SHARP THRESHOLDS FOR HYPERGRAPH REGRESSIVE RAMSEY NUMBERS

LORENZO CARLUCCI, GYESIK LEE, AND ANDREAS WEIERMANN

ABSTRACT. The f-regressive Ramsey number $R^{\text{reg}}_f(d,n)$ is the minimum N such that every colouring of the d-tuples of an N-element set mapping each $x_1,\dots,x_d$ to a colour $\le f(x_1)$ contains a min-homogeneous set of size n, where a set is called min-homogeneous if every two d-tuples from this set that have the same smallest element get the same colour. If f is the identity, then we are dealing with the standard regressive Ramsey numbers as defined by Kanamori and McAloon. In this paper we classifiy the growth-rate of the regressive Ramsey numbers for hypergraphs in dependence of the growth-rate of the parameter function f. The growth-rate has to be measured against the scale of fast-growing Hardy functions F_α indexed by towers of exponentiation in base ω. Our results give a sharp classification of the thresholds at which the f-regressive Ramsey numbers undergoe a drastical change in growth-rate. The case of graphs has been treated of Lee, Kojman, Omri and Weiermann. We extend their results to hypergraphs of arbitrary dimension. From the point of view of logic, our results classify the provability of the Regressive Ramsey Theorem for hypergraphs of fixed dimension with respect to the subsystems of Peano Arithmetic with restricted induction principles.

1. Introduction

Let $\mathbb{N}$ denote the set of all natural numbers including 0. A number $d \in \mathbb{N}$ is identified with the set $\{n \in \mathbb{N} : n < d\}$, and the set $\{0, 1, \dots, d-1\}$ may also be sometimes denoted by $[d]$. The set of all d*-element subsets* of a set X is denoted by $[X]^d$. For a function $C : [X]^d \to \mathbb{N}$ we write $C(x_1, \dots, x_d)$ for $C(\{x_1, \dots, x_d\})$ under the assumption that $x_1 < \cdots < x_d$. Let $f : \mathbb{N} \to \mathbb{N}$ be a number-theoretic function. A function $C : [X]^d \to \mathbb{N}$ is called f*-regressive* if for all $s \in [X]^d$ such that $f(\min(s)) > 0$ we have $C(s) < f(\min(s))$. When f is the identity function we just say that C is regressive. A set H is *min-homogeneous* for C if for all $s, t \in [H]^d$ with $\min(s) = \min(t)$ we have $C(s) = C(t)$. We write

$$X \overset{\min}{\to} (m)^d_f$$

if for all f-regressive $C : [X]^d \to \mathbb{N}$ there exists $H \subseteq X$ s.t. $card(H) = m$ and H is min-homogeneous for C. In case $d = 2$, we just write $X \overset{\min}{\to} (m)_f$. We denote by

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$(\mathrm{KM})^d_f$ the following statement.

$$(\forall m)(\exists \ell)[\ell \overset{\min}{\to} (m)^d_f],$$

and abbreviate $(\forall d)[(\mathrm{KM})^d_f]$ as $(\mathrm{KM})_f$. Using a compactness argument and the Canonical Ramsey Theorem of Erdös and Rado, Kanamori and McAloon [6] proved that $(\mathrm{KM})_f$ is true for every choice of f. For f the identity function, the theorem has the notable property of being a Gödel sentence for Peano Arithmetic [6] and is known as the Regressive Ramsey Theorem. It is equivalent to the famous Paris-Harrington Theorem (see [12, 2, 7]). The latter was the first example of a theorem from finite combinatorics that is undecidable in formal number theory. Not a few people consider the Regressive Ramsey Theorem to be more natural. Regressive Ramsey numbers for graphs have been investigated by Kojmans and Shelah [10]. They showed that $R^{\mathrm{reg}}(2, i)$ grows as the Ackermann function. More recently, Kojmans, Lee, Omri and Weiermann computed the sharp thresholds on the parameter function f at which the f-regressive Ramsey numbers cease to be Ackermannian and become primitive recursive [9]. In this paper we extend the results of [9] to hypergraphs of arbitrary dimension. We classify the thresholds on f at which the f-regressive Ramsey number undergoe an acceleration against the scale of fast-growing Hardy functions that naturally extends the Grzegorczyk hierarchy.

The main result of [9] can be stated as follows. Let $B : \mathbb{N} \to \mathbb{N}^+$ be unbounded and non-decreasing. Let $f_B(i) = (\log_{d-1}(i))^{1/B^{-1}(i)}$. Then the f_B-regressive Ramsey numbers for graphs are Ackermannian if and only if B is. For every f dominated by f_B, the f-regressive Ramsey number is primitive recursive if B is. To state our main results, we need to introduce the Hardy hierarchy of fast-growing functions [14, 15]. This hierarchy naturally extends the Grzegorczyk hierarchy of primitive recursive functions used in [9]. The hierarchy is indexed by (constructive, countable) ordinals below the ordinal ε_0. The indexing by ordinals allows long iterations and diagonalization. We use the fact that any ordinal α below ε_0 can be written uniquely in (Cantor) normal form as $\sum_{i=k}^{0} c_i \cdot \omega^{\alpha_i}$, where $\alpha > \alpha_k > \cdots > \alpha_0$ and $c_i \geq 1$. We fix an assignment of fundamental sequences to ordinals below ε_0. A fundamental sequence for a limit ordinal λ is an infinite sequence $(\lambda_n)_{n\in\mathbb{N}}$ of smaller ordinals whose supremum is λ. We define the assignment $\cdot[\cdot] : \varepsilon_0 \times \mathbb{N} \to \varepsilon_0$ as follows by case distinction on the structure of the normal form of a limit ordinal α. $\alpha[x] = \gamma + \omega^{\lambda[x]}$, if $\alpha = \gamma + \omega^\lambda$ with λ limit. $\alpha[x] := \gamma + \omega^\beta \cdot x$, if $\alpha = \gamma + \omega^{\beta+1}$. We also set $\varepsilon_0[x] := \omega_{x+1}$, where $\omega_0(x) := x$, $\omega_{d+1}(x) := \omega^{\omega_d(x)}$ and $\omega_d := \omega_d(1)$. For technical reasons we extend the assignment to non-limit ordinals as follows: $(\beta+1)[x] := \beta$ and $0[x] := 0$. If f is a function and $d \geq 0$ we denote by f^d the d-th iteration of f, with $f^0(x) := x$. The fast-growing function is defined as follows, by induction on α.

$$\begin{aligned} F_0(x) &:= x+1 \\ F_{\alpha+1}(x) &:= F_\alpha^{(x+1)}(x) \\ F_\lambda(x) &:= F_{\lambda[x]}(x) \end{aligned}$$

The Hardy hierarchy is well-known in the study of formal systems of Arithmetic, where it can be used to classify the functions that have a proof of totality in the system. The correspondence is - roughly - as follows. A recursive function has a proof of totality in Peano arithmetic if and only if it is primitive recursive in F_α, for some $\alpha < \varepsilon_0$. For $d \geq 1$, a recursive function has a proof of totality in the subsystem of

Peano arithmetic with induction restricted to formulas of quantifier complexity Σ_d (i.e., predicates starting with d alternations of existential and universal quantifiers $\exists x_1 \forall x_2 \dots$) if and only if it is primitive recursive in F_{ω_d}. Also, $F_{\omega_{d+1}}$ eventually dominates all functions that are primitive recursive in F_α for all $\alpha < \omega_{d+1}$, and F_{ε_0} eventually dominates all functions that are primitive recursive in F_{ω_d} for all $d \in \mathbb{N}$. Thus, each new level of exponentiation corresponds to a drastical change in growth-rate as well as in logical complexity.

Lee obtained in his Ph.D. thesis [11] the following result. For hypergraphs of dimension $d+1$, the $\log_d$-regressive Ramsey numbers are primitive recursive, but the $\log_{d-2}$-regressive Ramsey numbers grow as fast as F_{ω_d}. This kind of drastical change in growth rate and proof complexity has been dubbed a "phase-transition" by Weiermann and it turned out to be a pervasive phenomenon in first-order arithmetic (see [17] for a survey). Lee conjectured that also $\log_{d-1}$-regressive Ramsey numbers, and $(\log_{d-1})^{1/\ell}$-regressive Ramsey numbers, for every ℓ, grow as fast as F_{ω_d}. Our results imply that Lee's conjecture is true and that we can also replace the constant ℓ with any function growing slower than the inverse of F_{ω_d}.

Theorem 1.1 (Upper bounds)**.** *Let $d \geq 1$. Let $B : \mathbb{N} \to \mathbb{N}^+$ be unbounded and non-decreasing. Let $f : \mathbb{N} \to \mathbb{N}$ be such that for every i, $f(i) \leq (\log_{d-1}(i))^{1/B^{-1}(i)}$. If B is bounded by a function primitive recursive in F_α for some $\alpha < \omega_d$, then the same is true of $R_f^{\mathrm{reg}}(d+1, \cdot)$. If B is primitive recursive in F_α for some $\alpha < \omega_d$, then the same is true of $R_f^{\mathrm{reg}}(d+1, \cdot)$.*

Theorem 1.2 (Lower bounds)**.** *Let $d \geq 1$. Let $B : \mathbb{N} \to \mathbb{N}^+$ be unbounded and non-decreasing. Let $f_B(i) = (\log_{d-1})^{1/B^{-1}(i)}$. If B grows as F_{ω_d} then $R_{f_B}^{\mathrm{reg}}(d+1, \cdot)$ eventually dominates F_α for all $\alpha < \omega_d$.*

In logical terms, this means that proving the f-regressive Ramsey theorem for hypergraphs of dimension $d+1$ *necessarily* requires induction on Σ_{d+1}-formulas if and only if f grows as f_B with $B(i) = F_{\omega_d}$.

2. Upper Bounds

In this section we show the upper bounds on f-regressive Ramsey numbers for $f(n) \leq (\log_{d-1}(n))^{1/F_\alpha^{-1}(n)}$ for $\alpha < \varepsilon_0$. Essentially, the bound for standard Ramsey functions from Erdös-Rado's [3] is adapted to the case of regressive functions.

Definition 2.1. *Let $C \colon [\ell]^d \to k$ be a coloring. Call a set H* s-homogeneous *for C if for any s-element set $U \subseteq H$ and for any $(d-s)$-element sets V, $W \subseteq H$ such that* $\max U < \min\{\min V, \min W\}$, *we have*

$$C(U \cup V) = C(U \cup W).$$

$(d-1)$-homogeneous sets are called end-homogeneous*.*

Note that 0-homogeneous sets are homogeneous and 1-homogeneous sets are min-homogeneous. Let

$$X \to_s \langle m \rangle_k^d$$

denote that given any coloring $C \colon [X]^d \to k$, there is H s-homogeneous for C such that $card(H) \geq m$. The following lemma shows a connection between s-homogeneity and homogeneity.

Lemma 2.2. *Let $s \leq d$ and assume*

(1) $\ell \to_s \langle p \rangle^d_k$,
(2) $p - d + s \to (m - d + s)^s_k$.

Then we have

$$\ell \to (m)^d_k.$$

Proof. Let $C\colon [\ell]^d \to k$ be given. Then assumption 1 implies that there is $H \subseteq \ell$ such that $|H| = p$ and H is s-homogeneous for C. Let $z_1 < \cdots < z_{d-s}$ be the last $d - s$ elements of H. Set $H_0 := H \setminus \{z_1, \ldots, z_{d-s}\}$. Then $card(H_0) = p - d + s$. Define $D\colon [H_0]^s \to k$ by

$$D(x_1, \ldots, x_s) := C(x_1, \ldots, x_s, z_1, \ldots, z_{d-s}).$$

By assumption 2 there is Y_0 such that $Y_0 \subseteq H_0$, $card(Y_0) = m - d + s$, and homogeneous for D. Hence $D \restriction [Y_0]^s = e$ for some $e < k$. Set $Y := Y_0 \cup \{z_1, \ldots, z_{d-s}\}$. Then $card(Y) = m$ and Y is homogeneous for C. Indeed, we have for any sequence $x_1 < \cdots < x_d$ from Y

$$C(x_1, \ldots, x_d) = C(x_1, \ldots, x_s, z_1, \ldots, z_{d-s}) = D(x_1, \ldots, x_s) = e.$$

The proof is complete. □

Given d, s such that $s \leq d$ define $R^s_\mu(d, \cdot, \cdot)\colon \mathbb{N}^2 \to \mathbb{N}$ by

$$R^s_\mu(d, k, m) := \min\{\ell\colon \ell \to_s \langle m \rangle^d_k\}.$$

Then

- $R^0_\mu(1, k, m - d + 1) = k \cdot (m - d) + 1$,
- $R^d_\mu(d, k, m) = R^s_\mu(d, 1, m) = m$,
- $R^s_\mu(d, k, d) = d$,
- $R^s_\mu(d, k, m) \leq R^{s-1}_\mu(d, k, m)$ for any $s > 0$.

R^s_μ are called *Ramsey functions.* Then the standard Ramsey function for d-hypergraphs and two colors - which we denote by $R(d, k, m)$ - coincides with $R^0_\mu(d, k, m)$ and $R^{\text{reg}}_{f_k}(d, m) = R^1_\mu(d, k, m)$ where f_k is the constant function with value k. Define a binary operation $*$ by putting, for positive natural numbers x and y,

$$x * y := x^y.$$

Further, we put for $p \geq 3$

$$x_1 * x_2 * \cdots * x_p := x_1 * (x_2 * (\cdots * (x_{p-1} * x_p) \cdots))$$

Erdös and Rado [3] gave an upper bound for $R(d, k, m)$: Given d, k, m such that $k \geq 2$ and $m \geq d \geq 2$, we have

$$R(d, k, m) \leq k * (k^{d-1}) * (k^{d-2}) * \cdots * (k^2) * (k \cdot (m - d) + 1).$$

Theorem 2.3 ($\mathrm{I}\Sigma_1$)**.** *Let $2 \leq d \leq m$, $0 < s \leq d$, and $2 \leq k$.*

$$R^s_\mu(d, k, m) \leq k * (k^{d-1}) * (k^{d-2}) * \cdots * (k^{s+1}) * (m - d + s) * s.$$

In particular, for $s = 1$, we have $R^{\text{reg}}_{f_k}(2, m) \leq k^{m-1}$, where f_k is the constant function with value k.

