linguistic utterances during their usual linguistic activities (both oral and written). As will be seen later, the formal concept of a *general* correction grammar can be equivalently expressed in terms of programs that initially exclude all candidate items but are allowed a finite number of mind-changes before choosing whether to include in or exclude from the language each particular candidate item (in other words, correction grammars are limiting recursive functions — that initially output 0 for exclusion). For example, a correction grammar $\langle g_1, g_2\rangle$ for the language $L = L_1 - L_2$, as described above, can be equivalently thought of as an algorithm that initially excludes each item x and, then, it can change its mind up to twice before giving its final answer as to whether x is excluded or included. This description of correction grammars can be seen as modeling the observable self-correcting behaviour of humans.

The observable need for self-correction might be blamed on an error in the rule itself or on a performance error, i.e., an error of rule application. The idea of correction grammars explores the theoretical possibility that the cause of self-correcting behaviour depends on the form of the rules themselves, rather than, e.g., on an error in the rule or in rule application.

If it is the case that the self-correcting behaviour of humans depends on the fact that humans internally represent the target language as a correction grammar and not as a standard grammar, it is natural to ask: is there some *learning* gain that compensates for the need of self-corrections?

[4] W_i is the i-th r.e. set, where i codes a program for generating or for accepting W_i.

We investigate a formal version of the latter question, in the context of computability-theoretic learning theory (or learning in the limit) [23, 25]: a learning machine (an algorithmic device) receives as input a sequence $e_1, e_2, \ldots$ of all and only the elements of an r.e. language L (any such sequence is called a *text* for L) and outputs a corresponding infinite sequence $g_1, g_2, \ldots$ of grammars that *may* generate L. Several criteria of successful learning of a language can and have been studied. The most basic one is *Explanatory Learning from Text* (**TxtEx**-learning): the machine is required to output, past some point, one and the same correct grammar for the input language. A more liberal (and more powerful) criterion is *Behaviourally Correct Learning from Text* (**TxtBc**-learning): the machine is required to output, past some point, only correct grammars, though possibly infinitely many syntactically distinct ones. Both criteria feature learning *in the limit* (the machine does not know if and when it has converged) and require success of the learner on *any* order of presentation of the data. Since learning a single r.e. language is trivial in this model, the simultaneous learning of *classes* of r.e. languages is studied. Note that the conjectures of learning machine as just above are standard type-0 grammars [24], or, equivalently, r.e. indices for the language [39]. The formal version of the above question is now the following: what is the power of an algorithmic learning machine that outputs correction grammars instead of r.e. indices for *r.e.* languages?

One of the main results of the present paper (Theorem 21, Section 3.2) implies that learning correction grammars for classes of r.e. languages is more powerful than learning ordinary r.e. indices in the **TxtEx**-model. The increase in power is so strong that there are classes of *recursive* languages that are **TxtEx**-learnable by a machine that outputs correction grammars but not by any **TxtBc**-learner conjecturing r.e. indices — and this even if the learner is presented with full graphs of characteristic functions of the languages in the class. Theorem 21 in fact is stronger: it shows the advantages of learning $(k + 1)$-correction grammars over learning k-correction grammars for recursive languages.

The next, mathematically natural step is to extend the concept of a correction grammar into the *constructive transfinite*. The idea here is to use ordinals to bound the number of corrections that the grammar can make. To do this rigorously, we use and explain below the idea of counting-down from *notations* for (countable, constructive) transfinite ordinals. *For example*, it can be shown that counting down corrections allowed from any notation for the smallest infinite ordinal $\omega = 0 < 1 < 2 < \ldots$ is equivalent to declaring algorithmically, at the time a first correction is made, the *finite* number of *further* corrections to be allowed. This is more powerful than just initially setting the finite number of corrections allowed.

Intuitively, *ordinals* are representations of well-orderings. The *constructive ordinals* are just those that have a program (called a *notation*) in some system which specifies how to build them (lay them out end to end, so to speak). We will employ, as our system of notations, Kleene's general system O [28–30, 39]. This system has at least one notation for each constructive ordinal and comes with Kleene's standard, useful order relation $<_o$ on the notations in O. $<_o$ naturally embeds into the ordering of the corresponding constructive ordinals (i.e., if u is a notation for α and v is a notation for β and $u <_o v$, then $\alpha < \beta$).

The idea of using constructive notations to perform *algorithmic* count-down from transfinite (constructive) ordinals is widely used in Proof Theory (e.g., to measure the strength of formal systems and classify their provably total functions) [42, 38], and in Term Rewriting (e.g., to prove termination of rewrite systems) [7, 43]. In Learning Theory, this idea has been introduced by Freivalds and Smith in [22]. They used ordinal notations, for example, for algorithmically counting down the mind-changes of inductive inference procedures. This allowed for handling and studying constructive, "transfinite" bounds on mind-changes of inductive inference machines.

In the present paper, we use notations to bound the number of corrections allowed to a correction grammar. Equivalently, beginning with each item not included, we bound the number of mind-changes on the way to convergence of (total) procedures for limiting computable 0-1 valued functions (see [1] for another use of ordinal notations in the context of inductive inference *of functions*). We formalize the concept of a u-correction grammar, where u is a notation in O for some constructive ordinal. To do this, we use concepts from the Ershov Hierarchy [17–19]. The Ershov Hierarchy is based on effective iteration of set-theoretic difference on r.e. sets including up into the constructive transfinite. A correction grammar for an r.e. set will be a "description" of the r.e. set as belonging to some level of the Ershov Hierarchy. We will build on recent work by Case and Royer [13] who obtained succinctness results for correction grammars and developed useful, *uniform* numberings (i.e., programming systems) for the relevant classes of the Ershov Hierarchy. Using these programming systems has the following advantage: it is shown in [13] that these programming systems are *acceptable* [39, 31].[5] Since our results are *independent* of which acceptable system is used for the Ershov classes, we obtain that our results hold for *all* acceptable systems.

From the perspective of the Cognitive Science interpretation of the present investigation, the following should be observed. Although the definition of general correction grammars makes use of the notion of (constructive) transfinite ordinals, the algorithmic count-down always consists of a *finite* number of steps down along a (possibly very intricate) countable *well-ordering*. Thus, the general notion of correction grammar still reflects the behaviour of a learner that is allowed to correct its linguistic output *a finite number* of times.

On the other hand, the correction grammars model has no pretension of modeling human self-correcting behaviour in more detail. For example, there is no control on *how late* a correction may occur, and it should be observed that this is already true for the base case of correction grammars, i.e., grammars that are allowed only one correction on each candidate element of the language.

The existence of a hierarchy of more and more powerful learning criteria, increasing in the ordinal notation used to count corrections of a correction grammar is one of the main results of the present paper (Theorem 19, Section 3.1). We show that, if u and v are two notations for two

[5] By definition, the *acceptable* programming systems for a class are those which contain a universal simulator and into which all other universal programming systems for the class can be compiled. Acceptable systems are characterized as universal systems with an algorithmic substitutivity principle called S-m-n [39, 31, 40, 13]. Acceptable systems also satisfy self-reference principles such as Recursion Theorems [39, 31, 40, 13].

constructive ordinals α and β, respectively, such that $u <_o v$ (which implies $\alpha < \beta$), then **TxtEx**-learning correction grammars that count-down from v is more powerful than **TxtEx**-learning correction grammars that count-down from u. That is, there are classes of r.e. languages, even classes of recursive languages, that can be **TxtEx**-learned by conjecturing correction grammars that count-down from v but not by conjecturing correction grammars that count-down from u — and this even if the learner is required to be successful only on the full graphs of the characteristic functions of the languages in the class, instead of on arbitrary texts for those languages.

Surprisingly, Theorem 29 and Theorem 35 show that the following collapse occurs: any class of r.e. languages that is **TxtEx**-learnable or **TxtBc**-learnable by a learner outputting u-correction grammars, for any notation u for a transfinite ordinal, is already **TxtBc**-learnable by a learner outputting w-correction grammars, where w is a notation for the smallest infinite ordinal ω. Hence, there is a learning power tradeoff between, on the one hand, employing u-corrections, where u is a notation for a very large transfinite ordinal, with **TxtEx**-learning which nicely features only *one* correct correction grammar in the limit and, on the other hand, stopping at "ω" corrections but, then, paying the price of infinitely many distinct correction grammars in the limit. Several other similar collapsing results are proved.

The paper is organized as follows. In Section 2 we introduce notations and definitions needed for the rest of the paper. This section includes general recursion-theoretic notation (Section 2.1), a quick treatment of Kleene's O (Section 2.2), the definition of the Ershov Hierarchy (Section 2.3), and the basics of Gold-style Learning Theory, culminating in the formal definition of learning general correction grammars (Section 2.4).

Next, in Section 3, we first prove our general hierarchy result (Theorem 19): **TxtEx**-learning correction grammars that count-down from (constructive notations for) transfinite ordinals yields an infinite hierarchy of more and more powerful learning criteria.

We then show (Theorem 21) that, for each finite level of this hierarchy, the result can be strengthened as follows: **TxtEx**-learning $n+1$-correction grammars is sometimes more powerful than **TxtBc**-learning n-correction grammars for every natural number n.

In Section 4, we present some surprising collapsing results, showing that the hierarchy results of Section 3 are best possible. For example, we show (Theorem 29) that every class that is **TxtEx**-learnable by conjecturing u-correction grammars, where u is a notation for any constructive ordinal, is already **TxtBc**-learnable using correction grammars that count-down from any notation for ω, the least infinite ordinal.

Section 5 contains some miscellaneous further results regarding learning slightly anomalous programs (Section 5.1) and program size complexity restricted learning (Section 5.2).

Section 6 contains concluding remarks, announcement of further results, open problems, and suggestions for future work.

2 Preliminaries

This section contains terminology for the rest of the paper, a presentation of Kleene's ordinal notation system O, and general background on the Ershov Hierarchy. These technical concepts and tools are then used to define formally the criteria of learning correction grammars that count-down from (notations for constructive) transfinite ordinals.

2.1 Notation and Recursion Theory Background

Any unexplained recursion theoretic notation is from [39]. The symbol $\mathbf{N}$ denotes the set of natural numbers, $\{0, 1, 2, 3, \ldots\}$. The symbols $\emptyset$, $\subseteq$, $\subset$, $\supseteq$, $\supset$, and $\mathbf{\Delta}$ denote empty set, subset, proper subset, superset, proper superset, and symmetric difference, respectively. The cardinality of a set S is denoted by $\mathrm{card}(S)$. $\mathrm{card}(S) \leq *$ denotes that S is finite. We use the convention $n < *$ for all $n \in \mathbf{N}$. The maximum and minimum of a set are denoted by $\max(\cdot), \min(\cdot)$, respectively, where $\max(\emptyset) = 0$ and $\min(\emptyset) = \infty$. $L_1 =^n L_2$ means that $\mathrm{card}(L_1 \mathbf{\Delta} L_2) \leq n$ and L_1 and L_2 are called n-variants. $L_1 =^* L_2$ means that $\mathrm{card}(L_1 \mathbf{\Delta} L_2) \leq *$, i.e., is finite; in this case L_1 and L_2 are called finite-variants.

