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Shult Sets and Translation Ovoids of the Hermitian Surface

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Abstract

Starting with carefully chosen sets of points in the Desarguesian affine plane $AG(2, q^2)$ and using an idea first formulated by E. Shult, several infinite families of translation ovoids of the Hermitian surface are constructed. Various connections with locally Hermitian 1–spreads of $\mathcal{Q}^-(5, q)$ and semifield spreads of $PG(3, q)$ are also discussed.

1 Introduction

Let $\mathcal{Q}^-(5, q)$ be the elliptic quadric of $PG(5, q)$. A *spread* $\mathcal{S}$ of $\mathcal{Q}^-(5, q)$ is a partition of the point-set of $\mathcal{Q}^-(5, q)$ into lines. Let L be a fixed line of such a spread $\mathcal{S}$. For every other line M of $\mathcal{S}$, the subspace $\langle L, M\rangle$ has dimension three and intersects $\mathcal{Q}^-(5, q)$ in a non–singular hyperbolic quadric. Let $\mathcal{R}_{L,M}$ be the regulus of $\langle L, M\rangle \cap \mathcal{Q}^-(5, q)$ containing the lines L and M. The spread $\mathcal{S}$ is said to be *locally Hermitian* with respect to L if for any line M of $\mathcal{S}$, different from L, the regulus $\mathcal{R}_{L,M}$ is contained in $\mathcal{S}$. This forces the spread $\mathcal{S}$ to be a union of q^2 reguli pairwise meeting in the line L. The spread $\mathcal{S}$ is called *Hermitian* if it is locally Hermitian with respect to all lines of $\mathcal{S}$.

Let $\perp$ be the orthogonal polarity of $PG(5, q)$ defined by the quadratic form associated with $\mathcal{Q}^-(5, q)$. For each spread $\mathcal{S}$ of $\mathcal{Q}^-(5, q)$ which is locally Hermitian with respect to one of its lines L, one can associate a spread of the 3–dimensional projective space $\Lambda := L^\perp \cong PG(3, q)$ in the following way. Let M be any line of $\mathcal{S}$ different from L. Then, the line $m_{L,M} := \langle L, M\rangle^\perp$ of Λ is skew to $\langle L, M\rangle$ and hence skew to L. Moreover, the set of lines $S_\Lambda := \{m_{L,M} | M \in \mathcal{S}, M \neq L\} \cup \{L\}$ turns out to be a spread of Λ. It should be noted that in this process the lines of the regulus $\mathcal{R}_{L,M}$ other than L collapse to a single line.

In $PG(3, q^2)$ a *Hermitian surface* is defined to be the set of all isotropic points of a non–degenerate unitary polarity, and it will be denoted by $\mathcal{H} = \mathcal{H}(3, q^2)$. The number of points on the surface is $(q^3 + 1)(q^2 + 1)$. Any line of $PG(3, q^2)$ meets $\mathcal{H}$ in 1, $q + 1$ or $q^2 + 1$ points. The latter lines are the *generators* of $\mathcal{H}$, and they are $(q + 1)(q^3 + 1)$ in number. The intersections of size $q + 1$ are Baer sublines and are often called *chords*, whereas the lines meeting $\mathcal{H}$ in a Baer subline are called *hyperbolic lines*. Lines meeting $\mathcal{H}$ in one point are called *tangent lines*. Through each point P of $\mathcal{H}$ there pass exactly $q + 1$ generators, and these generators are coplanar. The plane containing these generators, say π_P, is the polar plane of P with respect to the unitary polarity defining $\mathcal{H}$. The tangent lines through P are precisely the remaining $q^2 - q$ lines of π_P incident with P, and π_P is also called the *tangent plane* of $\mathcal{H}$ at P. Every plane of $PG(3, q^2)$ which is not a tangent plane of $\mathcal{H}$ meets $\mathcal{H}$ in a *Hermitian curve* $\mathcal{H}(2, q^2)$, and is called a *secant plane* of $\mathcal{H}$.

An *ovoid* $\mathcal{O}$ of $\mathcal{H}$ is a set of $q^3 + 1$ points which has exactly one common point with every generator of $\mathcal{H}$. The intersection of the Hermitian surface $\mathcal{H}$ with any of its secant planes, namely a Hermitian curve $\mathcal{H}(2, q^2)$, is easily seen to be an ovoid (called the *classical ovoid*). Many nonclassical ovoids of $\mathcal{H}$ are now known to exist. An ovoid $\mathcal{O}$ of $\mathcal{H}$ is called a *translation ovoid* with respect to one of its points P if there is a collineation group of $\mathcal{H}$ fixing the point P, leaving invariant all the generators through P, and acting regularly on the points of $\mathcal{O} \setminus \{P\}$. The first two examples of translation ovoids appearing in print were the classical ovoid and the one obtained from a Baer elliptic quadric $\mathcal{Q}^-(3, q)$ embedded in $\mathcal{H}$ which is permutable with $\mathcal{H}$ (see [6]). In the latter example, the q^2 secant lines to the Baer elliptic quadric through any one of its points produce q^2 hyperbolic lines whose union meets the Hermitian surface in a translation ovoid. The importance of translation ovoids arises from the connection with locally Hermitian spreads of $\mathcal{Q}^-(5, q)$, as described above, and the resulting spreads of $PG(3, q)$. It is now known that there is an intimate connection between translation ovoids of $\mathcal{H}$ and semifields of dimension two over their left nucleus (see Theorem 3.3 of [2]).