Proof. The proof construction below generalizes Erdös and Rado [3]. We shall work with s-homogeneity instead of homogeneity.

Let X be a finite set. In the following construction we assume that $card(X)$ is large enough. How large it should be will be determined after the construction has been defined. Throughout this proof the letter Y denotes subsets of X such that $card(Y) = d-2$.

Let $C\colon [X]^d \to k$ be given and $x_1 < \ldots < x_{d-1}$ the first $d-1$ elements of X. Given $x \in X \setminus \{x_1, \ldots, x_{d-1}\}$ put

$$C_{d-1}(x) := C(x_1, \ldots, x_{d-1}, x).$$

Then $\operatorname{Im}(C_{d-1}) \subseteq k$, and there is $X_d \subseteq X \setminus \{x_1, \ldots, x_{d-1}\}$ such that C_{d-1} is constant on X_d and

$$card(X_d) \geq k^{-1} \cdot (card(X) - d + 1).$$

Let $x_d := \min X_d$ and given $x \in X_d \setminus \{x_d\}$ put

$$C_d(x) := \prod\{C(Y \cup \{x_d, x\})\colon Y \subseteq \{x_1, \ldots, x_{d-1}\}\}.$$

Then $\operatorname{Im}(C_d) \subseteq k * \binom{d-1}{d-2}$, and there is $X_{d+1} \subseteq X_d \setminus \{x_d\}$ such that C_d is constant on X_{d+1} and

$$card(X_{d+1}) \geq k^{-\binom{d-1}{d-2}} \cdot (card(X_d) - 1)\,.$$

Generally, let $p \geq d$, and suppose that $x_1, \ldots, x_{p-1}$ and X_d, $X_{d+1}, \ldots, X_p$ have been defined, and that $X_p \neq \varnothing$. Then let $x_p := \min X_p$ and for $x \in X_p \setminus \{x_p\}$ put

$$C_p(x) := \prod\{C(Y \cup \{x_p, x\})\colon Y \subseteq \{x_1, \ldots, x_{p-1}\}\}.$$

Then $\operatorname{Im}(C_p) \subseteq k * \binom{p-1}{d-2}$, and there is $X_{p+1} \subseteq X_p \setminus \{x_p\}$ such that C_p is constant on X_{p+1} and

$$card(X_{p+1}) \geq k^{-\binom{p-1}{d-2}} \cdot (card(X_p) - 1)\,.$$

Now put

$$\ell := 1 + R^s_\mu(d-1, k, m-1).$$

Then $\ell \geq m \geq d$. If $card(X)$ is sufficiently large, then $X_p \neq \varnothing$, for all p such that $d \leq p \leq \ell$, so that $x_1, \ldots, x_\ell$ exist. Note also that $x_1 < \cdots < x_\ell$. For $1 \leq \rho_1 < \cdots < \rho_{d-1} < \ell$ put

$$D(\rho_1, \ldots, \rho_{d-1}) := C(x_{\rho_1}, \ldots, x_{\rho_{d-1}}, x_\ell).$$

By definition of ℓ there is $Z \subseteq \{1, \ldots, \ell-1\}$ such that Z is s-homogeneous for D and $card(Z) = m-1$. Finally, we put

$$X' := \{x_\rho\colon \rho \in Z\} \cup \{x_\ell\}.$$

We claim that X' is min-homogeneous for C. Let

$$H := \{x_{\rho_1}, \ldots, x_{\rho_d}\} \quad \text{and} \quad H' = \{x_{\eta_1}, \ldots, x_{\eta_d}\}$$

be two subsets of X' such that $\rho_1 = \eta_1, \ldots, \rho_s = \eta_s$ and

$$1 \leq \rho_1 < \cdots < \rho_d \leq \ell, \quad 1 \leq \eta_1 < \cdots < \eta_d \leq \ell.$$

Since x_{ρ_d}, $x_\ell \in X_{\rho_d}$, we have $C_{\rho_{d-1}}(x_{\rho_d}) = C_{\rho_{d-1}}(x_\ell)$ and hence

$$C(x_{\rho_1}, \ldots, x_{\rho_{d-1}}, x_{\rho_d}) = C(x_{\rho_1}, \ldots, x_{\rho_{d-1}}, x_\ell).$$

Similarly, we show that

$$C(x_{\eta_1}, \ldots, x_{\eta_{d-1}}, x_{\eta_d}) = C(x_{\eta_1}, \ldots, x_{\eta_{d-1}}, x_\ell).$$

In addition, since $\{x_{\rho_1}, \ldots, x_{\rho_{d-1}}\} \cup \{x_{\eta_1}, \ldots, x_{\eta_{d-1}}\} \subseteq X'$, we have

$$D(\rho_1, \ldots, \rho_{d-1}) = D(\eta_1, \ldots, \eta_{d-1}),$$

i.e.,

$$C(x_{\rho_1},\dots,x_{\rho_{d-1}},x_\ell)=C(x_{\eta_1},\dots,x_{\eta_{d-1}},x_\ell).$$

This means that $C(H)=C(H')$ and proves that X' is maz-homogeneous for C. This implies that X' is min-homogeneous for C.

We now return to the question of how large $card(X)$ should be in order to ensure that the construction above can be carried through.

Set

$$\begin{aligned} t_d &:= k^{-1}\cdot(card(X)-d+1),\\ t_{p+1} &:= k^{-\binom{p-1}{d-2}}\cdot(t_p-1) \qquad (d\le p<\ell). \end{aligned}$$

Then we require that $t_\ell>0$, where

$$\begin{aligned} t_\ell &= k^{-\binom{\ell-2}{d-2}}\cdot\left(k^{-\binom{\ell-3}{d-2}}\cdot\left(\cdots\left(k^{-\binom{d-1}{d-2}}\cdot\left(t_d-1\right)\right)\cdots\right)-1\right)\\ &= k^{-\binom{\ell-2}{d-2}-\cdots-\binom{d-1}{d-2}}\cdot t_d-k^{-\binom{\ell-2}{d-2}-\cdots-\binom{d-1}{d-2}}-\cdots-k^{-\binom{\ell-2}{d-2}-\binom{\ell-3}{d-2}}-k^{-\binom{\ell-2}{d-2}}. \end{aligned}$$

Since $k=k^{\binom{d-2}{d-2}}$ a sufficient condition on $card(X)$ is then

$$card(X)-d+1>k^{\binom{\ell-3}{d-2}+\cdots+\binom{d-2}{d-2}}+k^{\binom{\ell-4}{d-2}+\cdots+\binom{d-2}{d-2}}+\cdots+k^{\binom{d-2}{d-2}}.$$

A possible value is

$$card(X)=d+\sum_{p=d-1}^{\ell-2}k^{\binom{p}{d-1}},$$

so that

$$\begin{aligned} R^s_\mu(d,k,m) &\le d+\sum_{p=d-1}^{\ell-2}k^{\binom{p}{d-1}} \quad\le\quad d+\sum_{p=d-1}^{\ell-2}k^{p^{d-1}}\\ &\le d+\sum_{p=d-1}^{\ell-2}\left(k^{(p+1)^{d-1}}-k^{p^{d-1}}\right)\\ &= d+k^{(\ell-1)^{d-1}}-k^{(d-1)^{d-1}}\\ &\le k^{(\ell-1)^{d-1}}\\ &= k^{R^s_\mu(d-1,k,m-1)^{d-1}}. \end{aligned}$$

Hence

$$R^s_\mu(d,k,m)*d \quad\le\quad (k^d)*R^s_\mu(d-1,k,m-1)*(d-1).$$

After $(d-s)$ times iterated applications of the inequality we get

$$\begin{aligned} R^s_\mu(d,k,m)*d &\le (k^d)*(k^{d-1})*\cdots*(k^{s+1})*R^s_\mu(s,k,m-d+s)*s\\ &= (k^d)*(k^{d-1})*\cdots*(k^{s+1})*(m-d+s)*s. \end{aligned}$$

This completes the proof. □

Remark 2.4. *Lemma 26.4 in* [1] *gives a slight sharper estimate for* $s=d-1:$

$$R^{d-1}_\mu(d,k,m)\le d+\sum_{i=d-1}^{m-2}k^{\binom{i}{d-1}}$$

Corollary 2.5. *Let $2 \le d \le m$ and $2 \le k$. Let f_k be the constant function with value k.*

$$R^{\rm reg}_{f_k}(d,m) \le k * (k^{d-1}) * (k^{d-2}) * \cdots * (k^2) * (m-d+1).$$

Now we come back to f-regressiveness and prove the key upper bound of the present section.

Lemma 2.6. *Given $d \ge 1$ and $\alpha \le \varepsilon_0$ set $f^d_\alpha(i) := \lfloor \sqrt[F_\alpha^{-1}(i)]{\log_d(i)} \rfloor$. Then there exist p, $q \in \mathbb{N}$ depending (primitive-recursively) on d and α such that, for all m,*

$$R^{\rm reg}_{f^{d-1}_\alpha}(d+1,m) \le 2_{d-1}^{F_\alpha(q)^{m+p}}$$

Proof. Given d, α and m, let p be such that $d < p$, and for every x

$$2_{d-1}^{x^{m+d}+1} + x \le 2_{d-1}^{x^{m+p}}.$$

Let $q > p$ be so large that

$$(k) * (k^d) * \cdots * (k^2) * (m-d) < 2_{d-1}^{F_\alpha(q)^{m+d}+1}, \tag{2.1}$$

with $k := \lfloor F_\alpha(q)^{(m+p)/q} \rfloor + 1$. Now set

$$\ell := 2_{d-1}^{F_\alpha(q)^{m+d}+1} + F_\alpha(q) \le 2_{d-1}^{F_\alpha(q)^{m+p}} =: N.$$

Let $C\colon [N]^{d+1} \to \mathbb{N}$ be any f^d_α-regressive function and

$$D\colon [F_\alpha(q),\ell]^{d+1} \to \mathbb{N}$$

be defined from C by restriction. Then for any $y \in [F_\alpha(q),\ell]$, we have

$$\begin{aligned}\sqrt[F_\alpha^{-1}(y)]{\log_{d-1}(y)} &\le \sqrt[F_\alpha^{-1}(F_\alpha(q))]{\log_{d-1}(2_{d-1}(F_\alpha(q)^{m+p}))}\\ &= \sqrt[q]{F_\alpha(q)^{m+p}}.\end{aligned}$$

Hence

$$\mathrm{Im}(D) \subseteq \lfloor F_\alpha(q)^{(m+p)/q} \rfloor + 1,$$

i.e., D is a ($\lfloor F_\alpha(q)^{(m+p)/q} \rfloor + 1$)-colouring.

By Corollary 2.5 and inequality 2.1 above, there is an $H \subseteq N$ min-homogeneous for D, hence for C, such that $card(H) \ge m$.

□

Theorem 2.7. (1) *$R^{\rm reg}_{\log^*}(\cdot,\cdot)$ is primitive recursive.*

(2) *Let $d \ge 1$. $R^{\rm reg}_{\log_d}(d+1,\cdot)$ is primitive recursive.*
(3) *Let $d \ge 1$. Let $\alpha < \omega_d$. Then $R^{\rm reg}_{\sqrt[F_\alpha^{-1}(\cdot)]{\log_{d-1}(\cdot)}}(d+1,\cdot)$ is primitive recursive in F_{ω_d}.*

Proof. (1) Let $m \ge d \ge 1$ be given. Choose x so large that $k = x+m$ satisfies

$$k * (k^{d-1}) * (k^{d-2}) * \cdots * (k^2) * (m-d+1) < 2_d^{x+m},$$

and $\ell := 2_d^{x+m}$ satisfies

$$\log^* \ell \le k.$$

Thus, any $\log^*$-regressive coloring of $[\ell]^d$ is a k-coloring. We claim that $R^{\rm reg}_{\log^*}(d,m) \le \ell$. Let $C\colon [\ell]^d \to \mathbb{N}$ be $\log^*$-regressive. By Theorem 2.3 we can find an $H \subseteq \ell$ min-homogeneous for C such that $card(H) \ge m$.

(2) Let d, $m \geq 1$ be given. Let x be such that for $k := x + m$ and $\ell := 2_d^{x+m}$ we have

$$k * (k^d) * (k^{d-1}) * \cdots * (k^2) * (m - d) < 2_d^{x+m},$$

and

$$\lfloor \log_d(\ell) \rfloor \leq k.$$

Thus any $\log_d$-regressive coloring of $[\ell]^{d+1}$ is a k-coloring. We claim that $R^{\mathrm{reg}}_{\log_d}(d+1, m) \leq \ell$. Let $C\colon [\ell]^{d+1} \to \mathbb{N}$ be $\log_d$-regressive. By Theorem 2.3 we can find an $H \subseteq \ell$ min-homogeneous for C such that $card(H) \geq m$.

(3) The assertion follows from Lemma 2.6. □

It is also possible to work with variable iterations to obtain an upper bound for the Kanamori-McAloon principle with unbounded dimensions, as shown in Lee [11].

Lemma 2.8. *Given $d \geq 2$ and $\alpha \leq \varepsilon_0$ let $f_\alpha(i) := |i|_{F_\alpha^{-1}(i)}$. Then*

$$R^{\mathrm{reg}}_{f_\alpha}(d, m) \leq 2_{d+1}(F_\alpha(m)),$$

where m is large enough.

