Combinatorial invariance of Kazhdan-Lusztig polynomials for short intervals in the symmetric group

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OVERVIEW

1. Preliminaries
2. Main result
3. Drawing the Bruhat order: the diagram of (x, y)
4. From the diagram to the poset structure of $[x, y]$
5. From the diagram to the polynomial $\widetilde{R}_{x,y}(q)$
6. Proof sketch
7. Explicit formulas

1. PRELIMINARIES

1.1 Coxeter groups

W : Coxeter group $\qquad$ S : set of generators

Set of *reflections*: $\quad T = \{wsw^{-1} : w \in W,\ s \in S\}$.

Let $w \in W$. *Length* of w:

$$\ell(w) = \min\{k : w \text{ is a product of } k \textit{ generators}\}.$$

Absolute length of w:

$$a\ell(w) = \min\{k : w \text{ is a product of } k \textit{ reflections}\}.$$

Bruhat graph of W (BG): directed graph with W as vertex set and

$$x \to y \quad \Leftrightarrow \quad y = xt, \text{ with } t \in T, \text{ and } \ell(x) < \ell(y).$$

Edge supposed labelled by the reflection t: $x \xrightarrow{t} y$

Bruhat order of W: partial order on W defined by

$$x \leq y \quad \Leftrightarrow \quad x = x_0 \to x_1 \to \cdots \to x_k = y.$$

W, with the Bruhat order, is a graded poset with rank function ℓ.

Let $x, y \in W$, with $x < y$. The *length* of the pair (x, y) is

$$\ell(x, y) = \ell(y) - \ell(x).$$

1.2 The symmetric group

$\mathbf{N} = \{1, 2, 3, \ldots\}$, $\quad [n] = \{1, 2, \ldots, n\} \quad (n \in \mathbf{N})$,

$[n, m] = \{n, n+1, \ldots, m\} \quad (n, m \in \mathbf{N}$, with $n \leq m)$.

Denote by S_n the *symmetric group* over n elements:

$$S_n = \{x : [n] \to [n] \text{ bijection}\}.$$

S_n is a Coxeter group, with generators $\{s_1, s_2, \ldots, s_{n-1}\}$, where

$$s_i = (i, i+1) \quad \forall i \in [n-1].$$

1.3 Polynomials associated with W

Theorem There exists a unique family of polynomials

$$\{R_{x,y}(q)\}_{x,y\in W} \subseteq \mathbf{Z}[q]$$

such that

1. $R_{x,y}(q) = 0$, if $x \not\leq y$;
2. $R_{x,y}(q) = 1$, if $x = y$;
3. if $x < y$ and $s \in S$ is such that $ys \lhd y$ then

$$R_{x,y}(q) = \begin{cases} R_{xs,ys}(q), & \text{if } xs \lhd x, \\ qR_{xs,ys}(q) + (q-1)R_{x,ys}(q), & \text{if } xs \rhd x. \end{cases}$$

They are called the *R-polynomials* of W.

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2. $P_{x,y}(q) = 1$, if $x = y$;

3. if $x < y$ then $\deg(P_{x,y}(q)) < \ell(x,y)/2$ and

$$q^{\ell(x,y)} P_{x,y}\left(q^{-1}\right) - P_{x,y}(q) = \sum_{x < z \leq y} R_{x,z}(q)\, P_{z,y}(q).$$

They are called the *Kazhdan-Lusztig polynomials* of W.

1.4 Applications

Kazhdan-Lusztig polynomials play a crucial role in

- algebraic geometry of Schubert varieties;
- topology of Schubert varieties;
- representation theory of semisimple algebraic groups;
- representation theory of Hecke algebras.

1.5 Combinatorial interpretation

Proposition There exists a unique family of polynomials

$$\{\widetilde{R}_{x,y}(q)\}_{x,y \in W} \subseteq \mathbf{Z}_{\geq 0}[q]$$

such that

$$R_{x,y}(q) = q^{\frac{\ell(x,y)}{2}} \widetilde{R}_{x,y}\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)$$

for every $x, y \in W$.

They are called the *$\widetilde{R}$-polynomials* of W.

Proposition There is a bijection

$$\begin{array}{ccc} \text{(positive roots) } \Phi^+ & \leftrightarrow & T \text{ (reflections)} \\ \alpha & \mapsto & t_\alpha \end{array}$$

Definition A *reflection ordering* on T is a total ordering $\prec$ such that

$$\forall \alpha, \beta \in \Phi^+, \quad \forall \lambda, \mu \in \mathbf{R}^+, \quad \text{with } \lambda\alpha + \mu\beta \in \Phi^+$$

$$t_\alpha \prec t_\beta \quad \Rightarrow \quad t_\alpha \prec t_{\lambda\alpha+\mu\beta} \prec t_\beta.$$

Proposition A reflection ordering on T always exists.

$Paths(x,y)$: set of paths in BG from x to y.

$\Delta = (x_0, x_1, \ldots, x_k) \in Paths(x,y)$ has *length* $|\Delta| = k$.

Let $\prec$ be a fixed reflection ordering on T.

A path $x_0 \xrightarrow{t_1} x_1 \xrightarrow{t_2} \cdots \xrightarrow{t_k} x_k$ is *increasing* if $t_1 \prec t_2 \prec \cdots \prec t_k$.

$Paths^{\prec}(x,y)$: set of increasing paths in BG from x to y.

Theorem [Dyer] Let $x, y \in W$, with $x < y$. Then

$$\widetilde{R}_{x,y}(q) = \sum_{\Delta \in Paths^{\prec}(x,y)} q^{|\Delta|}.$$

1.6 Absolute length of a pair

Definition Let $x, y \in W$, with $x < y$. The *absolute length* of (x, y), denoted by $a\ell(x, y)$, is the (oriented) distance between x and y in BG.

Corollary Let $x, y \in W$, $x < y$. Set $\ell = \ell(x, y)$ and $a\ell = a\ell(x, y)$. Then

$$\widetilde{R}_{x,y}(q) = q^{\ell} + c_{\ell-2}\, q^{\ell-2} + \cdots + c_{a\ell+2}\, q^{a\ell+2} + c_{a\ell}\, q^{a\ell},$$

where, $\forall k \in [a\ell, \ell - 2]$, with $k \equiv \ell$ (2)

$$c_k = |\{\Delta \in Paths^{\prec}(x, y) : |\Delta| = k\}| \geq 1.$$

Proposition [Dyer] The absolute length $a\ell(x, y)$ is a combinatorial invariant, that is, it depends only on the poset structure of $[x, y]$.

1.7 Combinatorial invariance conjecture

Conjecture [Lusztig] [Dyer] The Kazhdan-Lusztig polynomials are combinatorial invariants. In other words, if W_1, W_2 are Coxeter groups and $x, y \in W_1$, with $x < y$, and $u, v \in W_2$, with $u < v$, then

$$[x, y] \cong [u, v] \quad \Rightarrow \quad P_{x,y}(q) = P_{u,v}(q).$$

Equivalent to the same statement for R- and $\widetilde{R}$-polynomials.

Known to be true if $[x, y]$ is a lattice or if $\ell(x, y) \leq 4$.

Theorem [Brenti, Caselli, Marietti] True for $x = u = e$.

2. MAIN RESULT

2.1 Some notation

Let W be a Coxeter group and let $x, y \in W$, with $x < y$.

Number of *atoms* and *coatoms* of $[x, y]$:

$$a(x, y) = |\{z \in [x, y] : x \lhd z\}| \quad \text{and} \quad c(x, y) = |\{z \in [x, y] : z \lhd y\}|.$$

Introduce the *capacity* of $[x, y]$:

$$cap(x, y) = \min\{a(x, y), c(x, y)\}.$$

Denote by $\mathcal{B}_k$ the *boolean algebra* of rank k, that is, the family $\mathcal{P}([k])$ of all subsets of $[k]$ partially ordered by inclusion.

2.2 Main result

Theorem Let $x, y \in S_n$, for some n, with $x < y$ and $\ell(x, y) = 5$. Set $a = a(x, y)$, $c = c(x, y)$ and $cap = cap(x, y)$. Then

$$\widetilde{R}_{x,y}(q) = \begin{cases} q^5 + 2q^3 + q, & \text{if } \{a, c\} = \{3, 4\}, \\ q^5 + 2q^3, & \text{if } a = c = 3, \\ q^5 + q^3, & \text{if } cap \in \{4, 5\} \text{ but } [x, y] \ncong \mathcal{B}_5, \\ q^5, & \text{if } cap \in \{6, 7\} \text{ or } [x, y] \cong \mathcal{B}_5. \end{cases}$$

Corollary Let $x, y \in S_n$, with $x < y$ and $\ell(x, y) = 5$, and $u, v \in S_m$, with $u < v$ and $\ell(u, v) = 5$, for some n and m. Then

$$[x, y] \cong [u, v] \quad \Rightarrow \quad P_{x,y}(q) = P_{u,v}(q).$$

Proposition Let $x, y \in W$, with $x < y$. Then

$$\sum_{x \leq z \leq y} (-1)^{\ell(x,z)} R_{x,z}(q) R_{z,y}(q) = 0.$$

In particular, if $\ell(x, y)$ is even,

$$R_{x,y}(q) = \frac{1}{2} \sum_{x < z < y} (-1)^{\ell(x,z)-1} R_{x,z}(q) R_{z,y}(q).$$

Corollary Let $x, y \in S_n$, with $x < y$ and $\ell(x, y) = 6$, and $u, v \in S_m$, with $u < v$ and $\ell(u, v) = 6$, for some n and m. Then

$$[x, y] \cong [u, v] \quad \Rightarrow \quad P_{x,y}(q) = P_{u,v}(q).$$

3. DRAWING THE BRUHAT ORDER

3.1 Denoting permutations

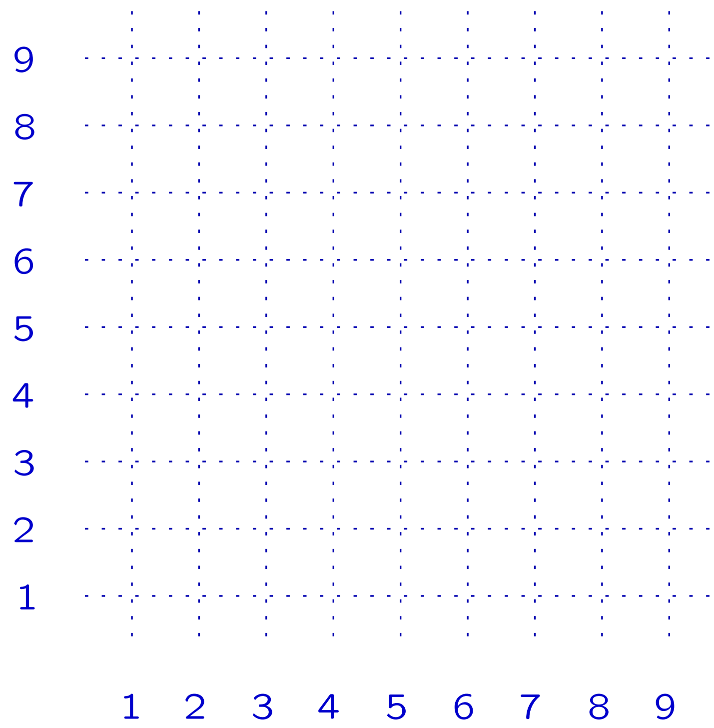
Denote a permutation $x \in S_n$ using the *one-line notation*:

$$x = x_1 x_2 \ldots x_n \quad \text{means} \quad x(i) = x_i \quad \forall i \in [n].$$

The *diagram* of $x \in S_n$ is the subset of $\mathbf{N}^2$ so defined:

$$Diag(x) = \{(i, x(i)) : i \in [n]\}.$$

Example $x = 315472986 \in S_9$. Diagram of x:



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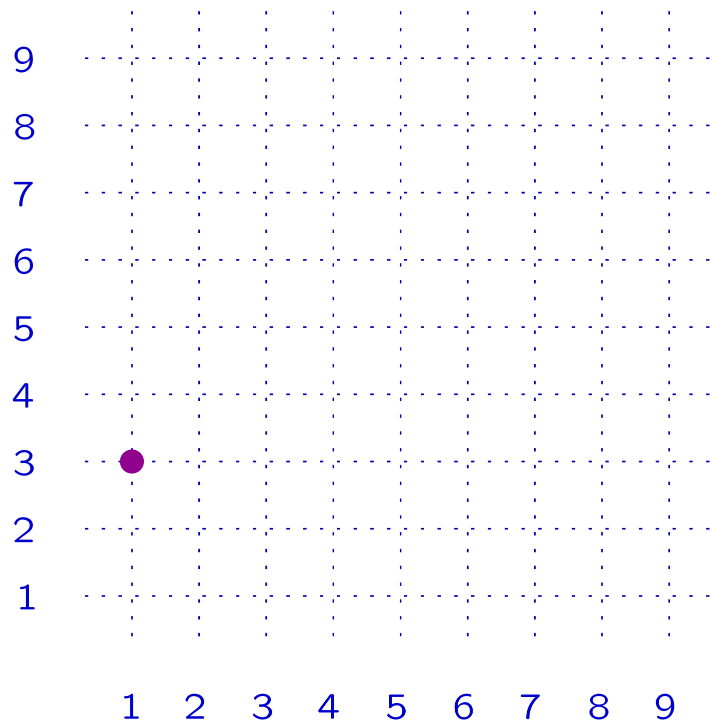

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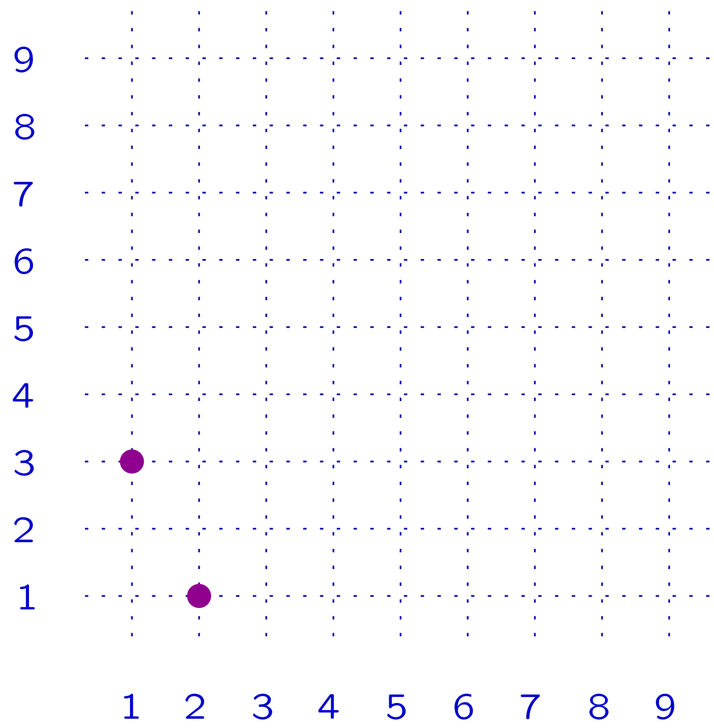

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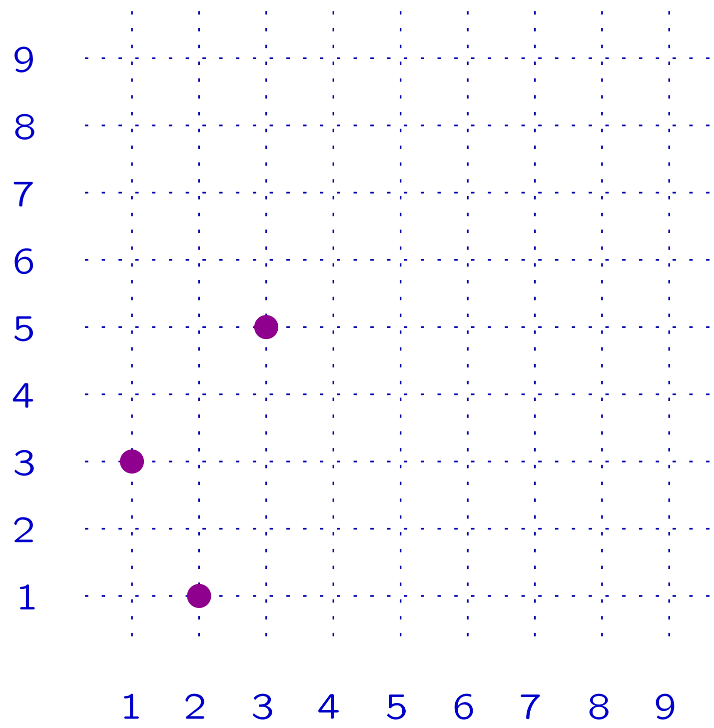