To develop these connections, consider the Klein correspondence κ between the lines of $PG(3, q^2)$ and the points of $\mathcal{Q}^+(5, q^2)$ via Plücker coordinates. The generators of a Hermitian surface $\mathcal{H}$ correspond to the points of a Baer elliptic quadric $\mathcal{Q}^-(5, q)$ (see [4]) under κ. Moreover, an ovoid $\mathcal{O}$ of

$\mathcal{H}$ corresponds to a spread $\mathcal{S}$ of $\mathcal{Q}^-(5,q)$. More precisely, each of the q^3+1 lines of $\mathcal{S}$ consists of the $q+1$ points corresponding under κ to the $q+1$ generators of $\mathcal{H}$ through some point of $\mathcal{O}$. In this way a chord contained in the ovoid $\mathcal{O}$ will correspond to a regulus contained in the spread $\mathcal{S}$. Thus the spread $\mathcal{S}$ is locally Hermitian with respect to some line L if and only if the corresponding ovoid $\mathcal{O}$ is comprised of q^2 Baer sublines (lying on q^2 distinct hyperbolic lines) sharing a point P, where P corresponds to L as described above.

Moreover, if such a locally Hermitian spread $\mathcal{S}$ admits a collineation group fixing the line L pointwise and acting regularly on the remaining lines of $\mathcal{S}$, then the corresponding ovoid $\mathcal{O}$ is a translation ovoid, and vice–versa. Looking at the spread S_Λ of $L^\perp \cong PG(3,q)$ associated with the locally Hermitian spread $\mathcal{S}$, the induced group action shows that this spread will be a semifield spread when the ovoid $\mathcal{O}$ is a translation ovoid.

It should also be noted that the spread $\mathcal{S}$ of $\mathcal{Q}^-(5,q)$ is Hermitian if and only if the ovoid $\mathcal{O}$ of $\mathcal{H}$ is classical.

2 Shult sets

In [8] Shult describes several interesting geometric constructions, and poses a number of open problems. At the very beginning of that paper Shult gives the following construction.

Let $A = AG(2,q^2)$ be a Desarguesian affine plane, and let l_∞ be the line at infinity. Thus $l_\infty \cong PG(1,q^2)$. Let π denote the standard completion of A to a projective plane on the point-set $A \cup l_\infty$. Obviously the points of A correspond to the affine points of π. We will call a subset F of the point-set of A a *Shult set* (after [8]) if it satisfies the following conditions:

(i) $|F| = q^2$,

(ii) there exists a Baer subline H of l_∞ such that any secant line of F (that is, any line of π which intersects F in at least two points) meets l_∞ in a point that is not in H.

There are two "natural" examples of Shult sets. The first one is obtained by choosing F to be any affine line of A, and then choosing H to be any Baer subline of l_∞ not containing the point at infinity for this affine line. This is the so-called *classical case*. The second example is the following.

Select a Baer subplane π_0 of π that intersects l_∞ at a Baer subline $l_{\infty,0}$, and then choose H to be any Baer subline on l_∞ disjoint from $l_{\infty,0}$. Thus $A_0 := \pi_0 \setminus l_{\infty,0}$ is an affine Baer subplane of A whose affine lines have their points at infinity on $l_{\infty,0}$. Hence the point-set of A_0 is also a Shult set, the so-called *semiclassical case*

Suppose that A, F and H are given as above in the definition of a Shult set. Let π^* be the dual plane of π, and let P denote the projective point of π^* corresponding to the line l_∞ under duality. Then the duality takes the Baer subline H of π to a cone H^* of π^* with vertex P, whose ruling lines correspond to the points of H. That is, H^* is a Baer subpencil. Let $\mathcal{F}$ be the collection of lines of π^* corresponding to the point-set F of π under duality. Then,

$(P1)$ $\pi^* \cong PG(2,q^2)$ is a projective plane with a degenerate Hermitian variety H^*, which is a Baer subpencil with vertex P.

$(P2)$ $\mathcal{F}$ is a collection of q^2 lines of π^*, none passing through P.

$(P3)$ Any two distinct lines of $\mathcal{F}$ intersect at a point of $\pi^* \setminus H^*$.

Now embed the plane π^* in a 3–dimensional projective space $PG(3,q^2)$ containing a Hermitian surface $\mathcal{H}$ in such a way that $\mathcal{H} \cap \pi^* = H^*$. Denote by ρ the unitary polarity associated with $\mathcal{H}$, and let

$$O(F) := O(\mathcal{F}) := \bigcup \{L^\rho |\, L \in \mathcal{F}\}.$$

Since any line of $\mathcal{F}$ is a hyperbolic line of $\mathcal{H}$, lying on $\pi^* = P^\rho = \pi_P$, we see that $O(F)$ is the union of q^2 hyperbolic lines pairwise meeting at the point P of $\mathcal{H}$. Moreover, from property $(P3)$ it follows that the plane spanned by any pair of lines of $O(F)$ is not a tangent plane of $\mathcal{H}$. Hence, the set $O(F)$ is an ovoid of $\mathcal{H}$. In fact, as discussed above, this ovoid corresponds under κ to a spread $\mathcal{S}$ of $\mathcal{Q}^-(5,q)$ which is locally Hermitian with respect to the line L corresponding to the point P. In particular, in the classical case all of the q^2 lines of $\mathcal{F}$ pass through a non–isotropic point Q in the plane P^ρ, and $O(F)$ is the Hermitian curve (classical ovoid) defined by the non–tangent plane $Q^\perp$.

Conversely, given an ovoid $\mathcal{O}$ of a Hermitian surface $\mathcal{H}$ consisting of the union of q^2 hyperbolic lines of $\mathcal{H}$ through an isotropic point P, the corresponding polar lines lie on the tangent plane π_P to $\mathcal{H}$ at P. Let $\mathcal{F}$ denote the set consisting of such polar lines. Note that any two of these lines intersect at a non–isotropic point of π_P. Thus, in the dual plane of π_P such a set $\mathcal{F}$ corresponds to a set of q^2 points disjoint from the dual line of the point P, and such that any secant line to this set is skew to a fixed Bear subline (which is the dual of the $q+1$ isotropic lines through P). Thus we have shown that Shult sets are in one–to–one correspondence with ovoids of $\mathcal{H}$ consisting of q^2 Baer sublines (on q^2 distinct hyperbolic lines) pairwise meeting at a point of $\mathcal{H}$.