Proof. Given α, d, m define ℓ, N by

$$\ell := 2_d(F_\alpha(m)) + F_\alpha(m) \leq 2_{d+1}(F_\alpha(m)) =: N.$$

Let $C\colon [N]^d \to \mathbb{N}$ be any f_α-regressive function and

$$D\colon [F_\alpha(m), \ell]^d \to \mathbb{N}.$$

be defined from C by restriction. Then for any $y \in [F_\alpha(m), \ell]$ we have

$$\begin{aligned} |y|_{F_\alpha^{-1}(y)} &\leq |2_{d+1}(F_\alpha(m))|_{F_\alpha^{-1}(F_\alpha(m))} \\ &= |2_{d+1}(F_\alpha(m))|_m \\ &< F_\alpha(m) \end{aligned}$$

if $m > d + 1$. Hence,

$$\mathrm{Im}(g) \subseteq F_\alpha(m).$$

In addition, we have for $k := F_\alpha(m)$

$$(k) * (k^{d-1}) * \cdots * (k^2) * (m - d + 1) < 2_d(F_\alpha(m))$$

if m is large enough. By Theorem 2.3 we find H min-homogeneous for D, hence for C, such that $card(H) \geq m$. □

Theorem 2.9. *$R^{\mathrm{reg}}_{|\cdot|_{F_\alpha^{-1}(\cdot)}}$ is primitive recursive in F_{ε_0} for all $\alpha < \varepsilon_0$.*

Proof. The claim follows directly from Lemma 2.8. □

3. Lower Bounds

In this section we prove the lower bounds on the f-regressive Ramsey numbers for $f(n) = (\log_{d-1}(n))^{1/F_{\omega_d}^{-1}(n)}$, for all $d \geq 1$. The key arguments in subsection 3.4 are a non-trivial adaptation of Kanamori-McAloon's [6], Section 3. Before being able to apply those arguments we need to develop, by bootstrapping, some relevant bounds for the parametrized Kanamori-McAloon principle. This is done in subsection 3.3 by adapting the idea of the Stepping-up Lemma in [5]. We begin with the base case $d = 1$ which is helpful for a better understanding of the coming general cases. The following subsection 3.1, covering the base case $d = 1$ of our main result, is already done in [11, 9].

3.1. **Ackermannian Ramsey functions.** Throughout this subsection m denotes a *fixed* positive natural number. Set

$$h_\omega(i) := \lfloor \sqrt[F_\omega^{-1}(i)]{i} \rfloor \quad \text{and} \quad h_m(i) := \lfloor \sqrt[m]{i} \rfloor .$$

Define a sequence of strictly increasing functions $f_{m,n}$ for as follows:

$$f_{m,n}(i) := \begin{cases} i+1 & \text{if } n = 0, \\ f_{m,n-1}^{(\lfloor \sqrt[m]{i} \rfloor)}(i) & \text{otherwise.} \end{cases}$$

Note that $f_{m,n}$ are strictly increasing.

Lemma 3.1. *$R_{h_m}^{\mathrm{reg}}(2, R(2, c, i+3)) \geq f_{m,c}(i)$ for all c and i.*

Proof. Let $k := R(2, c, i+3)$ and define a function $C_m : [R_{h_m}^{\mathrm{reg}}(2, k)]^2 \to \mathbb{N}$ as follows:

$$C_m(x, y) := \begin{cases} 0 & \text{if } f_{m,c}(x) \leq y, \\ \ell & \text{otherwise,} \end{cases}$$

where the number ℓ is defined by

$$f_{m,p}^{(\ell)}(x) \leq y < f_{m,p}^{(\ell+1)}(x)$$

where $p < c$ is the maximum such that $f_{m,p}(x) \leq y$. Note that C_m is h_m-regressive since $f_{m,p}^{(\lfloor \sqrt[m]{x} \rfloor)}(x) = f_{m,p+1}(x)$. Let H be a k-element subset of $R_{h_m}^{\mathrm{reg}}(2, k)$ which is min-homogeneous for C_m. Define a c-coloring $D_m : [H]^2 \to c$ by

$$D_m(x, y) := \begin{cases} 0 & \text{if } f_{m,c}(x) \leq y, \\ p & \text{otherwise,} \end{cases}$$

where p is as above. Then there is a $(i+3)$-element set $X \subseteq H$ homogeneous for D_m. Let $x < y < z$ be the last three elements of X. Then $i \leq x$. Hence, it suffices to show that $f_{m,c}(x) \leq y$ since $f_{m,c}$ is an increasing function.

Assume $f_{m,c}(x) > y$. Then $f_{m,c}(y) \geq f_{m,c}(x) > z$ by the min-homogeneity. Let $C_m(x, y) = C_m(x, z) = \ell$ and $D_m(x, y) = D_m(x, z) = D_m(y, z) = p$. Then

$$f_{m,p}^{(\ell)}(x) \leq y < z < f_{m,p}^{(\ell+1)}(x).$$

By applying $f_{m,p}$ we get the contradiction that $z < f_{m,p}^{(\ell+1)}(x) \leq f_{m,p}(y) \leq z$. □

We are going to show that $R_{h_m}^{\mathrm{reg}}(2, \cdot)$ is not primitive recursive. This will be done by comparing the functions $f_{m,n}$ with the Ackermann function.

Lemma 3.2. *Let $i \geq 4^m$ and $\ell \geq 0$.*

(1) $(2i+2)^m < f_{m,\ell+2m^2}(i)$ *and* $f_{m,\ell+2m^2}((2i+2)^m) < f^{(2)}_{m,\ell+2m^2}(i)$.
(2) $F_n(i) < f^{(2)}_{m,n+2m^2}(i)$.

Proof. (1) By induction on k it is easy to show that $f_{m,k}(i) > (\lfloor \sqrt[m]{i} \rfloor)^k$ for any $i > 0$. Hence for $i \geq 4^m$

$$f_{m,2m^2}(i) > (\lfloor \sqrt[m]{i} \rfloor)^{2m^2} \geq (\lfloor \sqrt[m]{i} \rfloor)^{m^2} \cdot 2^{m^2+m} \geq (\sqrt[m]{i+1})^{m^2} \cdot 2^m = (2i+2)^m$$

since $2 \cdot \lfloor \sqrt[m]{i} \rfloor \geq \sqrt[m]{i+1}$. The second claim follows from the first one.

(2) By induction on n we show the claim. If $n = 0$ it is obvious. Suppose the claim is true for n. Let $i \geq 4^m$ be given. Then by induction hypothesis we have $F_n(i) \leq f^{(2)}_{m,n+2m^2}(i)$. Hence

$$F_{n+1}(i) \leq F_n^{(i+1)}(i) \leq f^{(2i+2)}_{m,n+2m^2}(i) \leq f_{m,n+2m^2+1}((2i+2)^m) < f^{(2)}_{m,n+2m^2+1}(i).$$

The induction is now complete. □

Corollary 3.3. $F_n(i) \leq f_{m,n+2m^2+1}(i)$ *for any* $i \geq 4^m$.

Theorem 3.4. $R^{\rm reg}_{h_m}(2,\cdot)$ *and* $R^{\rm reg}_{h_\omega}(2,\cdot)$ *are not primitive recursive.*

Proof. Lemma 3.1 and Corollary 3.3 imply that $R^{\rm reg}_{h_m}(2,\cdot)$ is not primitive recursive. For the second assertion we claim that

$$N(i) := R^{\rm reg}_{h_\omega}(2, R(2, i+2i^2+1, 4^i+3)) > F_\omega(i)$$

for all i. Assume to the contrary that $N(i) \leq A(i)$ for some i. Then for any $\ell \leq N(i)$ we have $A^{-1}(\ell) \leq i$, hence $\sqrt[i]{\ell} \leq \sqrt[A^{-1}(\ell)]{\ell}$. Hence

$$\begin{array}{rcl} R^{\rm reg}_{h_\omega}(2, R(2, i+2i^2+1, 4^i+3)) & \geq & R^{\rm reg}_{h_i}(2, R(2, i+2i^2+1, 4^i+3)) \\ & \geq & f_{i,i+2i^2+1}(4^i) \\ & > & F_\omega(i) \end{array}$$

by Lemma 3.1 and Corollary 3.3. Contradiction! □

Now we are ready to begin with the general cases.

3.2. **Fast-growing hierarchies.** We introduce some variants of the fast-growing hierarchy and prove that they are still fast-growing, meaning they match-up with the original hierarchy.

Definition 3.5. *Let* $d > 0, c > 1$ *be natural numbers. Let* ϵ *be a real number such that* $0 < \epsilon \leq 1$.

$$\begin{array}{rcl} B_{\epsilon,c,d,0}(x) & := & 2_d^{\lfloor \log_d(x) \rfloor^c} \\ B_{\epsilon,c,d,\alpha+1}(x) & := & B^{\lfloor \epsilon \cdot \sqrt[c]{\log_d(x)} \rfloor}_{\epsilon,c,d,\alpha}(x) \\ B_{\epsilon,c,d,\lambda(x)} & := & B_{\epsilon,c,d,\lambda[\lfloor \epsilon \cdot \sqrt[c]{\log_d(x)} \rfloor]}(x) \end{array}$$

In the following we abbreviate $B_{\epsilon,c,d,\alpha}$ by B_α when ϵ, c, d are fixed.

Lemma 3.6. *Let* c, d, ϵ *be as above. For all* $x > 0$
(1) $B_{i+1}(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1) \rfloor^c}) \geq 2_d^{\lfloor \epsilon^{-1}\cdot(F_i(x)+1) \rfloor^c}$ *for all* $i \in \omega$ *and* $x > 0$.
(2) $B_\alpha(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1) \rfloor^c}) \geq 2_d^{\lfloor \epsilon^{-1}\cdot(F_\alpha(x)+1) \rfloor^c}$ *for all* $\alpha \geq \omega$ *and* $x > 0$.

Proof. (1) We claim that $B_0^m(x) = 2_d^{\lfloor \log_d(x) \rfloor^{c^m}}$ for $m > 0$. Proof by induction on m. The base case holds trivially. For the induction step we calculate:

$$\begin{aligned} B_0^{m+1}(x) &= B_0(B_0^m(x)) \\ &= 2_d^{\lfloor \log_d(B_0^m(x)) \rfloor^c} \\ &= 2_d^{\lfloor \log_d(2_d^{\lfloor \log_d(x) \rfloor^{c^m}}) \rfloor^c} \\ &= 2_d^{\lfloor \lfloor \log_d(x) \rfloor^{c^m} \rfloor^c} \\ &= 2_d^{\lfloor \log_d(x) \rfloor^{c^{m+1}}} \end{aligned}$$

We now claim that $B_{i+1}(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1) \rfloor^c}) \geq 2_d^{\lfloor 2\epsilon^{-1}\cdot(F_i(x)+1) \rfloor^c}$. Proof by induction on i. For $i = 0$ we obtain

$$\begin{aligned} B_1(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1) \rfloor^c}) &= B_0^{\lfloor \epsilon\cdot \sqrt[c]{\log_d(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1) \rfloor^c})} \rfloor}(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1) \rfloor^c}) \\ &= B_0^{\lfloor \epsilon\cdot \lfloor 2\epsilon^{-1}\cdot(x+1) \rfloor \rfloor}(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1) \rfloor^c}) \\ &\geq B_0^{x+1}(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1) \rfloor^c}) \\ &= 2_d^{\lfloor \log_d(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1) \rfloor^c}) \rfloor^{c^{x+1}}} \\ &= 2_d^{\lfloor \lfloor 2\epsilon^{-1}\cdot(x+1) \rfloor^c \rfloor^{c^{x+1}}} \\ &= 2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1) \rfloor^{c^{x+2}}} \\ &\geq 2_d^{\lfloor 2\epsilon^{-1}\cdot(F_0(x)+1) \rfloor^c} \end{aligned}$$

since $x > 0$ and $c > 1$. For the induction step we compute

$$\begin{aligned} B_{i+1}(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1) \rfloor^c}) &= B_i^{\lfloor \epsilon\cdot \sqrt[c]{\log_d(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1) \rfloor^c})} \rfloor}(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1) \rfloor^c}) \\ &\geq B_i^{x+1}(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1) \rfloor^c}) \\ &\geq B_i^{x}(2_d^{\lfloor 2\epsilon^{-1}\cdot(F_{i-1}(x)+1) \rfloor^c}) \\ &\geq B_i^{x-1}(2_d^{\lfloor 2\epsilon^{-1}\cdot(F_{i-1}^2(x)+1) \rfloor^c}) \\ &\geq \cdots \\ &\geq 2_d^{\lfloor 2\epsilon^{-1}\cdot(F_{i-1}^{x+1}(x)+1) \rfloor^c} \\ &= 2_d^{\lfloor 2\epsilon^{-1}\cdot(F_i(x)+1) \rfloor^c} \end{aligned}$$

(2) We prove the claim by induction on $\alpha \geq \omega$. Let $\alpha = \omega$. We obtain

$$\begin{aligned} B_\omega(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1) \rfloor^c}) &= B_{\omega\left[\epsilon\cdot \sqrt[c]{\log_d(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1) \rfloor^c})}\right]}(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1) \rfloor^c}) \\ &\geq B_{x+1}(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1) \rfloor^c}) \\ &\geq 2_d^{\lfloor 2\epsilon^{-1}\cdot(F_x(x)+1) \rfloor^c} \\ &= 2_d^{\lfloor 2\epsilon^{-1}\cdot(F_\omega(x)+1) \rfloor^c} \end{aligned}$$

For the successor case $\alpha + 1$ we compute

$$\begin{aligned} B_{\alpha+1}(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1)\rfloor^c}) &= B_\alpha^{\lfloor \epsilon\cdot\sqrt[c]{\log_d(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1)\rfloor^c})}\rfloor}(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1)\rfloor^c}) \\ &= B_\alpha^{x+1}(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1)\rfloor^c}) \\ &= B_\alpha^{x}(B_\alpha(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1)\rfloor^c})) \\ &\geq B_\alpha^{x}(2_d^{\lfloor 2\epsilon^{-1}\cdot(F_\alpha(x)+1)\rfloor^c}) \\ &\geq \cdots \\ &\geq 2_d^{\lfloor 2\epsilon^{-1}\cdot(F_\alpha^{x+1}(x)+1)\rfloor^c} \\ &\geq 2_d^{\lfloor 2\epsilon^{-1}\cdot(F_{\alpha+1}(x)+1)\rfloor^c} \end{aligned}$$

If λ is a limit we obtain

$$\begin{aligned} B_\lambda(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1)\rfloor^c}) &= B_{\lambda\left[\lfloor \epsilon\cdot\sqrt[c]{\log_d(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1)\rfloor^c})}\rfloor\right]}(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1)\rfloor^c}) \\ &\geq B_{d,\lambda[x+1]}(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1)\rfloor^c}) \\ &\geq 2_d^{\lfloor 2\epsilon^{-1}\cdot(F_{\lambda[x+1]}(x)+1)\rfloor^c} \\ &= 2_d^{\lfloor 2\epsilon^{-1}\cdot(F_\lambda(x)+1)\rfloor^c} \end{aligned}$$

□

Theorem 3.7. *Let $d > 0, c > 1$ be natural numbers. Let $0 < \epsilon \leq 1$.*
(1) *$B_{\epsilon,c,d,\omega}$ eventually dominates all primitive recursive functions.*
(2) *$B_{\epsilon,c,d,\omega_d}$ eventually dominates F_α for all $\alpha < \omega_d$.*

Proof. Obvious by Lemma 3.6. □

3.3. **Bootstrapping.** In this section we show how suitable iterations of the Regressive Ramsey theorem for $(d+1)$-hypergraphs and parameter function $f(x) = \sqrt[c]{\log_{d-1}(x)}$ (for constant c) can be used to obtain min-homogeneous sets whose elements are "spread apart" with respect to the function $2_{d-1}(\log_{d-1}(x)^c)$ (i.e., $B_{\epsilon,c,d-1,0}$). This fact will be used next (Proposition 3.21) to show that one can similarly obtain from the same assumption even sparser sets (essentially sets whose elements are "spread apart" with respect to the function $F_{\omega_{d-1}^c}$).