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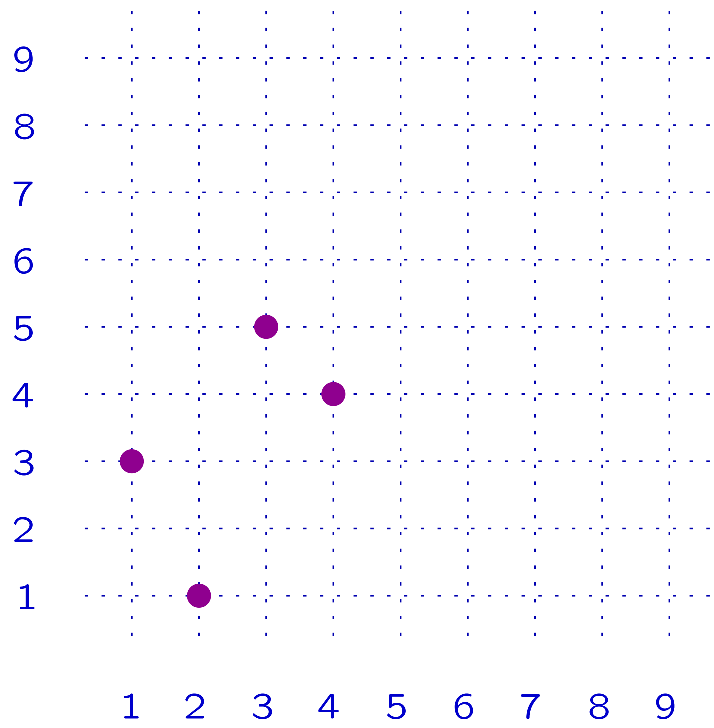

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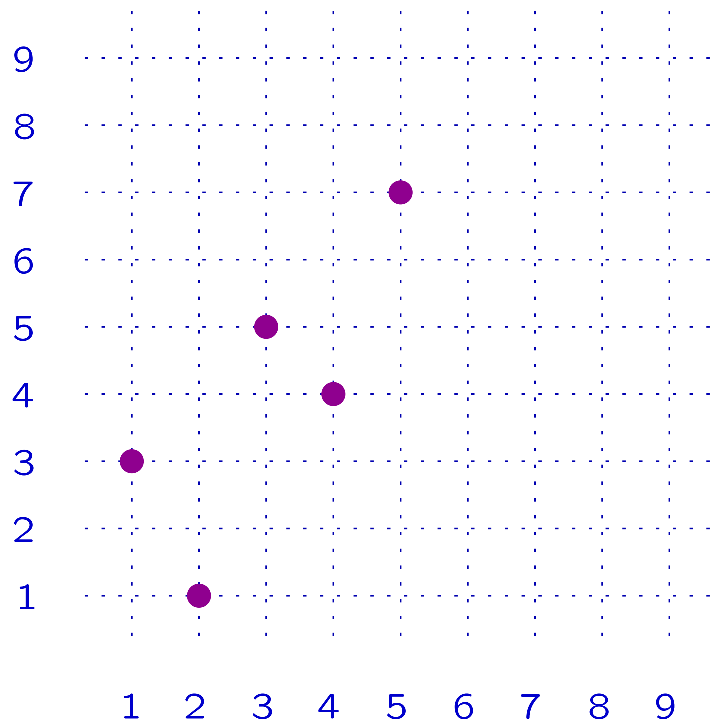

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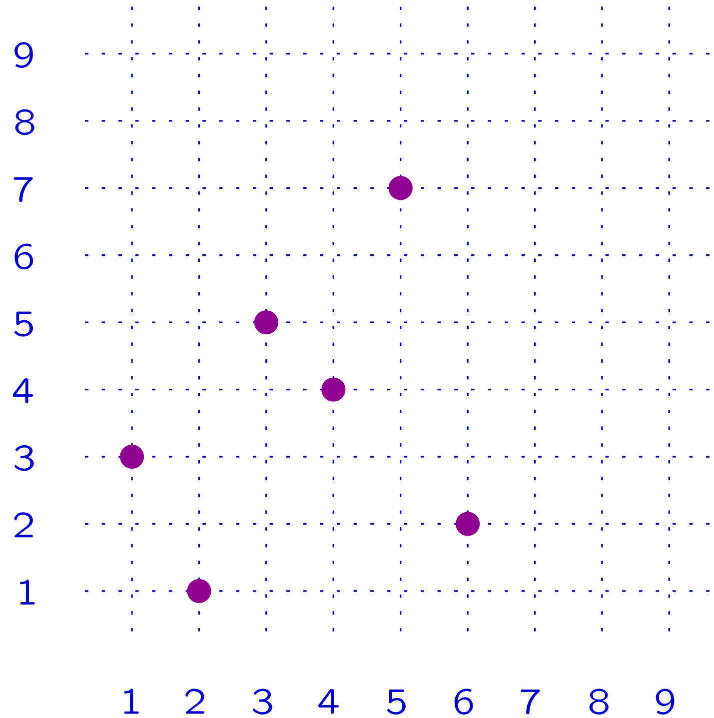

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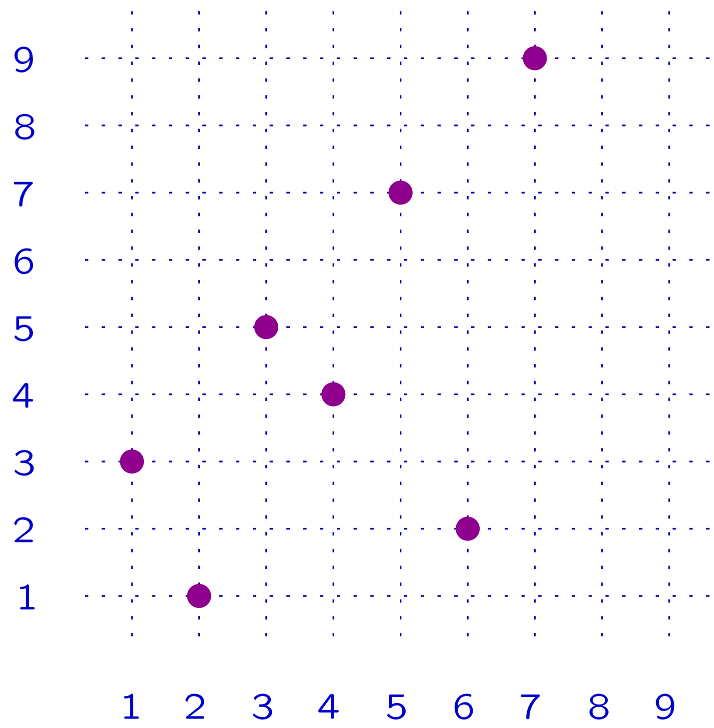

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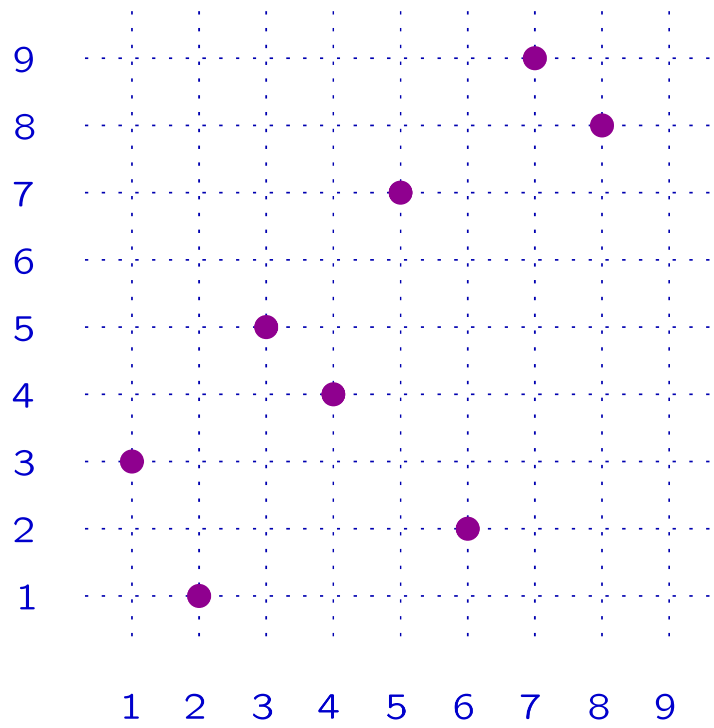

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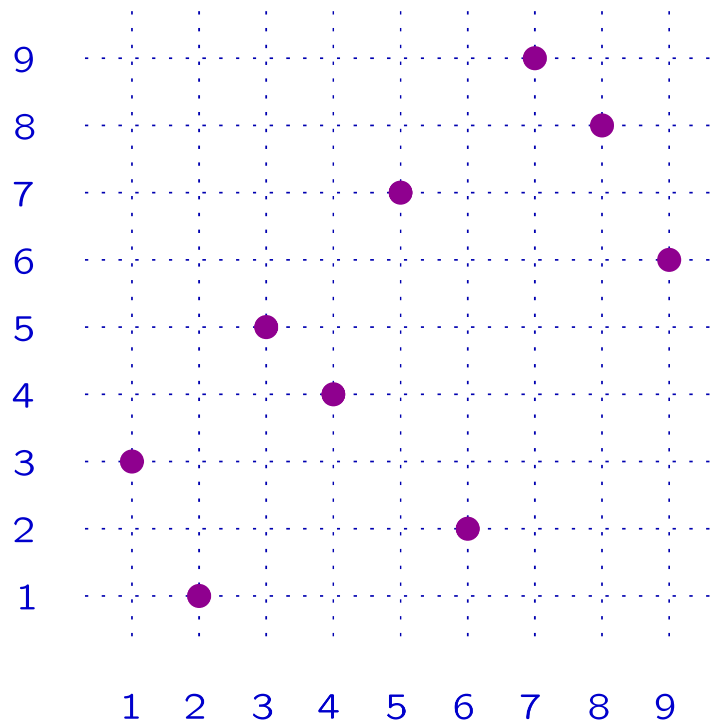

3.2 Length in the symmetric group

Let $x \in S_n$. Number of *inversions* of x:

$$inv(x) = |\{(i,j) \in [n]^2 : i < j,\, x(i) > x(j)\}|.$$

Proposition Let $x \in S_n$. Then

$$\ell(x) = inv(x).$$

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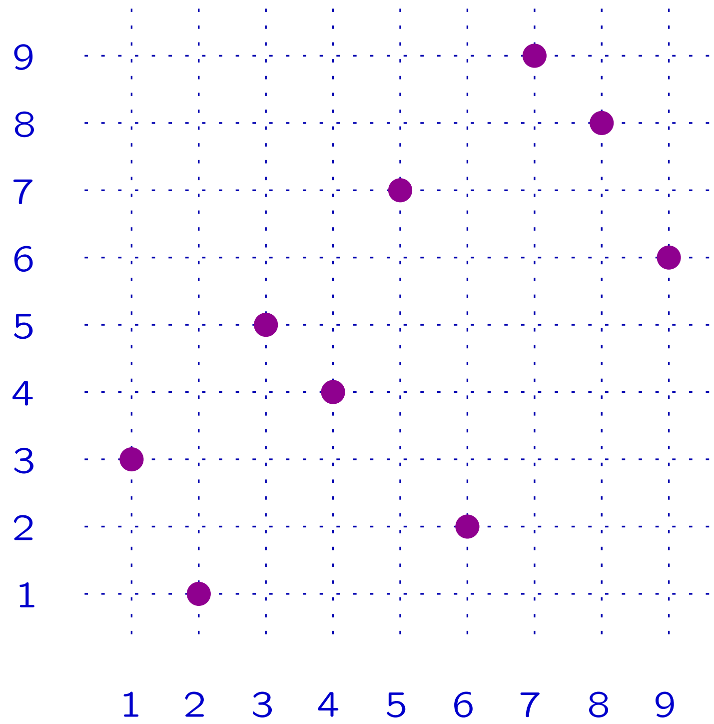

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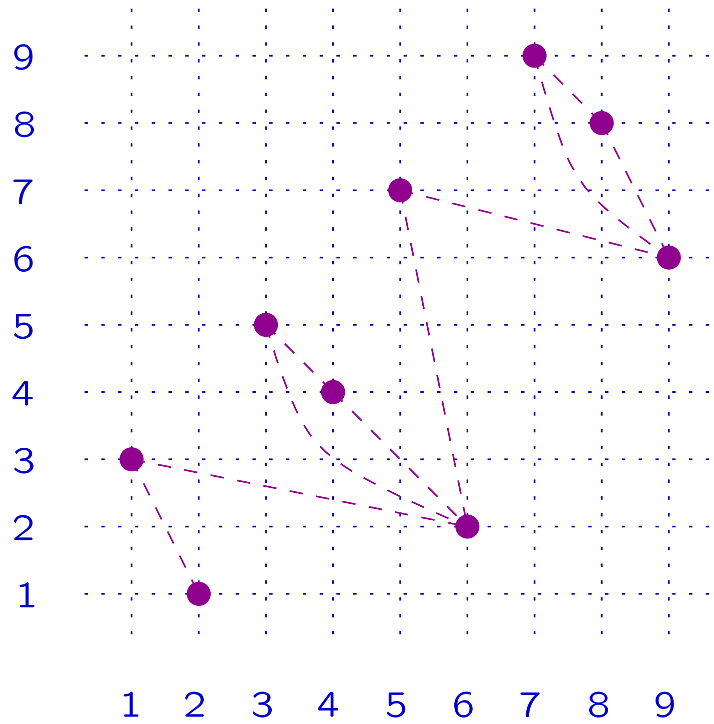
$$\ell(x) = inv(x) = 10$$



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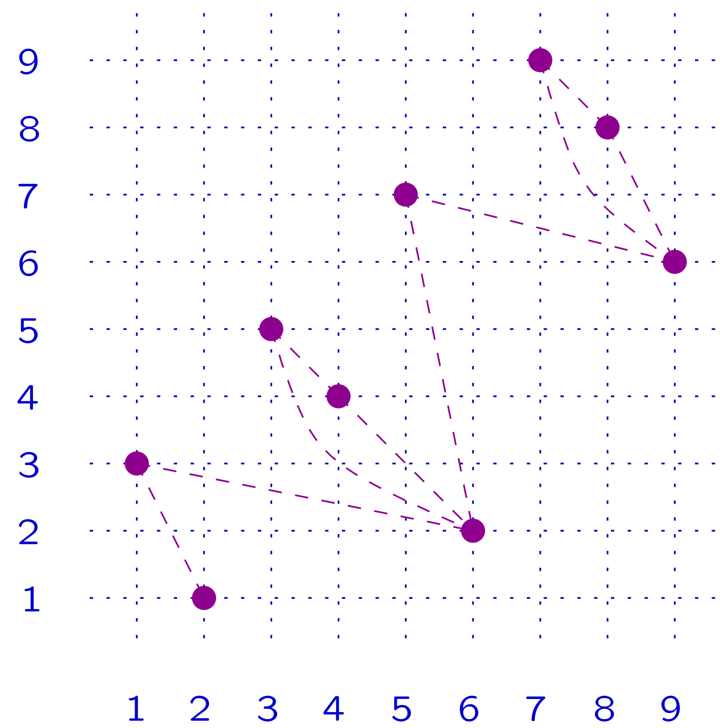
Let $x, y \in S_n$, with $x < y$. Then

$$\ell(x,y) = inv(y) - inv(x).$$



3.3 Bruhat order in the symmetric group

Let $x \in S_n$. $\forall (h,k) \in [n]^2$ set

$$x[h,k] = |\{i \in [h] : x(i) \in [k,n]\}.$$

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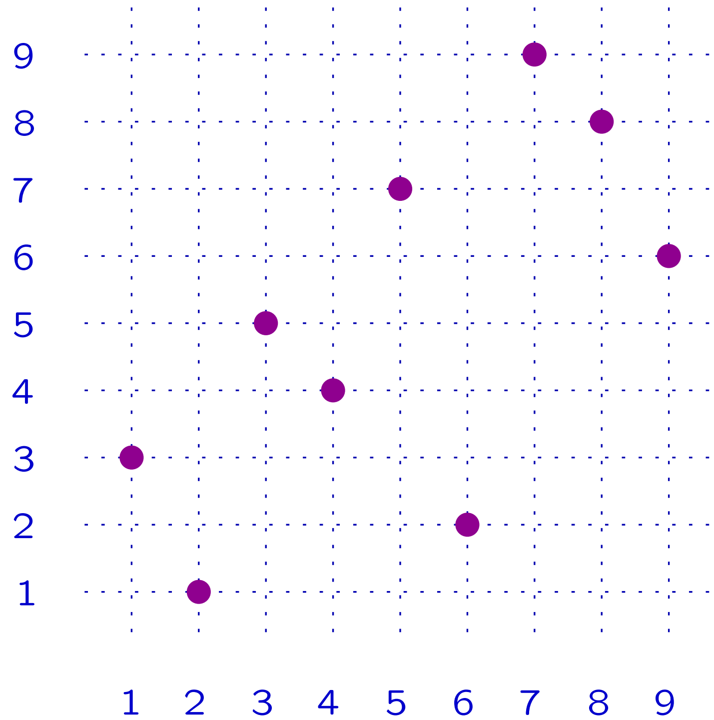

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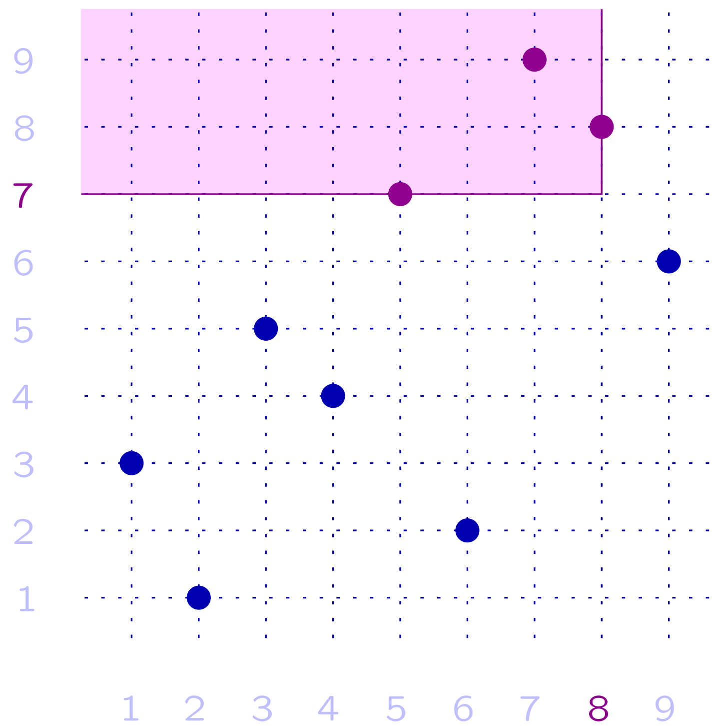
$$x[8,7] = 3$$