We let $\langle \cdot, \cdot \rangle$ stand for Cantor's computable, bijective mapping $\langle x, y \rangle = \frac{1}{2}(x+y)(x+y+1)+x$ from $\mathbf{N} \times \mathbf{N}$ onto $\mathbf{N}$ [39]. Note that $\langle \cdot, \cdot \rangle$ is monotonically increasing in both of its arguments. We define $\pi_1(\langle x, y \rangle) = x$ and $\pi_2(\langle x, y \rangle) = y$.

By φ we denote a fixed *acceptable* programming system for the partial-recursive functions mapping $\mathbf{N}$ to $\mathbf{N}$ [39]. By φ_i we denote the partial-recursive function computed by the program number i in the φ-system. We assume that multiple arguments are coded in some standard way [39] and suppress the explicit coding. By Φ we denote an arbitrary fixed Blum complexity measure [6, 24] for the φ-system. A partial recursive function $\Phi(\cdot, \cdot)$ is said to be a Blum complexity measure for φ, if and only if the following two conditions are satisfied:

(a) for all i and x, $\Phi(i, x)\downarrow$ if and only if $\varphi_i(x)\downarrow$.

(b) the predicate: $P(i, x, t) \equiv \Phi(i, x) \leq t$ is decidable.

By convention we use Φ_i to denote the partial recursive function $x \rightarrow \Phi(i, x)$. Intuitively, $\Phi_i(x)$ may be thought as the number of steps it takes to compute $\varphi_i(x)$. $\varphi_{i,s}$ denotes the complexity-bounded version of φ_i, that is, $\varphi_{i,s}(x) = \varphi_i(x)$, if $x < s$ and $\Phi_i(x) < s$; $\varphi_{i,s}(x)$ is undefined otherwise.

Formally, W_i denotes $\mathrm{domain}(\varphi_i)$. That is, W_i is the set of all numbers on which the φ-program i halts. This treats i as an *acceptor* program for W_i [24]. By $W_{i,s}$ we denote the set $\mathrm{domain}(\varphi_{i,s}) = \{x < s \mid \Phi_i(x) < s\}$. χ_L denotes the characteristic function of L. We say that p is a limiting recursive program for a total function f if φ_p is a total function, and for all x, $\lim_{t\to\infty} \varphi_p(x, t) = f(x)$.

The symbol $\mathcal{E}$ will denote the set of all r.e. languages. The symbol L ranges over $\mathcal{E}$. By $\overline{L}$, we denote the complement of L, that is $\mathbf{N} - L$. The symbol $\mathcal{L}$ ranges over subsets of $\mathcal{E}$.

$\forall^\infty$ denotes for all but finitely many.

2.2 Constructive Ordinals and Kleene's $\boldsymbol{O}$

We proceed informally (for a detailed treatment see [39, 3]). A *system of notation* S is a collection of programs (*S-notations*) each of which specifies a structured algorithmic description of some ordinal. Specifically, the notations are programs for building, or laying down end-to-end, the denoted ordinal. An ordinal is called *constructive* when it has a notation in some system of notation.

A *system of notation* S consists of a subset N_S of $\mathbf{N}$ (the set of S-notations), and a mapping $S[\cdot]$ from N_S to an initial segment of the ordinals, such that:

- For $x \in N_S$, the properties of being a notation for 0, a notation for a successor ordinal and a notation for a limit ordinal are recursively decidable.
- There is a partial computable function p (the *predecessor* function) such that, if $x \in N_S$ is a notation for a successor ordinal, then $p(x)$ is defined and is a notation for the immediate predecessor of the ordinal $S[x]$ denoted by x. I.e., $S[p(x)] + 1 = S[x]$.
- There exists a partial computable function q (the *fundamental sequence* function) such that for every notation $x \in N_S$ for a limit ordinal, the function mapping y to $q(x, y)$ is total and $S[q(x, 1)] < S[q(x, 2)] < \ldots$ has limit $S[x]$.

A system of notation S is *acceptable* if any other system is recursively order-preservingly embeddable in it. Formally, if, for any other system of notation S', there exists a partial computable τ such that, for all S'-notation x, $\tau(x)$ is defined and is an S-notation, and $S'[x] \leq S[\tau(x)]$; Furthermore, if $x <_{S'} y$, then $\tau(x) <_S \tau(y)$, where $<_{S'}$ and $<_S$ are ordering relation among S' and S notation respectively. Each acceptable system of notation assigns at least one notation to every constructive ordinal. A system of notation S is *univalent* if $S[\cdot]$ is injective. It is known that every acceptable system *fails* to be univalent (see [39]).

Kleene [28–30, 39] developed a general acceptable system of notation O. Every constructive ordinal has at least one notation in O. O is endowed with a relation $<_o$ on notations that naturally embeds in the ordering of the corresponding constructive ordinals: for all O-notations u, v, if $u <_o v$ then $O[u] < O[v]$.

We will not need much of the particular features of O in what follows, but it is nonetheless necessary to refer to *some* system of notations to define precisely our general concept of learning by correction grammars. The use of O is advantageous in that it provides a general system containing at least one notation for each constructive ordinal. This will allow us to state our general Hierarchy Theorem with satisfactory generality and uniformity.

In Kleene's system O, 2^0 is (by definition) the *notation* for the *ordinal* 0. If u is a notation for the immediate predecessor of a successor ordinal, then a notation for that successor ordinal is (by definition) 2^u. We omit the details of the definition of $<_o$. Suppose $\varphi_p(0), \varphi_p(1), \varphi_p(2), \ldots$ are each notations in $<_o$ order (see [39]). Suppose, then, that the corresponding ordinals are longer and longer initial segments of some limit ordinal which is their sup. For example, some such p generates the respective notations for $0, 1, 2, \ldots$ in $<_o$ order, and ω is the sup of this sequence. In general, then, p essentially describes how to build the limit ordinal which is the sup of the ordinals with notations $\varphi_p(0), \varphi_p(1), \varphi_p(2), \ldots$. A notation for this limit ordinal is (by definition) $3 \cdot 5^p$. A sequence of notations for larger and larger ordinals is also called a *fundamental sequence* for their limit. The assignment of fundamental sequences to limit ordinals is an essential ingredient of an ordinal notation system. Clearly limit ordinals have infinitely many notations, different ones for different generating p's. Nothing else is a notation. We define

'$x =_o y$' to mean '$x, y \in O$ and $x = y$'. As in the literature on constructive ordinals, we use '$x \leq_o y$' for '$x <_o y \vee x =_o y$', '$x \geq_o y$' to mean '$y \leq_o x$ and '$x >_o y$' to mean '$y <_o x$'. We also recall that the mapping $O[\cdot] : O \to$ the set of (constructive) ordinals, is defined as follows [28, 30, 39]

$$\begin{aligned} O[1] &= 0; \\ O[2^u] &= O[u] + 1; \\ O[3 \cdot 5^p] &= \lim_{n \to \infty} O[\varphi_p(n)], \end{aligned}$$

where, for the latter, for each n, $\varphi_p(n) <_o \varphi_p(n+1) \in O$.

For all $x, y \in O$, it is true that, if $x <_o y$ then $O[x] < O[y]$. It is also true that, for all $y \in O$, if $O[y] = \beta$, then for every $\alpha < \beta$, there is an x such that $x <_o y$ and $O[x] = \alpha$. If $u \in O$ and $O[u] = \alpha$, then we say that u is *for* α.

We shall use the following properties of O in later proofs.

Lemma 1 (Some properties of O, [39]).

1. *For every $n \in \mathbf{N}$ there exists a unique O-notation for n. This notation will be denoted by $\underline{n}$.*
2. *For every $v \in O$, $\{u \mid u <_o v\}$ is a univalent system of notations for the corresponding initial segment of the ordinals.*
3. *There exists an r.e. set Z such that $\{u \mid u <_o v\} = \{u \mid \langle u, v\rangle \in Z\}$, for each $v \in O$.*
4. *There exists a computable mapping $+_o : \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{N}$ such that, for every $u, v \in O$,* (i) *$u +_o v \in O$,* (ii) *$O[u +_o v] = O[u] + O[v]$, and* (iii) *if $v \neq 0$ then $u <_o u +_o v$.*

In the rest of this paper, u, v, w denote elements in O.

2.3 The Ershov Hierarchy

In the present subsection we introduce the Ershov Hierarchy [17–19], and give the definition of a *particular* acceptable universal programming system W^u for each level of the hierarchy, due to Case and Royer in [13].[6]

Our presentation of the Ershov Hierarchy is in terms of count-down functions from O-notations for constructive ordinals. Similar presentations, differing from but equivalent to the one originally given by Ershov, can be found in [3, 16].

Definition 2 (Count-Down Function). A computable function $F : \mathbf{N} \times \mathbf{N} \to O$ is a *count-down function* if for all x and t, $F(x, t+1) \leq_o F(x, t)$.

For a binary function $h(\cdot, \cdot)$ we write $h(x, \infty)$ for the limit $\lim_{t \to \infty} h(x, t)$. For a set A, we denote by χ_A the characteristic function of A.

[6] This system was created in part to make sure there *is* such an acceptable system. The construction of the W_u's is also nicely uniform in $u \in O$.

Definition 3 (Ershov Hierarchy). $A \in \Sigma_u^{-1}$ if and only if there exists a computable function $h : \mathbf{N} \times \mathbf{N} \to \{0,1\}$ and a count-down function F such that, for all $x, t \in \mathbb{N}$,

(i) $\chi_A(x) = h(x, \infty)$,
(ii) $h(x, 0) = 0$ and $F(x, 0) \leq_o u$,
(iii) $h(x, t+1) \neq h(x, t) \Rightarrow F(x, t+1) <_o F(x, t)$.

In this case we say that h and F *witness* $A \in \Sigma_u^{-1}$.

Note that $\Sigma_{\underline{0}}^{-1} = \{\emptyset\}$. Definition 3 immediately implies that $u <_o v \Rightarrow \Sigma_u^{-1} \subseteq \Sigma_v^{-1}$. The containment is in fact proper, so that one speaks of the Ershov Hierarchy.

The next lemma spells out the correspondence between a description of a set in terms of finite differences of r.e. sets and in terms of count-down functions.

Lemma 4 (Case and Royer, [13]). *$X \in \Sigma_{u +_o \underline{1}}^{-1}$ if and only if there exists $Y \in \mathcal{E}$, $\exists Z \in \Sigma_u^{-1}$ such that $Z \subseteq Y$ and $X = Y - Z$.*

Corollary 5. *Let $n \geq 1$. $X \in \Sigma_{\underline{n}}^{-1}$ if and only if there exists r.e. sets $Y_1 \supseteq \ldots \supseteq Y_n$ such that $X = Y_1 - (Y_2 - (\ldots - (Y_{n-1} - Y_n) \ldots))$.*

We will now proceed to define, for each $u \in O$, an acceptable universal numbering or programming system W^u for Σ_u^{-1}. This numbering is due to Case and Royer [13]. The proof that it gives an acceptable numbering is omitted here.