However, an ovoid arising from a Shult set will not necessarily admit an appropriate collineation group to make it a translation ovoid. For instance, for odd q, one can take as your set F in $AG(2,q^2)$ the origin, the points on the x–axis corresponding to the nonzero squares in $GF(q^2)$, and the points on the y–axis corresponding to the nonsquares in $GF(q^2)$. The slopes for this set of q^2 affine points will consist of $\{0,\infty\}$ and the nonsquares in $GF(q^2)$. The Baer subline on the line at infinity corresponding to those elements z in $GF(q^2)$ such that $z^{q+1} = 1$ will contain none of these slope points as all such elements z will be nonzero squares in $GF(q^2)$. Hence F is a Shult set, although the associated ovoid will not have an appropriate translation group of order q^3, as can be seen for small values of q by direct Magma [5] computations. In order to obtain a translation ovoid, we need to start with Shult set F admitting a (translation) group of the plane acting regularly on it and fixing pointwise the Baer subline H. Also it should be noted that inequivalent ovoids of $\mathcal{H}$ may arise from the same Shult set by choosing different Baer sublines H, as we shall soon see.

We close this section with some observations about Baer subpencils.

Proposition 2.1. *Using the above notation, consider an ovoid $\mathcal{O}$ arising from a Shult set. Suppose there are q chords of $\mathcal{O}$ through P that lie in some plane, say σ, and suppose further that the q hyperbolic lines containing these chords together with $t = \sigma \cap \pi_P$ form a Baer subpencil. Then the resulting spread S_Λ of $L^\perp \cong PG(3,q)$ contains a regulus through the line L.*

Proof. Follows from properties of the Klein correspondence. □

Theorem 2.2. *Let S_Λ be the spread of $L^\perp \cong PG(3,q)$ arising from a Shult set F as described above. Then S_Λ is a regular spread if and only if F is classical or semiclassical.*

Proof. Follows from the above proposition and Theorem 3.1 in [7]. □

3 The Semiclassical Case

We let K denote the finite field $GF(q^2)$, and let K_q be its subfield of order q. We begin by looking at semiclassical Shult sets when q is odd. Using the notation of the previous section, let $A = AG(2, q^2)$ and let l_∞ be its line at infinity. Complete A to a projective plane $\pi = PG(2, q^2)$ with homogeneous coordinates (x_1, x_2, x_3) in such a way that l_∞ has equation $x_1 = 0$. Let $F \cong AG(2, q)$ be an affine Baer subplane of A. Initially, we will assume that F is embedded in canonical position; that is, using left–normalized coordinates for uniqueness, let $F = \{(1, a, b) : a, b \in K_q\}$. Hence the slope points (points at infinity) for F are $l_{\infty,0} = \{(0, 0, 1)\} \cup \{(0, 1, b) : b \in K_q\}$. To construct an ovoid of $\mathcal{H}$ using the method described above, we must choose a Baer subline H of l_∞ that is disjoint from $l_{\infty,0}$. There are many ways to do this, and we first choose H so that it is orthogonal to $l_{\infty,0}$. In particular, we choose $H = \{(0, 1, z) : z \in K \mid z^{q+1} = w\}$, where w is a primitive element of K_q. As w is a nonsquare of K_q, we see that $H \cap l_{\infty,0} = \emptyset$. Moreover, the Baer involutions of these two sublines commute, and hence the Baer sublines are orthogonal in the sense of orthogonal circles in the associated Miquelian inversive plane. Note that, up to equivalence, there is only one way to choose such an H since the Baer sublines disjoint from and orthogonal to $l_{\infty,0}$ form a single orbit under its stabilizer in $PGL(2, q^2)$.

Taking the dual projective plane π^* and embedding this plane in $PG(3, q^2)$ so that π^* has equation $x_0 = 0$, we may take as our Hermitian surface $\mathcal{H}$ the one with equation $x_0 x_1^q + x_0^q x_1 + x_2^{q+1} - w x_3^{q+1} = 0$. In this way $\mathcal{H}$ meets π^* in the degenerate Hermitian variety (Baer subpencil) which is the dual of the Baer subline H. In particular, $\pi^* = P^\rho$ where $P = (0, 1, 0, 0)$ and ρ is the unitary polarity associated with $\mathcal{H}$. Applying ρ to the lines of F^*, we obtain q^2 hyperbolic lines passing through P. Intersecting these lines with $\mathcal{H}$, we obtain q^2 chords whose union is an ovoid $\mathcal{O}_1$ of $\mathcal{H}$ as described in the previous section. Straightforward computations show that

$$\mathcal{O}_1 = \{(0, 1, 0, 0)\} \cup \{(1, \alpha, a, -w^{-1}b) : a, b \in K_q,\ \alpha \in K \mid \alpha + \alpha^q = w^{-1}b^2 - a^2\}.$$

To deduce that $\mathcal{O}_1$ is indeed a translation ovoid, we must compute the stabilizer of $\mathcal{O}_1$ in $Stab(\mathcal{H}) \cong PGU(4, q^2)$.