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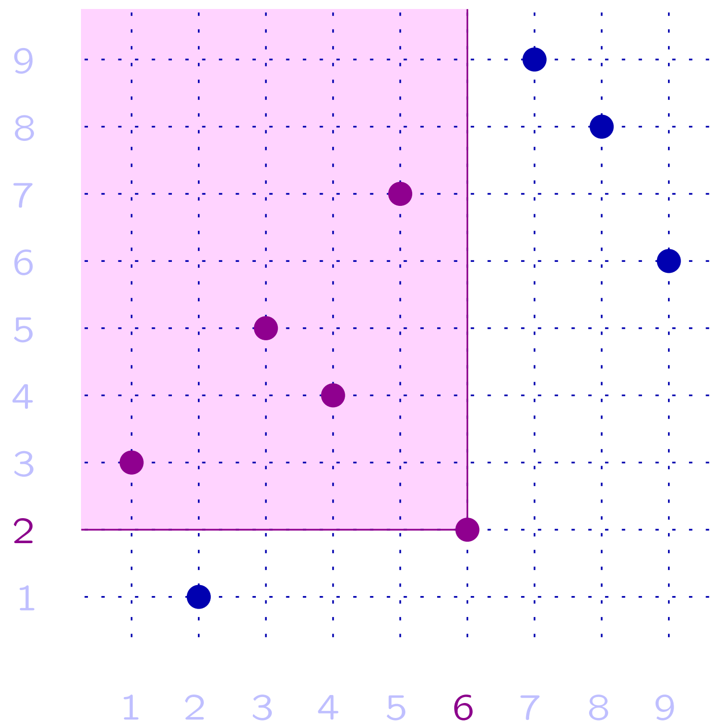
Example $x = 315472986 \in S_9$.

$$x[8,7] = 3$$

$$x[6,2] = 5$$



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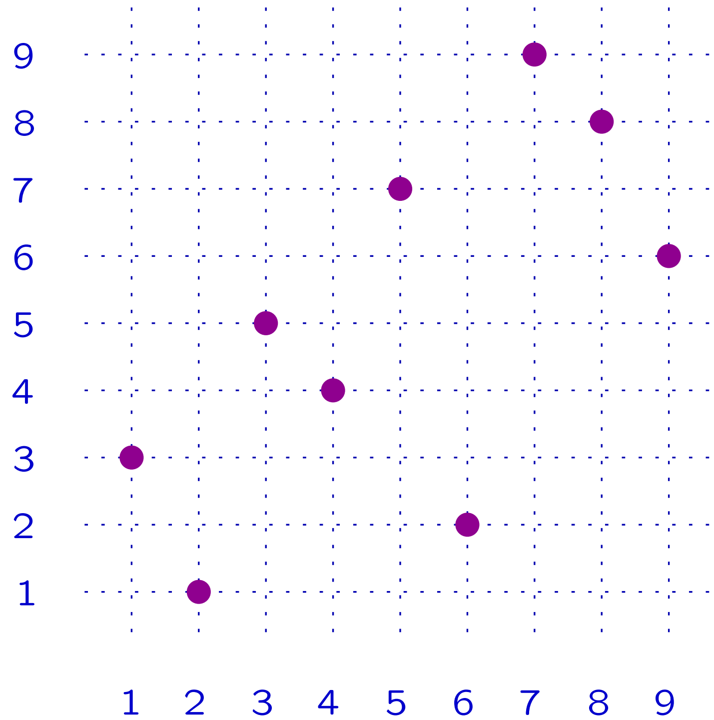

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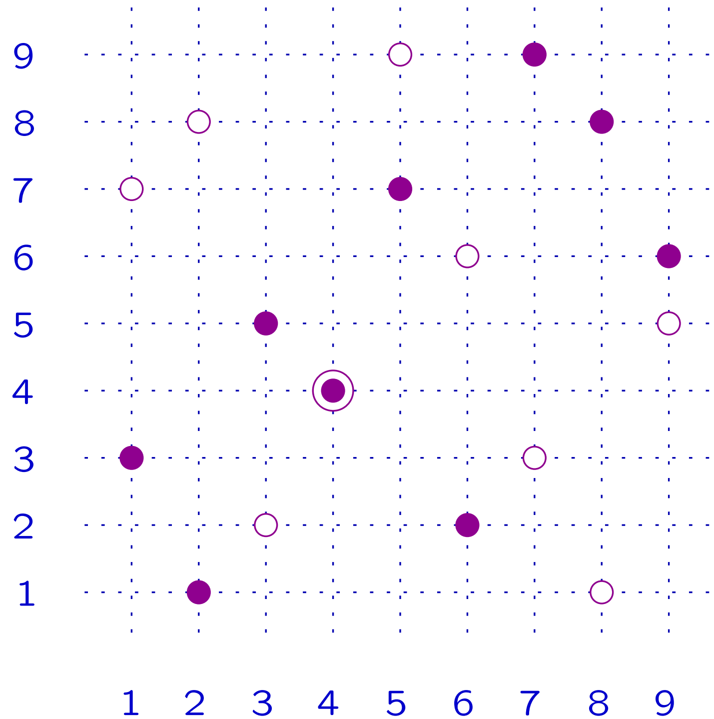
Example $x = 315472986$ ($\bullet$)

$y = 782496315$ ($\circ$)



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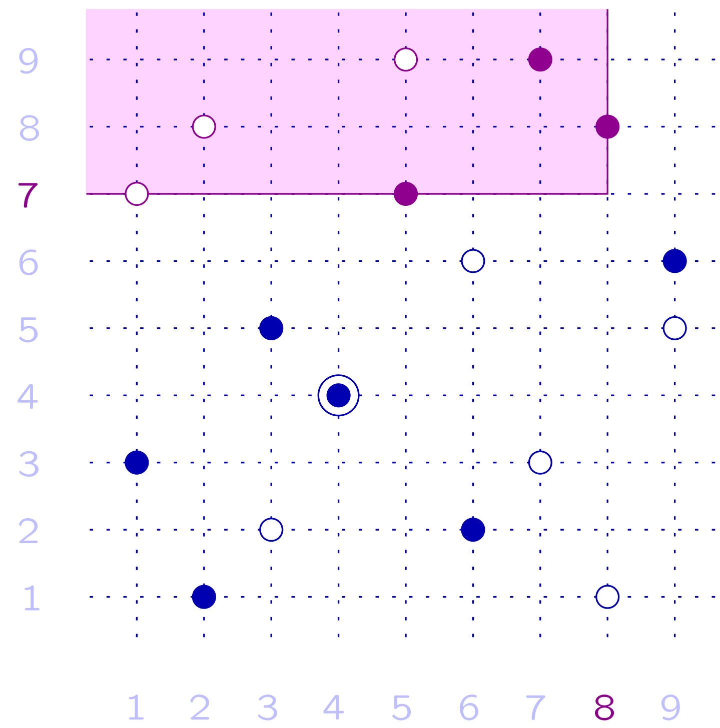
$$(x,y)[8,7] = 0$$



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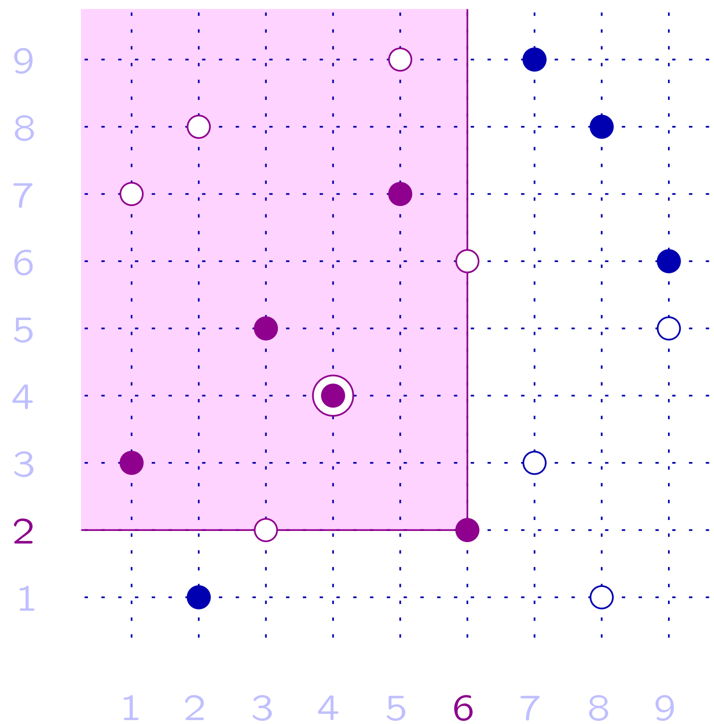
$$(x,y)[6,2] = 1$$



Theorem Let $x, y \in S_n$. Then

$$x \leq y \quad \Leftrightarrow \quad (x,y)[h,k] \geq 0, \quad \forall (h,k) \in [n]^2.$$

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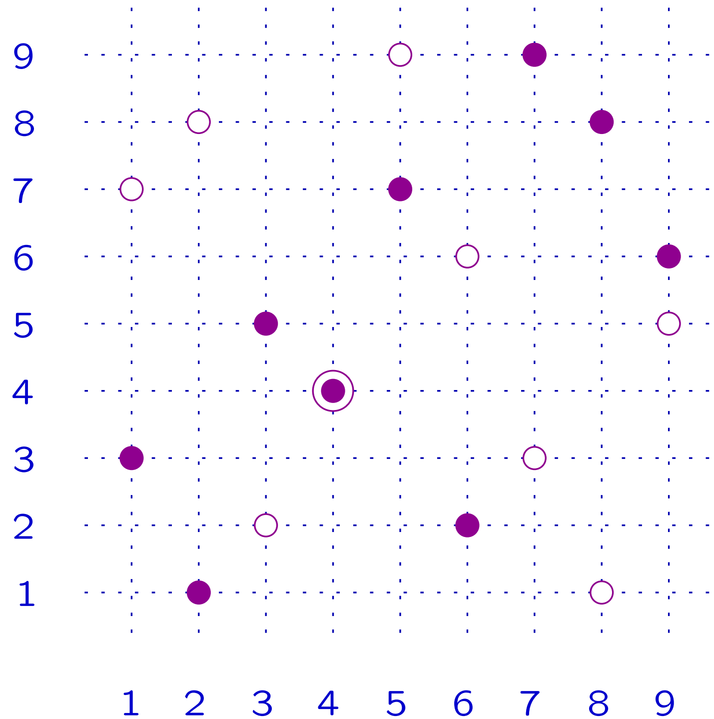

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Example $x = 315472986$ ($\bullet$)

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$$(x,y)[h,k] \geq 0, \quad \forall (h,k) \in [\mathbf{9}]^2$$

9	0	0	0	0	1	1	0	0	0
8	0	1	1	1	2	2	1	0	0
7	1	2	2	2	2	2	1	0	0
6	1	2	2	2	2	3	2	1	0
5	1	2	1	1	1	2	1	0	0
4	1	2	1	1	1	2	1	0	0
3	0	0	0	0	0	1	1	0	0
2	0	1	1	1	1	1	1	0	0
1	0	0	0	0	0	0	0	0	0
	1	2	3	4	5	6	7	8	9

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Example $x = 315472986$ ($\bullet$)

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$$(x,y)[h,k] \geq 0, \quad \forall (h,k) \in [\mathbf{9}]^2$$

$$\Downarrow$$

$$x < y$$

	1	2	3	4	5	6	7	8	9
9	0	0	0	0	1	1	0	0	0
8	0	1	1	1	2	2	1	0	0
7	1	2	2	2	2	2	1	0	0
6	1	2	2	2	2	3	2	1	0
5	1	2	1	1	1	2	1	0	0
4	1	2	1	1	1	2	1	0	0
3	0	0	0	0	0	1	1	0	0
2	0	1	1	1	1	1	1	0	0
1	0	0	0	0	0	0	0	0	0

Extend the notation: $\forall (h,k) \in \mathbf{R}^2$ set

$$x[h,k] = |\{i \in [h] : x(i) \in [k,n]\}|, \quad (x,y)[h,k] = y[h,k] - x[h,k].$$

Definition Let $x, y \in S_n$. The *multiplicity mapping* of (x,y) is

$$(h,k) \in \mathbf{R}^2 \mapsto (x,y)[h,k] \in \mathbf{Z}.$$

Definition Let $x, y \in S_n$, with $x < y$. The *support* of (x,y) is

$$\Omega(x,y) = \{(h,k) \in \mathbf{R}^2 : (x,y)[h,k] > 0\}.$$

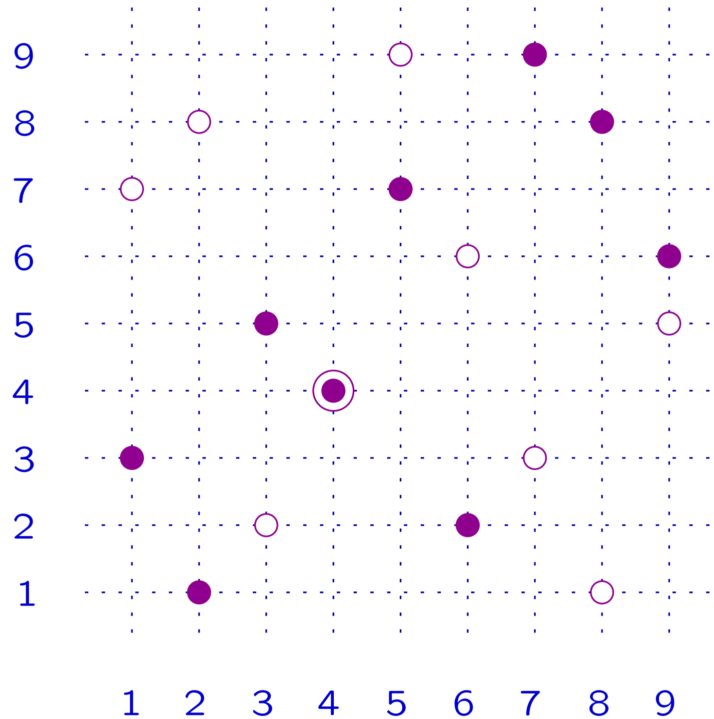
3.4 Diagram of a pair of permutations

Definition Let $x, y \in S_n$. The *diagram* of (x, y) is the collection of:

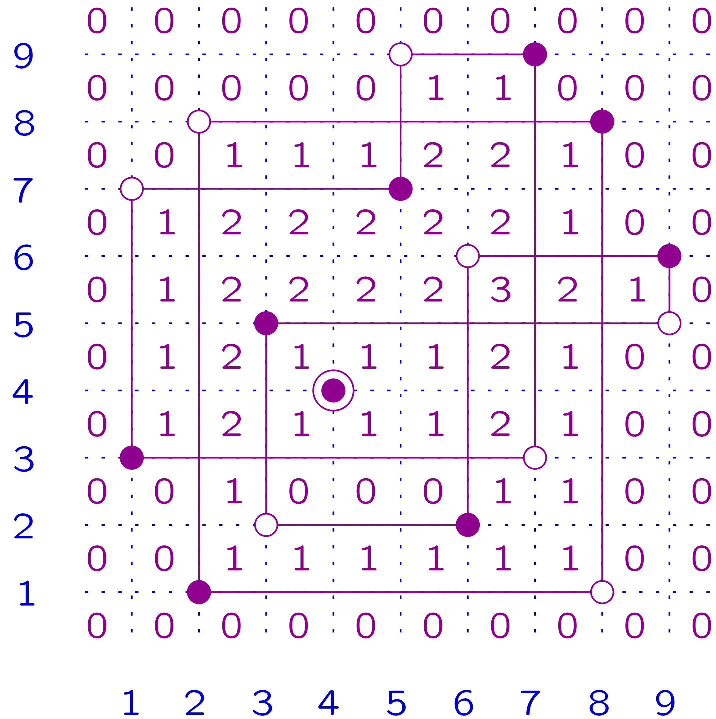
1. the diagram of x;
2. the diagram of y;
3. the multiplicity mapping $(h, k) \mapsto (x, y)[h, k]$.