For the sake of clarity we work out the proofs of the main results of the present section for the base cases $d = 2$ and $d = 4$ in detail in section 3.3.1 before generalizing them in section 3.3.2. We hope that this will improve the readability of the arguments.

Definition 3.8. *We say that a set X if f-*sparse *if and only if for all $a, b \in X$ we have $f(a) \leq b$. We say that two elements a, b of a set X are n-*apart *if and only if there exist $e_1, \ldots, e_n$ from X such that $a < e_1 < \cdots < e_n < b$. We say that a set is (f, n)-*sparse *if and only if for all $a, b \in X$ such that a and b are n-apart we have $f(a) \leq b$.*

Definition 3.9. *Let X be a set of cardinality $> m \cdot k$. We define X/m as the set $\{x_0, x_m, x_{2m}, \ldots, x_{k\cdot m}\}$, where x_i is the $(i+1)$-th smallest element of X.*

Thus, if a set X is (f, m)-sparse of cardinality $> k \cdot m$ we have that X/m is f-sparse and has cardinality $> k$.

3.3.1. *$B_{\epsilon,2,1,0}$-sparse min-homogeneous sets - Base Cases.* Given $P : [\ell]^d \to \mathbb{N}$ we call $X \subseteq \ell$ max-homogeneous for P if for all $U, V \in [X]^d$ with $\max(U) = \max(V)$ we have $P(U) = P(V)$.

Let $\mathrm{MIN}^d_k(m) := R_\mu(d, k, m)$, i.e., the least natural number ℓ such that for all partitions $P : [\ell]^d \to k$ there is a min-homogeneous $Y \subseteq \ell$ such that $card(Y) \geq m$. Let $\mathrm{MAX}^d_k(m)$ be the least natural number ℓ such that for all partitions $P : [\ell]^d \to k$ there is a max-homogeneous $Y \subseteq \ell$ such that $card(Y) \geq m$.

Let $k \geq 2$ and $m \geq 1$. Given an integer $a < k^m$ let $a = k^{m-1} \cdot a(m-1) + \cdots + k^0 \cdot a(0)$ be in the unique representation with $a(m-1), \ldots, a(0) \in \{0, \ldots, k-1\}$. Then $D^{(k,m)} : [k^m]^2 \to m$ is defined by

$$D^{(k,m)}(a, b) := \max\{j : a(j) \neq b(j)\}.$$

Lemma 3.10. *Let $k \geq 2$ and $m \geq 1$.*

(1) $\mathrm{MIN}^2_{k\cdot m}(m+2) > k^m$.
(2) $\mathrm{MAX}^2_{k\cdot m}(m+2) > k^m$.

Proof. Let us show the first item. Define $R_1 : [k^m]^2 \to k \cdot m$ as follows.

$$R_1(a, b) := k \cdot D(a, b) + b(D(a, b)),$$

where $D := D^{(k,m)}$. Assume $Y = \{a_0, \ldots, a_\ell\}$ with $a_0 < \ldots < a_\ell$ is min-homogeneous for R_1. We claim $\ell \leq m$. Let $c_i := D(a_i, a_{i+1})$, $i < \ell$. Since $m > c_0$ it is sufficient to show $c_{i+1} < c_i$ for every $i < \ell - 1$.

Fix $i < \ell - 1$. We have $D(a_i, a_{i+1}) = D(a_i, a_{i+2})$ since $R_1(a_i, a_{i+1}) = R_1(a_i, a_{i+2})$ by min-homogeneity. Hence for any $j > D(a_i, a_{i+1})$ we have $a_i(j) = a_{i+1}(j) = a_{i+2}(j)$ which means $c_i \geq c_{i+1}$. Moreover, $R_1(a_i, a_{i+1}) = R_1(a_i, a_{i+2})$ further yields $a_{i+1}(D(a_i, a_{i+1})) = a_{i+2}(D(a_i, a_{i+2}))$, hence $c_i = c_{i+1}$ cannot be true, since $a_{i+1}(D(a_{i+1}, a_{i+2})) \neq a_{i+2}(D(a_{i+1}, a_{i+2}))$.

For the proof of the second item define $R'_1 : [k^m]^2 \to k \cdot m$ as follows.

$$R'_1(a, b) := k \cdot D(a, b) + a(D(a, b)),$$

where $D := D^{(k,m)}$. Assume $Y = \{a_0, \ldots, a_\ell\}$ with $a_0 < \ldots < a_\ell$ is max-homogeneous for R'_1. We claim $\ell \leq m$. Let $c_i := D(a_i, a_{i+1})$, $i < \ell$. Since $m > c_{\ell-1}$ it is sufficient to show $c_{i+1} > c_i$ for every $i < \ell - 1$.

Fix $i < \ell - 1$. We have $D(a_i, a_{i+2}) = D(a_{i+1}, a_{i+2})$ since $R'_1(a_i, a_{i+2}) = R'_1(a_{i+1}, a_{i+2})$ by max-homogeneity. Hence for any $j > D(a_{i+1}, a_{i+2})$ we have $a_i(j) = a_{i+1}(j) = a_{i+2}(j)$ which means $c_i \leq c_{i+1}$. Moreover, $R'_1(a_i, a_{i+2}) = R'_1(a_{i+1}, a_{i+2})$ further yields $a_i(D(a_i, a_{i+2})) = a_{i+1}(D(a_{i+1}, a_{i+2}))$, hence $c_i = c_{i+1}$ cannot be true, since $a_i(D(a_i, a_{i+1})) \neq a_{i+1}(D(a_i, a_{i+1}))$. □

Lemma 3.11. *Let $k, m \geq 2$.*

(1) $\mathrm{MIN}^3_{2k\cdot m}(2m+4) > 2^{k^m}$.
(2) $\mathrm{MAX}^3_{2k\cdot m}(2m+4) > 2^{k^m}$.

Proof. (1) Let $k, m \geq 2$ be positive integers and put $e := k^m$. Let R_1 and R'_1 be the partitions from Lemma 3.10. Define $R_2 : [2^e]^3 \to 2k \cdot m$ as follows:

$$R_2(u, v, w) := \begin{cases} R_1(D(u, v), D(v, w)) & \text{if } D(u, v) < D(v, w), \\ k \cdot m + R'_1(D(v, w), D(u, v)) & \text{if } D(u, v) > D(v, w), \end{cases}$$

where $D := D^{(2,e)}$. The case $D(u,v) = D(v,w)$ does not occur since we developed u, v, w with respect to base 2. Let $Y \subseteq 2^e$ be min-homogeneous for R_2. We claim $card(Y) < 2m+4$.

Assume $card(Y) \geq 2m+4$. Let $\{u_0, \ldots, u_{2m+3}\} \subseteq Y$ be min-homogeneous for R_2. We shall provide a contradiction. Let $d_i := D(u_i, u_{i+1})$ for $i < 2m+3$.

Case 1: Assume there is some r such that $d_r < \ldots < d_{r+m+1}$. We claim that $Y' := \{d_r, \ldots, d_{r+m+1}\}$ is min-homogeneous for R_1 which would contradict Lemma 3.10.

Note that for all i, j with $r \leq i < j \leq r+m+2$ we have

$$D(u_i, u_j) = \max\{D(u_i, u_{i+1}), \ldots, D(u_{j-1}, u_j)\}.$$

We have therefore for $r \leq i < j \leq r+m+1$

$$R_1(d_i, d_j) = R_1(D(u_i, u_{i+1}), D(u_{i+1}, u_{j+1})) = R_2(u_i, u_{i+1}, u_{j+1}).$$

By min-homogeneity of Y we obtain similarly

$$R_2(u_i, u_{i+1}, u_{j+1}) = R_2(u_i, u_{i+1}, u_{p+1}) = R_1(d_i, d_p)$$

for all i, j, p such that $r \leq i < j < p \leq r+m+1$.

Case 2: Assume there is some r such that $d_r > \ldots > d_{r+m+1}$. We claim that $Y' := \{d_{r+m+1}, \ldots, d_r\}$ is max-homogeneous for R_1' which would contradict Lemma 3.10.

Assume $r \leq i < j < p \leq r+m+1$, hence $u_i < u_j < u_p$ and $d_p < d_j < d_i$. Note that we also have $d_j = D(u_j, u_p)$ and $d_i = D(u_i, u_p)$. Hence

$$k \cdot m + R_1'(d_p, d_j) = k \cdot m + R_1'(D(u_p, u_{p+1}), D(u_j, u_p)) = R_2(u_j, u_p, u_{p+1}).$$

By min-homogeneity we obtain

$$\begin{aligned} k \cdot m + R_1'(d_p, d_i) &= k \cdot m + R_1'(D(u_p, u_{p+1}), D(u_i, u_p)) \\ &= R_2(u_i, u_p, u_{p+1}) \\ &= R_2(u_i, u_j, u_{j+1}) \\ &= k \cdot m + R_1'(d_j, d_i). \end{aligned}$$

Case 3: There is a local maximum of the form $d_i < d_{i+1} > d_{i+2}$. Note then that $D(u_i, u_{i+2}) = d_{i+1}$. Hence we obtain the following contradiction using the min-homogeneity: $k \cdot m > R_1(d_i, d_{i+1}) = R_2(u_i, u_{i+1}, u_{i+2}) = R_2(u_i, u_{i+2}, u_{i+3}) = k \cdot m + R_1'(d_{i+2}, d_{i+1}) \geq k \cdot m$.

Case 4: Cases 1 to 3 do not hold. Then there must be two local minima. But then inbetween we have a local maximum and we are back in Case 3.

(2) Similar to the first claim. Define R_2' just by interchanging R_1 and R_1' and argue as above interchanging the role of min-homogeneous and max-homogeneous sets. □

Lemma 3.12. *Let $k, m \geq 2$.*

(1) $\mathrm{MIN}^4_{4k \cdot m}(2(2m+4)+2) > 2^{2^{k^m}}$.
(2) $\mathrm{MAX}^4_{4k \cdot m}(2(2m+4)+2) > 2^{2^{k^m}}$.

Proof. (1) Let $k, m \geq 2$ be positive integers and put $\ell := 2^{k^m}$. Let R_2 and R_2' be the partitions from the Lemma 3.11. Let $D := D^{(2,\ell)}$. Then define $R_3 : [2^\ell]^4 \to 4k \cdot m$ as follows:

$R_3(u, v, w, x) :=$

$$\begin{cases} R_2(D(u, v), D(v, w), D(w, x)) & \text{if } D(u, v) < D(v, w) < D(w, x) \\ 2k \cdot m + R_2'(D(w, x), D(v, w), D(u, v)) & \text{if } D(u, v) > D(v, w) > D(w, x) \\ 0 & \text{if } D(u, v) < D(v, w) > D(w, x) \\ 2k \cdot m & \text{if } D(u, v) > D(v, w) < D(w, x) \end{cases}$$

The cases $D(u, v) = D(v, w)$ or $D(v, w) = D(w, x)$ don't occur since we developed u, v, w, x with respect to base 2.

Let $Y \subseteq 2^\ell$ be min-homogeneous for R_3. We claim $card(Y) \leq 2(2m + 4) + 1$. Let $Y = \{u_0, \dots, u_h\}$ be min-homogeneous for R_3, where $h := 2(2m + 4) + 1$. Put $d_i := D(u_i, u_{i+1})$ and $g := 2m + 3$.

Case 1: Assume that there is some r such that $d_r < \dots < d_{r+g}$. We claim that $Y' := \{d_r, \dots, d_{r+g}\}$ is min-homogeneous for R_2 which would contradict Lemma 3.11.