Let us denote by φ^{TM} an acceptable programming system for the partial computable functions based on a coding of deterministic multi-tape Turing Machines. By standard results [39], φ^{TM} is an acceptable programming system for the partial computable functions.

For each x and i in $\mathbf{N}$, let $\Phi_i^{\mathrm{TM}}(x)$ be the runtime of Turing Machine i on input x. It is easy to check that $\Phi_i^{\mathrm{TM}}(x)$ is a Blum Complexity Measure [6] for φ^{TM}. It is easy to arrange the numerical TM coding so that $\{\langle x, i, t\rangle \mid \Phi_i^{\mathrm{TM}}(x) \leq t\}$ is primitive recursively decidable (see, for example, [40]).

From Lemma 1 part 3, we know that there exists an r.e. set Z such that

$$\{u : \langle u, v\rangle \in Z\} = \{u : u <_o v\},$$

for each $v \in O$. In what follows, let z_0 be a fixed φ^{TM} program for accepting Z.

Definition 6 (Convenient Function). Let F be a count-down function. Then F is *convenient* (relative to z_0) if

$$(\forall x)(\forall t)[F(x, t+1) <_o F(x, t) \Rightarrow \Phi_{z_0}^{\mathrm{TM}}(\langle F(x, t+1), F(x, t)\rangle) \leq t].$$

We say that h and F *conveniently witness* $A \in \Sigma_u^{-1}$ when h and F witness $A \in \Sigma_u^{-1}$ and F is convenient.

Let ψ be a *standard* programming system for the primitive recursive functions, (i.e., one for which the S-m-n Theorem, Recursion Theorems, etc. all hold) [40].

Definition 7. Let $u \in O$. We say that i, j, x are *u-consistent through t* when $\psi_i(x, 0) = 0$, $\psi_j(x, 0) =_o u$ and, for each $t' < t$,

(i) $\psi_i(x, t'+1) \in \{0, 1\}$,
(ii) $\psi_j(x, t'+1) \neq \psi_j(x, t') \Leftrightarrow \Phi^{\mathrm{TM}}_{z_0}(\psi_j(x, t'+1), \psi_j(x, t')) \leq t'$,
(iii) $\psi_i(x, t'+1) \neq \psi_i(x, t') \Rightarrow \psi_j(x, t'+1) <_o \psi_j(x, t')$.

Definition 8. Let $u \in O$. For each i, j, x, t let

$$h^u(i, j, x, t) = \begin{cases} 0 & \text{if } i, j, x \text{ are not } u\text{-consistent through } 0; \\ \psi_i(x, t') & \text{otherwise, where} \\ & t' \text{ is the greatest number } \leq t \\ & \text{such that } i, j, x \text{ are } u\text{-consistent through } t'. \end{cases}$$

For each $i, j \in \mathbf{N}$, let

$$W^u_{\langle i,j \rangle} = \{x : h^u(i, j, x, \infty) = 1\}.$$

We call p a *u-correction grammar* for W^u_p, and we are herein interested in such grammars for r.e. (and recursive) languages.

We observe that $h^u(i, j, x, t)$ is double recursive (see [36]) as a function of u, i, j, x, t, while, for each u, $h^u(i, j, x, t)$ is primitive recursive as a function of i, j, x, t.[7]

Theorem 9 (Case and Royer, [13]).

(a) *W^u is an acceptable universal numbering for Σ^{-1}_u.*
(b) *The S-m-n and the Recursion Theorems hold for W^u.*

In particular, by the previous Theorem, the Kleene Recursion Theorem holds in the W^u-system.[8] This means that, given the specification of a W^u-system task p, there exists an e such that program e in the W^u-system makes a self-copy (i.e., computes a copy of its own code) and applies that task p to this self-copy (and, of course, to its external input). In proofs below, for convenience, we will give the description of what such an e does with its self-copy in an *informal* system. In particular we will describe task-relevant functions h and F, *each informally in terms of* e, such that F is a count-down function and h and F witness that $\{x \mid h(x, \infty) = 1\} \in \Sigma^{-1}_u$ (Definition 3). In each such case, we make sure this latter set *is* u-r.e. Implicitly, then, we invoke the acceptability of W^u to obtain a translation of the informal description involving h and F into the W^u-system to get what e really does in the formal W^u-system with its self-copy (and its external input). In practice, though, we will merely describe informally what such a formal e does with its self-copy, and not mention again the invoking of acceptability to get a translation of what such an e does more formally with its self-copy, etc.

[7] As noted in [13], it is possible, using methods from [40], to make $h^u(i, j, x, t)$ polynomial-space computable as a function of u, i, x, t and, for every u, to make $h^u(i, j, x, t)$ linear-time computable as a function of i, j, x, t.
[8] For the φ-system the Kleene Recursion Theorem is described in [39, Page 214]. Herein we employ this self-reference principle for the W_u systems, $u \in O$.

We adopt, for the sake of readability, the following notational convention:

$$\theta_p^u(x,t) = h^u(\pi_1(p), \pi_2(p), x, t).$$

We now have all the tools needed to define the general learning criteria of learning u-correction grammars.

2.4 Learning Criteria

We now present concepts from language learning theory (see [25]) and then formally define learning correction grammars.

The next definition introduces the concept of a *sequence* of data.

Definition 10 (Sequence).

(a) A *sequence* σ is a mapping from an initial segment of $\mathbf{N}$ into $(\mathbf{N} \cup \{\#\})$. The empty sequence is denoted by λ.

(b) The *content* of a sequence σ, denoted $\mathrm{content}(\sigma)$, is the set of natural numbers in the range of σ.

(c) The *length* of σ, denoted by $|\sigma|$, is the number of elements in σ. So, $|\lambda| = 0$.

(d) For $n \leq |\sigma|$, the initial sequence of σ of length n is denoted by $\sigma[n]$. So, $\sigma[0]$ is λ.

Intuitively, the pause-symbol $\#$ represents a pause in the presentation of data. We let σ, τ and γ range over finite sequences. We denote the sequence formed by the concatenation of τ at the end of σ by $\sigma \diamond \tau$ or by $\sigma\tau$. Sometimes we abuse the notation and use σx to denote the concatenation of sequence σ and the sequence of length 1 which contains the element x. SEQ denotes the set of all finite sequences.

Definition 11 (Texts, [23]).

(a) A *text* T for a language L is a mapping from $\mathbf{N}$ into $(\mathbf{N} \cup \{\#\})$ such that L is the set of natural numbers in the range of T. $T(i)$ represents the $(i+1)$-st element in the text.

(b) The *content* of a text T, denoted by $\mathrm{content}(T)$, is the set of natural numbers in the range of T; that is, the language which T is a text for.

(c) $T[n]$ denotes the finite initial sequence of T with length n.

Definition 12 (Learning Machine, [23]). A *learning machine* (or just *learner*) is an algorithmic device which computes a mapping from SEQ into $\mathbf{N}$.

We let $\mathbf{M}$ range over learning machines. We note that, without loss of generality, for all criteria of learning discussed in this paper, a learner $\mathbf{M}$ may be assumed to be total. $\mathbf{M}(T[n])$ denotes the hypothesis of the learner $\mathbf{M}$ after it has seen the first n members of T. $\mathbf{M}(T) = e$ denotes that $\mathbf{M}$ converges on T to e, that is $\mathbf{M}(T[n]) = e$, for all but finitely many n.

There are several criteria for a learning machine to be successful on a language. Below we define some of them.

Definition 13 (Explanatory Learning, [12, 23]).

Suppose $a \in \mathbf{N} \cup \{*\}$.

(a) $\mathbf{M}$ $\mathbf{TxtEx}^a$*-identifies a text* T just in case $(\exists i \mid W_i =^a \text{content}(T))$ $(\forall^\infty n)[\mathbf{M}(T[n]) = i]$.

(b) $\mathbf{M}$ $\mathbf{TxtEx}^a$*-identifies an r.e. language* L (written: $L \in \mathbf{TxtEx}^a(\mathbf{M})$) just in case $\mathbf{M}$ $\mathbf{TxtEx}^a$-identifies each text for L.

(c) $\mathbf{M}$ $\mathbf{TxtEx}^a$*-identifies a class* $\mathcal{L}$ *of r.e. languages* (written: $\mathcal{L} \subseteq \mathbf{TxtEx}^a(\mathbf{M})$) just in case $\mathbf{M}$ $\mathbf{TxtEx}^a$-identifies each language from $\mathcal{L}$.

(d) $\mathbf{TxtEx}^a = \{\mathcal{L} \subseteq \mathcal{E} \mid (\exists \mathbf{M})[\mathcal{L} \subseteq \mathbf{TxtEx}^a(\mathbf{M})]\}$.

Definition 14 (Behaviourally Correct Learning, [12, 33]).

Suppose $a \in \mathbf{N} \cup \{*\}$.

(a) $\mathbf{M}$ $\mathbf{TxtBc}^a$*-identifies a text* T just in case $(\forall^\infty n \in \mathbf{N})[W_{\mathbf{M}(T[n])} =^a \text{content}(T)]$.

(b) $\mathbf{M}$ $\mathbf{TxtBc}^a$*-identifies an r.e. language* L (written: $L \in \mathbf{TxtBc}^a(\mathbf{M})$) just in case $\mathbf{M}$ $\mathbf{TxtBc}^a$-identifies each text for L.

(c) $\mathbf{M}$ $\mathbf{TxtBc}^a$*-identifies a class* $\mathcal{L}$ *of r.e. languages* (written: $\mathcal{L} \subseteq \mathbf{TxtBc}^a(\mathbf{M})$) just in case $\mathbf{M}$ $\mathbf{TxtBc}^a$-identifies each language from $\mathcal{L}$.

(d) $\mathbf{TxtBc}^a = \{\mathcal{L} \subseteq \mathcal{E} \mid (\exists \mathbf{M})[\mathcal{L} \subseteq \mathbf{TxtBc}^a(\mathbf{M})]\}$.

For learning criteria $\mathbf{I}$, $a = 0$, we often write $\mathbf{I}$ instead of $\mathbf{I}^a$. It is well-known that $\mathbf{TxtEx}^a \subset \mathbf{TxtBc}^a$ (see [25]). With an abuse of terminology we sometimes refer to any criterion that requires syntactic (resp. semantic) convergence in the limit as **Ex**- (resp. **Bc**-) learning.

We collect some well-known facts about learning with anomalies in the following theorem (see [25] for references).

Theorem 15.

Suppose $n, m \in \mathbf{N}$.