Analog definition in [Kassel, Lascoux, Reutenauer, 2003]

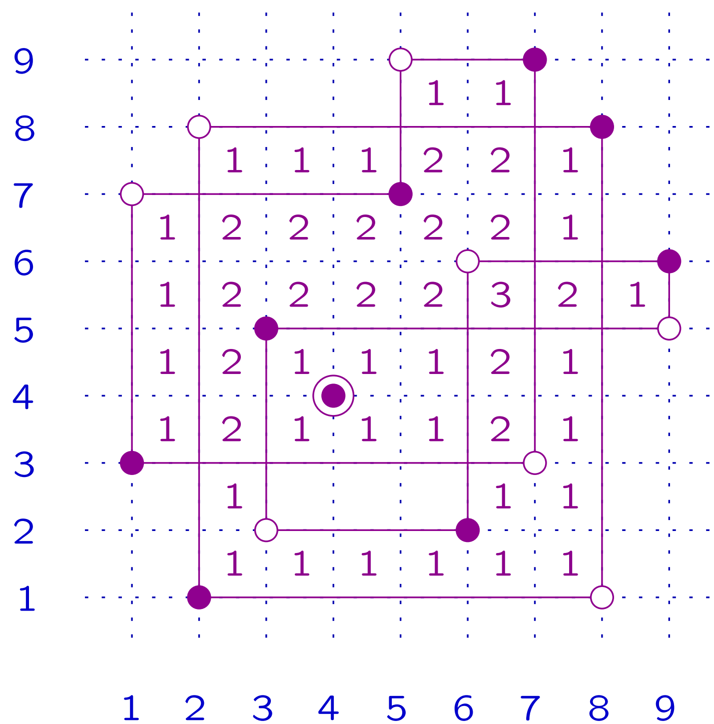
Example $x = 315472986$ ($\bullet$), $y = 782496315$ ($\circ$).



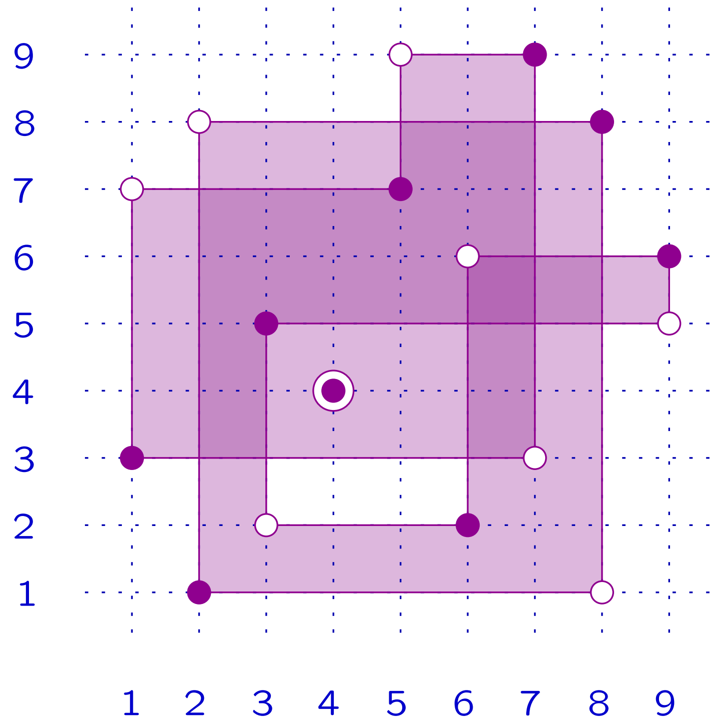
Example $x = 315472986$ (•), $y = 782496315$ (∘).



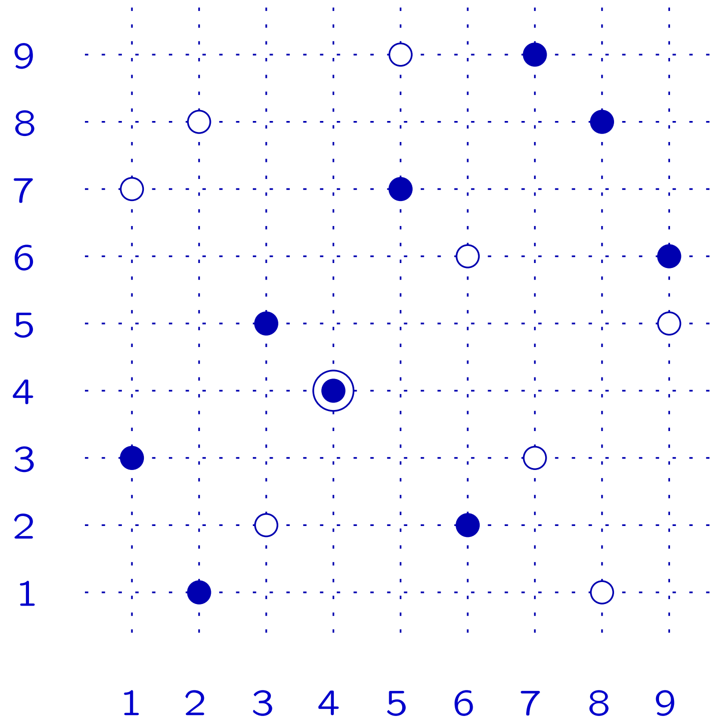
Example $x = 315472986$ (•), $y = 782496315$ (∘).



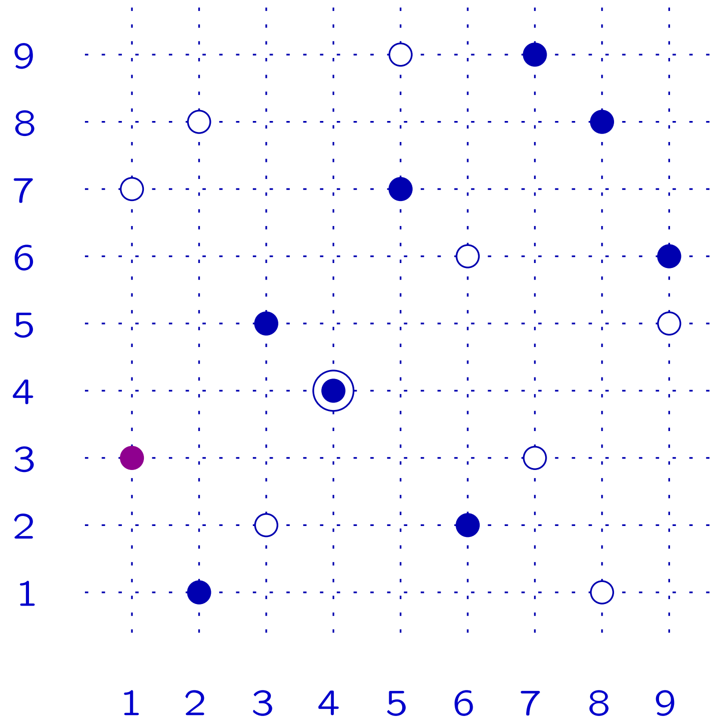
Example $x = 315472986$ ($\bullet$), $y = 782496315$ ($\circ$).



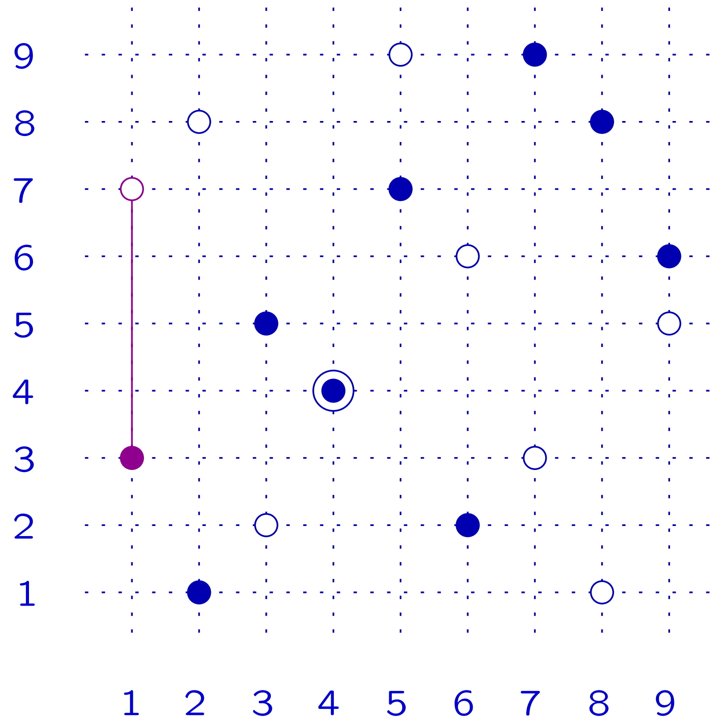
Example $x = 315472986$ ($\bullet$), $y = 782496315$ ($\circ$).



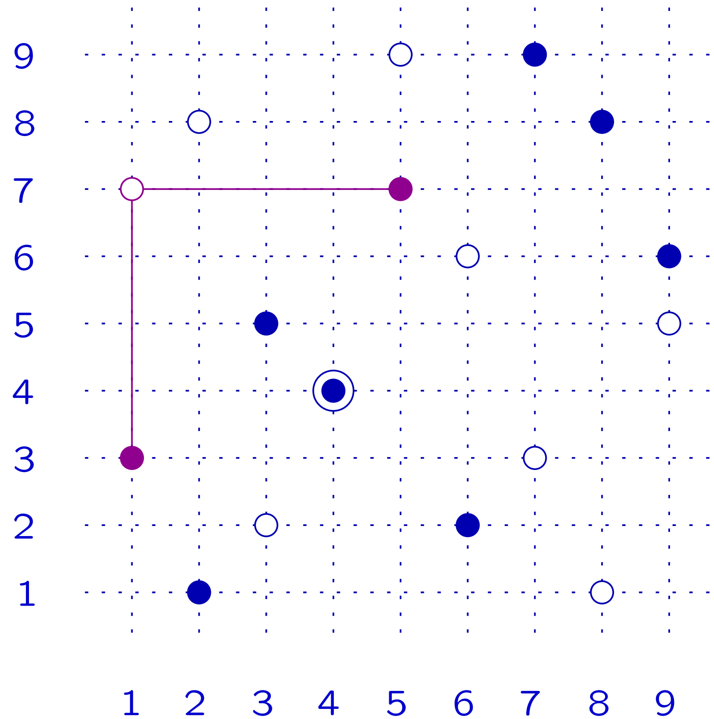
Example $x = 315472986$ ($\bullet$), $y = 782496315$ ($\circ$).



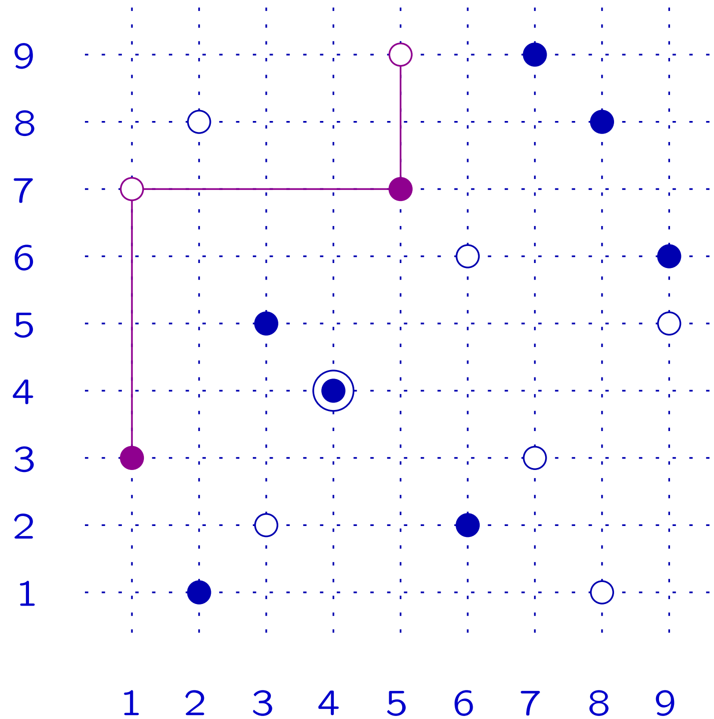
Example $x = 315472986$ (•), $y = 782496315$ (○).



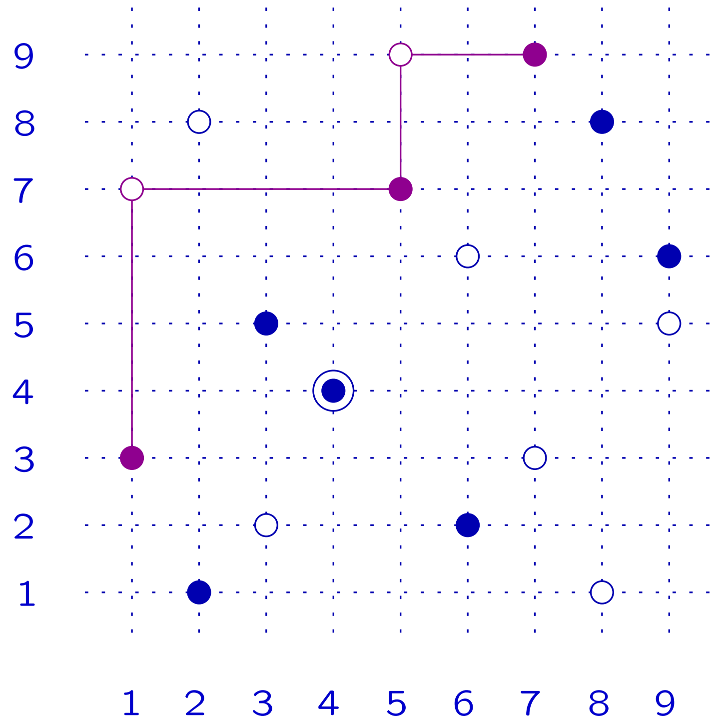
Example $x = 315472986$ ($\bullet$), $y = 782496315$ ($\circ$).



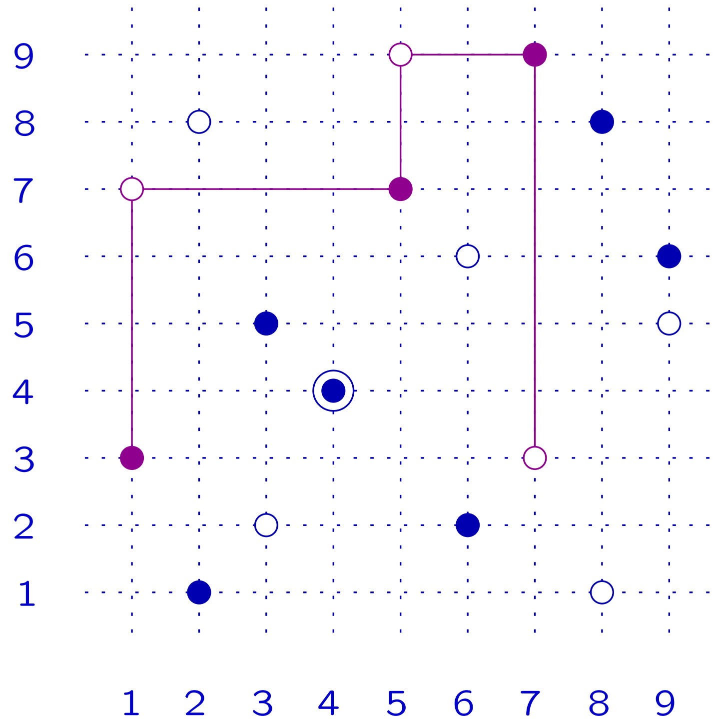
Example $x = 315472986$ (•), $y = 782496315$ (○).



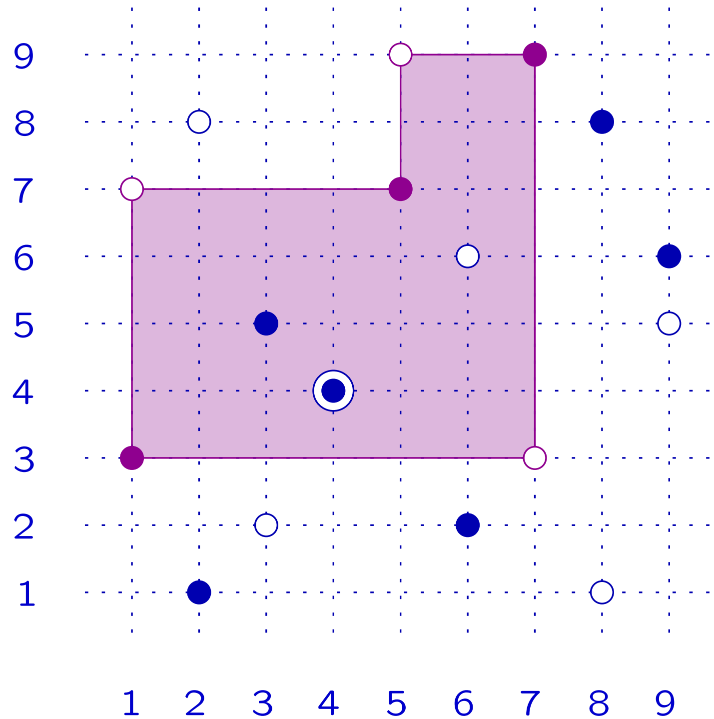
Example $x = 315472986$ (•), $y = 782496315$ (◦).



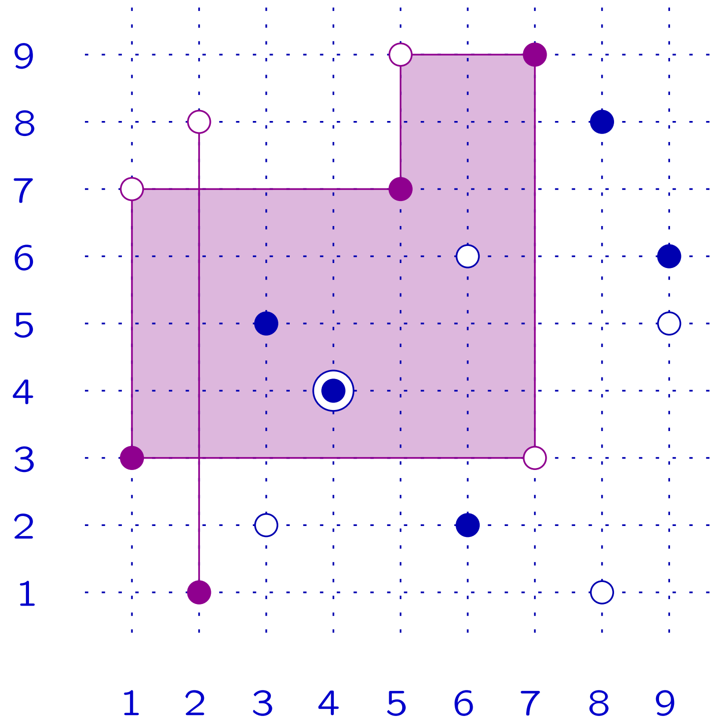
Example $x = 315472986$ ($\bullet$), $y = 782496315$ ($\circ$).