Note again that for $r \leq i < j \leq r + g + 1$ we have

$$D(u_i, u_j) = \max\{D(u_i, u_{i+1}), \dots, D(u_{j-1}, u_j)\} = D(u_{j-1}, u_j).$$

Therefore for $r \leq i < p < q \leq r + g$

$$\begin{aligned} R_2(d_i, d_p, d_q) &= R_2(D(u_i, u_{i+1}), D(u_{i+1}, u_{p+1}), D(u_{p+1}, u_{q+1})) \\ &= R_3(u_i, u_{i+1}, u_{p+1}, u_{q+1}). \end{aligned}$$

By the same pattern we obtain for $r \leq i < u < v \leq r + g$

$$\begin{aligned} R_2(d_i, d_u, d_v) &= R_2(D(u_i, u_{i+1}), D(u_{i+1}, u_{u+1}), D(u_{u+1}, u_{v+1})) \\ &= R_3(u_i, u_{i+1}, u_{u+1}, u_{v+1}). \end{aligned}$$

By min-homogeneity of Y for R_3 we obtain then $R_2(d_i, d_p, d_q) = R_2(d_i, d_u, d_v)$. Thus Y' is min-homogeneous for R_2.

Case 2: Assume that there is some r such that $d_r > \dots > d_{r+g}$. We claim that $Y' := \{d_{r+g}, \dots, d_r\}$ is max-homogeneous for R_2' which would contradict Lemma 3.11.

Then for $r \leq i < p < q \leq r + g$

$$\begin{aligned} 2k \cdot m + R_2'(d_q, d_p, d_i) &= 2k \cdot m + R_2'(D(u_{p+1}, u_{q+1}), D(u_{i+1}, u_{p+1}), D(u_i, u_{i+1})) \\ &= R_3(u_i, u_{i+1}, u_{p+1}, u_{q+1}). \end{aligned}$$

By the same pattern we obtain for $r \leq i < u < v \leq r + g$

$$\begin{aligned} 2k \cdot m + R_2'(d_v, d_u, d_i) &= 2k \cdot m + R_2'(D(u_{u+1}, u_{v+1}), D(u_{i+1}, u_{u+1}), D(u_i, u_{i+1})) \\ &= R_3(u_i, u_{i+1}, u_{u+1}, u_{v+1}). \end{aligned}$$

By min-homogeneity of Y for R_3 we obtain then $R_2'(d_q, d_p, d_i) = R_2'(d_v, d_u, d_i)$. Thus Y' is max-homogeneous for R_2'.

Case 3: There is a local maximum of the form $d_i < d_{i+1} > d_{i+2}$. Then we obtain the following contradiction using the min-homogeneity

$$\begin{aligned} 0 &= R_3(u_i, u_{i+1}, u_{i+2}, u_{i+3}) \\ &= R_3(u_i, u_{i+2}, u_{i+3}, u_{i+4}) \\ &\geq 2k \cdot m \end{aligned}$$

since $D(u_i, u_{i+2}) = d_{i+1} > d_{i+2}$.

Case 4: Cases 1 to 3. do not hold. Then there must be two local minima. But then inbetween we have a local maximum and we are back in Case 3.

(2) Similar to the first claim. Define R'_3 just by interchanging R_2 and R'_2 and argue interchanging the role of min-homogeneous and max-homogeneous sets. □

We now show how one can obtain sparse min-homogeneous sets for certain functions of dimension 3 from the bounds from Lemma 3.11. It will be clear that the same can be done for functions of dimension 4 using the bounds from Lemma 3.12. In section 3.3.2 we will lift the bounds and the sparseness results to the general case.

Lemma 3.13. *Let* $f(i) := \lfloor \sqrt{\log(i)} \rfloor$. *Let* $\ell := 2^{(16\cdot 17+1)^2}$. *Then there exists an* f*-regressive partition* $P : [\mathbb{N}]^3 \to \mathbb{N}$ *such that if* Y *is min-homogeneous for* P *and of cardinality not below* $3\ell - 1$, *then we have* $2^{(\log(a))^2} \leq b$ *for all* $a, b \in \bar{Y}/4$, *where*

$$\bar{Y} := Y \setminus (\{\textit{the first } \ell \textit{ elements of } Y\} \cup \{\textit{the last } \ell - 2 \textit{ elements of } Y\}).$$

Proof. Let $u_0 := 0$, $u_1 = \ell$ and $u_{i+1} := \mathrm{MIN}^3_{f(u_i)-1}(\ell+1) - 1$ for $i > 0$. Notice that $u_i < u_{i+1}$. This is because $u_i \geq 2^{(16\cdot 17+1)^2}$ implies by Lemma 3.11, letting $m = 8$,

$$\begin{aligned} u_{i+1} &= \mathrm{MIN}^3_{f(u_i)-1}(\ell+1) - 1 \\ &\geq \mathrm{MIN}^3_{f(u_i)-1}(20) - 1 \\ &\geq 2^{\lfloor \frac{f(u_i)-1}{16} \rfloor^8} \\ &> 2^{f(u_i)^4} \\ &= 2^{\log(u_i)^2} \\ &\geq u_i \end{aligned}$$

Let $G_0 : [u_1]^3 \to 1$ be the constant function with the value 0 and for $i > 0$ choose $G_i : [u_{i+1}]^3 \to f(u_i) - 1$ such that every G_i-min-homogeneous set $Y \subseteq u_{i+1}$ satisfies $card(Y) < \ell + 1$. Let $P : [\mathbb{N}]^3 \to \mathbb{N}$ be defined as follows:

$$P(x_0, x_1, x_2) := \begin{cases} G_i(x_0, x_1, x_2) + 1 & \text{if } u_i \leq x_0 < x_1 < x_2 < u_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then P is f-regressive by the choice of the G_i. Assume that $Y \subseteq \mathbb{N}$ is min-homogeneous for P and $card(Y) \geq 3\ell - 1$ and $\bar{Y}$ is as described, i.e., $card(\bar{Y}) \geq \ell + 1$. If $\bar{Y} \subset [u_i, u_{i+1}[$ then $\bar{Y}$ is G_i-min-homogeneous hence $card(\bar{Y}) \leq \ell$ which is excluded. Hence each interval $[u_i, u_{i+1}[$ contains at most two elements from Y since we have omitted the last $\ell - 2$ elements from Y.

If a, b are in $\bar{Y}/4$. Then there are $e_1, e_2, e_3 \in \bar{Y}$ such that $a < e_1 < e_2 < e_3 < b$, and so there exists an $i \geq 1$ such that $a \leq u_i < u_{i+1} \leq b$. Hence $b \geq u_{i+1} \geq 2^{f(u_i)^4} \geq 2^{\log(a)^2}$ as above by Lemma 3.11. □

We just want to remark that $2^{(16\cdot 17+1)^2}$ is not the smallest number which satisfies Lemma 3.13.

3.3.2. *$B_{\epsilon,c,d,0}$-sparse min-homogeneous sets - Generalization.* We now show how the above results Lemma 3.12 and Lemma 3.13 can be generalized to arbitrary dimension. Let g_d be defined inductively as follows. $g_0(x) := x$, $g_{d+1}(x) := 2 \cdot g_d(x) + 2$. Thus

$$g_d(x) := \underbrace{2(\ldots(2(2}_{d} x + 2) + 2)\ldots) + 2,$$

i.e., d iterations of the function $x \mapsto 2x + 2$.

Lemma 3.14. *Let $d \geq 1$ and $k, m \geq 2$.*

(1) $\mathrm{MIN}^{d+1}_{2^{d-1}k \cdot m}(g_{d-2}(2m+4)) > 2_{d-1}(k^m)$.
(2) $\mathrm{MAX}^{d+1}_{2^{d-1}k \cdot m}(g_{d-2}(2m+4)) > 2_{d-1}(k^m)$.

Proof Sketch. By a simultaneous induction on $d \geq 1$. The base cases for $d \leq 2$ are proved in Lemma 3.10 and Lemma 3.11. Let now $d \geq 2$. The proof is essentially the same as the previous ones.

Let $R_d : [2_{d-1}(k^m)]^{d+1} \to 2^{d-1}k \cdot m$ (or $R'_d : [2_{d-1}(k^m)]^{d+1} \to 2^{d-1}k \cdot m$) be a partition such that every min-homogeneous set for R_d (or max-homogeneous set for R'_d) is of cardinality $< g_{d-2}(2m+4)$.

We define then $R_{d+1} : [2^{k^m}_d]^{d+2} \to 2^d k \cdot m$ as follows. $R_{d+1}(x_1, \ldots, x_{d+2}) :=$

$$\begin{cases} R_d(d(x_1,x_2),\ldots,d(x_{d+1},x_{d+2})) & \text{if } d(x_1,x_2) < \cdots < d(x_{d+1},x_{d+2}), \\ 2^{d-1}k \cdot m + R'_d(d(x_{d+2},x_{d+1}),\ldots,d(x_2,x_1)) & \text{if } d(x_1,x_2) > \cdots > d(x_{d+1},x_{d+2}), \\ 0 & \text{if } d(x_1,x_2) < d(x_2,x_3) > d(x_3,x_4) \\ 2^{d-1}k \cdot m & \text{else.} \end{cases}$$

And $R'_{d+1} : [2^{k^m}_d]^{d+2} \to 2^d k \cdot m$ is defined similarly by interchanging R_d and R'_d. Now we can argue analogously to Lemma 3.12. □

We now state the key result of the present section, the Sparseness Lemma. Let $f(i) := \lfloor \sqrt[c]{\log_{d-1}(i)} \rfloor$. We show how an f-regressive function P of dimension $d+1$ can be defined such that all large min-homogeneous sets are $(2^{(\log_{d-1}(\cdot))^c}_{d-1}, 3)$-sparse.

Lemma 3.15 (Sparseness Lemma)**.** *Given $c \geq 2$ and $d \geq 1$ let $f(i) := \lfloor \sqrt[c]{\log_{d-1}(i)} \rfloor$. And define $m := 2c^2$, $n := 2^{d-1} \cdot m$, and $\ell := 2_{d-1}((n \cdot (n+1)+1)^c)$. There exists an f-regressive partition $P_{c,d} : [\mathbb{N}]^{d+1} \to \mathbb{N}$ such that, if Y is*

- *min-homogeneous for $P_{c,d}$ and*
- *$card(Y) \geq 3\ell - 1$,*

then we have $2^{(\log_{d-1}(a))^c}_{d-1} \leq b$ for all $a, b \in \bar{Y}/4$, where

$$\bar{Y} := Y \setminus (\{\text{the first } \ell \text{ elements of } Y\} \cup \{\text{the last } \ell - 2 \text{ elements of } Y\}).$$

Proof. Let $u_0 := 0, u_1 := \ell$ and $u_{i+1} := \mathrm{MIN}^{d+1}_{f(u_i)-1}(\ell+1) - 1$. Notice that $u_i < u_{i+1}$. This is because $u_i \geq \ell$ implies by Lemma 3.14

$$\begin{aligned} u_{i+1} &= \mathrm{MIN}^{d+1}_{f(u_i)-1}(\ell+1) - 1 \\ &\geq \mathrm{MIN}^{d+1}_{f(u_i)-1}(g_{d-2}(2m+4)) - 1 \\ &\geq 2_{d-1}^{\lfloor \frac{f(u_i)-1}{2^{d-1} \cdot m} \rfloor^m} \\ &> 2_{d-1}^{f(u_i)^{m/2}} \\ &= 2^{\log(u_i)^c} \\ &\geq u_i \end{aligned}$$

Note that $\ell > g_{d-2}(2m+4)$. Let $G_0 : [u_1]^{d+1} \to 1$ be the constant function with value 0 and for $i > 0$ choose $G_i : [u_{i+1}]^{d+1} \to f(u_i) - 1$ such that every G_i-min-homogeneous set $Y \subseteq u_{i+1}$ satisfies $card(Y) \leq \ell$. Let $P : [\mathbb{N}]^{d+1} \to \mathbb{N}$ be defined as follows:

$$P_{c,d}(x_0, \ldots, x_d) := \begin{cases} G_i(x_0, \ldots, x_d) + 1 & \text{if } u_i \leq x_0 < \cdots < x_d < u_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

Then $P_{c,d}$ is f-regressive by choice of the G_i's. Assume $Y \subseteq \mathbb{N}$ is min-homogeneous for $P_{c,d}$ and $card(Y) \geq 3\ell - 1$. Let $\bar{Y}$ be as described, i.e., $card(\bar{Y}) \geq \ell + 1$. If $\bar{Y} \subseteq [u_i, u_{i+1}[$ for some i then $\bar{Y}$ is min-homogeneous for G_i, hence $card(\bar{Y}) \leq \ell$, which is impossible. Hence each interval $[u_i, u_{i+1}[$ contains at most two elements from $\bar{Y}$, since we have omitted the last $\ell - 2$ elements of Y.

Given $a, b \in \bar{Y}/4$ let $e_1, e_2, e_3 \in \bar{Y}$ such that $a < e_1 < e_2 < e_3 < b$. Then there exists an $i \geq 1$ such that $a \leq u_i < u_{i+1} \leq b$. Hence $b \geq u_{i+1} \geq 2^{f(u_i)^{m/2}} \geq 2^{\log(a)^c}$ as above by Lemma 3.14. □

3.4. **Capturing, glueing, compressing.** Given $c \geq 2$ and $d \geq 1$ let $f_{c,d}(x) := \lfloor \sqrt[c]{\log_d(x)} \rfloor$. We first want to show that the regressive Ramsey function $R^{\text{reg}}_{f_{c,d-1}}(d+1, \cdot)$ eventually dominates $B_{\epsilon,c,d,\omega^c_{d-1}}$ (for suitable choices of ϵ). Now let $f_{\omega_d,d-1}$ be $\lfloor \sqrt[B^{-1}_{\omega_d}(\cdot)]{\log_{d-1}} \rfloor$. We will conclude that the regressive Ramsey function $R^{\text{reg}}_{f_{\omega_d,d-1}}(d+1, \cdot)$ eventually dominates B_{ω_d}. From the viewpoint of logic this implies that the Regressive Ramsey theorem for $(d+1)$-hypergraphs with parameter function $f_{\omega_d,d-1}$ cannot be proved without induction on predicates with $(d+2)$ alternations of existential and universal quantifiers.