(a) $\mathbf{TxtEx} \subset \mathbf{TxtEx}^1 \subset \ldots \subset \mathbf{TxtEx}^*$,

(b) $\mathbf{TxtEx}^* - \bigcup_{n \in \mathbf{N}} \mathbf{TxtEx}^n$,

(c) $\mathbf{TxtBc} \subset \mathbf{TxtBc}^1 \subset \ldots \subset \mathbf{TxtBc}^*$,

(d) $\mathbf{TxtEx}^*$ and $\mathbf{TxtBc}$ are incomparable,

(e) $\{L \mid L =^{2m+1} \mathbf{N}\} \in (\mathbf{TxtEx}^{2m+1} - \mathbf{TxtBc}^m)$,

(f) $\mathbf{TxtEx}^{2m} \subseteq \mathbf{TxtBc}^m$.

When we only require that a learner is successful when fed the graph of the characteristic function of the language instead of any text, we obtain the concept of *learning from informant* (see [23]). For an informant I, we denote by $I[n]$, the first n elements of I. A canonical informant for a language L is $(0, \chi_L(0)), (1, \chi_L(1)), (2, \chi_L(2)), \ldots$. We often identify χ_L with the canonical informant for L. For a characteristic function f, we use $f[n]$ to denote the initial segment $(0, f(0)), (1, f(1)), \ldots, (n-1, f(n-1))$.

Using **Inf** instead of **Txt** in the name of any learning criterion indicates that the requirement of learning *from texts* is substituted by the requirement of learning *from informant*. It is well-known that more can be learned from informant than from text (see [25]).

We can now formally introduce learning by correction grammars. Intuitively, a $\mathbf{Cor}^u\mathbf{I}$-learner, where $\mathbf{I}$ is any learning criterion, is a successful $\mathbf{I}$-learner when its conjectures are interpreted as u-correction grammars.

Definition 16 (Learning Correction Grammars). Let $u \in O$, $a \in \mathbf{N} \cup \{*\}$.

(a) $\mathbf{Cor}^u\mathbf{TxtEx}^a$ is the collection of all classes $\mathcal{L}$ of *r.e.* languages such that there exists an $\mathbf{M}$ such that $(\forall L \in \mathcal{L})(\forall$ texts T for $L)(\exists i)[W_i^u =^a L \wedge (\forall^\infty n)[\mathbf{M}(T[n]) = i]]$ — in this case we say that $\mathcal{L} \subseteq \mathbf{Cor}^u\mathbf{TxtEx}^a(\mathbf{M})$ or $\mathcal{L}$ is $\mathbf{Cor}^u\mathbf{TxtEx}^a$-identified by $\mathbf{M}$.

(b) $\mathbf{Cor}^u\mathbf{TxtBc}^a$ is the collection of all classes $\mathcal{L}$ of r.e. languages such that there exists an $\mathbf{M}$ such that $(\forall L \in \mathcal{L})(\forall$ texts T for $L)(\forall^\infty n)[W^u_{\mathbf{M}(T[n])} =^a L]$ — in this case we say that $\mathcal{L} \subseteq \mathbf{Cor}^u\mathbf{TxtBc}^a(\mathbf{M})$ or $\mathcal{L}$ is $\mathbf{Cor}^u\mathbf{TxtBc}^a$-identified by $\mathbf{M}$.

It is important to note that, while the Ershov Hierarchy goes well beyond the r.e. languages, we are interested primarily in the *r.e.* languages and their learnability with respect to u-correction grammars. This implies that, for example, a $\mathbf{Cor}^u\mathbf{TxtEx}$-learner outputs "descriptions" of an *r.e.* language as a member of the u-th level of the Ershov Hierarchy (whereas the latter level also contains non-r.e. sets, if u denotes an ordinal larger than 1).

3 Hierarchy Results

In this section we prove some hierarchy results about learning correction grammars. Each of these separation results is witnessed by a class of recursive languages.

Our first main result (Theorem 19) shows that an increase in learning power is obtained — in the context of $\mathbf{TxtEx}$-learning correction grammars — when the number of corrections allowed is counted by (notations for) larger and larger constructive transfinite ordinals.

Next (Theorem 21) we prove a strengthening of this hierarchy for all finite levels: for all $n \in \mathbf{N}$, there are classes of *recursive* languages that can be $\mathbf{TxtEx}$-learned by a learner conjecturing $\underline{k+1}$-correction grammars that cannot be $\mathbf{TxtBc}$-learned by any learner conjecturing $\underline{k}$-correction grammars (not even from informant). We will show in Section 4 that this strengthening is best possible: it cannot be extended beyond the ω-th level of the hierarchy.

3.1 The General $\mathbf{Cor}^u\mathbf{TxtEx}$ Hierarchy

We will prove that for all $u, v \in O$ such that $u <_o v$ there exist classes of recursive languages that are learnable by a $\mathbf{TxtEx}$-learner that outputs v-correction grammars but such that no $\mathbf{TxtEx}$-learner can learn those classes using u-correction grammars, even if presented with informants instead of texts. The general case will follow from the following result on the successor case.

Notation: We use $h(\cdot, s)$ to denote the function which maps x to $h(x, s)$.

Theorem 17. *For all $n \in \mathbf{N}$, $u \in O$,* $\mathbf{Cor}^{u+_o\underline{1}}\mathbf{TxtEx} - \mathbf{Cor}^u\mathbf{InfEx}^n \neq \emptyset$.

Proof. Let $\mathcal{L} = \{L$ recursive $\mid L \neq \emptyset \wedge W^{u+_o\underline{1}}_{\min(L)} = L\}$. Clearly $\mathcal{L} \in \mathbf{Cor}^{u+_o\underline{1}}\mathbf{TxtEx}$. Suppose by way of contradiction that $\mathbf{M}$ $\mathbf{Cor}^u\mathbf{InfEx}^n$-identifies $\mathcal{L}$.

By the Kleene Recursion Theorem in the system $W^{u+_o\underline{1}}$ there exists an e such that $W_e^{u+_o\underline{1}} = \{x \mid h(x,\infty) = 1\}$, where h is a function informally defined in stages below (we will have that $e = \min(\{x \mid h(x,\infty) = 1\})$). Along with h we informally define another function F, such that F is a count-down function and h and F witness that $\{x \mid h(x,\infty) = 1\} \in \Sigma^{-1}_{u+_o\underline{1}}$ (Definition 3).

Initially, $h(x,0) = 0$ for all x; $h(y,1) = 0$, for $y < e$, and $h(y,1) = 1$, for $y \geq e$. $F(y,0) = u+_o\underline{1}$ and $F(y,1) = u$, for all y. Let $x_1 = e+1$. Go to stage 1 (we start with stage 1 for ease of notation).

We will have the invariants that, at the start of stage s,

(1) for $x > x_s + n$, $h(x,s) = 1$ and $F(x,s) = u$.
(2) for $x < x_s$, for all $t > s$, $h(x,t) = h(x,s)$.
(3) For all $x_s \leq x \leq x_s + n$, either
  (3a) for $i = \mathbf{M}(h(\cdot,s)[x_s])$,
    $h(x,s) = 1 - \theta_i^u(x,s)$, and
    $F(x,s) = \psi_{\pi_2(i)}(x,s)$, or
  (3b) $h(x,s) = 1$, $F(x,s) = u$ (in this case $x_s \neq x_{s-1}$, where we take $x_0 = 0$).

Stage s
1. If there exists a z, $x_s + n < z \leq s$, such that $\mathbf{M}(h(\cdot,s)[z]) \neq \mathbf{M}(h(\cdot,s)[x_s])$, then
    Let $x_{s+1} = z$.
    For all x, let $h(x,s+1) = h(x,s)$ and $F(x,s+1) = F(x,s)$.
    Go to stage $s+1$.
2. Else,
  2.1 Let $i = \mathbf{M}(h(\cdot,s)[x_s])$.
  2.2 For $x_s \leq x \leq x_s + n$, let
      $h(x,s+1) = 1 - \theta_i^u(x,s+1)$, and
      $F(x,s+1) = \psi_{\pi_2(i)}(x,s+1)$.
    (Note that above change is valid, based on invariant 3 above).
  2.3 For $x < x_s$ or $x > x_s + n$, let
      $h(x,s+1) = h(x,s)$, and
      $F(x,s+1) = F(x,s)$.
  2.4 Let $x_{s+1} = x_s$.
    Go to stage $s+1$.

End stage s

It is easy to see that the invariants are satisfied. We now consider two cases.

Case 1: $\lim_{s\to\infty} x_s$ is infinite.

In this case, clearly, the function mapping x to $\lim_{t\to\infty} h(x,t)$ is a recursive function, and on this function $\mathbf{M}$ makes infinitely many mind-changes. Furthermore, clearly, $\lim_{t\to\infty} h(x,t)$ is a characteristic function for a language in $\mathcal{L}$.

Case 2: $\lim_{s\to\infty} x_s$ is finite.

Suppose $\lim_{t\to\infty} x_t = z = x_s$. In this case, clearly, the function mapping x to $\lim_{t\to\infty} h(x,t)$ is a recursive function, and a characteristic function for a language (say L) in $\mathcal{L}$. Let χ_L denote the characteristic function of L and let $\mathbf{M}(\chi_L)$ denote $\mathbf{M}$'s final conjecture when the input informant is χ_L. We have that $\mathbf{M}(\chi_L) = \mathbf{M}(\chi_L[z])$, as the condition in step 1 did not succeed beyond stage s. Furthermore, using invariant (3), $\mathbf{M}(\chi_L[z])$ makes errors on inputs x, for $x_s \le x \le x_s + n$.

From both the above cases, we have that e is a $(u +_o \underline{1})$-correction grammar for a language in $\mathcal{L}$ which is not $\mathbf{Cor}^u\mathbf{InfEx}^n$-identified by $\mathbf{M}$. $\square$

Corollary 18. *For all $n \in \mathbf{N}$, for all $v \in O$ such that v is a notation for a limit ordinal, for all $u <_o v$, $\mathbf{Cor}^v\mathbf{TxtEx} - \mathbf{Cor}^u\mathbf{InfEx}^n \neq \emptyset$.*

Proof. If v is a notation for a limit ordinal and $u <_o v$ then $u +_o \underline{1} <_o v$. But $\mathbf{Cor}^u\mathbf{TxtEx} \subset \mathbf{Cor}^{u+_o\underline{1}}\mathbf{TxtEx}$ by Theorem 17, and obviously $\mathbf{Cor}^{u+_o\underline{1}}\mathbf{TxtEx}$ is included in $\mathbf{Cor}^v\mathbf{TxtEx}$. $\square$

By Theorem 17 and Corollary 18, we have the following Hierarchy Theorem. As a corollary we obtain an even stronger version.

Theorem 19. *For all $u, v \in O$, if $u <_o v$ then for all $n \in \mathbf{N}$*
(a) $\mathbf{Cor}^u\mathbf{TxtEx}^n \subset \mathbf{Cor}^v\mathbf{TxtEx}^n$.
(b) $\mathbf{Cor}^u\mathbf{InfEx}^n \subset \mathbf{Cor}^v\mathbf{InfEx}^n$.