Example $x = 315472986$ (•), $y = 782496315$ (∘).



Example $x = 315472986$ (•), $y = 782496315$ (∘).



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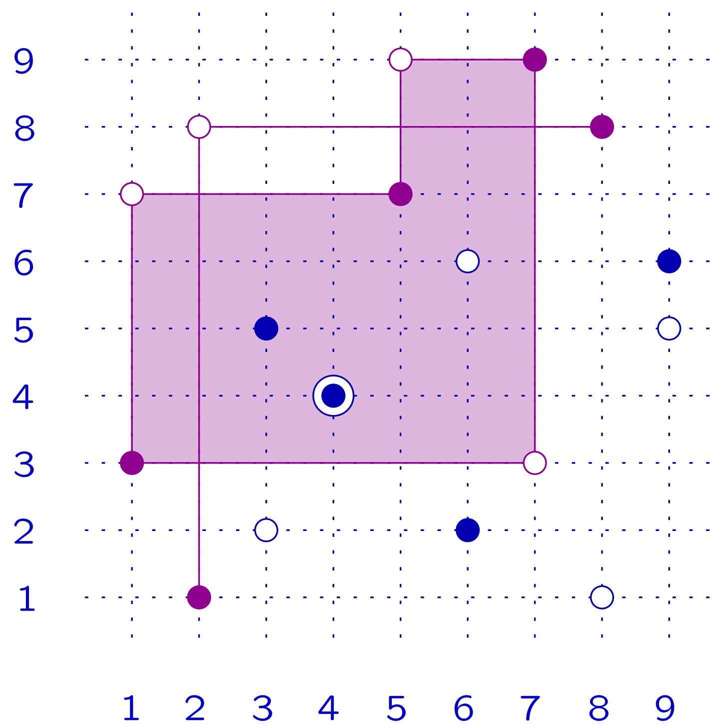
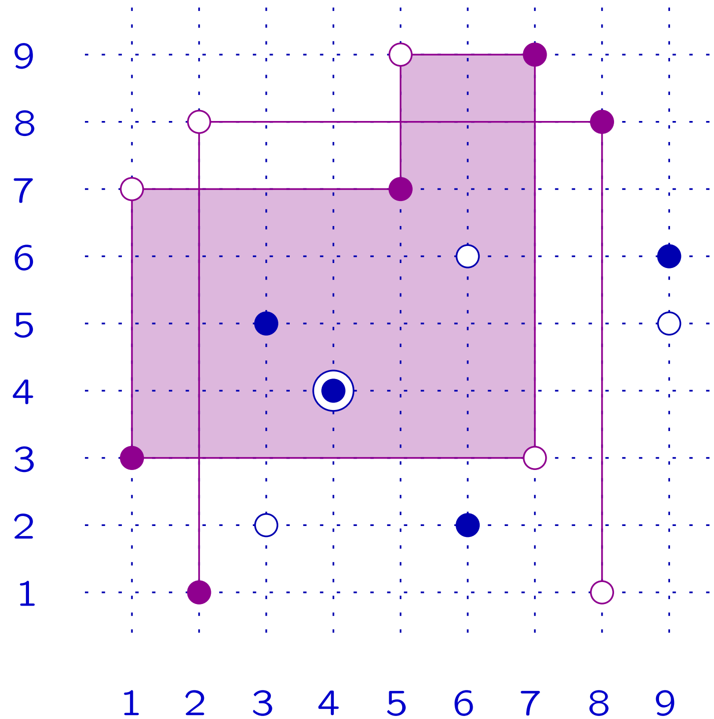
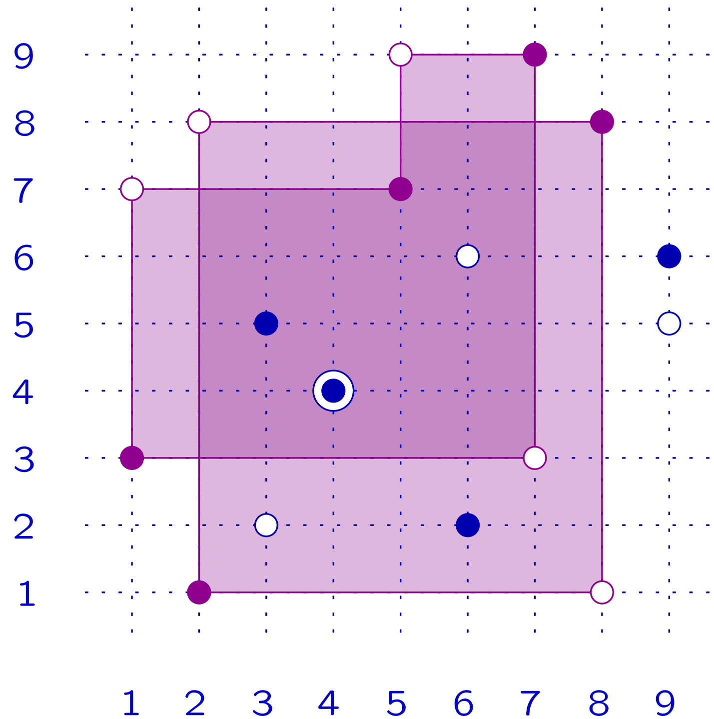


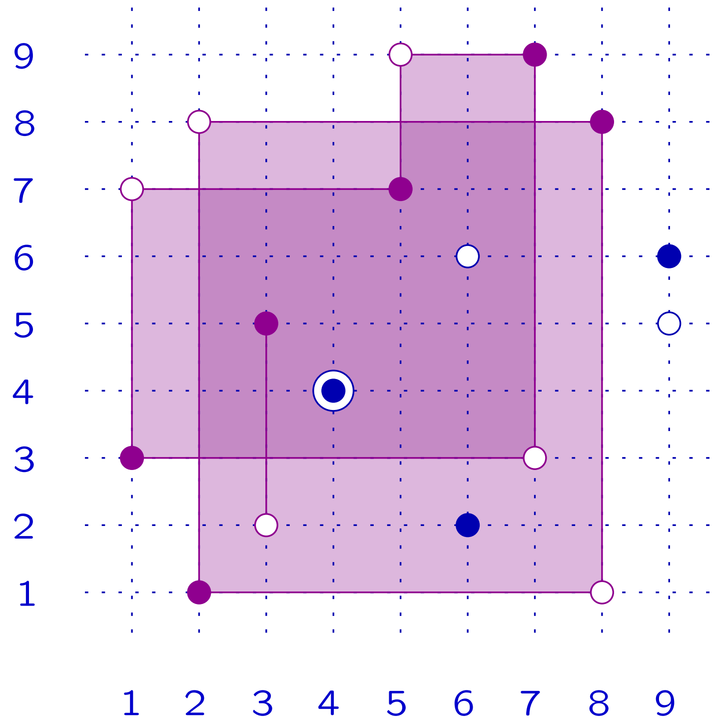
Example $x = 315472986$ (●), $y = 782496315$ (○).



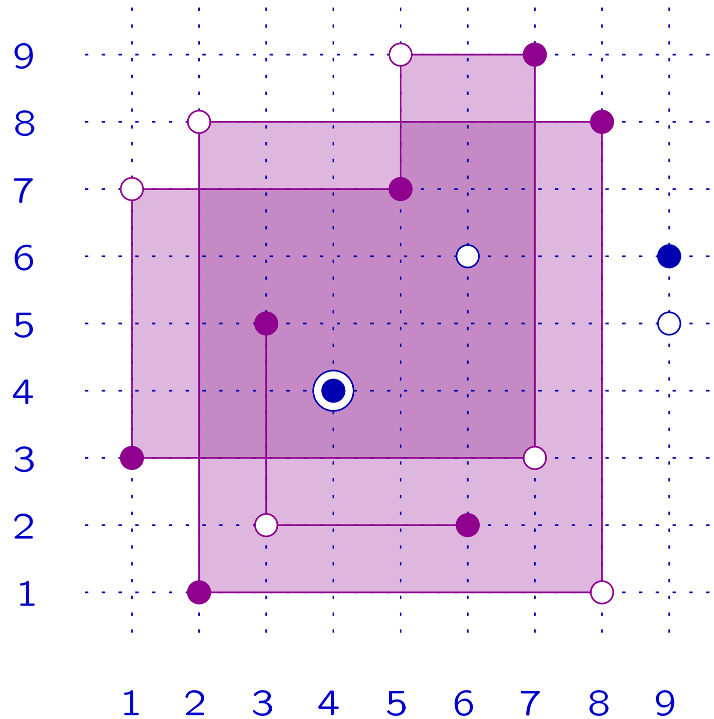
Example $x = 315472986$ ($\bullet$), $y = 782496315$ ($\circ$).



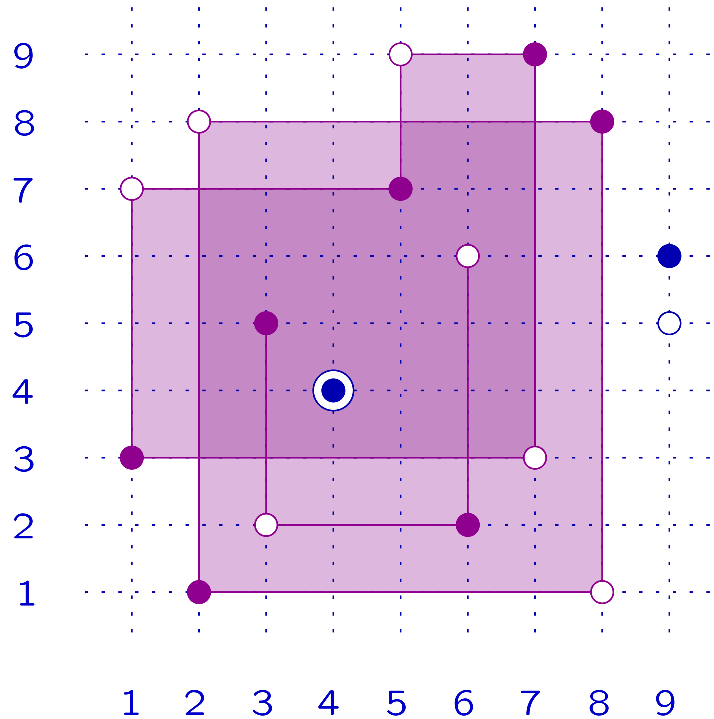
Example $x = 315472986$ (•), $y = 782496315$ (∘).



Example $x = 315472986$ (•), $y = 782496315$ (∘).



Example $x = 315472986$ (●), $y = 782496315$ (○).



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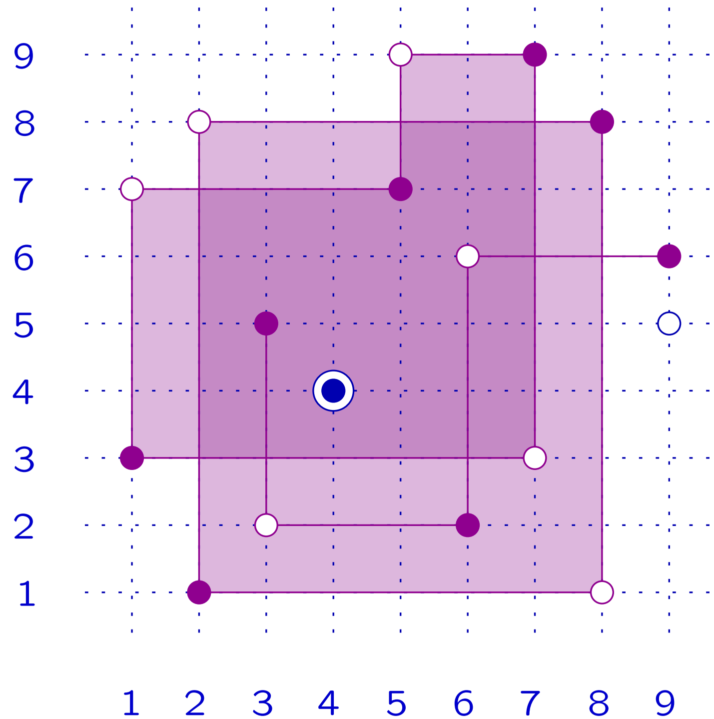

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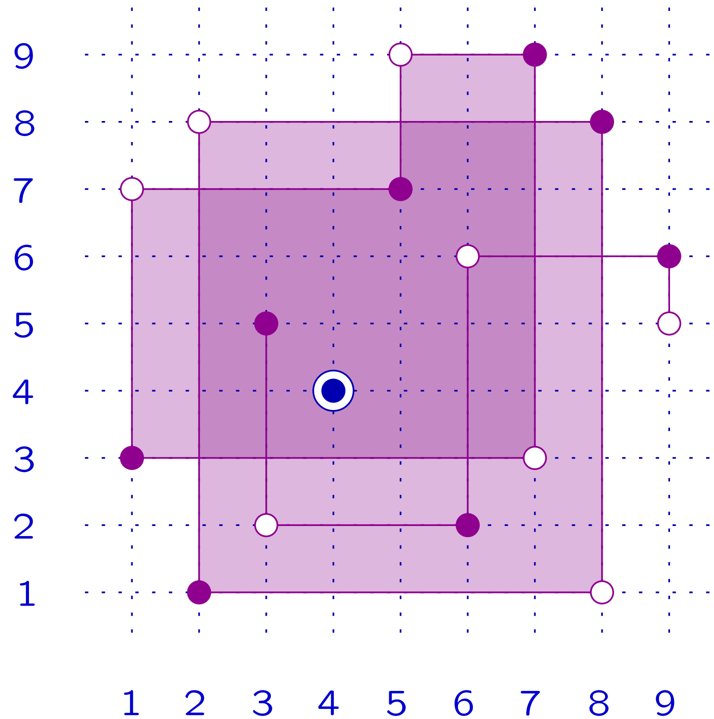
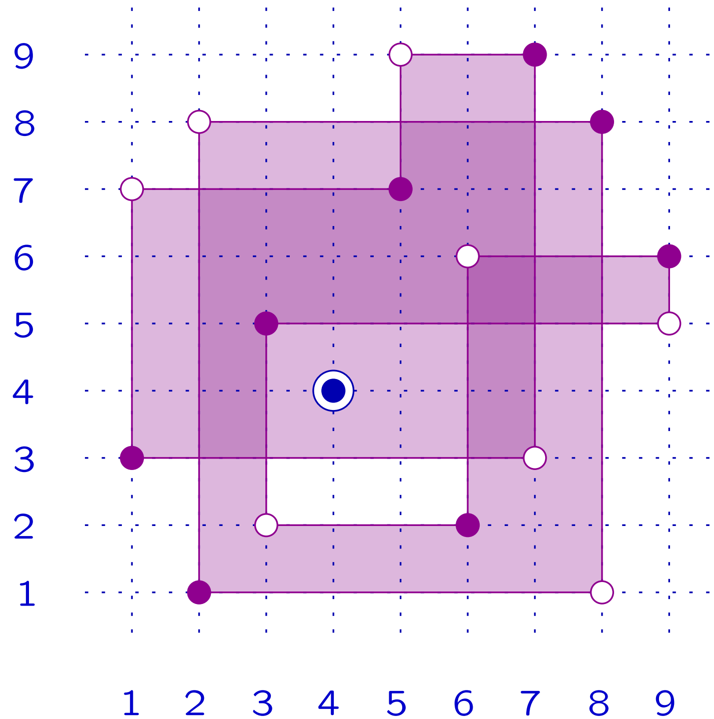
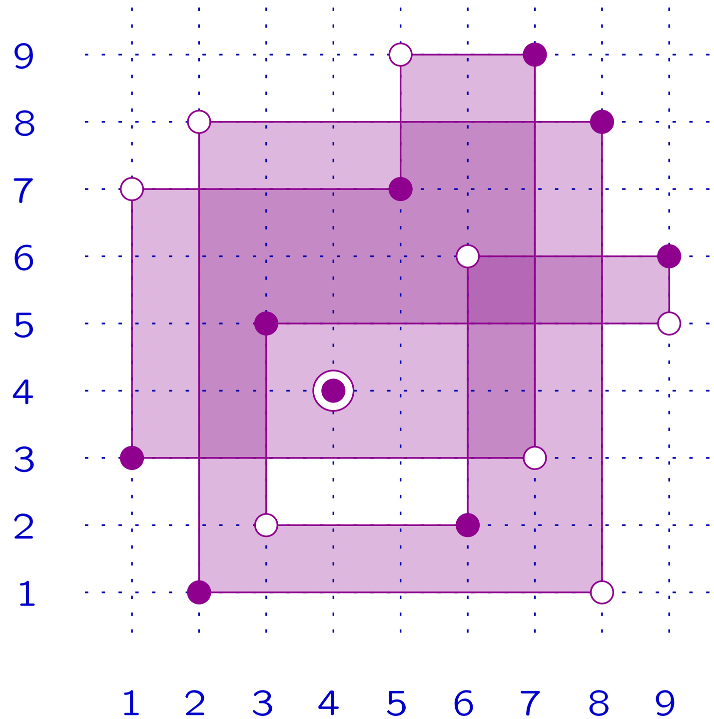


Example $x = 315472986$ ($\bullet$), $y = 782496315$ ($\circ$).