3.4.1. *$B_{\omega^c_d}$-sparse min-homogeneous sets.* We begin by recalling the definition of the "step-down" relation on ordinals from [7] and some of its properties with respect to the hierarchies defined in Section 3.2.

Definition 3.16. *Let $\alpha < \beta \leq \varepsilon_0$ Then $\beta \longrightarrow_n \alpha$ if for some sequence $\gamma_0, \ldots, \gamma_k$ of ordinals we have $\gamma_0 = \beta, \gamma_{i+1} = \gamma_i[n]$ for $0 \leq i < k$ and $\gamma_k = \alpha$.*

We first recall the following property of the $\longrightarrow_n$ relation. It is stated and proved as Corollary 2.4 in [7].

Lemma 3.17. *Let $\beta < \alpha < \varepsilon_0$. Let $n > i$. If $\alpha \longrightarrow_i \beta$ then $\alpha \longrightarrow_n \beta$.*

Proposition 3.18. *Let $\alpha \leq \varepsilon_0$. For all $c \geq 2, d \geq 1$, let $f(x) = \lfloor \sqrt[c]{\log_d(x)} \rfloor$. Let $0 < \epsilon \leq 1$. Then we have the following.*

1. *If $f(n) > f(m)$ then $B_{\epsilon,c,d,\alpha}(n) > B_{\epsilon,c,d,\alpha}(m)$.*
2. *If $\alpha = \beta + 1$ then $B_{\epsilon,c,d,\alpha}(n) \geq B_{\epsilon,c,d,\beta}(n)$; if $\epsilon \cdot f(n) \geq 1$ then $B_{\epsilon,c,d,\alpha}(n) > B_{\epsilon,c,d,\beta}(n)$.*
3. *If $\alpha \longrightarrow_{\lfloor \epsilon \cdot f(n) \rfloor} \beta$ then $B_{\epsilon,c,d,\alpha}(n) \geq B_{\epsilon,c,d,\beta}(n)$.*

Proof. Straightforward from the proof of Proposition 2.5 in [7]. □

We denote by $T_{\omega^c_d,n}$ the set $\{\alpha : \omega^c_d \longrightarrow_n \alpha\}$. We recall the following bound from [7], Proposition 2.10.

Lemma 3.19. *Let $n \geq 2$ and $c, d \geq 1$. Then*

$$card(T_{\omega^c_d,n}) \leq 2_{d-1}(n^{6c}).$$

Observe that, by straightforward adaptation of the proof of Lemma 3.19 (Proposition 2.10 in [7]), we accordingly have $card(T_{\omega_d^c, f(n)}) \leq 2_{d-1}(f(n)^{6c})$ for f a non-decreasing function and all n such that $f(n) \geq 2$.

Definition 3.20. *Let τ be a function of type k. We say that τ is* weakly monotonic on first arguments *on X (abbreviated* w.m.f.a.*) if for all $s, t \in [X]^k$ such that $\min(s) < \min(t)$ we have $\tau(s) \leq \tau(t)$.*

In the rest of the present section, when ϵ, c, d are fixed and clear from the context, B_α stands for $B_{\epsilon,c,d,\alpha}$ for brevity.

Proposition 3.21 (Capturing). *Given $c, d \geq 2$ let $\epsilon = \sqrt[6c]{1/3}$. Put*

$$\begin{aligned} f(x) &:= \left\lfloor \sqrt[c]{\log_{d-1}(x)} \right\rfloor \\ g(x) &:= \left\lfloor \sqrt[6c^2]{\log_{d-1}(x)} \right\rfloor \\ h(x) &:= \left\lfloor \sqrt[6c]{\frac{1}{3}} \cdot \sqrt[6c^2]{\log_{d-1}(x)} \right\rfloor \end{aligned}$$

Then there are functions $\tau_1 : [\mathbb{N}]^2 \to \mathbb{N}$ $2_{d-2}\left(\frac{1}{3}f\right)$-regressive, $\tau_2 : [\mathbb{N}]^2 \to \mathbb{N}$ f-regressive, $\tau_3 : [\mathbb{N}]^2 \to 2$ so that the following holds: If $H \subseteq \mathbb{N}$ is of cardinality > 2 and s.t.

- (a) *H is min-homogeneous for τ_1,*
- (b) *$\forall s, t \in [H]^2$ if $\min(s) < \min(t)$ then $\tau_1(s) \leq \tau_1(t)$ (i.e., τ_1 is w.m.f.a. on H),*
- (c) *H is $2_{d-1}^{\lfloor \log_{d-1}(\cdot)^c \rfloor}$-sparse (i.e., $B_{\epsilon,c,d-1,0}$-sparse),*
- (d) $\min(H) \geq h^{-1}(2)$,
- (e) *H is min-homogeneous for τ_2, and*
- (f) *H is homogeneous for τ_3,*

then for any $x < y$ in H we have $B_{\epsilon,c,d-1,\omega_{d-1}^c}(x) \leq y$ (i.e., H is $B_{\epsilon,c,d-1,\omega_{d-1}^c}$-sparse).

Proof. Define a function τ_1 as follows.

$$\tau_1(x,y) := \begin{cases} 0 & \text{if } B_{\omega_{d-1}^c}(x) \leq y \text{ or } h(x) < 2, \\ \xi \dot{-} 1 & \text{otherwise, where } \xi = \min\{\alpha \in T_{\omega_{d-1}^c, h(x)} : y < B_\alpha(x)\}. \end{cases}$$

$\xi \dot{-} 1$ means 0 if $\xi = 0$ and β if $\xi = \beta + 1$. We have to show that τ_1 is well-defined. First observe that the values of τ_1 can be taken to be in $\mathbb{N}$ since, by Lemma 3.19, we can assume an order preserving bijection between $T_{\omega_{d-1}^c, h(x)}$ and $2_{d-2}^{h(x)^{6c}}$:

$$\tau_1(x,y) < 2_{d-2}(h(x)^{6c}) = 2_{d-2}^{\left(\sqrt[6c]{\frac{1}{3}}\sqrt[6c^2]{\log_{d-1}(x)}\right)^{6c}} = 2_{d-2}^{\left(\frac{1}{3}\sqrt[c]{\log_{d-1}(x)}\right)}.$$

In the following we will only use properties of values of τ_1 that can be inferred by this assumption.

Let $\xi = \min\{\alpha \in T_{\omega_{d-1}^c, h(x)} : y < B_\alpha(x)\}$. Suppose that the minimum ξ is a limit ordinal, call it λ. Then, by definition of the hierarchy, we have

$$B_\lambda(x) = B_{\lambda[h(x)]}(x) > y.$$

But $\lambda[h(x)] < \lambda$ and $\lambda[h(x)] \in T_{\omega_{d-1}^c, h(x)}$, against the minimality of λ.

Define a function τ_2 as follows.

$$\tau_2(x,y) := \begin{cases} 0 & \text{if } B_{\omega^c_{d-1}}(x) \leq y \text{ or } h(x) < 2, \\ k-1 & \text{otherwise, where } B^{k-1}_{\tau_1(x,y)}(x) \leq y < B^k_{\tau_1(x,y)}(x). \end{cases}$$

If $\xi = \min\{\alpha \in T_{\omega^c_{d-1},h(x)} : y < B_\alpha(x)\} = 0$, i.e., $B_0(x) > y$, then $\tau_2(x,y) = 0$. On the other hand, if $\xi > 0$ then one observes that $k - 1 < \epsilon \cdot \sqrt[c]{\log_{d-1}(x)}$ by definition of τ_1 and of B, so that τ_2 is f-regressive.

Define a function τ_3 as follows.

$$\tau_3(x,y) := \begin{cases} 0 & \text{if } B_{\omega^c_{d-1}}(x) \leq y \text{ or } h(x) < 2, \\ 1 & \text{otherwise.} \end{cases}$$

Suppose H is as hypothesized. We show that τ_3 takes constant value 0. This implies the $B_{\omega^c_{d-1}}$-sparseness since $h(\min(H)) \geq 2$. Assume otherwise and let $x < y < z$ be in H. Note first that by the condition (c)

$$\min\{\alpha \in T_{\omega^c_{d-1},h(x)} : y < B_\alpha(x)\} > 0 \quad \text{and hence} \quad \tau_2(x,y) > 0.$$

By hypotheses on H, $\tau_1(x,y) = \tau_1(x,z)$, $\tau_2(x,y) = \tau_2(x,z)$, $\tau_1(x,z) \leq \tau_1(y,z)$. We have the following, by definition of τ_1, τ_2.

$$B^{\tau_2(x,z)}_{\tau_1(x,z)}(x) \leq y < z < B^{\tau_2(x,z)+1}_{\tau_1(x,z)}(x).$$

This implies that $B^{\tau_2(x,z)+1}_{\tau_1(x,z)}(x) \leq B_{\tau_1(x,z)}(y)$, by one application of $B_{\tau_1(x,z)}$.

We now show that $\tau_1(y,z) \longrightarrow_{h(y)} \tau_1(x,z)$. We know $\tau_1 \in T_{\omega^c_{d-1},h(x)}$, i.e., $\omega^c_{d-1} \longrightarrow_{h(x)} \tau_1(x,z)$. Since $x < y$ implies $h(x) \leq h(y)$ we have $\omega^c_{d-1} \longrightarrow_{h(y)} \tau_1(x,z)$. But since $\tau_1(y,z) \in T_{\omega^c_{d-1},h(y)}$ and $\tau_1(y,z) \geq \tau_1(x,z)$ by hypotheses on H, we can conclude that $\tau_1(y,z) \longrightarrow_{h(y)} \tau_1(x,z)$.

Hence, by Lemma 3.17 and Proposition 3.18.(3), we have $B_{\tau_1(x,z)}(y) \leq B_{\tau_1(y,z)}(y)$, and we know that $B_{\tau_1(y,z)}(y) \leq z$ by definition of τ_1. So we reached the contradiction $z < z$. □

A comment about the utility of Proposition 3.21. If, assuming $(\mathrm{KM})^{d+1}_{\lfloor \sqrt[c]{\log_{d-1}} \rfloor}$, we are able to infer the existence of a set H satisfying the conditions of Proposition 3.21, then we can conclude that $R^{\mathrm{reg}}_{\lfloor \sqrt[c]{\log_{d-1}} \rfloor}(d+1,\cdot)$ eventually dominates $B_{\omega^c_{d-1}}$. In fact, suppose that there exists a M such that for almost all x there exists a set H satisfying the conditions of Proposition 3.21 and such that $H \subseteq R^{\mathrm{reg}}_{\lfloor \sqrt[c]{\log_{d-1}} \rfloor}(d+1, x+M)$, which means that such an H can be found as a consequence of $(\mathrm{KM})^{d+1}_{\lfloor \sqrt[c]{\log_{d-1}} \rfloor}$. Also suppose that, for almost all x we can find such an H of cardinality $\geq x+2$. Then for such an $H = \{h_0, \ldots, h_k\}$ we have $k \geq x+1$, $h_{k-1} \geq x$ and, by Proposition 3.21 $h_k \geq B_{\omega^c_{d-1}}(h_{k-1})$. Hence we can show that $R^{\mathrm{reg}}_{\lfloor \sqrt[c]{\log_{d-1}} \rfloor}(d+1,\cdot)$ has eventually dominates $B_{\omega^c_{d-1}}$:

$$R^{\mathrm{reg}}_{\lfloor \sqrt[c]{\log_{d-1}} \rfloor}(d+1, x+M) \geq h_k \geq B_{\omega^c_{d-1}}(h_{k-1}) \geq B_{\omega^c_{d-1}}(x)$$

In the following we show how to obtain a set H as in Proposition 3.21 using the Regressive Ramsey Theorem for $(d+1)$-hypergraphs with parameter function $\lfloor \sqrt[c]{\log_{d-1}} \rfloor$.

3.4.2. *Glueing and logarithmic compression of f-regressive functions.* We here collect some tools that are needed to combine or glue distinct f-regressive functions in such a way that a min-homogeneous set (or a subset thereof) for the resulting function is min-homogeneous for each of the component functions. Most of these tools are straightforward adaptations of analogous results for regressive partitions from [6].

The first simple lemma (Lemma 3.22 below) will help us glue the partition ensuring sparseness obtained by the Sparseness Lemma 3.15 with some other relevant function introduced below. Observe that one does not have to go to an higher dimension if one is willing to give up one square root in the regressiveness condition.

Lemma 3.22. *Let $P : [\mathbb{N}]^n \to \mathbb{N}$ be $Q : [\mathbb{N}]^n \to \mathbb{N}$ be $\lfloor \sqrt[2c]{\log_k} \rfloor$-regressive functions. And define $(P \otimes Q) : [\mathbb{N}]^n \to \mathbb{N}$ as follows:*

$$(P \otimes Q)(x_1, \ldots, x_n) := P(x_1, \ldots, x_n) \cdot \lfloor \sqrt[2c]{\log_k(x_1)} \rfloor + Q(x_1, \ldots, x_n)$$

Then $(P \otimes Q)$ is $\lfloor \sqrt[c]{\log_k} \rfloor$-regressive and if H is min-homogeneous for $(P \otimes Q)$ then H is min-homogenous for P and for Q.

Proof. We show that $(P \otimes Q)$ is $\sqrt[c]{\log_k}$-regressive:

$$\begin{aligned}
(P \otimes Q)(\vec{x}) &= P(\vec{x}) \cdot \lfloor \sqrt[2c]{\log_k(x_1)} \rfloor + Q(\vec{x}) \\
&\leq (\sqrt[2c]{\log_k(x_1)} - 1) \cdot \sqrt[2c]{\log_k(x_1)} + (\sqrt[2c]{\log_k(x_1)} - 1) \\
&= \sqrt[c]{\log_k(x_1)} - 1 \\
&< \lfloor \sqrt[c]{\log_k(x_1)} \rfloor
\end{aligned}$$

We show that if H is min-homogeneous for $(P \otimes Q)$ then H is min-homogeneous for both P and Q. Let $x < y_2 \cdots < y_n$ and $x < z_2 < \cdots < z_n$ be in H. Then $(P \otimes Q)(x, \vec{y}) = (P \otimes Q)(x, \vec{z})$. Then we show $a := P(x, \vec{y}) = P(x, \vec{z}) =: c$ and $c := Q(x, \vec{y}) = Q(x, \vec{z}) =: d$.