Corollary 20. *For all $v \in O$, $\mathbf{Cor}^v\mathbf{TxtEx} - \bigcup_{u<_o v} \mathbf{Cor}^u\mathbf{InfEx}^n \neq \emptyset$.*

Proof. Let $\mathcal{L}_u$ denote $\mathcal{L}$ as defined in the proof of Theorem 17 (for u as in the statement of Theorem 17). Let $\mathcal{L}'_u = \{\{\langle u, x\rangle \mid x \in L\} \mid L \in \mathcal{L}_u\}$. Consider $\mathcal{L} = \bigcup_{u<_o v} \mathcal{L}'_u$. It is easy to see that $\mathcal{L} \in \mathbf{Cor}^v\mathbf{TxtEx}$. Suppose now that $\mathcal{L} \in \bigcup_{u<_o v} \mathbf{Cor}^u\mathbf{InfEx}^n$. Let $u <_o v$ be such that $\mathcal{L} \in \mathbf{Cor}^u\mathbf{InfEx}^n$. Then, $\mathcal{L}'_u \in \mathbf{Cor}^u\mathbf{InfEx}^n$, and thus $\mathcal{L}_u \in \mathbf{Cor}^u\mathbf{InfEx}^n$. This contradicts Theorem 17. $\square$

3.2 The Finite Levels: a Strong Hierarchy

In order to measure the increase in learning power unveiled by the previous results, it is natural to ask: are there classes that can be $\mathbf{TxtEx}$-learned by guessing a $(u +_o \underline{1})$-correction grammar but such that no learner guessing u-correction grammars can learn those classes even if it is allowed to conjecture infinitely many syntactically distinct but correct conjectures in the limit? Our next result shows that the answer is positive for all the finite levels of the correction-grammars hierarchy. In the next section we will show that it is *impossible* to obtain the analogous strengthening of the hierarchy for all levels of the $\mathbf{Cor}^u\mathbf{TxtEx}$-hierarchy.

Theorem 21. *For $k \in \mathbf{N}$, $\mathbf{Cor}^{\underline{k+1}}\mathbf{TxtEx} - \mathbf{Cor}^{\underline{k}}\mathbf{InfBc} \neq \emptyset$.*

Proof. Let $\mathcal{L} = \{L \text{ recursive} \mid L \neq \emptyset \wedge W^{\underline{k+1}}_{\min(L)} = L\}$.

Clearly, $\mathcal{L} \in \mathbf{Cor}^{\underline{k+1}}\mathbf{TxtEx}$. Now suppose by way of contradiction that $\mathcal{L} \in \mathbf{Cor}^{\underline{k}}\mathbf{TxtBc}$ as witnessed by $\mathbf{M}$.

By the Kleene Recursion Theorem in the system $W^{\underline{k+1}}$, there exists an e such that $W_e^{\underline{k+1}} = \{x \mid h(x,\infty)=1\}$, where h can be informally defined in stages as follows. We will ensure that $h(x,\cdot)$ changes its mind for any x at most $k+1$-times. Thus, the definition of a function F such that h and F witness $\{x \mid h(x,\infty)=1\} \in \Sigma^{-1}_{\underline{k+1}}$ is implicit in our construction.

We will also define finite sets $S^0 \subseteq S^1 \ldots$. Intuitively, these sets denote the values of x whose membership in $L = W_e^{\underline{k+1}}$ has been frozen. In other words, for all s, for all $x \in S^s$, for all $t \geq s$, $h(x,t) = h(x,s)$.

Notation: Let $MC(h,x,s) = \mathrm{card}(\{t < s \mid h(x,t) \neq h(x,t+1)\})$ (thus, $MC(h,x,s)$ denotes the number of mind changes in the sequence $h(x,0), h(x,1), \ldots, h(x,s)$). Similarly, let $MCP(i,x,s) = \mathrm{card}(\{t < s \mid \theta_i^{\underline{k}}(x,t) \neq \theta_i^{\underline{k}}(x,t+1)\})$.

We will have the following invariants for each s.

(1) For all $x \notin S^{s+1}$, $MC(h,x,s+1) \leq 1 + MCP(i,x,s)$, where $i = \mathbf{M}(h(\cdot,s+1)[x])$.

(2) For all $x \in S^{s+1}$, $MC(h,x,s+1) \leq k+1$.

(3) If $x \notin S^{s+1}$ and $x \leq s$, then $h(x,s+1) \neq \theta^{\underline{k}}_{\mathbf{M}(h(\cdot,s+1)[x])}(x,s)$.

Initially, $h(x,0) = 0$, for all x; $h(x,1) = 0$, for $x < e$ and $h(x,1) = 1$ for all $x \geq e$. Let $S^1 = \{x \mid x \leq e\}$. Clearly, the invariants are satisfied in the beginning. Go to stage $s = 1$ (we start with stage 1, for ease of notation).

Begin Stage s:

1. If there exists an $x \leq s$, $x \notin S^s$ such that $\theta^{\underline{k}}_{\mathbf{M}(h(\cdot,s)[x])}(x,s) = h(x,s)$, then pick the least such x and go to step 2. Otherwise, go to step 3.
   (* For $i = \mathbf{M}(h(\cdot,s)[x])$, note that invariant (1) implies that $MC(h,x,s) \leq 1 + MCP(i,x,s-1) \leq 1 + MCP(i,x,s)$. Thus, $\theta^{\underline{k}}_{\mathbf{M}(h(\cdot,s)[x])}(x,s) = h(x,s)$, implies, $MC(h,x,s) \leq MCP(i,x,s)$. Thus, step 2 modification of $h(x,s+1)$ preserves invariant (1). *)
2. Let $h(x,s+1) = 1 - h(x,s)$. For $y \neq x$, let $h(y,s+1) = h(y,s)$.
   Let $S^{s+1} = S^s \cup \{y < x \mid MC(h,y,s+1) < MC(h,x,s+1)\} \cup \{y \mid x < y \leq s\}$.
   (* Intuitively, $\{y < x \mid MC(h,y,s+1) < MC(h,x,s+1)\}$ is added to S^{s+1}, as these y's had too few mind changes, and we need to freeze them to maintain recursiveness of $W_e^{\underline{k+1}}$. Set $\{y \mid x < y \leq s\}$, is added to S^{s+1} as the diagonalizations done up to now for these y are no longer valid due to a change in the membership of x; thus, to maintain invariant (1) and (3) we need to place such y into S^{s+1}. *)
   Go to stage $s+1$.
3. For all x, let $h(x,s+1) = h(x,s)$, and let $S^{s+1} = S^s$.
   Go to stage $s+1$.

End Stage s

It is easy to verify that the invariants are satisfied. Also, using invariants (1), (2) we have that $\{x \mid h(x,\infty)=1\} \in \Sigma^{-1}_{\underline{k+1}}$. Let L be the language for which h is the limiting characteristic function. We will show below that L is recursive. Thus, $L \in \mathcal{L}$. We now argue that L is not $\mathbf{Cor}^{\underline{k}}\mathbf{TxtBc}$-identified by $\mathbf{M}$.

Let $k' \leq k+1$ be maximal such that there are infinitely many inputs x for which $MC(h, x, \infty) = \lim_{t\to\infty} MC(h, x, t) = k'$. Let s be the largest stage such that $MC(h, z, s+1) > MC(h, z, s) \geq k'$ for some z. Such a largest stage s exists by the maximality of k'.

Note that if $x > z$, and $MC(h, x, t+1) = k' > MC(h, x, t)$, for some $t > s$, then for all $y < x$, for all $t' > t$, $h(y, t') = h(y, t)$, as either $MC(h, y, t) \geq k'$, or y will be placed in S^{t+1} at stage t. It follows that all such x are not in $\bigcup_{s' \in \mathbf{N}} S^{s'}$, and thus $\theta^{\underline{k}}_{\mathbf{M}(\chi_L[x])}(x) \neq h(x, \infty)$, by invariant (3). Thus, $L \notin \mathbf{Cor}^{\underline{k}}\mathbf{InfBc}$.

Recursiveness of $L = W_e^{\underline{k+1}}$ follows because — except for finitely many elements on which h has $> k'$ mind-changes — once we find an element x on which h has k' mind-changes, we know the membership in $W_e^{\underline{k+1}}$ for all $y \leq x$. $\square$

The following corollary is immediate.

Corollary 22. $\mathbf{Cor}^{\underline{k+1}}\mathbf{TxtBc} - \mathbf{Cor}^{\underline{k}}\mathbf{InfBc} \neq \emptyset$.

We observe that the proof of Theorem 21 essentially also shows the following.

Theorem 23. *For* $k \in \mathbf{N}$, $\mathbf{Cor}^{\underline{k+1}}\mathbf{TxtEx} - \mathbf{Cor}^{\underline{k}}\mathbf{InfEx}^* \neq \emptyset$.

As an obvious corollary, we have a strong hierarchy with respect to $\mathbf{TxtEx}^*$-learning: $\mathbf{Cor}^{\underline{1}}\mathbf{TxtEx}^* \subset \mathbf{Cor}^{\underline{2}}\mathbf{TxtEx}^* \subset \ldots$.

The proof of Theorem 21 can also be generalized to show the following.

Theorem 24. *For* $k \in \mathbf{N}$, *for all* $m \in \mathbf{N}$, $\mathbf{Cor}^{\underline{k+1}}\mathbf{TxtEx} - \mathbf{Cor}^{\underline{k}}\mathbf{InfBc}^m \neq \emptyset$.

Proof. (Sketch) The idea would be to use a class $\mathcal{L}'$ formed by $(m+1)(k+1)$-cylindrification of the languages in the class $\mathcal{L}$ used in Theorem 21. It is easy to verify that $\mathcal{L}'$ is in $\mathbf{Cor}^{\underline{k+1}}\mathbf{TxtEx}$ (as $\mathcal{L} \in \mathbf{Cor}^{\underline{k+1}}\mathbf{TxtEx}$). Also, one can show that if $\mathcal{L}' \in \mathbf{Cor}^{\underline{k}}\mathbf{InfBc}^m$, then $\mathcal{L} \in \mathbf{Cor}^{\underline{k}}\mathbf{InfBc}$, contradicting Theorem 21. $\square$

In Section 4 we will show that — surprisingly — the strengthenings of the general hierarchy Theorem 17 given by Theorems 21, 23, and 24 cannot be generalized to the transfinite levels of the hierarchy!