Example $x = 315472986$ ($\bullet$), $y = 782496315$ ($\circ$).



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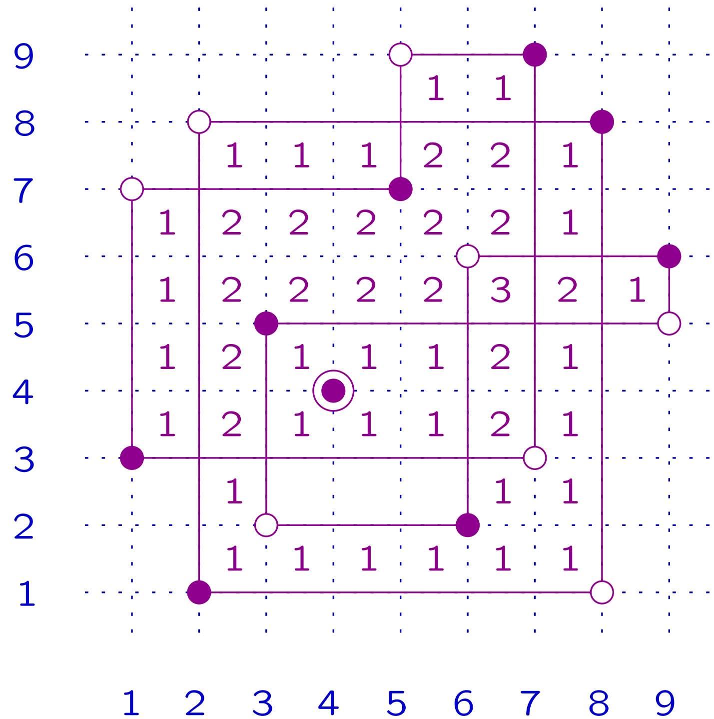

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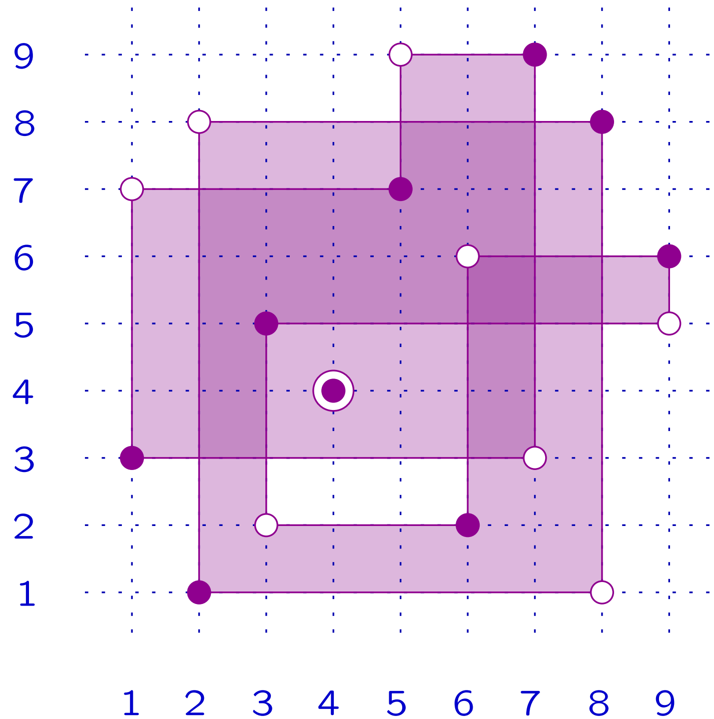


4. FROM THE DIAGRAM TO $[x, y]$

4.1 Symmetries

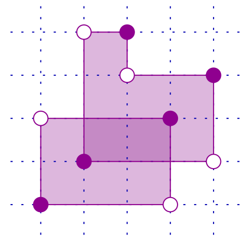
Let W be a Coxeter group.

The mapping $x \mapsto x^{-1}$ is an isomorphism of the Bruhat order.

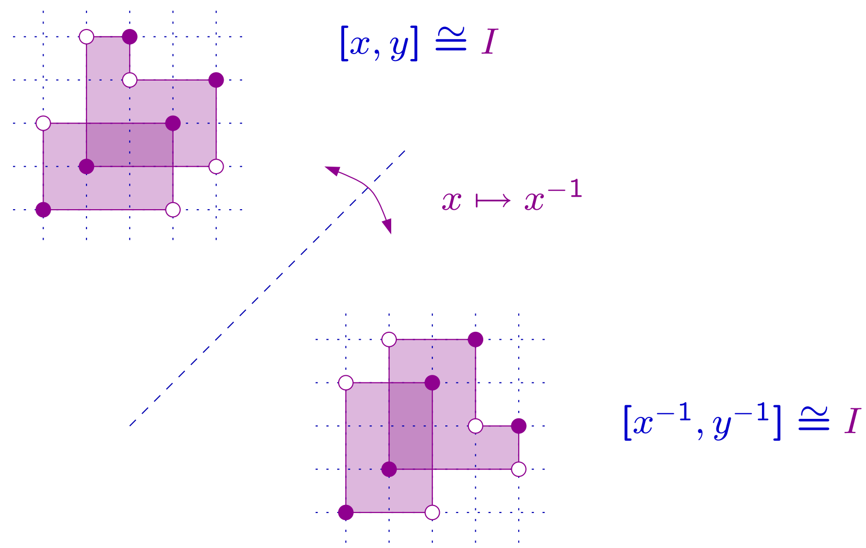
If W is finite, then it has a maximum, denoted by w_0, and

$x \mapsto xw_0$ and $x \mapsto w_0x$ are anti-isomorphisms

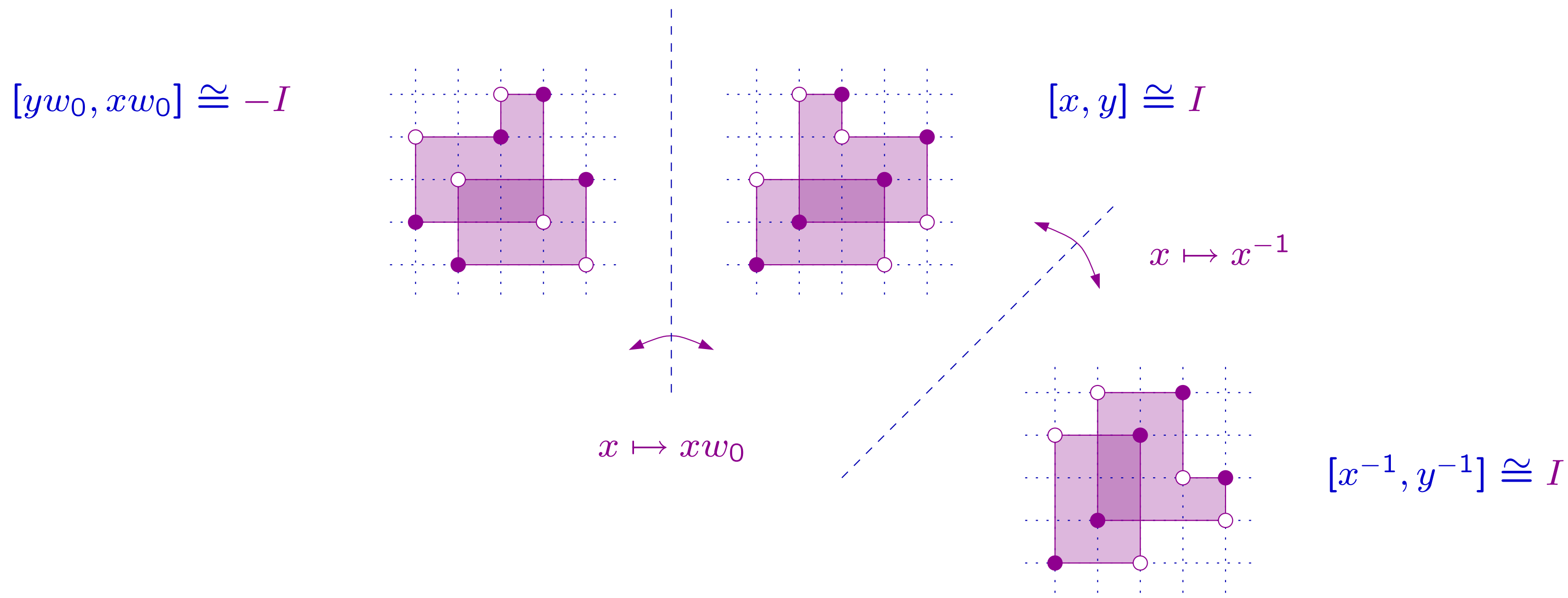
$x \mapsto w_0xw_0$ is an isomorphism.

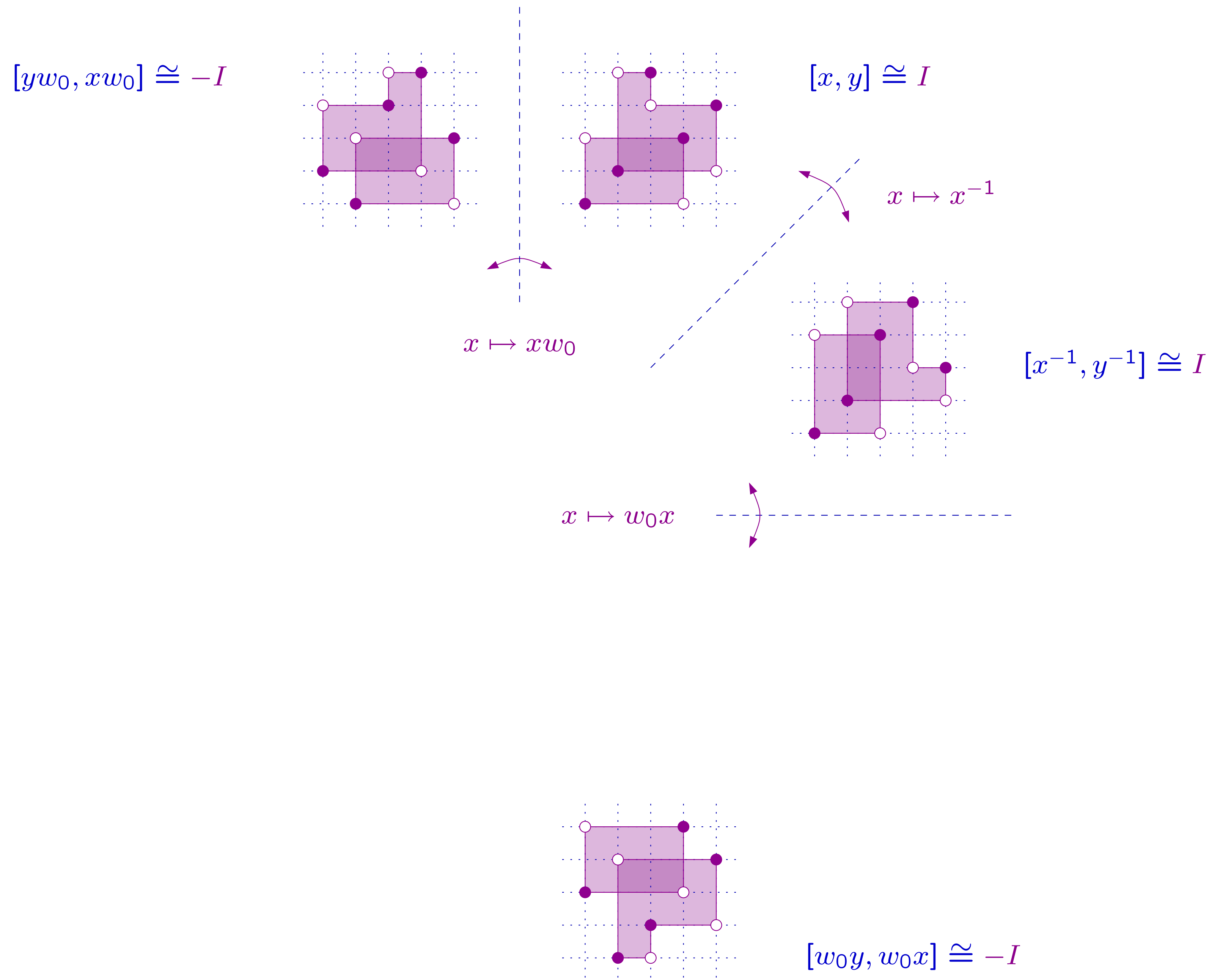


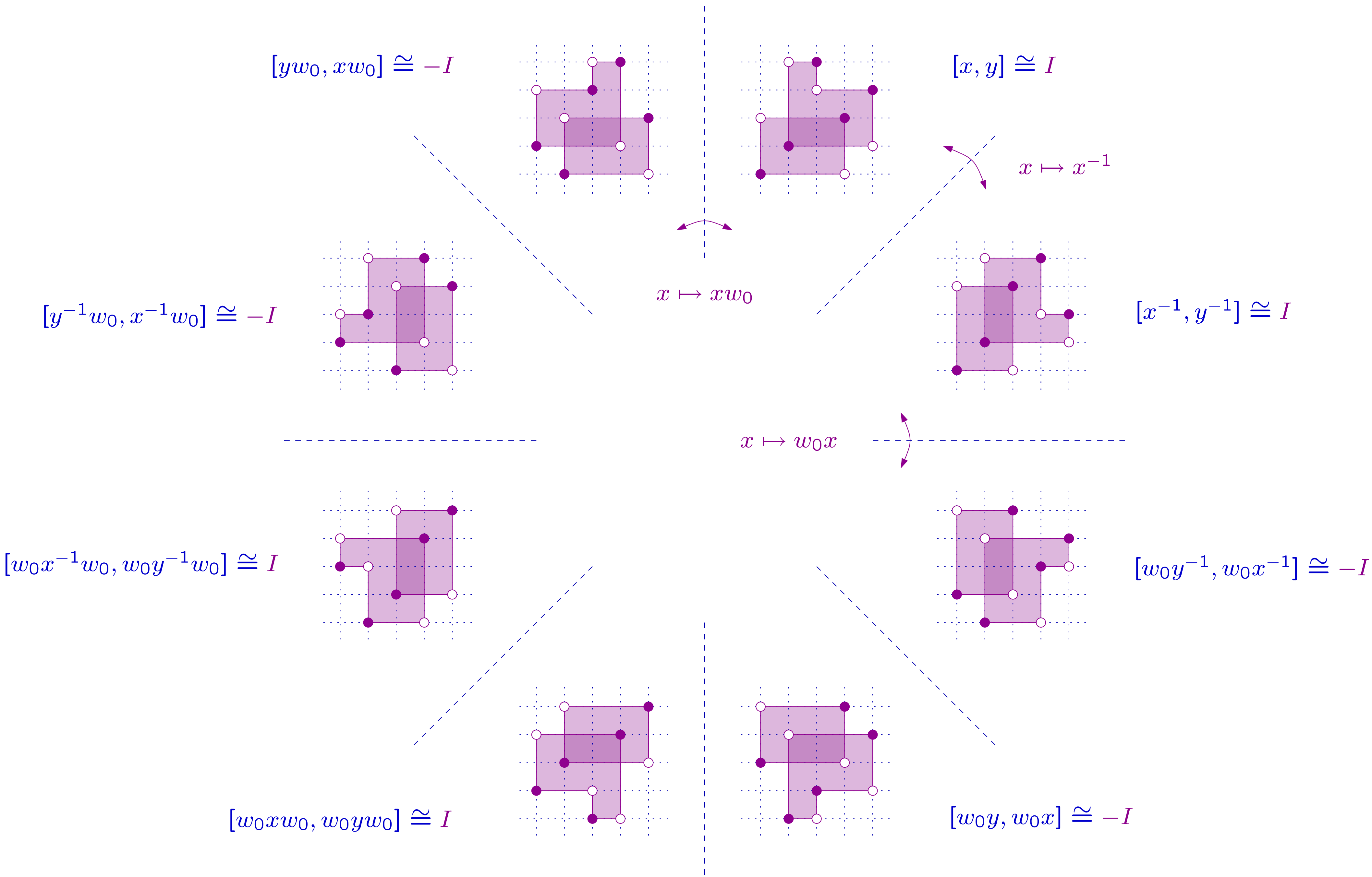
$[x, y] \cong I$



$[x, y] \cong I$
$x \mapsto x^{-1}$
$[x^{-1}, y^{-1}] \cong I$


[yw0, xw0] ≅ −I
[x, y] ≅ I
x ↦ x−1
x ↦ xw0
[x−1, y−1] ≅ I


$[yw_0, xw_0] \cong -I$
$[x,y] \cong I$
$x \mapsto x^{-1}$
$[x^{-1}, y^{-1}] \cong I$
$x \mapsto xw_0$
$x \mapsto w_0x$
$[w_0y, w_0x] \cong -I$



$[yw_0, xw_0] \cong -I$
$[x, y] \cong I$
$x \mapsto x^{-1}$
$[y^{-1}w_0, x^{-1}w_0] \cong -I$
$x \mapsto xw_0$
$[x^{-1}, y^{-1}] \cong I$
$x \mapsto w_0x$
$[w_0x^{-1}w_0, w_0y^{-1}w_0] \cong I$
$[w_0y^{-1}, w_0x^{-1}] \cong -I$
$[w_0xw_0, w_0yw_0] \cong I$
$[w_0y, w_0x] \cong -I$