If $w := \lfloor \sqrt[2c]{\log_k(x_1)} \rfloor = 0$ then it is obvious since $a = b = 0$. Assume now $w > 0$. Then $a \cdot w + b = c \cdot w + d$. This, however, implies that $a = c$ and $b = d$, since $a, b, c, d < w$. □

The next two results are adaptations of Lemma 3.3 and Proposition 3.6 of Kanamori-McAloon [6] for f-regressiveness (for any choice of f). Lemma 3.23 is used in [6] for a different purpose, and it is quite surprising how well it fits in the present investigation. Essentially, it will be used to obtain, from an 2^f_{d-2}-regressive of dimension 2, an f-regressive function of dimension $d - 2$ such that both have almost same min-homogeneous sets. Each iteration of the following Lemma costs one dimension.

Lemma 3.23. *If $P : [\mathbb{N}]^n \to \mathbb{N}$ is f-regressive, then there is a $\bar{P} : [\mathbb{N}]^{n+1} \to \mathbb{N}$ f-regressive s.t.*

(i) *$\bar{P}(s) < 2\log(f(\min(s))) + 1$ for all $s \in [\mathbb{N}]^{n+1}$, and*
(ii) *if $\bar{H}$ is min-homogeneous for $\bar{P}$ then $H = \bar{H} - (f^{-1}(7) \cup \{\max(\bar{H})\})$ is min-homogeneous for P.*

Proof. Write $P(s) = (y_0(s), \ldots, y_{d-1}(s))$ where $d = \log(f(\min(s)))$. Define $\bar{P}$ on $[N]^{n+1}$ as follows.

$$\bar{P}(x_0, \ldots, x_n) := \begin{cases} 0 & \text{if either } f(x_0) < 7 \text{ or } \{x_0, \ldots, x_n\} \\ & \text{is min-homogeneous for } P, \\ 2i + y_i(x_0, \ldots, x_{n-1}) + 1 & \text{otherwise, where } i < \log(f(x_0)) \\ & \text{is the least s.t. } \{x_0, \ldots, x_n\} \\ & \text{is not min-homogeneous for } y_i. \end{cases}$$

Then $\bar{P}$ is f-regressive and satisfies (i). We now verify (ii). Suppose that $\bar{H}$ is min-homogeneous for $\bar{P}$ and H is as described. If $\bar{P}|[H]^{n+1} = \{0\}$ then we are done, since then all $\{x_0, \ldots, x_n\} \in [H]^{n+1}$ are min-homogeneous for P. Suppose then that there are $x_0 < \cdots < x_n$ in H s.t. $\bar{P}(x_0, \ldots, x_n) = 2i + y_i(x_0, \ldots, x_{n-1}) + 1$. Given $s, t \in [\{x_0, \ldots, x_n\}]^n$ with $\min(s) = \min(t) = x_0$ we observe that

$$\bar{P}(s \cup \max(\bar{H})) = \bar{P}(x_0, \ldots, x_n) = \bar{P}(t \cup \max(\bar{H}))$$

by min-homogeneity. But then $y_i(s) = y_i(t)$, a contradiction. □

The next proposition allows one to glue together a finite number of f-regressive functions into a single f-regressive. This operation costs one dimension.

Proposition 3.24. *There is a primitive recursive function $p : \mathbb{N} \to \mathbb{N}$ such that for any $n, e \in \mathbb{N}$, if $P_i : [\mathbb{N}]^n \to \mathbb{N}$ is f-regressive for every $i \leq e$ and $P : [\mathbb{N}]^{n+1} \to \mathbb{N}$ is f-regressive, there are $\rho_1 : [\mathbb{N}]^{n+1} \to \mathbb{N}$ f-regressive and $\rho_2 : [\mathbb{N}]^{n+1} \to 2$ such that if $\bar{H}$ is min-homogeneous for ρ_1 and homogeneous for ρ_2, then*

$$H = \bar{H} \setminus (\max\{f^{-1}(7), p(e)\} \cup \{\max(\bar{H})\})$$

is min-homogeneous for each P_i and for P.

Proof. Note that given any $k \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that for all $x \geq m$

$$(2\log(f(x)) + 1)^{k+1} \leq f(x).$$

Let $p(k)$ be the least such m.

For each P_i, let $\bar{P}_i$ be obtained by an application of the Lemma 3.23. Define $\rho_2 : [\mathbb{N}]^{n+1} \to 2$ as follows.

$$\rho_2(s) := \begin{cases} 0 & \text{if } \bar{P}_i(s) \neq 0 \text{ for some } i \leq e, \\ 1 & \text{otherwise.} \end{cases}$$

Define $\rho_1 : [\mathbb{N}]^{n+1} \to \mathbb{N}$ f-regressive as follows.

$$\rho_1(s) := \begin{cases} \langle \bar{P}_0(s), \ldots, \bar{P}_e(s) \rangle & \text{if } \rho_2(s) = 0 \text{ and } \min(s) \geq p(e), \\ P(s) & \text{otherwise.} \end{cases}$$

Observe that ρ_1 can be coded as a f-regressive function by choice of $p(\cdot)$.

Suppose $\bar{H}$ is as hypothesized and H is as described. If ρ_2 on $[H]^{n+1}$ were constantly 0, we can derive a contradiction as in the proof of the previous Lemma. Thus ρ_2 is constantly 1 on $[H]^{n+1}$ and therefore $\rho_1(s) = P(s)$ for $s \in [H]^{n+1}$ and the proof is complete. □

The following proposition is an f-regressive version of Proposition 3.4 in Kanamori-McAloon [6]. It is easily seen to hold for any choice of f, but we include the proof for completeness. This proposition will allow us to find a min-homogeneous set on which τ_1 from Proposition 3.21 is weakly monotonic increasing on first arguments. The cost for this is one dimension.

Proposition 3.25. *If $P : [\mathbb{N}]^n \to \mathbb{N}$ is f-regressive, then there are $\sigma_1 : [\mathbb{N}]^{n+1} \to \mathbb{N}$ f-regressive and $\sigma_2 : [\mathbb{N}]^{n+1} \to 2$ such that if H is of cardinality $> n+1$, min-homogeneous for σ_1 and homogeneous for σ_2, then $H \setminus \{\max(H)\}$ is min-homogeneous for P and for all $s, t \in [H]^n$ with $\min(s) < \min(t)$ we have $P(s) \leq P(t)$.*

Proof. Define $\sigma_1 : [\mathbb{N}]^{n+1} \to \mathbb{N}$ as follows:

$$\sigma_1(x_0, \ldots, x_n) := \min(P(x_0, \ldots, x_{n-1}), P(x_1, \ldots, x_n))$$

Obviously σ_1 is f-regressive since P is f-regressive. Define $\sigma_2 : [\mathbb{N}]^{n+1} \longrightarrow \mathbb{N}$ as follows:

$$\sigma_2(x_0, \ldots, x_n) := \begin{cases} 0 & \text{if } P(x_0, \ldots, x_{n-1}) \leq P(x_1, \ldots, x_n), \\ 1 & \text{otherwise} \end{cases}$$

Now let H be as hypothesized. Suppose first that σ_2 is constantly 0 on $[H]^{n+1}$. Then weak monotonicity is obviously satisfied. We show that $H \setminus \{\max(H)\}$ is min-homogeneous for P as follows. Let $x_0 < x_1 \cdots < x_{n-1}$ and $x_0 < y_1 < \cdots < y_{n-1}$ be in $H \setminus \{\max(H)\}$. Since σ_2 is constantly 0 on H, we have $F(x_0, x_1, \ldots, x_{n-1}) \leq F(x_1, \ldots, x_{n-1}, \max(H))$, and $F(x_0, y_1, \ldots, y_{n-1}) \leq F(y_1, \ldots, y_{n-1}, \max(H))$. Since H is also min-homogeneous for σ_1, we have

$$\sigma_1(x_0, x_1, \ldots, x_{n-1}, \max(H)) = \sigma_1(x_0, y_1, \ldots, y_{n-1}, \max(H)).$$

Thus, $F(x_0, x_1, \ldots, x_{n-1}) = F(x_0, y_1, \ldots, y_{n-1})$.

Assume by way of contradiction that σ_2 is constantly 1 on $[H]^{n+1}$. Let $x_0 < \cdots < x_{n+1}$ be in H. Then, by two applications of σ_2 we have

$$F(x_0, \ldots, x_{n-1}) > F(x_1, \ldots, x_n) > F(x_2, \ldots, x_{n+1}),$$

so that $\sigma_1(x_0, \ldots, x_n) = F(x_1, \ldots, x_n)$ while $\sigma_1(x_0, x_2, \ldots, x_{n+1}) = F(x_2, \ldots, x_{n+1})$, against the min-homogeneity of H for σ_1. □

3.4.3. *Putting things together.* Now we have all ingredients needed for the lower bound part of the sharp threshold result. Figure 1 below is a scheme of how we will put them together to get the desired result. It illustrates, besides the general structure of the argument, how the need for Kanamori-McAloon principle for hypergraphs of dimension $d+1$ arises when dealing with the ω_d-level of the fast-growing hierarchy.

Given f let $\bar{f}_k$ be defined as follows: $\bar{f}_0(x) := f(x)$, $\bar{f}_{k+1}(x) := 2\log(\bar{f}_k(x)) + 1$. Thus,

$$\bar{f}_k(x) := 2\log(2\log(\ldots(2\log(f(x)) + 1)\ldots) + 1) + 1,$$

with k iterations of $2\log(\cdot) + 1$ applied to f.

Let $f(x) = \lfloor \sqrt[c]{\log_{d-1}} \rfloor$ and $f'(x) = 2_\ell(1/3 \cdot f(x))$, $\ell = d-2$. Observe then that $\bar{f}'_\ell$ is eventually dominated by f, so that an $\bar{f}'_\ell$-regressive function is also f-regressive

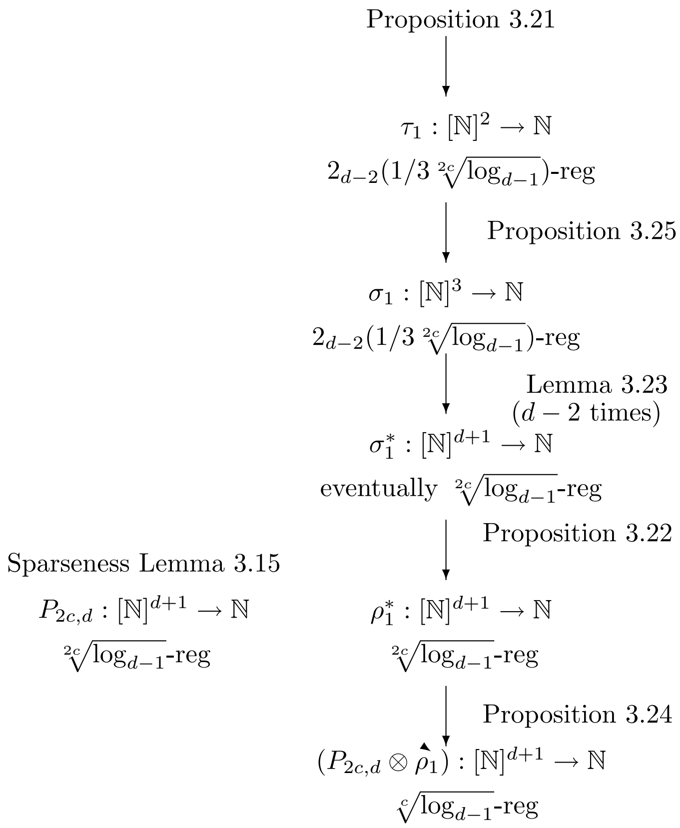

FIGURE 1. Scheme of the lower bound proof

if the arguments are large enough. Let m be such that $\lfloor \sqrt[c]{\log_{d-1}(x)} \rfloor \geq \bar{f}'_\ell(x)$ for all $x \geq m$. We have

$$R_f^{\text{reg}}(d+1, x+m) \geq R_{\bar{f}'_\ell}^{\text{reg}}(d+1, x).$$

We summarize the above argument in the following Lemma.

Lemma 3.26. *If h eventually dominates g then*

$$R_h^{\text{reg}}(d, x+m) \geq R_g^{\text{reg}}(d, x),$$

where m is such that $h(x) \geq g(x)$ for all $x \geq m$.

Proof Sketch. If G is g-regressive then define G' on the same interval by letting $G'(i) = 0$ if $i \leq m$ and $G'(i) = G(i)$ otherwise. Then G' is h-regressive. If H' is min-homogeneous for G' and $card(H') \geq x+m$ then $H = H' - \{\text{first } m \text{ elements of } H'\}$ is min-homogeneous for G and of cardinality $\geq x$. $\square$

The next Theorem shows that $R^{\text{reg}}_f(d+1,\cdot)$, with $f(x) = \lfloor \sqrt[c]{\log_{d-1}(x)} \rfloor$, has eventually dominates $B_{\epsilon,c,d-1,\omega^c_{d-1}}(x)$. As a consequence - using Lemma 3.6 - we will obtain the desired lower bound in terms of F_{ω_d}.

Theorem 3.27 (in $\mathrm{I}\Sigma_1$)**.** *Given $c, d \geq 2$ let $f(x) = \lfloor \sqrt[c]{\log_{d-1}(x)} \rfloor$. Then for all x*

$$R^{\text{reg}}_f(d+1, 12x + K(c,d)) > B_{\epsilon,2c,d-1,\omega^{2c}_{d-1}}(x),$$

where $\epsilon = \sqrt[12c]{1/3}$ and $K : \mathbb{N}^2 \to \mathbb{N}$ is a primitive recursive function.