Theorem 3.1. *Let $\mathcal{O}_1$ be the ovoid of $\mathcal{H}$ constructed above, and let*

$$G_1 = \left\{ \begin{bmatrix} 1 & x & u & t \\ 0 & 1 & 0 & 0 \\ 0 & -u & 1 & 0 \\ 0 & wt & 0 & 1 \end{bmatrix} : u, t \in K_q,\ x \in K \mid x + x^q = wt^2 - u^2 \right\}.$$

Then G_1 is an elementary abelian p–group of order q^3, where q is a power of the odd prime p, such that G_1 leaves invariant the Hermitian surface $\mathcal{H}$, the ovoid $\mathcal{O}_1$, all generators of $\mathcal{H}$ through P, and acts regularly on the points of $\mathcal{O}_1 \setminus \{P\}$. In particular, $\mathcal{O}_1$ is a translation ovoid.

Proof. Straightforward computations. □

Magma [5] computations (for small q) indicate that the full linear stabilizer of $\mathcal{O}_1$ in $PGU(4, q^2)$ has order $2q^3(q^2 - 1)$. In fact, $\mathcal{O}_1$ is projectively equivalent to the translation ovoid arising from a permutable Baer elliptic quadric embedded in $\mathcal{H}$, as described in Section 1.

Theorem 3.2. *The translation ovoid $\mathcal{O}_1$ of Theorem 3.1 arises from a permutable Baer elliptic quadric embedded in $\mathcal{H}$ and containing P by taking the union of all the chords joining P to another point of this quadric.*

Proof. For each value of a and b in the equation of $\mathcal{O}_1$, choose α so that it is also in the subfield K_q of K. The resulting subset of $\mathcal{O}_1$, namely

$$\mathcal{E}_1 = \{(0, 1, 0, 0)\} \cup \{(1, \frac{1}{2}(w^{-1}b^2 - a^2), a, -w^{-1}b) : a, b \in K_q\}$$

is an elliptic quadric $\mathcal{Q}^-(3, q)$ of the Baer subspace in canonical position whose equation is $2x_0x_1 + x_2^2 - wx_3^2 = 0$, where w is a nonsquare in K_q. The symmetric matrix representing this quadric is exactly the same as the Hermitian matrix representing $\mathcal{H}$, and hence the two polarities clearly commute. Moreover, each chord of $\mathcal{O}_1$ is a secant line to this elliptic quadric through P, and thus $\mathcal{E}_1$ can serve as a "base" for the translation ovoid $\mathcal{O}_1$. □

We next choose a Baer subline H disjoint from $l_{\infty,0}$ which is not orthogonal to $l_{\infty,0}$. Again, this can be done in only one way, up to equivalence. To make the coordinates as nice as possible, we actually keep the same Baer subline H, and hence extend to the same Hermitian surface $\mathcal{H}$, but take a different embedding for our affine Baer subplane. That is, we take $F = \{(1, a+b\epsilon, a-b\epsilon) : a, b \in K_q\}$, where ϵ is chosen in K such that $\epsilon^q = -\epsilon$ and $\epsilon^2 = w$, where again w is the primitive element of K_q used in the equation for H. This time $l_{\infty,0} = \{(0,1,z) : z \in K \mid z^{q+1} = 1\}$ is disjoint from H but not orthogonal to H, as the respective Baer involutions do not commute. The resulting ovoid is not equivalent to the ovoid of Theorem 3.1.

Theorem 3.3. *Let $\mathcal{O}_2$ be the ovoid arising from the semiclassical Shult set for odd $q = p^e$ when the Baer sublines H and $l_{\infty,0}$ are not orthogonal. Then $\mathcal{O}_2$ is a translation ovoid whose translation group G_2 of order q^3 is nonabelian of exponent p. Moreover, this ovoid can be constructed with a non–permutable Baer elliptic quadric as its base.*

Proof. Using H and F as above, straightforward computations show that

$$\mathcal{O}_2 = \{0,1,0,0)\} \cup \{(1, \alpha, a+b\epsilon, -w^{-1}a + w^{-1}b\epsilon) : a, b \in K_q,\ \alpha \in K \mid \alpha + \alpha^q = (w^{-1} - 1)(a^2 - wb^2)\}.$$

The Hermitian surface $\mathcal{H}$ is the same as in Theorem 3.1. More straightforward computations show that

$$G_2 = \left\{ \begin{bmatrix} 1 & x & -y & w^{-1}y^q \\ 0 & 1 & 0 & 0 \\ 0 & y^q & 1 & 0 \\ 0 & y & 0 & 1 \end{bmatrix} : x, y \in K \mid x + x^q = (w^{-1} - 1)y^{q+1} \right\}$$

is a nonabelian group of order q^3 and exponent p stabilizing the Hermitian surface $\mathcal{H}$, the ovoid $\mathcal{O}_2$, and all the generators of $\mathcal{H}$ through the point P, while acting regularly on the points of $\mathcal{O}_2 \setminus \{P\}$. Thus $\mathcal{O}_2$ is a translation ovoid. It is not equivalent to the ovoid $\mathcal{O}_1$ of $\mathcal{H}$ as its translation group G_2 is nonabelian.

One can again obtain a nice "base" for this ovoid by always choosing the second coordinate to be in the subfield K_q. To see that this base is a Baer elliptic quadric $\mathcal{Q}^-(3,q)$, we need to define the appropriate Baer subspace of $PG(3,q^2)$ containing this set of points. To do so, we take all K_q–linear combinations of the basis vectors $\{(1,0,0,0), (0,1,0,0), (0,0,1,-w^{-1}), (0,0,\epsilon, w^{-1}\epsilon)\}$. Using left–normalized K_q–coordinates with respect to this basis, we see that our "ovoidal base" can be expressed as

$$\mathcal{E}_2 = \{(0,1,0,0)\} \cup \{(1, \frac{1}{2}(w^{-1} - 1)(a^2 - wb^2), a, b) : a, b \in K_q\}.$$

To see that this is an elliptic quadric in our Baer subspace, simply observe that it satisfies the equation $2x_0x_1 + (1 - w^{-1})x_2^2 + (1 - w)x_3^2 = 0$ and note that w is a nonsquare in K_q. Thus $\mathcal{O}_2$ also can be obtained by taking the points of $\mathcal{H}$ lying on the secants to $\mathcal{E}_2$, which are necessarily hyperbolic lines, through the point P. However, this time the Baer elliptic quadric $\mathcal{E}_2$ is not permutable with the Hermitian surface $\mathcal{H}$, as direct computation shows the two polarities do not commute. □

Magma [5] computations indicate (for small q) that the full stabilizer in $PGU(4, q^2)$ has order $q^3(q^2 - 1)$, one half as large as for $\mathcal{O}_1$. We also mention that there are no subgroups of order q^2 in the stabilizer that act regularly on the chords of $\mathcal{O}_2$ through P. This is significant difference with the permutable case.