4.2 Covering relation

Definition Let $x \in S_n$. A *rise* of x is a pair (i, j), with

$$i < j \quad \text{and} \quad x(i) < x(j).$$

A rise (i, j) of x is *free* if there is no $k \in \mathbf{N}$, with

$$i < k < j \quad \text{and} \quad x(i) < x(k) < x(j).$$

Proposition Let $x, y \in S_n$. Then

$$x \lhd y \quad \Leftrightarrow \quad y = x(i, j), \quad \text{with } (i, j) \text{ free rise of } x.$$

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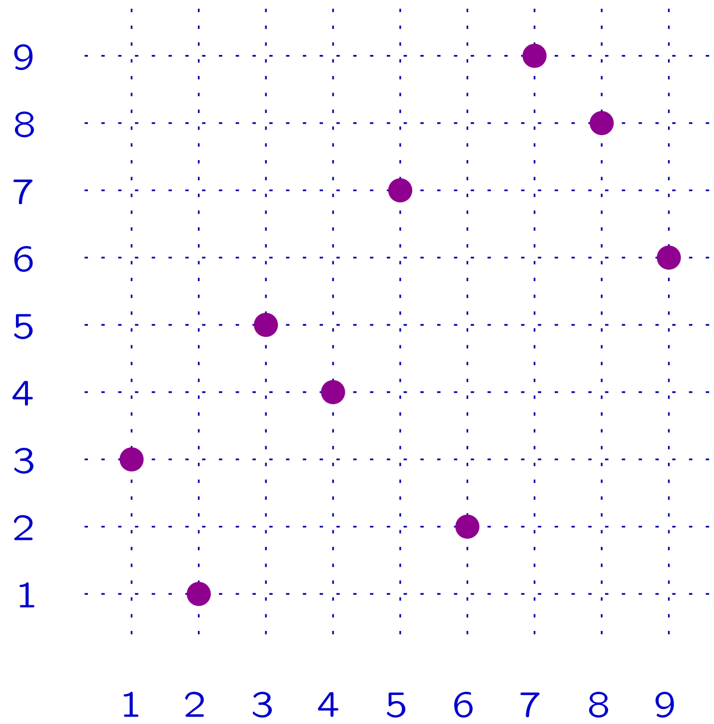
Example $x = 315472986$ ($\bullet$)

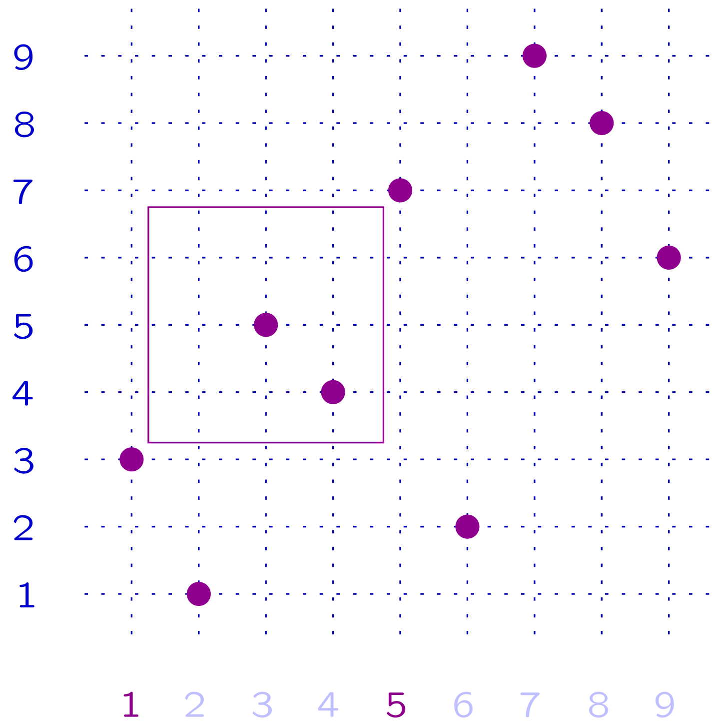


Proposition Let $x, y \in S_n$. Then

$$x \lhd y \quad \Leftrightarrow \quad y = x(i,j), \quad \text{with } (i,j) \text{ free rise of } x.$$

Example $x = 315472986$ $(\bullet)$

$(1,5)$ non-free rise of x

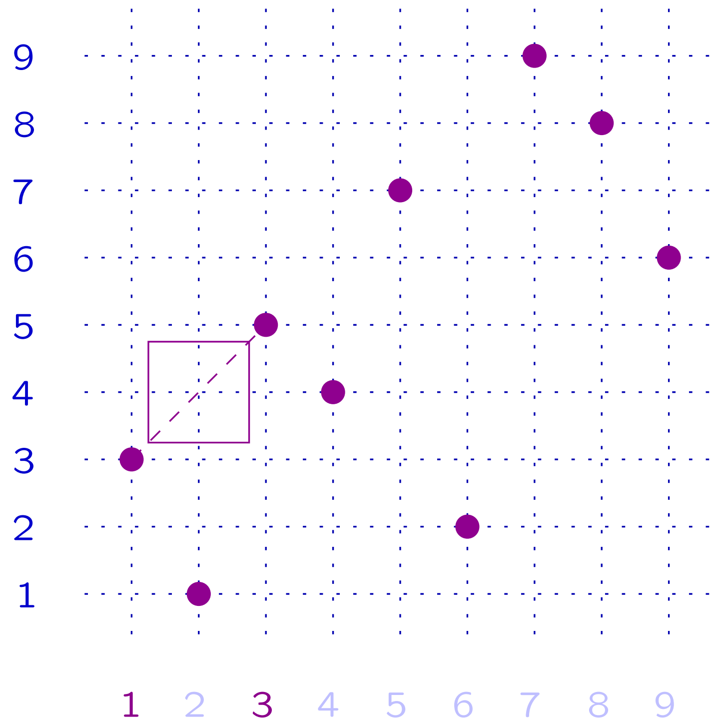


Proposition Let $x, y \in S_n$. Then

$$x \lhd y \quad \Leftrightarrow \quad y = x(i, j), \quad \text{with } (i, j) \text{ free rise of } x.$$

Example $x = 315472986$ ($\bullet$)

$(1, 3)$ free rise of x



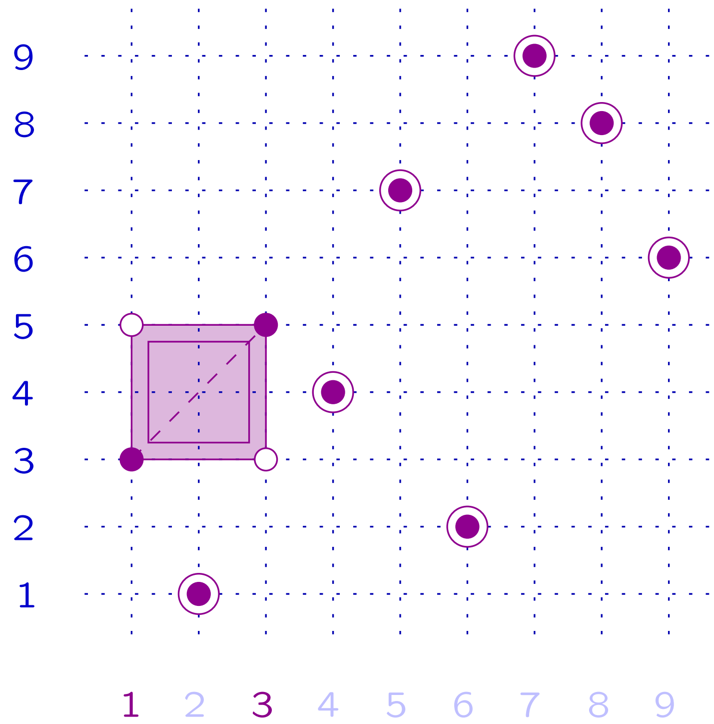
Proposition Let $x, y \in S_n$. Then

$$x \lhd y \quad \Leftrightarrow \quad y = x(i,j), \quad \text{with } (i,j) \text{ free rise of } x.$$

Example $x = 315472986$ $(\bullet)$

$(1,3)$ free rise of x

$y = x(1,3)$ $(\circ)$



Proposition Let $x, y \in S_n$. Then

$$x \lhd y \quad \Leftrightarrow \quad y = x(i,j), \quad \text{with } (i,j) \text{ free rise of } x.$$

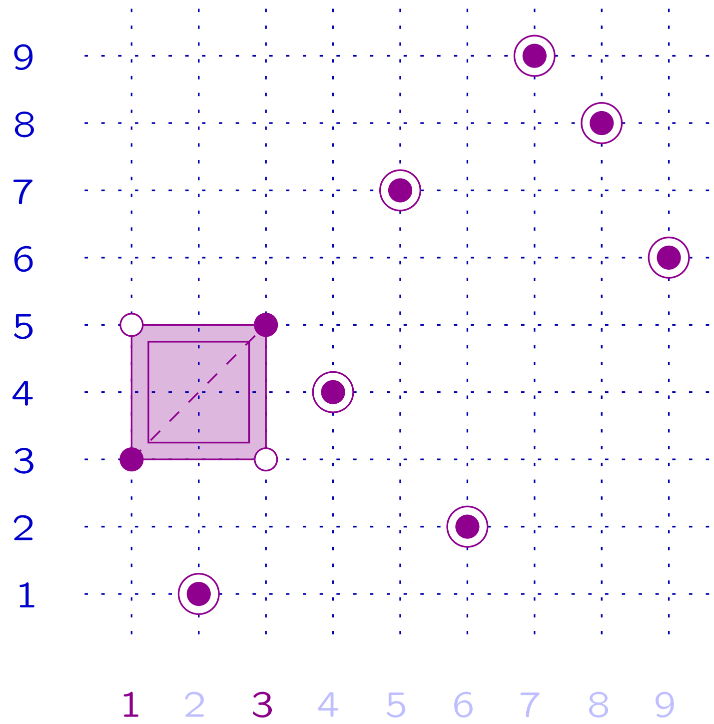
Example $x = 315472986$ $(\bullet)$

$(1,3)$ free rise of x

$y = x(1,3)$ $(\circ)$

$$\Downarrow$$

$$x \lhd y$$

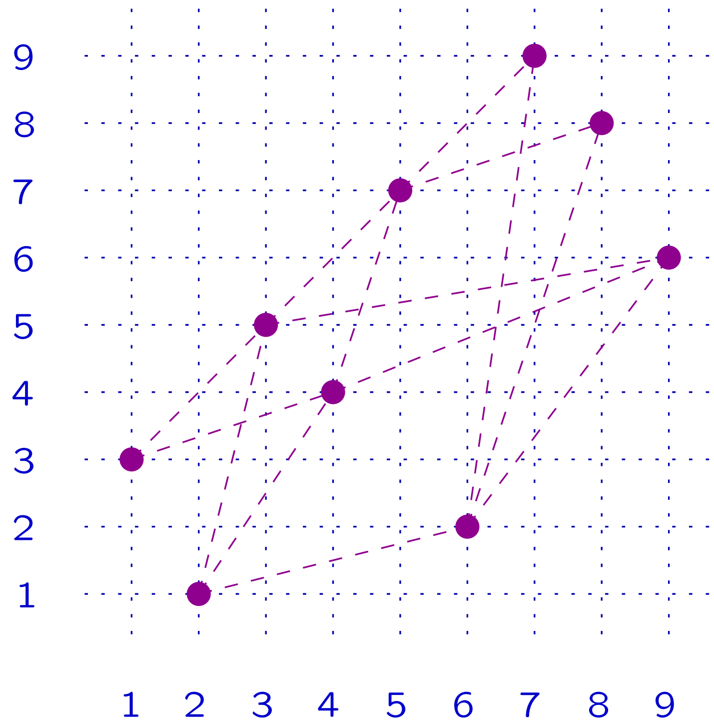


Proposition Let $x, y \in S_n$. Then

$$x \lhd y \quad \Leftrightarrow \quad y = x(i,j), \quad \text{with } (i,j) \text{ free rise of } x.$$

Example $x = 315472986$ $(\bullet)$

x has 14 free rises:



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$$x \lhd y \quad \Leftrightarrow \quad y = x(i,j), \quad \text{with } (i,j) \text{ free rise of } x.$$

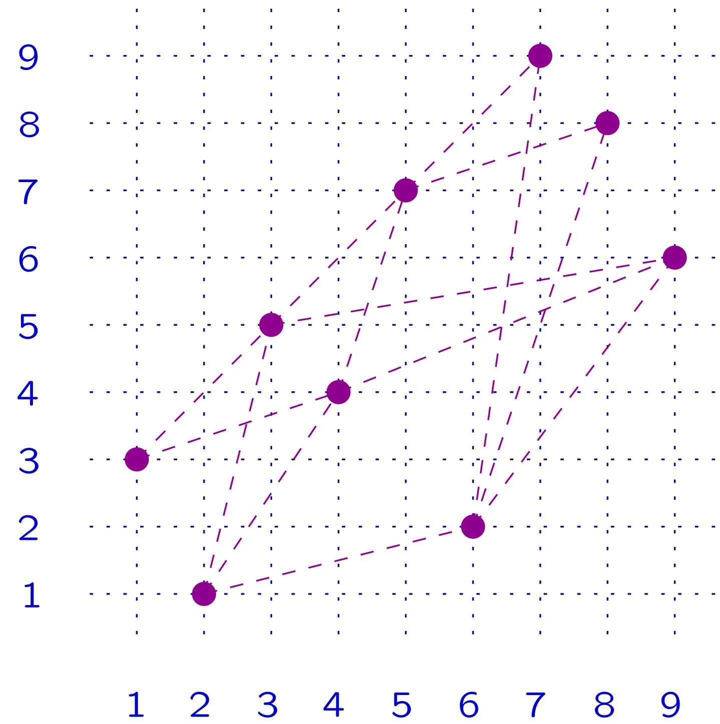
Example $x = 315472986$ $(\bullet)$

x has 14 free rises

$\Downarrow$

x is covered by

14 permutations



4.3 Atoms and coatoms

Definition Let (i, j) be a free rise of x. The *rectangle associated* is

$$Rect_x(i, j) = \{(h, k) \in \mathbf{R}^2 : i \leq h < j\ ,\ x(i) < k \leq x(j)\}.$$

Let $x, y \in S_n$, with $x < y$. A free rise (i, j) of x is *good* w.r.t. y if

$$Rect_x(i, j) \subseteq \Omega(x, y).$$

Proposition Let $x, y \in S_n$, with $x < y$. Then

z atom of $[x, y]$ $\Leftrightarrow$ $z = x(i, j)$, with (i, j) free rise of x, good with respect to y.

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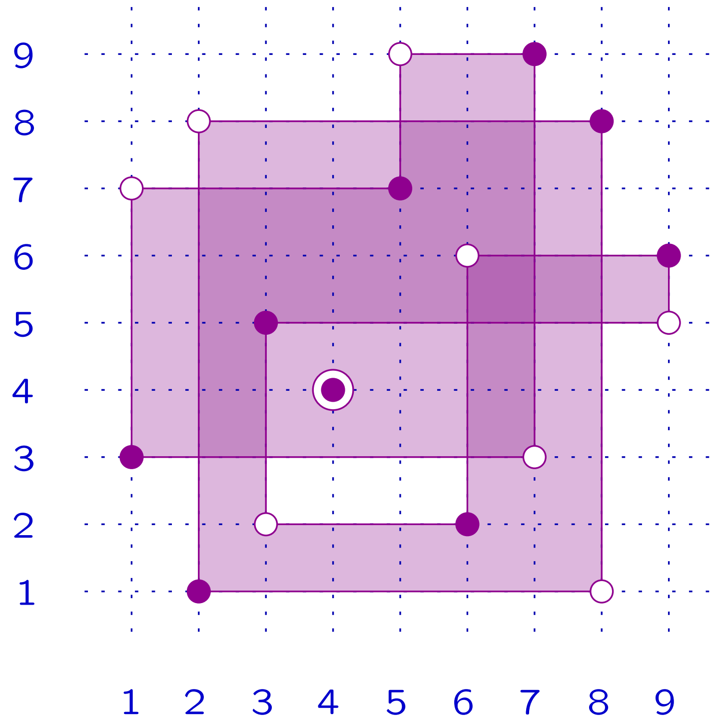
Example $x = 315472986$ ($\bullet$)

$y = 782496315$ ($\circ$)



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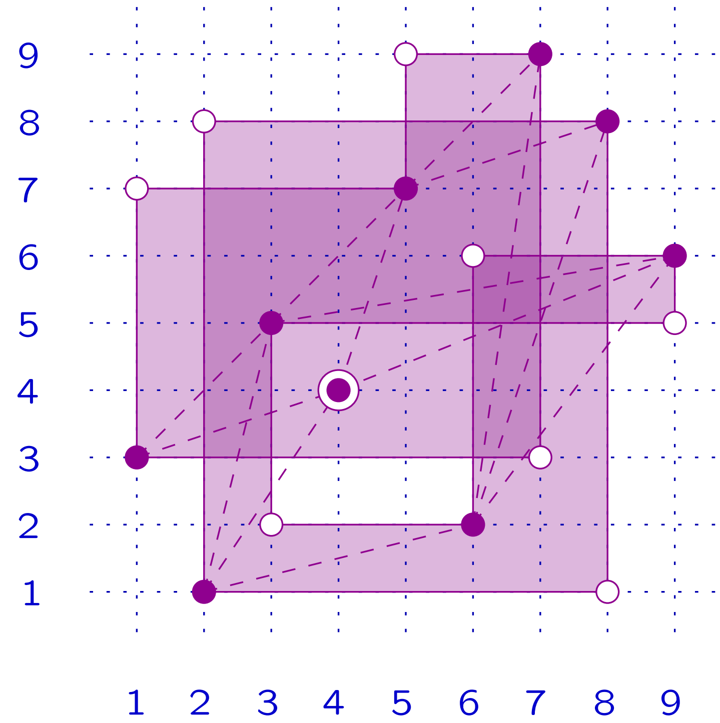
Example $x = 315472986$ ($\bullet$)