Proof. Let $\hat{f}(x) := \lfloor \sqrt[2c]{\log_{d-1}(x)} \rfloor$ and $q(x) := 2_{d-2}(\frac{1}{3}\hat{f}(x))$. Then $\bar{q}_{d-2}$ is eventually dominated by $\hat{f}$, so there is a number r such that for all $x \geq r$ we have $\bar{q}_{d-2}(x) \leq \hat{f}(x)$. Let $D(c,d)$ be the least such r. Notice that $D : \mathbb{N}^2 \to \mathbb{N}$ is primitive recursive.

Let $h(x) := \lfloor \sqrt[12c]{1/3} \cdot \sqrt[24c^2]{\log_{d-1}(x)} \rfloor$. Now we are going to show that for all x

$$R^{\text{reg}}_f(d+1, 3\ell' - 1) > B_{\epsilon,2c,d-1,\omega^{2c}_{d-1}}(x),$$

where $\ell' = \ell + 4x + 4d + 4D(c,d) + 7$, $\ell = 2_{d-1}((n \cdot (n+1) + 1)^{2c})$, $n = 2^{d-1} \cdot m$, where m is the least number such that $m \geq 2(2c)^2$, and

$$\ell \geq \max(\{\hat{f}^{-1}(7), h^{-1}(2), p(0)\} \cup \{\bar{q}_k^{-1}(7) : k \leq d-3\}),$$

where $p(\cdot)$ is as in Proposition 3.24. The existence of such an m depends primitive recursively on c, d. Notice that the Sparseness Lemma 3.15 functions for any such m with respect to $\hat{f}$. We just remark that one should not wonder about how one comes to the exact numbers above. They just follows from the following construction of the proof.

Let τ_1, τ_2, τ_3 be the functions defined in Proposition 3.21 with respect to $\hat{f}$. Observe that τ_1 is $2_{d-2}(\frac{1}{3}\hat{f})$-regressive and τ_2 is $\hat{f}$-regressive.

Let σ_1, σ_2 be the functions obtained by Proposition 3.25 applied to τ_1. Observe that σ_1 is $2_{d-2}(\frac{1}{3}\hat{f})$-regressive, i.e., q-regressive.

Let $\sigma_1^* : [\mathbb{N}]^{d+1} \to \mathbb{N}$ be the function obtained by applying Proposition 3.23 to σ_1 $d-2$ times. Observe that σ_1^* is *eventually* $\hat{f}$-regressive by the same argument as above.

Define $\hat{\sigma}_1^* : [\mathbb{N}]^{d+1} \to \mathbb{N}$ as follows:

$$\hat{\sigma}_1^* := \begin{cases} 0 & \text{if } x < D(c,d), \\ \sigma_1^*(x) & \text{otherwise.} \end{cases}$$

Then $\hat{\sigma}_1^*$ is $\hat{f}$-regressive such that if H is min-homogeneous for $\hat{\sigma}_1^*$ then

$$H \setminus \{\text{first } D(c,d) \text{ elements of } H\}$$

is min-homogeneous for σ_1^*.

Let ρ_1 and ρ_2 be the functions obtained by applying Proposition 3.24 to the $\hat{f}$-regressive functions $\hat{\sigma}_1^*$ and τ_2 (the latter trivially lifted to dimension d). Observe that ρ_1 is $\hat{f}$-regressive.

Now let $(P_{2c,d} \otimes \rho_1)$ be obtained, as in Lemma 3.22, from ρ_1 and the partition $P_{2c,d} : [\mathbb{N}]^{d+1} \longrightarrow \mathbb{N}$ from the Sparseness Lemma 3.15 with respect to $\hat{f}$. Observe that, by Lemma 3.22, we have that $(P_{2c,d} \otimes \rho_1)$ is $\sqrt[c]{\log_{d-1}}$-regressive, i.e., f-regressive.

Now x be given. Let $H \subseteq R_f^{\mathrm{reg}}(d+1, 3\ell'-1)$ be such that

$$card(H) > 3\ell' - 1$$

and H is min-homogeneous for $(P_{2c,d} \otimes \rho_1)$ and homogeneous for ρ_2, for σ_2 and for τ_3. This is possible since the Finite Ramsey Theorem is provable in $\mathrm{I}\Sigma_1$. Notice that H is then min-homogeneous for $P_{2c,d}$ and for ρ_1.

Now we follow the process just above in the reverse order to get a set which satisfies the conditions of the Capturing Proposition 3.21.

Define first H_0 and H_1 by:

$$\begin{aligned} H_0 &:= H \setminus (\{\text{first } \ell \text{ elements of } H\} \cup \{\text{last } \ell - 2 \text{ elements of } H\}), \\ H_1 &:= H_0/4 \end{aligned}$$

Then for all $a, b \in H_1$ such that $a < b$ we have $2_{d-1}^{(\log_{d-1}(a))^{2c}} \leq b$ by Lemma 3.15. Notice that

$$\begin{aligned} card(H_0) &\geq \ell' + 1, \\ card(H_1) &\geq \lfloor (\ell'+1)/4 \rfloor + 1. \end{aligned}$$

Since H_1 is also min-homogeneous for ρ_1 (and ρ_2) we have by Proposition 3.24 that H_2 defined by

$$H_2 := H_1 \setminus (\max\{\hat{f}^{-1}(7), p(0)\} \cup \{\max(H_1)\}) = H_1 \setminus \{\max(H_1)\}$$

is min-homogeneous for $\hat{\sigma}_1^*$ and for τ_2, and

$$card(H_2) \geq \lfloor (\ell'+1)/4 \rfloor .$$

Let

$$H_3 := H_2 \setminus \{ \text{ first } D(c,d) \text{ elements of } H_2\}.$$

Then H_3 is also min-homogeneous for σ_1^* (and obviously still min-homogeneous for τ_2, homogeneous for ρ_2, for σ_2 and for τ_3). Also, we have

$$card(H_3) \geq \lfloor (\ell'+1)/4 \rfloor - D(c,d).$$

By Lemma 3.23 we have that H_4 defined by

$$\begin{aligned} H_4 &:= H_3 \setminus (\max\{\bar{q}_k^{-1}(7) : k \leq d-3\} \cup \{\text{last } d-2 \text{ elements of } H_3\}) \\ &= H_3 \setminus \{\text{last } d-2 \text{ elements of } H_3\} \end{aligned}$$

is min-homogeneous for σ_1 (and σ_2), and

$$card(H_4) \geq \lfloor (\ell'+1)/4 \rfloor - D(c,d) - d + 2.$$

Now define H^* as follows:

$$H^* := H_4 \setminus \{\max H_4\}$$

Notice that $card(H_4) > 3$. Then by Proposition 3.25 H^* is min-homogeneous for τ_1 which is weakly monotonic on first arguments on $[H^*]^2$, and

$$card(H^*) \geq \lfloor (\ell'+1)/4 \rfloor - D(c,d) - d + 1 > x + 1.$$

The second inequality follows from the definition of ℓ'. Notice now that H^* satisfies all the conditions of the Capturing Proposition 3.21 with respect to $\hat{f}$.

Let $H^* = \{h_0, \ldots, h_k\}$ ($k \geq x+1$, so that $h_{k-1} \geq x$). Then, by Proposition 3.21, for all $a, b \in H^*$ such that $a < b$ we have $B_{\omega_{d-1}^c}(a) \leq b$.

$$R_f^{\mathrm{reg}}(d+1, 3\ell'-1) > h_k \geq B_{\epsilon, 2c, d-1, \omega_{d-1}^{2c}}(h_{k-1}) \geq B_{\epsilon, 2c, d-1, \omega_{d-1}^{2c}}(x),$$

where $\epsilon = \sqrt[12c]{1/3}$. The first inequality holds since we chose $H^* \subseteq R_f^{\mathrm{reg}}(d+1, \ell'-1)$. The second holds by Proposition 3.21. The third holds because $h_{k-1} \geq x$. □

Let us restate Theorem 3.27 in a somewhat simplified form. Given $c, d \geq 2$ set, from now on,

$$\hat{g}_{c,d}(x) := \sqrt[c]{\log_{d-1}(x)}$$

Theorem 3.28. *There are primitive recursive functions $h : \mathbb{N} \to \mathbb{N}$ and $K : \mathbb{N}^2 \to \mathbb{N}$ such that for all x and all $c, d \geq 2$*

$$R^{\mathrm{reg}}_{\hat{g}_{c,d}}(d+1, h(x) + K(c,d)) \geq B_{\epsilon,c,d-1,\omega^c_{d-1}}(x),$$

where $\epsilon = \sqrt[6c]{1/3}$.

Proof. By inspection of the proof of Theorem 3.27, and by the fact that, as proved in Theorem 3.7, $B_{c,d,\alpha}$ and $B_{2c,d,\alpha}$ have the same growth rate. □

Theorem 3.29. *Given $d \geq 2$ let $f(x) = \sqrt[F^{-1}_{\omega_d}(x)]{\log_{d-1}(x)}$. Then $R^{\mathrm{reg}}_f(d+1, \cdot)$ eventually dominates F_{ω_d}.*

Proof. First remember that, by Lemma 3.6, there is a primitive recursive function $r : \mathbb{N}^2 \to \mathbb{N}$ such that

$$B_{\omega^c_{d-1}}(r(c,x)) \geq F_{\omega^c_{d-1}}(x).$$

On the other hand by Theorem 3.28, we have that for all x

$$R^{\mathrm{reg}}_{\hat{g}_{c,d}}(d+1, h(x) + K(c,d)) > B_{\omega^c_{d-1}}(x)$$

for some primitive recursive functions h and K. Hence

$$R^{\mathrm{reg}}_{\hat{g}_{c,d}}(d+1, h(r(c,x)) + K(c,d)) > B_{\omega^c_{d-1}}(r(c,x)) > F_{\omega^c_{d-1}}(x).$$

We claim that

$$R^{\mathrm{reg}}_f(d+1, h(r(x,x)) + K(x,d)) > F_{\omega_d}(x)$$

for all x.

Assume it is false for some x and let

$$N(x) := R^{\mathrm{reg}}_f(d+1, h(r(x,x)) + K(x,d)).$$

Then for all $i \leq N(x)$ we have $F^{-1}_{\omega_d}(i) \leq x$ and so

$$f(i) = \sqrt[F^{-1}_{\omega_d}(i)]{\log_{d-1}(i)} \geq \sqrt[x]{\log_{d-1}(i)} = \hat{g}_x(i).$$

This implies that

$$\begin{aligned} R^{\mathrm{reg}}_f(d+1, h(r(x,x)) + K(x,d)) &\geq R^{\mathrm{reg}}_{\hat{g}_x}(d+1, h(r(x,x)) + K(x,d)) \\ &> F_{\omega^x_{d-1}}(x) \\ &= F_{\omega_d}(x). \end{aligned}$$

Contradiction! □

4. Concluding remarks

As a corollary of our main results one gets the following dichotomy.

Corollary 4.1. *Let $d, \ell \geq 1$.*

(1) *For all $n < d$, $R^{\text{reg}}_{\lfloor \sqrt[\ell]{\log_n(\cdot)} \rfloor}(d+1, \cdot)$ is primitive recursive in F_{ω_d},*
(2) *For all $n \geq d$, $R^{\text{reg}}_{\lfloor \sqrt[\ell]{\log_d(\cdot)} \rfloor}(d+1, \cdot)$ eventually dominates F_{ω_d}.*

This also proves Lee's conjecture and closes the gap between $d-2$ and d left open in [11].

Our result can also be used to classify the threshold for the full Regressive Ramsey theorem $(\forall d)(\mathrm{KM})^d_f$ with respect to F_{ε_0}.

Theorem 4.2. (1) *For all $\alpha < \varepsilon_0$, $R^{\text{reg}}_{|\cdot|_{F_\alpha^{-1}(\cdot)}}$ is primitive recursive in some F_β, with $\beta < \varepsilon_0$.*
(2) *$R^{\text{reg}}_{|\cdot|_{F_{\varepsilon_0}^{-1}(\cdot)}}$ eventually dominates F_{ε_0}.*

Proof. The upper bound is established in Theorem 2.9. Now let $f(x) = |x|_{F_{\varepsilon_0}^{-1}(x)}$. Note first that it follows from the proof of Theorem 3.29 that

$$R^{\text{reg}}_{|\cdot|_{d-1}}(d+1, s(c,d,x)) > F_{\omega^c_{d-1}}(x)$$

for some primitive recursive function s. This is because $\log_{d-1}$ and $|\cdot|_{d-1}$ have the same growth rate.

We claim that $R^{\text{reg}}_f(d+1, s(d-1,d,d-1)) > F_{\omega_d}(d-1)$ for all $d > 0$. Assume otherwise. Then there is a $d > 0$ such that

$$N(d) := R^{\text{reg}}_f(d+1, s(d-1,d,d-1)) \leq F_{\omega_d}(d-1) = F_{\omega^{d-1}_{d-1}}(d-1).$$

Then for all $i \leq N(d)$ we have $F^{-1}_{\omega_d}(i) \leq d-1$. This means

$$\begin{aligned} R^{\text{reg}}_f(d+1, s(d-1,d,d-1)) &\geq R^{\text{reg}}_{|\cdot|_{d-1}}(d+1, s(d-1,d,d-1)) \\ &> F_{\omega^{d-1}_{d-1}}(d-1). \end{aligned}$$

Contradiction! This implies the upper bound. □

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Università di Roma "La Sapienza", Department of Computer Science, Via Salaria 113, 00198 Roma, Italy
E-mail address: carlucci@di.uniroma1.it

National Institute of Advanced Industrial Science and Technology, Research Center for Information Security, 1-18-13 Sotokanda, Chiyoda-ku, Tokyo 101-0021, Japan
E-mail address: gyesik.lee@aist.go.jp

University of Ghent, Department of Pure Mathematics and Computer Algebra, Krijgslaan 281, B9000 Gent, Belgium
E-mail address: weierman@cage.ugent.be