Thus for q odd two inequivalent translation ovoids are obtained from the semiclassical Shult set, depending upon whether the Baer subline H is orthogonal or not to the Baer subline $l_{\infty,0}$. In both cases the translation ovoid has a base which is Baer elliptic quadric embedded in the Hermitian surface, but this elliptic quadric is permutable with $\mathcal{H}$ if and only if the two Baer sublines on l_∞ are orthogonal. This answers one of questions raised in [8]. We should also mention that the permutable Baer elliptic quadrics $\mathcal{Q}^-(3, q)$ embedded in $\mathcal{H}$ are the only known examples of *special sets* on $\mathcal{H}$, as defined in [8]. These are point sets of size $q^2 + 1$ with the property that any three distinct points of the set span a secant plane of $\mathcal{H}$. The non–permutable Baer elliptic quadrics embedded in $\mathcal{H}$, such as the base $\mathcal{E}_2$ for our translation ovoid $\mathcal{O}_2$, are not special sets as some triples of points on $\mathcal{E}_2$ span tangent planes to $\mathcal{H}$.

Next we look at semiclassical Shult sets for even q. Again we pick a convenient basis for the vector space K over its subfield K_q. Namely, pick some element $w \neq 1$ in K_q whose absolute trace is 1, and thus the polynomial $x^2 + x + w$ is irreducible over K_q. Letting β and β^q denote the two roots of this polynomial in K, we take $\{1, \beta\}$ as our basis for K over K_q. Note that $\beta + \beta^q = 1$ and $\beta^{q+1} = w$.

As in the last construction, we choose a non–canonical embedding for our affine Baer subplane F in $A = AG(2, q^2)$. Namely, we let $F = \{(1, a + b\beta, a + b\beta^q) : a, b \in K_q\}$, so that $l_{\infty,0} = \{(0, 1, z) : z \in K \mid z^{q+1} = 1\}$. We choose $H = \{(0, 1, z) : z \in K \mid z^{q+1} = w\}$ as our Baer subline on l_∞ disjoint from $l_{\infty,0}$. For even q, orthogonal Baer sublines of l_∞ must share exactly one point, and hence we cannot choose H to be orthogonal to l_∞. In fact, there is only one choice for our subline H, up to projective equivalence. We dualize and embed in $PG(3, q^2)$ as usual, so that the resulting Hermitian surface has equation $x_0x_1^q + x_0^qx_1 + x_2^{q+1} + wx_3^{q+1} = 0$. Direct computations show that the resulting ovoid of $\mathcal{H}$ is

$$\mathcal{O}_3 = \{(0,1,0,0)\} \cup \{(1, \alpha, a + b\beta, w^{-1}(a + b\beta^q)) : a, b \in K_q,\ \alpha \in K \mid \alpha + \alpha^q = (w^{-1} + 1)(a^2 + ab + wb^2)\}.$$

There is no convenient subset of this ovoid that can serve as a "base", as opposed to the situation for odd q. In particular, there is no Baer elliptic quadric $\mathcal{Q}^-(3, q)$ contained in $\mathcal{O}_3$, as any such quadric embedded in $\mathcal{H}$ must be a complete partial ovoid for even q (see [1], for instance). However, we can find a Baer elliptic quadric that will determine $\mathcal{O}_3$ in an analogous fashion to the previous two cases. This time the Baer elliptic quadric will have only two points on the Hermitian surface. Namely, let

$$\mathcal{E}_3 = \{(0,1,0,0)\} \cup \{(1, a^2 + ab + wb^2, a + b\beta, w^{-1}(a + b\beta^q)) : a, b \in K_q\}.$$

Then $\mathcal{E}_3$ is the Baer elliptic quadric obtained by intersecting the hyperbolic quadric of $PG(3, q^2)$ whose equation is $w^{-1}x_0x_1 + x_2x_3 = 0$ with the Baer subspace whose basis vectors are $\{(1,0,0,0), (0,1,0,0), (0,0,1,w^{-1}), (0,0,\beta, w^{-1}\beta^q)\}$. The q^2 secant lines to $\mathcal{E}_3$ through the point P meet the Hermitian surface $\mathcal{H}$ in chords whose union is the ovoid $\mathcal{O}_3$. Once again, this ovoid is indeed a translation ovoid.

Theorem 3.4. *The ovoid $\mathcal{O}_3$ is a translation ovoid of $\mathcal{H}$ whose translation group G_3 of order q^3 is nonabelian and of exponent 4.*

Proof. Direct computations show that

$$G_3 = \left\{ \begin{bmatrix} 1 & x & y & w^{-1}y^q \\ 0 & 1 & 0 & 0 \\ 0 & y^q & 1 & 0 \\ 0 & y & 0 & 1 \end{bmatrix} : x, y \in K \mid x + x^q = (w^{-1} + 1)y^{q+1} \right\}$$

is a nonabelian group of order q^3 and exponent 4 that stabilizes $\mathcal{H}$ and has the appropriate action on $\mathcal{O}_3$. □

Magma [5] computations indicate (for small q) that the full stabilizer of $\mathcal{O}_3$ in $PGU(4, q^2)$ has order $q^3(q^2 - 1)$, the same as for $\mathcal{O}_2$. Again there is no subgroup of order q^2 acting regularly on the chords through P.