3.3 TxtBc*-Learning Correction Grammars: a Partial Result

We close this section by showing a partial result on the existence of a hierarchy of learning criteria for finite correction grammars with respect to $\mathbf{TxtBc}^*$-learnability. The results of Section 3.2 imply, among other things, the existence of hierarchies $\mathbf{Cor}^{\underline{1}}\mathbf{TxtEx}^* \subset \mathbf{Cor}^{\underline{2}}\mathbf{TxtEx}^* \subset \ldots$ and $\mathbf{Cor}^{\underline{1}}\mathbf{TxtBc}^m \subset \mathbf{Cor}^{\underline{2}}\mathbf{TxtBc}^m \subset \ldots$ for every $m \in \mathbf{N}$ (recall that $\mathbf{TxtEx}^*$ and $\mathbf{TxtBc}$ are incomparable). $\mathbf{TxtBc}^*$-learning is one of the most powerful learning paradigms, allowing the learner to converge in the limit to infinitely many syntactically distinct grammars, each for some finite variant of the target language.

Our next result shows that, in this model, learning $\underline{2}$-correction grammars is more powerful than learning r.e. indices. It is open whether this result generalizes to all finite correction grammars. However, we will show in the next section that there can be no transfinite hierarchy for $\mathbf{Cor}^u\mathbf{TxtBc}^*$ learning above any notation for ω.

Theorem 25. $\mathbf{Cor}^{\underline{2}}\mathbf{TxtBc}^* - \mathbf{Cor}^{\underline{1}}\mathbf{TxtBc}^* \neq \emptyset$.

Proof. Let $L_i = \{\langle i, x\rangle \mid x \in \mathbf{N}\}$. We assume some ordering $\sigma_0, \sigma_1, \ldots$ of finite sequences. We abuse notation slightly to say $\sigma < n$ when $\sigma = \sigma_i$ for some $i < n$.

Let P_i be the following predicate:

$$(\forall \sigma \mid \text{content}(\sigma) \subseteq L_i)(\exists \tau \mid \text{content}(\tau) \subseteq L_i)(\exists y)(\forall x \geq y)[x \notin W_{\mathbf{M}_i(\sigma\tau)}];$$

Let $\mathcal{L}_i = \{L_i\}$, if P_i; $\mathcal{L}_i = \{S \mid \emptyset \subset S \subseteq L_i, \text{card}(S) < \infty\}$, otherwise. Let $\mathcal{L} = \bigcup_{i\in\mathbf{N}} \mathcal{L}_i$.

By definition $\mathcal{L}_i \not\subseteq \mathbf{TxtBc}^*(\mathbf{M}_i)$: if P_i holds then, $\mathbf{M}_i$ does not have a $\mathbf{TxtBc}^*$-locking sequence for L_i, a necessary requirement for $\mathbf{M}_i$ to $\mathbf{TxtBc}^*$-identify L_i (see [5, 25] for details about locking sequences); if P_i does not hold, then $\mathbf{M}_i$ does not $\mathbf{TxtBc}^*$-identify one of the finite subsets, content(σ), of L_i — the σ which witnessed failure of P_i.

We now show $\mathcal{L} \in \mathbf{Cor}^2\mathbf{TxtBc}^*$. Given i, define $g_i^n(z, 0) = 0$. $g_i^n(z, t+1) = 1$, if and only if $(\forall \sigma \leq n \mid \text{content}(\sigma) \subseteq L_i)(\exists \tau \leq z \mid \text{content}(\tau) \subseteq L_i)(\exists y \leq z)[(\forall x \mid y \leq x \leq t)[x \notin W_{\mathbf{M}(\sigma\tau),t}]$.

Note that $g_i^n(z, \cdot)$, for any z, changes its mind from 1 to 0 at most once, and never from 0 to 1 (except the initial change at $g_i^n(z, 1)$). Thus, grammar for computing g_i^n is — essentially — a 2-correction grammar.

Furthermore, if P_i holds, then for all n, for all but finitely many z, $g_i^n(z, t+1) = 1$ (for any z which exceeds the τ, y witnessing the P_i corresponding to each $\sigma \leq n$). On the other hand, if P_i is false, then let σ witness this failure. For each τ, y, let $t_{\tau,y}$ be the corresponding value such that for some x between y and $t_{\tau,y}$, $x \in W_{\mathbf{M}_i(\sigma\tau),t_{\tau,y}}$.

Then, for all $n \geq \sigma$, for all z, for the value of t being the maximal of $t_{\tau,y}$, for $\tau \leq z$ and $t \leq z$, we will have that $g_i^n(z, t+1)$ becomes 0.

Now by outputting g_i^n on inputs of length n, whose content is a non-empty subset of L_i, one has that $\mathcal{L} \in \mathbf{Cor}^{\underline{2}}\mathbf{TxtBc}^*$. □

4 Collapsing Results

In this section we show that Theorem 17 cannot be improved, along the lines of Section 3.2, for the transfinite levels of the $\mathbf{Cor}^u\mathbf{TxtEx}$-hierarchy: every $\mathbf{Cor}^u\mathbf{TxtEx}$-learnable class is already *behaviourally* learnable by a learner outputting grammars that make at most ω mind-changes. In the rest of the section we show analogous collapsing results involving $\mathbf{TxtEx}^*$, $\mathbf{TxtBc}^a$, for $a \in \mathbf{N} \cup \{*\}$.

Lemma 26. *Suppose L is an r.e. language. Given a program p for limiting recursively computing χ_L, one can effectively (in p, i, t) define g_i^t, such that*

(a) *for all but finitely many t, for all but finitely many i, g_i^t is the minimal φ-grammar for L, and*
(b) *for all t, the sequence $g_0^t, g_1^t, \ldots$ is a non-increasing sequence starting with t.*

Proof. Given a limiting recursive program p, let $g_0^t = t$ and let $g_{r+1}^t = \min(\{g_r^t\} \cup \{i < g_r^t \mid W_{i,t} \subseteq \{x \mid \varphi_p(x, t+r) = 1\}$ and $\{x < t \mid \varphi_p(x, t+r) = 1\} \subseteq W_{i,r}\})$.

Now suppose p is a limiting recursive program for χ_L and j is the minimal grammar for L. Let $t > j$ be a large enough number such that for all $i < j$, there exists an $x < t$, such that (i) $\varphi_p(x, \cdot)$ does not make a mind-change beyond t (that is for all $t' > t$, $\varphi_p(x, t') = \varphi_p(x, t)$), and (ii) $x \in W_i$ if and only if $x \in W_{i,t}$ and (iii) W_i and L differ at x.

It is then easy to verify that for all $t' > t$, $\lim_{r\to\infty} g_r^{t'}$ converges to j. □

Corollary 27. *Let w be any O-notation for ω, and $u \in O$. There exists a recursive function $h(\cdot, \cdot)$ such that, for any W^u-grammar q for an r.e. language L, for all but finitely many n, $h(q, n)$ is a W^w grammar for L.*

Proof. $h(q, n)$ is defined as follows. Let p be such that p is a limiting recursive program for χ_L. Note that p can be obtained effectively from q. Let g_i^t be as defined in Lemma 26 for p.

Let e_n be such that $\varphi_{e_n}(x, 0) = 0$ and $\varphi_{e_n}(x, s+1) = 1$, if and only if $x \in W_{g_s^n, s}$. Thus, by Lemma 26, for all but finitely many n, e_n is a limiting recursive program for χ_L, and $\varphi_{e_n}(x, \cdot)$ changes its mind at most $2n + 2$ times.

By acceptability of W^w, one can effectively get a W^w grammar i_n (from e_n, and thus from q, n) for $\{x \mid \lim_{t\to\infty} \varphi_{e_n}(x, t) = 1\}$. We are now done by defining $h(q, n) = i_n$ as above. □

Corollary 28. *Let w be any O-notation for ω, and $u \in O$. There exists a recursive function $h(\cdot)$ such that, for any W^u-grammar q for an r.e. language L, $h(q)$ is a W^w grammar for a finite variant of L.*

Proof. $h(q)$ is defined as follows. Let p be such that p is a limiting recursive program for χ_L. Note that p can be obtained effectively from q. Let g_i^t be as defined in Lemma 26 for p.

Let e be such that $\varphi_e(x, 0) = 0$ and $\varphi_e(x, s+1) = 1$, if and only if $x \in W_{g_s^x, s}$. Thus, by Lemma 26, for all but finitely many x, $\varphi_e(x, \infty) = \chi_L(x)$, and $\varphi_e(x, \cdot)$ changes its mind at most $2x + 2$ times.

By acceptability of W^w, one can effectively get a W^w grammar i (from e, and thus from q) for $\{x \mid \lim_{t\to\infty} \varphi_e(x, t) = 1\}$. We are now done by defining $h(q) = i$ as above. □

Theorem 29. *For all $u \in O$, for all O-notation w for ω, $\mathbf{Cor}^u\mathbf{TxtEx} \subseteq \mathbf{Cor}^w\mathbf{TxtBc}$.*

Proof. Let h be as defined in Corollary 27. Let $\mathbf{M}$ be $\mathbf{Cor}^u\mathbf{TxtEx}$-learner for $\mathcal{L}$. Let $\mathbf{M}'(T[n]) = h(\mathbf{M}(T[n]), n)$. Theorem now follows from Corollary 27. □

We now generalize Theorem 29 to show the following result.

Theorem 30. *For all $u \in O$, for all O-notation w for ω, for all $m \in \mathbf{N}$, $\mathbf{Cor}^u\mathbf{TxtEx}^{2m} \subseteq \mathbf{Cor}^w\mathbf{TxtBc}^m$.*

Proof. This uses a trick similar to the one used by [12] to show $\mathbf{TxtEx}^{2m} \subseteq \mathbf{TxtBc}^m$.

Let h be as defined in Corollary 27. Let pat be a recursive function such that $W^w_{pat(q,S,S')} = (W^w_q \cup S) - S'$, where S and S' are finite sets.

Let $\mathbf{M}$ be $\mathbf{Cor}^u\mathbf{TxtEx}$-learner for $\mathcal{L}$. Let $\mathbf{M}'(T[n])$ be defined as follows. Let $S_n = \text{content}(T[n])$. Let S'_n be a set of least m elements in $\{x \mid \theta^u_{\mathbf{M}(T[n])}(x,n) = 1\} - S_n$. Then let $\mathbf{M}'(T[n]) = pat(h(\mathbf{M}(T[n]), n), S_n, S'_n)$.

Suppose T is a text for a language L which is $\mathbf{Cor}^u\mathbf{TxtEx}^{2m}$-identified by $\mathbf{M}$. Suppose $\mathbf{M}(T)$ converges to q. Let $X = W^u_q - L$ and $Y = L - W^u_q$. Now, clearly, for all but finitely many n, the least m elements of X belong to S'_n (if X consists of $\leq m$ elements, then $X \subseteq S'_n$, for all but finitely many n). Also, by Corollary 27, for all but finitely many n, $h(q,n)$ is a W^w-grammar for W^u_q.

Case 1: X contains $> m$ elements. In this case, for all but finitely many n, $W^w_{pat(h(\mathbf{M}(T[n]),n),S_n,S'_n)} = (W^u_q \cup Y) - X'$, where X' contains the least m elements of X. Thus, for all but finitely many n, $pat(h(\mathbf{M}(T[n]), n), S_n, S'_n)$ is a W^w grammar for $\text{card}(X - X')$ (which is $\leq m$) variant of L.