Thus we obtain three inequivalent ovoids from the semiclassical Shult set. All of these are translation ovoids, but only one arises from a special set. In particular, we have shown the following characterization.

Theorem 3.5. *Let $\mathcal{O}$ be a translation ovoid of $\mathcal{H}$ and let S_Λ be the associated spread of $PG(3,q)$. Then S_Λ is regular if and only if $\mathcal{O}$ is classical or $\mathcal{O}$ arises from a Baer elliptic quadric $\mathcal{E}$ by taking all points on $\mathcal{H}$ lying on the secants to $\mathcal{E}$ through one of its points.*

We emphasize that the Baer elliptic quadric in the above theorem may or may not be embedded in $\mathcal{H}$, and if so, may or may not be permutable with $\mathcal{H}$. As a final comment in this section, we mention that in the classical case the stabilizer of a point in $PGL(2,q^2)$ acts transitively on the Baer sublines of l_∞ disjoint from that point, and hence there is only one choice, up to equivalence, for the Baer subline H. As previously mentioned, the resulting ovoid is a secant plane section of $\mathcal{H}$, namely $\mathcal{H}(2,q^2)$.

4 Some New Translation Ovoids

In this section we construct two new families of translation ovoids on $\mathcal{H}$. We will assume throughout this section that q is an odd square prime power. We write $q = p^{2e}$ for some odd prime p and some integer $e \geq 1$. As before, we let K denote the finite field $GF(q^2)$, and we let K_q be its subfield of order q. Similarly, we let $K_{\sqrt{q}}$ denote its subfield of order p^e, and we let Tr denote the trace from K to $K_{\sqrt{q}}$. That is, $\mathrm{Tr}(x) = x + x^{\sqrt{q}} + x^q + x^{q\sqrt{q}}$ for any $x \in K$.

Consider the set $F_T = \{(1, a, \mathrm{Tr}(a)) : a \in K\}$ of q^2 affine points in $\pi = PG(2,q^2)$, where we assume that l_∞ has equation $x_1 = 0$ as before. The secant lines to F_T meet l_∞ in the set $U_T = \{(0, 1, \mathrm{Tr}(b)/b) : b \in K \mid b \neq 0\}$ of slope points. We first need to find a Baer subline on l_∞ that is disjoint from U_T.

Proposition 4.1. *Let U_T be the above set of slope points for the set F_T. Then the Baer subline $H = \{(0,0,1)\} \cup \{(0,1,z) : z \in K \mid z + z^q = 4\}$ of l_∞ is disjoint from U_T.*

Proof. Suppose that $(0,1,\mathrm{Tr}(b)/b)$ is a point of H for some nonzero $b \in K$. Then $4b^{q+1} = \mathrm{Tr}(b)(b+b^q) = (b+b^q)[(b+b^q)+(b+b^q)^{\sqrt{q}}] = (b+b^q)^2 + (b+b^q)^{\sqrt{q}+1}$, and hence $(b+b^q)^2 - 4b^{b+1} = -(b+b^q)^{\sqrt{q}+1} \in K_{\sqrt{q}}$. In fact, $b + b^q \neq 0$ as $4 \neq 0$. Thus $(b+b^q)^2 - 4b^{q+1}$ is a nonzero element of $K_{\sqrt{q}}$, and hence a nonzero square in K_q. As in the previous section, take $\{1, \epsilon\}$ as a basis for K over K_q, where $\epsilon + \epsilon^q = 0$ and $\epsilon^2 = w$ is a primitive element of K_q. Expressing $b = r + s\epsilon$, where $r, s \in K_q$, the above computations show that $(b + b^q)^2 - 4b^{q+1} = 4r^2 - 4(r^2 - ws^2) = 4ws^2$ is a nonzero square in K_q, contradicting the fact that w must be a nonsquare in K_q. Thus $H \cap U_T = \emptyset$. □

Thus F_T is a Shult set, and we may apply the procedure described in Section 2. Namely, we dualize and embed in $PG(3, q^2)$, as usual, so that the resulting Hermitian surface $\mathcal{H}$ has equation $x_0x_1^q + x_0^qx_1 + x_2x_3^q + x_2^qx_3 + 4x_3^{q+1} = 0$. Note that the Hermitian surface must agree with the Baer subpencil H^* on π_P. Straightforward computations then show that the resulting ovoid $\mathcal{O}_T$ of $\mathcal{H}$ is

$$\mathcal{O}_T = \{(0,1,0,0)\} \cup \{(1, \alpha, \mathrm{Tr}(b) - 4b, b) : b, \alpha \in K \mid \alpha + \alpha^q = 4b^{q+1} - \mathrm{Tr}(b)(b + b^q)\}.$$

The important question is whether or not there is an appropriate collineation group acting to make this a translation ovoid.

Theorem 4.2. *The ovoid $\mathcal{O}_T$ is a translation ovoid of $\mathcal{H}$ whose translation group G_T of order q^3 is nonabelian of exponent p.*

Proof. Direct computations show that

$$G_T = \left\{ \begin{bmatrix} 1 & x & \mathrm{Tr}(y) - 4y & y \\ 0 & 1 & 0 & 0 \\ 0 & -y^q & 1 & 0 \\ 0 & -\mathrm{Tr}(y) & 0 & 1 \end{bmatrix} : x, y \in K \mid x + x^q = 4y^{q+1} - \mathrm{Tr}(y)(y + y^q) \right\}$$

is a nonabelian group of order q^3 and exponent p that stabilizes $\mathcal{H}$ and has the appropriate action on $\mathcal{O}_T$. □

When $q = 9$, the full stabilizer of $\mathcal{O}_T$ in $PGU(4, q^2)$ has order $2^2 3^6$. There is no subgroup of order q^2 in the stabilizer that acts transitively on the chords through P, and there does not seem to be any particularly nice description for a "base" of this ovoid.

In order to say something about the resulting semifield spread of $L^\perp \cong PG(3, q)$, we need to know something about the line intersections of the Shult set F_T.

Proposition 4.3. *The q^2 points of F_T can be partitioned equally on $\sqrt{q}$ lines of slope* 0 *in the affine plane $AG(2,q^2)$. For each such line the $q\sqrt{q}$ points of F_T are partitioned into $\sqrt{q}$ affine Baer sublines.*

Proof. The horizontal lines of $AG(2,q^2)$ to be used are those indexed by elements of $K_{\sqrt{q}}$. More precisely, for any fixed $t \in K_{\sqrt{q}}$, let m_t be the horizontal line consisting of those points with second coordinate equal to t. Then the (affinely represented) points of F_T lying on m_t are precisely $Y_t = \{(a,t) : a \in K \mid \mathrm{Tr}(a) = t\}$. Using the fact that $\mathrm{Tr}(a) = (a + a^q) + (a + a^q)^{\sqrt{q}}$, we let $u_1, u_2, \ldots, u_{\sqrt{q}}$ denote the elements of K_q whose trace from K_q to $K_{\sqrt{q}}$ is t. Then the $q\sqrt{q}$ collinear points of Y_t are partitioned into the $\sqrt{q}$ affine Baer sublines $B_i = \{(a, u_i) : a \in K \mid a + a^q = u_i\}$, for $i = 1, 2, \ldots, \sqrt{q}$. Allowing t to vary over all elements of $K_{\sqrt{q}}$, we thus partition all points of F_T. □

Theorem 4.4. *The semifield spread S_T of $L^\perp \cong PG(3,q)$ arising from the translation ovoid $\mathcal{O}_T$ is a semifield flock spread.*

Proof. From Proposition 4.3 we know that the q^2 chords of $\mathcal{O}_T$ lie on $\sqrt{q}$ planes through P, each such plane containing $q\sqrt{q}$ of these chords. Moreover, in any such plane σ the $q\sqrt{q}$ hyperbolic lines containing these chords of $\mathcal{O}_T$ are partitioned into $\sqrt{q}$ affine Baer subpencils, each one being completed to a Baer subpencil by adjoining the tangent line $t = \pi_P \cap \sigma$ of $\mathcal{H}$. Allowing σ to vary, we get q Baer subpencils whose q^2 hyperbolic lines contain the q^2 chords of $\mathcal{O}_T$ through P, one per line. Hence Proposition 2.1 implies that the spread S_T is a union of q reguli pairwise meeting in the line L, and therefore S_T arises from the flock of a quadratic cone by Theorem 3.1 in [7]. □

From Theorem 2.2 we know that S_T is not a regular spread and hence contains only the q reguli described in the above proof (see [7]). When $q = 9$, MAGMA [5] computations show that this spread is the one associated with the Dickson/Kantor/Knuth semifield. The full stabilizer of S_T in $PGL(4,9)$ has order $2^5 3^4$.

For our next family of translation ovoids we let f be any divisor of e, where again $q = p^{2e}$ is an odd square prime power. Consider the set $F_B = \{(1, a, a^{p^f}) : a \in K\}$ of q^2 affine points in $\pi = PG(2,q^2)$. The secant lines to F_B meet l_∞ in the set $U_B = \{(0, 1, b^{p^f - 1}) : b \in K \mid b \neq 0\}$ of slope points. Again our first task if to find a Baer subline on l_∞ that is disjoint from U_B.

Proposition 4.5. *Let U_B be the above set of slope points for the set F_B. Then the Baer subline $H = \{(0,0,1)\} \cup \{(0,1,z) : z \in K \mid z + z^q = 0\}$ of l_∞ is disjoint from U_B.*

Proof. Suppose that $(0,1,b^{p^f-1})$ is a point of H for some nonzero $b \in K$. Then $b^{(p^f-1)(q-1)} = -1$ and hence the multiplicative order of b, say $o(b)$, divides $\gcd(q^2-1, 2(q-1)(p^f-1))$. Hence $o(b)|2(q-1)\gcd(\frac{1}{2}(q+1), p^f-1)$. Since $q = p^{2e}$, we know that $\frac{1}{2}(q+1)$ is odd. Moreover, since $f|e$, we have $\gcd(q+1, p^f-1) = 2$. Therefore $\gcd(\frac{1}{2}(q+1), p^f-1) = 1$ and $o(b)|2(q-1)$. This implies that $b^{(p^f-1)(q-1)} = 1$, contradicting the above condition on b. Hence $H \cap U_B = \emptyset$. □

Thus F_B is also a Shult set, and we may apply the usual procedure to obtain an ovoid. This time the resulting Hermitian surface $\mathcal{H}$ has equation $x_0x_1^q + x_0^qx_1 + x_2x_3^q + x_2^qx_3 = 0$. Straightforward computations then show that the resulting ovoid $\mathcal{O}_B$ of $\mathcal{H}$ is

$$\mathcal{O}_B = \{(0,1,0,0)\} \cup \{(1,\alpha,b^{p^f},b) : b,\alpha \in K \mid \alpha + \alpha^q = -(b^{q+p^f} + b^{qp^f+1})\}.$$

Once again this turns out to be a translation ovoid.