Case 2: X contains $\leq m$ elements. In this case, for all but finitely many n, $W^w_{pat(h(\mathbf{M}(T[n]),n),S_n,S'_n)} = (W^u_q \cup Y) - S'_n$, where $X \subseteq S'_n$. Thus, for all but finitely many n $pat(h(\mathbf{M}(T[n]), n), S_n, S'_n)$ is a W^w grammar for $(m - \text{card}(X))$-variant of L.

From the above cases it follows that $\mathbf{M}'$ $\mathbf{Cor}^w\mathbf{TxtBc}^m$-identifies L. □

Note that we have a hierarchy with respect to $\mathbf{Ex}^*$ for all finite levels, by Theorem 23: $\mathbf{Cor}^{\underline{1}}\mathbf{TxtEx}^* \subset \mathbf{Cor}^{\underline{2}}\mathbf{TxtEx}^* \subset \ldots$. The next result shows that the hierarchy collapses at level ω.

Theorem 31. *For all $u \in O$, for all O-notation w for ω,* $\mathbf{Cor}^u\mathbf{TxtEx}^* \subseteq \mathbf{Cor}^w\mathbf{TxtEx}^*$.

Proof. Let h be as defined in Corollary 28. Let $\mathbf{M}$ be $\mathbf{Cor}^u\mathbf{TxtEx}$-learner for $\mathcal{L}$. Let $\mathbf{M}'(T[n]) = h(\mathbf{M}(T[n]))$. Theorem now follows from Corollary 28. □

The same construction as above gives an analogous collapsing result with respect to $\mathbf{TxtBc}^*$. It should be noted here that it is open whether there exists a hierarchy of learning $\underline{n}$-correction grammars, for $n \in \mathbf{N}$, with respect to the $\mathbf{TxtBc}^*$-model. The following result shows that there can be no hierarchy above the level ω.

Theorem 32. *For all $u \in O$, for all O-notation w for ω,* $\mathbf{Cor}^u\mathbf{TxtBc}^* \subseteq \mathbf{Cor}^w\mathbf{TxtBc}^*$.

One can also show the following.

Theorem 33. *For all $u \in O$,* $\mathbf{Cor}^u\mathbf{TxtEx}^* \subseteq \mathbf{TxtBc}^*$.

Proof. Suppose $\mathbf{M}$ is a $\mathbf{Cor}^u\mathbf{TxtEx}^*$-learner for $\mathcal{L}$. Then, on input $T[s]$, the learner $\mathbf{M}'$ first determines a p_s such that $\{x \mid \lim_{t\to\infty} \varphi_{p_s}(x,t) = 1\} = W^u_{\mathbf{M}(T[s])}$. Note that such a p_s can be found effectively from $T[s]$. Then, $\mathbf{M}'$ determines g^t_i as given by Lemma 26, for $p = p_s$. Then $\mathbf{M}'$ outputs a grammar p'_s for the language $\{x \mid x \in W_{g^s_x}\}$.

Suppose T is a text for L which is $\mathbf{Cor}^u\mathbf{TxtEx}$-identified by $\mathbf{M}$. Then $\mathbf{M}(T)$ converges to a W^u grammar q for a finite variant of L. Thus, by Lemma 26, for all but finitely many s, for all but finitely many i, g^s_i, as constructed on input $T[s]$ above, is a grammar for W^u_q. Thus, for all but finitely many s, for all but finitely many x, $x \in W_{p'_s}$ if and only if $x \in L$. □

We now prove a collapsing result at level ω for **TxtBc**-learning correction grammars. We also include anomalies in the treatment, to obtain a stronger result. Note that, by Theorem 24, for every $m \in \mathbf{N}$ there is a hierarchy $\mathbf{Cor}^1\mathbf{TxtBc}^m \subset \mathbf{Cor}^2\mathbf{TxtBc}^m \subset \ldots$ of learning *finite* correction grammars with respect to **TxtBc**-learning with anomalies.

Lemma 34. *Fix $m \in \mathbf{N}$. There exists a recursive function f such that for all z and texts T such that $W_z =^{2m} \mathrm{content}(T)$, for all but finitely many t, $f(z, T[t])$ is a grammar for m-variant of* $\mathrm{content}(T)$.

Proof. This is based on a technique of [12]. By S-m-n theorem, there exists a recursive f such that $f(z, T[n])$ may be defined as follows. Let $S_n = \mathrm{content}(T[n])$. Let S'_n be the set of least m elements in $W_{z,n} - S_n$ (if $W_{z,n} - S_n$ contains less than m elements then $S'_n = W_{z,n} - S_n$). Let $W_{f(z,T[n])} = (W_z \cup S_n) - S'_n$.

Let $Y = \mathrm{content}(T) - W_z$ and $X = W_z - \mathrm{content}(T)$. We now consider two cases.

Case 1: $\mathrm{card}(X) \geq m$. Let X' be the set of least m elements of X. Now, for all but finitely many t, $Y \subseteq S_t$ and $S'_t = X'$; thus, $W_{h(z,T[t])} \mathbf{\Delta} \mathrm{content}(T) = X - X'$, which is of cardinality at most m.

Case 2: $\mathrm{card}(X) \leq m$. Now, for all but finitely many t, $Y \subseteq S_t$ and $X \subseteq S'_t$; thus, $W_{h(z,T[t])} \mathbf{\Delta} \mathrm{content}(T) = S'_t - X$, which is of cardinality at most m.

Lemma follows from above cases. □

Theorem 35. *For all $u \in O$, for all O-notation w for ω, for all $m \in \mathbf{N}$, $\mathbf{Cor}^u\mathbf{TxtBc}^m \subseteq \mathbf{Cor}^w\mathbf{TxtBc}^m$.*

Proof. Suppose $\mathbf{M}$ is given. Define $\mathbf{M}'$ as follows. Suppose T is a text for a language L which is $\mathbf{Cor}^u\mathbf{TxtBc}^m$-identified by $\mathbf{M}$. Let

$$U^t_p = \{x \mid (\exists t' \geq t)[\theta^u_p(x, t') = 1]\},$$

$$U^t_{p,r} = \{x \mid (\exists t' \leq r)[\theta^u_p(x, t + t') = 1]\}.$$

Below, let p_i denote $\mathbf{M}(T[i])$. Let

$$g^{t,S,S'}_r = \min(\{t\} \cup \{j' < t \mid j' \in S \ \wedge \ (\forall i \in S')[\mathrm{card}(W_{j',t} - U^t_{p_i,r}) \leq m]\}).$$

Note that for fixed t, S, S', $g^{t,S,S'}_r$ is monotonically non-increasing in r.

Now for any t, let $S_{t,r} = \{i \leq t \mid \mathrm{content}(T[t]) \subseteq W_{i,r}\}$, and $S'_{t,r} = \{i \leq t \mid \mathrm{card}(\mathrm{content}(T[t]) - U^t_{p_i,r}) \leq m\}$. It is easy to verify that $S_{t,r}$ and $S'_{t,r}$ are monotonically non-decreasing (in r) and bounded in cardinality by $t + 1$. Let $S_t = \lim_{r\to\infty} S_{t,r} = \{i \leq t \mid \mathrm{content}(T[t]) \subseteq W_i\}$, and $S'_t = \lim_{r\to\infty} S'_{t,r} = \{i \leq t \mid \mathrm{card}(\mathrm{content}(T[t]) - U^t_{p_i}) \leq m\}$.

We will later show that,

> (*) for all but finitely many t, $\lim_{r\to\infty} g^{t,S_t,S'_t}_r$ converges to a grammar $z \leq j$ such that $W_z =^{2m} L$, where j is the minimal grammar for L. Here, z may be different for different (large enough) t's; however, all these z's are $\leq j$.

Let $i^t_r = g_r^{t,S_{t,r},S'_{t,r}}$. Note that $\lim_{r\to\infty} i^t_r = \lim_{r\to\infty} g_r^{t,S_{t,r},S'_{t,r}} = \lim_{r\to\infty} g_r^{t,S_t,S'_t}$. Thus, for all but finitely many t, $\lim_{r\to\infty} i^t_r = i^t$ is a grammar for $2m$-variant of L, and is bounded by the minimal grammar for L. Furthermore, for a fixed t, $\mathrm{card}(\{r \mid i^t_r \neq i^t_{r+1}\}) \leq (t+1) * (t+1) * (t+1)$.

Let f be as defined in Lemma 34. Define p'_t such that $\varphi_{p'_t}(x,0) = 0$, and for $t > 0$, $\varphi_{p'_t}(x,r) = 1$, if and only if $x \in W_{f(i^t_r, T[t]),r}$, and then let $\mathbf{M}'(T[t])$ be a W^w grammar for the language $\{x \mid \varphi_{p'_t}(x,\infty) = 1\}$. By acceptability of W^w, such a W^w grammar can be obtained effectively from p'_t. Now for all but finitely many t, for all but finitely many r, i^t_r is a grammar for $2m$-variant of L which is bounded by minimal grammar for L. Thus, by Lemma 34, for all but finitely many t, for all but finitely many r, $f(i^t_r, T[t])$ is a grammar for m-variant of L. Theorem thus follows from the below proof of (*).

We now show that (*) holds. Let j be the minimal grammar for L. Let k be minimal such that p_k is a u-correction grammar for a m-variant of L. Let t be so large that the following holds.

(a) $t > j$, (thus $j \in S_t$).
(b) $t > k$, (thus $k \in S'_t$).
(c) for all $i < j$, such that $W_i \not\supseteq L$, $\mathrm{content}(T[t]) \not\subseteq W_i$. Thus, $i \in S_t$ and $i < j$, implies $L \subset W_i$.
(d) for all i such that $\mathrm{card}(L - \{x \mid \theta^u_{p_i}(x,\infty) = 1\}) > m$, we have $i < t$ and $\mathrm{card}(\mathrm{content}(T[t]) - U^t_{p_i}) > m$; thus, $i \notin S'_t$. Hence, S'_t consists only of i such that $\mathrm{card}(L - W^u_{p_i}) \leq m$. Thus, $(\forall i \in S'_t)[\mathrm{card}(W_{j,t} - U^t_{p_i}) \leq m]$.
(e) for all $i < j$, such that $W_i \supset L$ and $\mathrm{card}(W_i - L) > 2m$, there exists at least $(m+1)$ x's in $W_{i,t}$ such that, for all $t' \geq t$, $\theta^u_{p_k}(x,t') = 0$.

It follows from (a) and (d) that $\lim_{r\to\infty} g_r^{t,S_t,S'_t}$ converges to a grammar $i \leq j$. Furthermore, using (b), (c), and (e), it follows that $W_i \supseteq L$, and $\mathrm{card}(W_i - L) \leq 2m$. □

5 Other Results

5.1 Trade-offs with Anomalies

In Section 4 we have proved a number of collapsing results for learning correction grammars. These results have shown that the transfinite hierarchy of learning correction grammars criteria is specific to the **TxtEx** model of learning. Now we consider another question: are there trade-offs between learning r.e. indices, learning u-correction grammars and the number of anomalies allowed? The following theorem shows that learning r.e. indices with more anomalies gives dramatic increase of learning power compared to learning correction grammars of any ordinal complexity with less anomalies allowed. The proofs are straightforward lifts from other contexts (see, e.g., [11]).