Theorem 4.6. *The ovoid $\mathcal{O}_B$ is a translation ovoid of $\mathcal{H}$ whose translation group G_B of order q^3 is nonabelian of exponent p.*

Proof. Direct computations show that

$$G_B = \left\{ \begin{bmatrix} 1 & x & y^{p^f} & y \\ 0 & 1 & 0 & 0 \\ 0 & -y^q & 1 & 0 \\ 0 & -y^{qp^f} & 0 & 1 \end{bmatrix} : x,y \in K \mid x + x^q = -(y^{q+p^f} + y^{qp^f+1}) \right\}$$

is a nonabelian group of order q^3 and exponent p that stabilizes $\mathcal{H}$ and has the appropriate action on $\mathcal{O}_B$. □

For the ovoid $\mathcal{O}_B$ it is particularly easy to describe other collineations in its stabilizer. Namely, the cyclic group of order $2(q-1)$ generated by the diagonal matrix $Diag(1, z^{q+p^f}, z^{p^f}, z)$ for some $z \in K$ of order $2(q-1)$ is easily seen to stabilize both $\mathcal{H}$ and $\mathcal{O}_B$. Moreover, this group normalizes G_B and meets G_B trivially. All these computations follow from the fact that $z^{qp^f+1} = z^{q+p^f}$. Hence we see that the stabilzer of $\mathcal{O}_B$ in $PGU(4,q^2)$

contains the semidirect product of G_B by a cyclic group of order $2(q-1)$. In practice, MAGMA [5] computations show that this is the full stabilizer for small values of q. Once again there does not appear to be any natural "base" for this translation ovoid.

Elementary computations show that every secant line of the Shult set F_B meets it in exactly p^f points, and hence $\mathcal{O}_B$ contains no Baer subpencils. Thus the resulting semifield spread S_B of $L^\perp \cong PG(3,q)$ contains no reguli through the line L. In practice, at least for small values of q, this spread is regulus–free. When $q=9$, Magma [5] computations show that this spread has a full stabilizer in $PGL(4,9)$ of order $2^7 3^4$. As the associated semifield of order 81 has all three nuclei equal to $GF(9)$, it is one of the Hughes–Kleinfeld semifields (see [3]).

5 Concluding Remarks

We conclude by making some general remarks about Shult sets in the plane $\pi = PG(2,q^2)$. By using the collineation group of π leaving invariant l_∞ and the elation group of π with axis l_∞, we may assume without loss of generality that the Baer subline H on l_∞ containing no slope point of our Shult set is in canonical position (so $H = \{(0,0,1)\} \cup \{(0,1,t) : t \in K_q\}$) and also that the point $(1,0,0)$ is in our Shult set. We will call such a Shult set *normalized*. Moreover, any such Shult set looks like $F = \{(1,a,f(a)) : a \in K\}$, where f is a permutation polynomial of K such that $\frac{f(a_1)-f(a_2)}{a_1-a_2} \notin K_q$ for any two distinct elements $a_1, a_2 \in K$. We will call a normalized Shult set a *translation* Shult set if it admits a translation group of the plane acting regularly on it. Thus for any translation Shult set the above permutation polynomial f must be additive and the associated translation group of the plane is $T = \left\{ \begin{bmatrix} 1 & a & f(a) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : a \in K \right\}$. Conversely, if f is an additive permutation polynomial of K such that $\frac{f(a)}{a} \notin K_q$ for all nonzero $a \in K$, then $F = \{(1,a,f(a)) : a \in K\}$ is a normalized translation Shult set with the above translation group T acting regularly on it.

Let F be a normalized translation Shult set as above. If q is even, then applying the procedure described in Section 2, we obtain the ovoid

$$\mathcal{O} = \{(1,b,f(a)^q,a^q) : a,b \in K \mid b + b^q = a^q f(a) + f(a)^q a\} \cup \{(0,1,0,0)\}$$

of the Hermitian surface $\mathcal{H}$ whose equation is $x_0x_1^q + x_0^qx_1 + x_2x_3^q + x_2^qx_3 = 0$. Straightforward computations show that this is a translation ovoid with associated translation group

$$G = \left\{ \begin{bmatrix} 1 & b & f(a)^q & a^q \\ 0 & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & f(a) & 0 & 1 \end{bmatrix} : a, b \in K \mid b + b^q = a^q f(a) + f(a)^q a \right\}.$$

If q is odd, choose $\epsilon \in K$ such that $\epsilon^q = -\epsilon$, and then the standard lifting procedure of Section 2 produces the ovoid

$$\mathcal{O} = \{(1, b, \epsilon^{-1}f(a)^q, -\epsilon^{-1}a^q) : a, b \in K \mid b+b^q = \epsilon^{-1}(a^q f(a) - f(a)^q a)\} \cup \{(0,1,0,0)\}$$

of the Hermitian surface whose equation is $x_0x_1^q + x_0^qx_1 + \epsilon x_2x_3^q - \epsilon x_2^qx_3 = 0$. This is a translation ovoid with associated translation group

$$G = \left\{ \begin{bmatrix} 1 & b & \epsilon^{-1}f(a)^q & -\epsilon^{-1}a^q \\ 0 & 1 & 0 & 0 \\ 0 & -a & 1 & 0 \\ 0 & -f(a) & 0 & 1 \end{bmatrix} : a, b \in K \mid b + b^q = \epsilon^{-1}(a^q f(a) - f(a)^q a) \right\}.$$

Thus we see that studying translation ovoids of $\mathcal{H}$ is equivalent to studying (normalized) translation Shult sets of the plane π, which in turn is equivalent to studying additive permutation polynomials f of $GF(q^2)$ such $\frac{f(a)}{a} \notin GF(q)$ for all nonzero $a \in GF(a^2)$. Explicitly describing such permutation polynomials and sorting out projective equivalences among the resulting translation ovoids is a nontrivial task. Our examples in Sections 3 and 4 were not normalized for ease of computation and to make the descriptions simpler.

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