Theorem 36. *For all $u \in O$, for all $n \in \mathbf{N}$,*
(a) $\mathbf{TxtEx}^{2n+1} - \mathbf{Cor}^u\mathbf{TxtBc}^n \neq \emptyset$.
(b) $\mathbf{TxtBc}^{n+1} - \mathbf{Cor}^u\mathbf{TxtBc}^n \neq \emptyset$.
(c) $\mathbf{TxtEx}^{n+1} - \mathbf{Cor}^u\mathbf{TxtEx}^n \neq \emptyset$.

(d) $\mathbf{TxtEx}^* - \bigcup_{n\in\mathbf{N}} \mathbf{Cor}^u\mathbf{TxtBc}^n \neq \emptyset$.
(e) $\mathbf{Cor}^u\mathbf{InfEx}^* \subseteq \mathbf{Cor}^u\mathbf{InfBc}$.
(f) *For w being O-notation for ω, $\mathcal{E} \in \mathbf{Cor}^w\mathbf{InfBc}$.*

Proof. The class $\mathcal{L} = \{L \in \mathcal{E} \mid L =^{2n+1} \mathbf{N}\}$ witnesses (a) and (b). The class $\mathcal{L} = \{L \in \mathcal{E} \mid L =^{n+1} \mathbf{N}\}$ witnesses (c). The class $\mathcal{L} = \{L \in \mathcal{E} \mid L =^* \mathbf{N}\}$ witnesses (d). (e) can be proven by patching the input, along the same lines as $\mathbf{InfEx}^* \subseteq \mathbf{InfBc}$ (see [12]). Harrington's proof of $\mathcal{R}$, the class all recursive functions, being in $\mathbf{Bc}^*$ (see [14]) can also be used to show (f). We omit the details. □

Essentially, the anomaly hierarchies $\{\mathbf{TxtI}^n\}_{\underline{n}\in\mathbf{N}}$, with $\mathbf{I} \in \{\mathbf{Ex}, \mathbf{Bc}\}$ are very stable:[9] e.g., (b) and (c) show that the extra learning power of allowing *one* more anomaly in the final conjecture overplays the power of learning correction grammars of *any* transfinite ordinal complexity u.

5.2 Learning Succinct Correction Grammars

In scientific inference, parsimony of explanations is considered highly desirable. Grammar size is one of many ways to measure parsimony of grammars [20, 26]. It is known, for computability-theoretic inductive inference, that requiring the final and correct grammars to be minimal size [41] is highly restrictive on inferring power [20, 21] (and that the resultant inferring or learning power is dependent on which acceptable programming system is employed). Also known is the adverse effect on learning power of requiring the final and correct grammars to be merely within a computable parsimony factor of minimal size grammars [27, 15] (but that the resulting inferring power is independent of the underlying acceptable programming system [20]). Hence, parsimony restrictions of even the weaker kind described just above limit inferring power.

For r.e. L, let $\mathrm{MinGram}(L)$ be the minimal i such that $W_i = L$. For $\mathcal{L} \in \mathbf{TxtEx}$ as witnessed by $\mathbf{M}$, if there is a computable function g such that, for every $L \in \mathcal{L}$, for all texts T for L, $\mathbf{M}(T) \leq g(\mathrm{MinGram}(L))$, then we say $\mathcal{L} \in \mathbf{TxtMEx}$ (as witnessed by $\mathbf{M}$ and g). In this setting we call g a *parsimony factor*. The final grammars of a $\mathbf{TxtMEx}$-learner are, in a certain sense, *nearly minimal-size*. Kinber [27] was the first to show that $\mathbf{TxtMEx} \subset \mathbf{TxtEx}$. For example, the class $\mathrm{Zero}^* = \{\{\langle x, f(x)\rangle \mid x \in \mathbf{N}\} \mid f$ is a recursive function and $(\forall^\infty x)[f(x) = 0]\}$ witnesses this separation [27]. Chen [15] later showed that Zero^* is not even in $\mathbf{TxtMEx}^n$ for every $n \in \mathbf{N}$.

By contrast, the next Theorem shows that the class Zero^* is learnable with nearly-minimal-size final conjectures if the learner uses basic correction grammars (i.e., $\underline{2}$-correction grammars) instead of standard grammars.

Theorem 37. *There exists a learner $\mathbf{M}$ which $\mathbf{Cor}^{\underline{2}}\mathbf{TxtEx}$-identifies Zero^*, and for some recursive function g, for all texts T for $L \in \mathrm{Zero}^*$, $\mathbf{M}(T) \leq g(\mathrm{MinGram}(L))$.*

Proof. Let e_1 be minimal grammar for $Z = \{\langle x, 0\rangle \mid x \in \mathbf{N}\}$. Let h be recursive, 1–1 and increasing function such that $h(i, j)$ is a $W^{\underline{2}}$ grammar for $W_i \boldsymbol{\Delta} W_j$ (clearly, there exists such a recursive function).

[9] An inspection of the original proofs shows that virtually any criterion that requires convergence in the limit to some kind of correct grammar for the target language can be beaten by allowing one more anomaly in the limit.

Define a learner $\mathbf{M}$ as follows. Given a text T for $L \in \mathrm{Zero}^*$, $\mathbf{M}$ finds, in the limit, the finite set S of elements in $L\boldsymbol{\Delta}Z$ (note that this Z can be determined, in the limit, as for all $x \in \mathbf{N}$, L contains a unique y such that $\langle x, y\rangle \in L$). From this S, $\mathbf{M}$ computes, in the limit, the minimal grammar e_2 for S. (Note that for finite sets, one can determine minimal φ-grammars in the limit). Then $\mathbf{M}$ on T converges to $h(e_1, e_2)$.

For $L \in \mathrm{Zero}^*$, it is easy to verify that $\mathrm{MinGram}(L\boldsymbol{\Delta}Z) \leq h'(\mathrm{MinGram}(L))$, for some recursive function h', and thus $\mathbf{M}(T) \leq h(e_1, h'(\mathrm{MinGram}(L))) \leq g(\mathrm{MinGram}(L))$, for some recursive function g. □

Note that in Theorem 37 above, the final conjecture of $\mathbf{M}$ is within a recursive factor of minimal grammars, not of minimal correction grammars. As a corollary, we have that one can learn very succinct correction grammars, when compared to ordinary grammars.

This can also be compared with a recent result by Case and Royer [13]: correction grammars for some r.e. languages can be recursive-in-K more succinct than ordinary grammars for them. However, it should be observed that the latter result is not the reason of our Theorem 37. The fact that no machine can learn nearly minimal-size r.e. indices for Zero^* but some machine can learn correction grammars that are within a recursive factor of the minimal r.e. index has to do with the learner's inability to get hold of succinct grammars, rather than with the existence of succinct grammars.

6 Conclusion

In the present paper we have investigated a new learning paradigm in which the learner outputs correction grammars instead of ordinary r.e. indices.

We have shown that learning correction grammars of larger and larger complexity enhances learning power over learning r.e. indices. In the context of **TxtEx**-learning, an infinite hierarchy of more and more powerful learning criteria is obtained (Theorem 19). The increase in learning power is measured by notations for constructive transfinite ordinals used for algorithmic count-down of corrections. If u and v are notations in O for transfinite ordinals and u is smaller than v in the notation system, then one can learn more — in the **TxtEx**-model — by conjecturing v-correction grammars than by conjecturing u-correction grammars.

For correction grammars with a finite number of correction, we have shown that the hierarchy can be strengthened: there are classes of languages that are **TxtEx**-learnable with $\underline{k+1}$-correction grammars such that no **Bc**-learner can learn those classes using $\underline{k}$-correction grammars, not even from the characteristic function of the graph of the language.

Surprisingly, we have shown that some interesting collapsing phenomena occur at the first transfinite level: for example, every class in $\mathbf{Cor}^u\mathbf{TxtEx}$ for $u \in O$ is already in $\mathbf{Cor}^w\mathbf{TxtBc}$ with w a notation for ω. The same is also true for every class in $\mathbf{Cor}^u\mathbf{TxtBc}$.

From the Cognitive Science perspective, our hierarchy results can be read as *suggesting* that the drawback of using correction grammars instead of standard ones (i.e., the need of self-corrections) may be compensated by an increase of learning power.

In this respect we note that many of our results can be adapted to vacillatory learning [10]. $\mathbf{TxtFex}^a_b$ with $b, a \in \mathbf{N} \cup \{*\}$ is the vacillatory learning criterion allowing the learner to vacillate between $\leq b$ a-variants of the target language, $\mathbf{Cor}^{u}\mathbf{TxtFex}_{b}^{a}$ is the version using correction grammars, $\mathbf{Cor}^{u}\mathbf{InfFex}_{b}^{a}$ is the version with informants. The hierarchy proof from [10] can be adapted to show that we have a hierarchy $\mathbf{Cor}^{u}\mathbf{Fex}_{1}^{a} \subset \mathbf{Cor}^{u}\mathbf{Fex}_{2}^{a} \subset \ldots$ of vacillatory learning correction grammars. A modification of the proof of Theorem 17 can be used to show that $\mathbf{Cor}^{u+_{o}\underline{1}}\mathbf{TxtEx} - \mathbf{Cor}^{u}\mathbf{InfFex}_{*}^{n} \neq \emptyset$. Similarly, proof of Theorem 21 (see also Theorem 23) yields $\mathbf{Cor}^{\underline{n+1}}\mathbf{TxtEx} - \mathbf{Cor}^{\underline{n}}\mathbf{InfFex}_{*}^{*} \neq \emptyset$. Proof of Theorem 31 yields $\mathbf{Cor}^{u}\mathbf{InfFex}_{b}^{*} \subseteq \mathbf{Cor}^{w}\mathbf{InfFex}_{b}^{*}$ and $\mathbf{Cor}^{u}\mathbf{TxtFex}_{b}^{*} \subseteq \mathbf{Cor}^{w}\mathbf{TxtFex}_{b}^{*}$, for w a notation for ω.[10]

The question whether there exists a hierarchy of learning *finite* correction grammars with respect to $\mathbf{TxtBc}^{*}$-learning is open. We showed that $\mathbf{TxtBc}^{*} \subset \mathbf{Cor}^{\underline{2}}\mathbf{TxtBc}^{*}$ and conjecture that the general result will require different methods of proofs.

Also of interest for the future is the investigation of "complexity" results along the lines of Theorem 37.

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[10] For $u > 1$, $u, b \in \mathbf{N}$, the exact relationship among different $\mathbf{Cor}^{u}\mathbf{InfFex}_{b}$ is open.

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