$y = 782496315$ ($\circ$)

Among the 14 free rises of x



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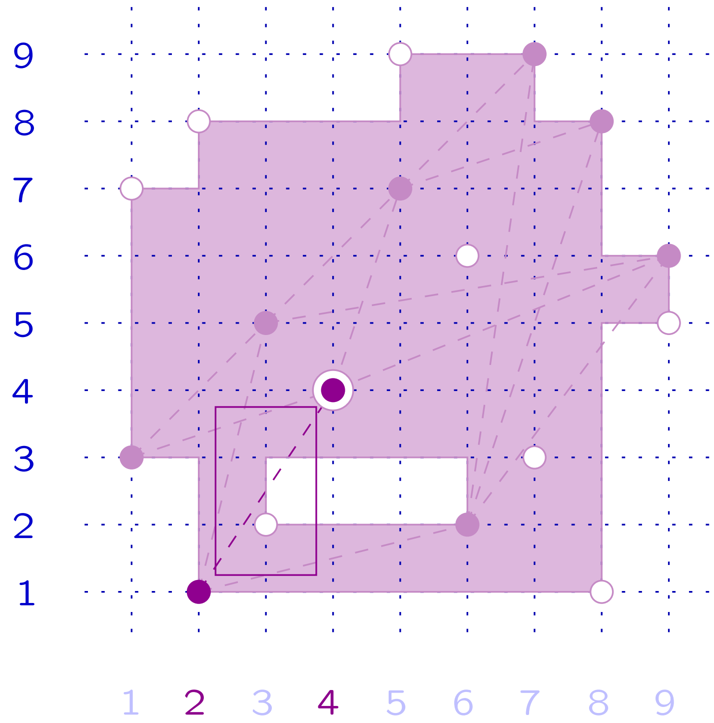
Example $x = 315472986$ ($\bullet$)

$y = 782496315$ ($\circ$)

Among the 14 free rises of x those non-good w.r.t. y are

(2, 4)



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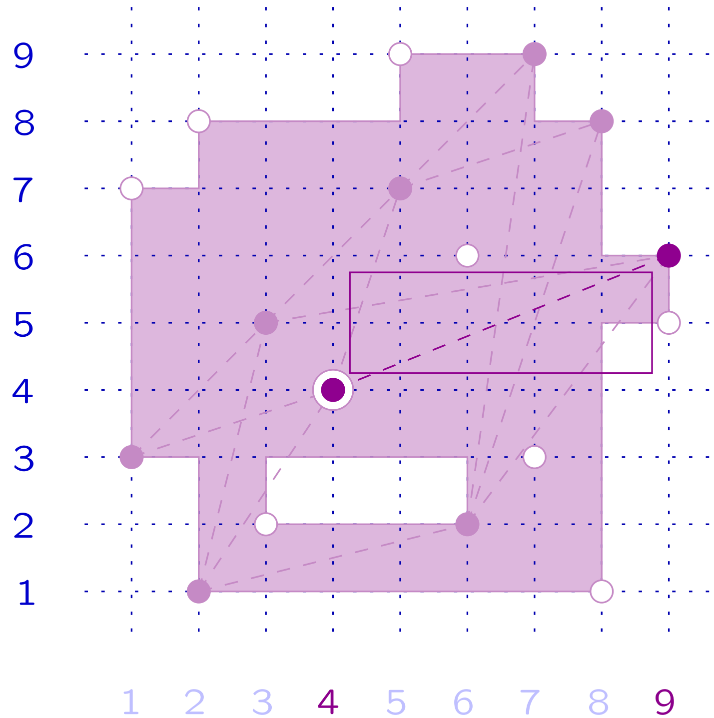
Example $x = 315472986$ ($\bullet$)

$y = 782496315$ ($\circ$)

Among the 14 free rises of x those non-good w.r.t. y are

(2, 4), (4, 9)



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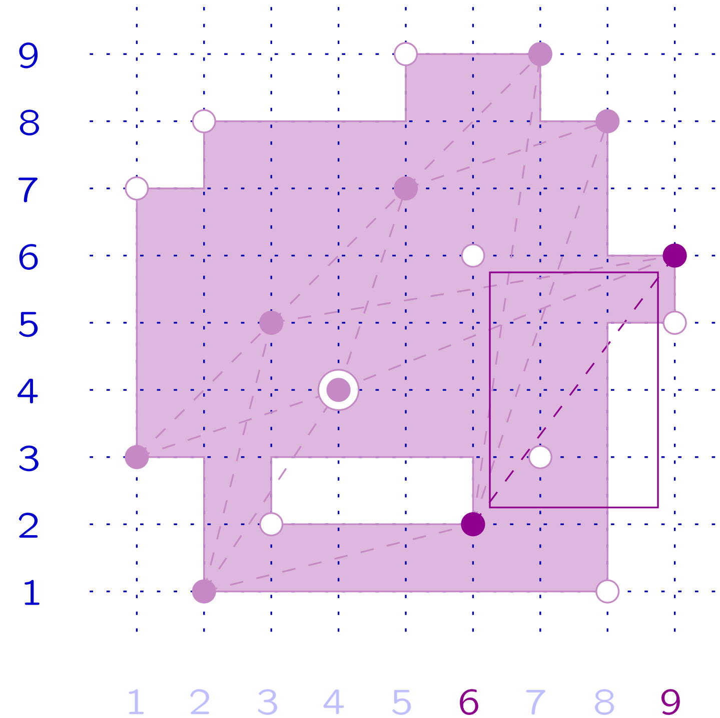
Example $x = 315472986$ ($\bullet$)

$y = 782496315$ ($\circ$)

Among the 14 free rises of x those non-good w.r.t. y are (2, 4), (4, 9) and (6, 9).



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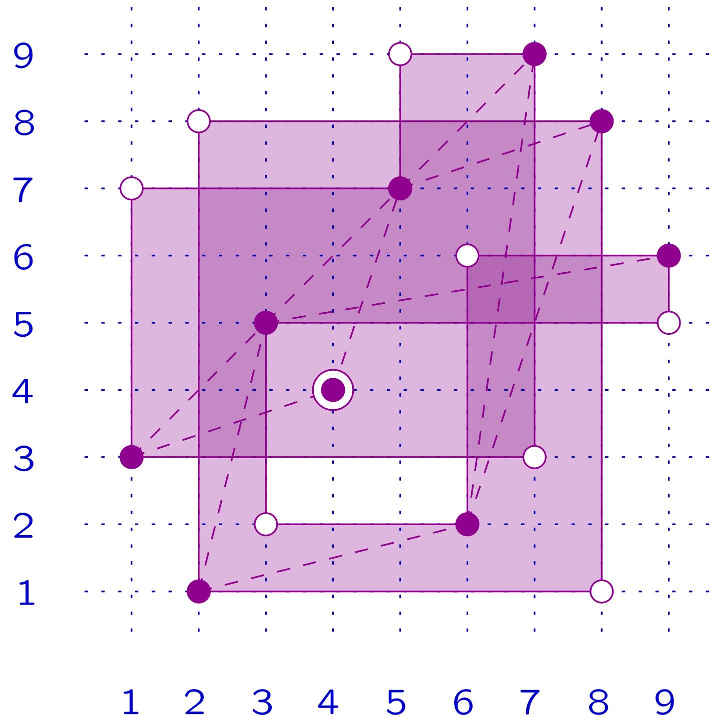
Example $x = 315472986$ ($\bullet$)

$y = 782496315$ ($\circ$)

x has 11 free rises

good w.r.t. y:



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Example $x = 315472986$ $(\bullet)$

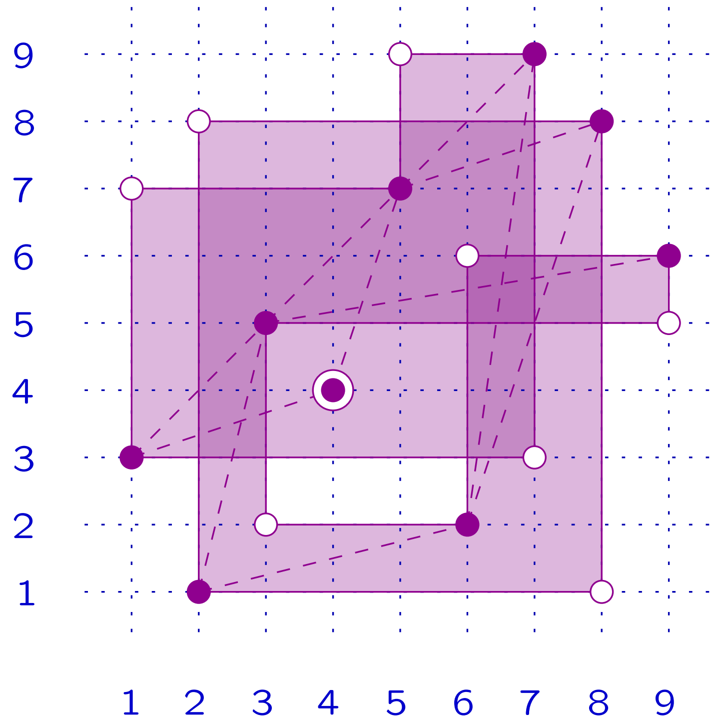
$y = 782496315$ $(\circ)$

x has 11 free rises

good w.r.t. y

$\Downarrow$

$[x, y]$ has 11 atoms



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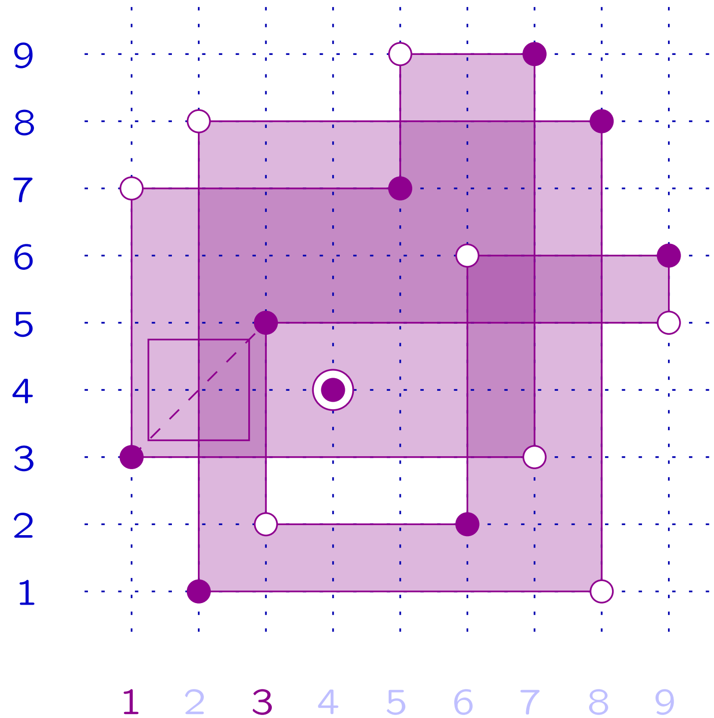
Example $x = 315472986$ ($\bullet$)

$y = 782496315$ ($\circ$)

$(1,3)$ free rise of x

good w.r.t. y



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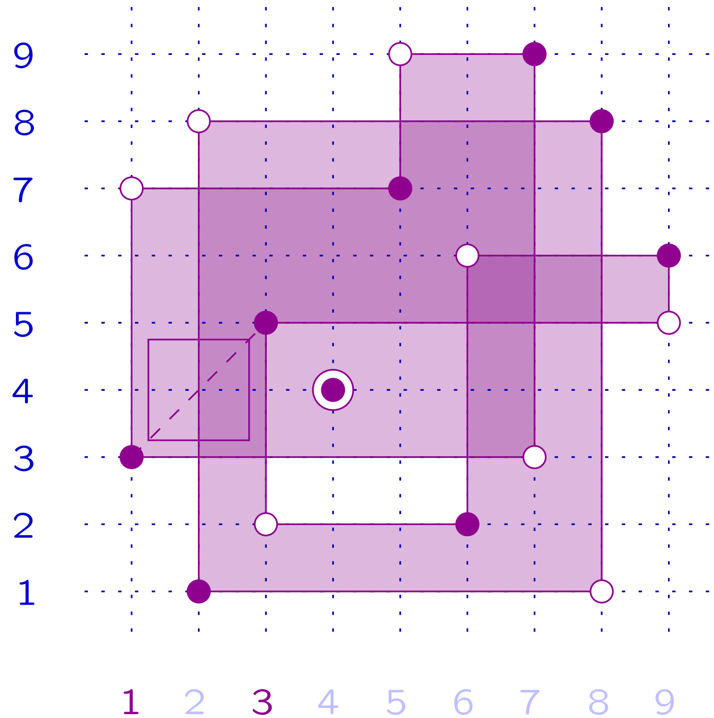
Example $x = 315472986$ ($\bullet$)

$y = 782496315$ ($\circ$)

$(1,3)$ free rise of x

good w.r.t. y

$z = x(1,3)$



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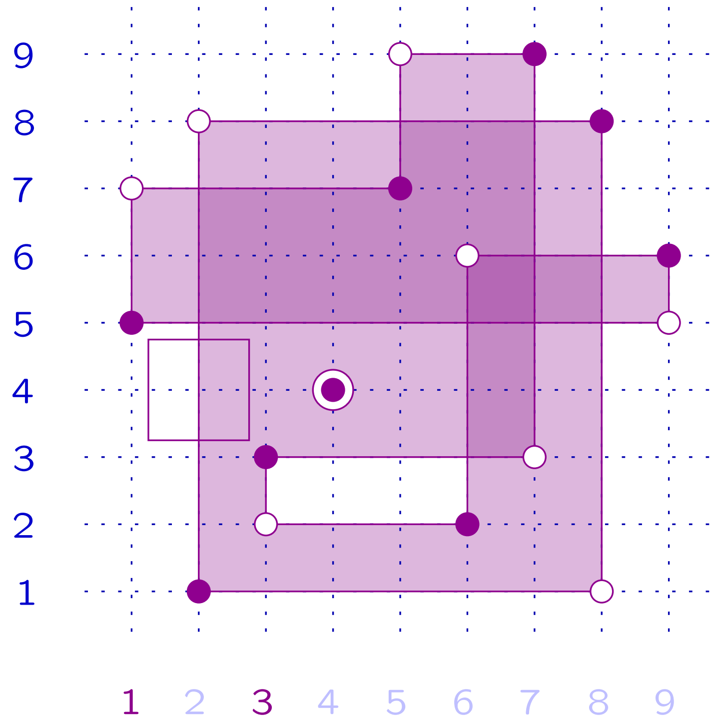
Example $x = 315472986$

$y = 782496315$ $(\circ)$

$(1, 3)$ free rise of x

good w.r.t. y

$z = x(1, 3)$ $(\bullet)$



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Example $x = 315472986$

$y = 782496315$ $(\circ)$

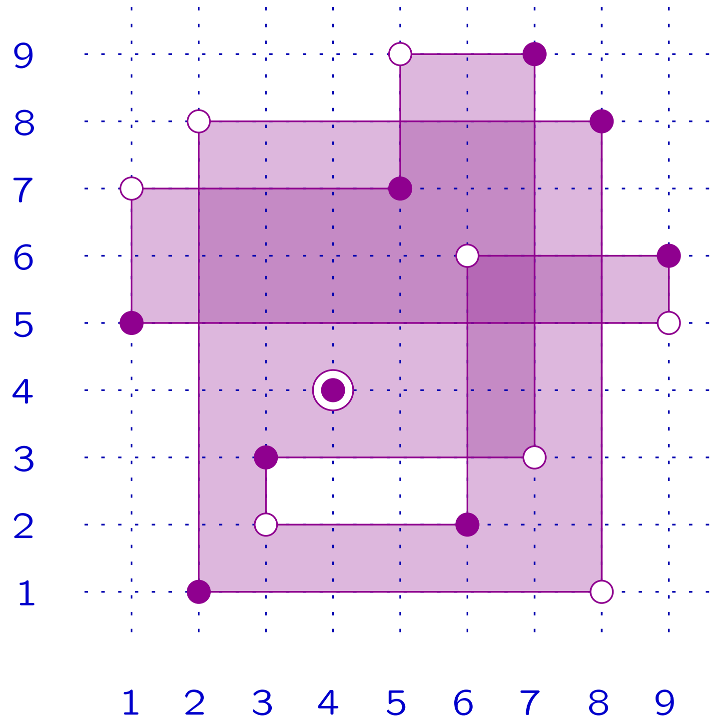
$(1, 3)$ free rise of x

good w.r.t. y

$z = x(1, 3)$ $(\bullet)$

$\Downarrow$

z atom of $[x, y]$



Proposition Let $x, y \in S_n$, with $x < y$. Then

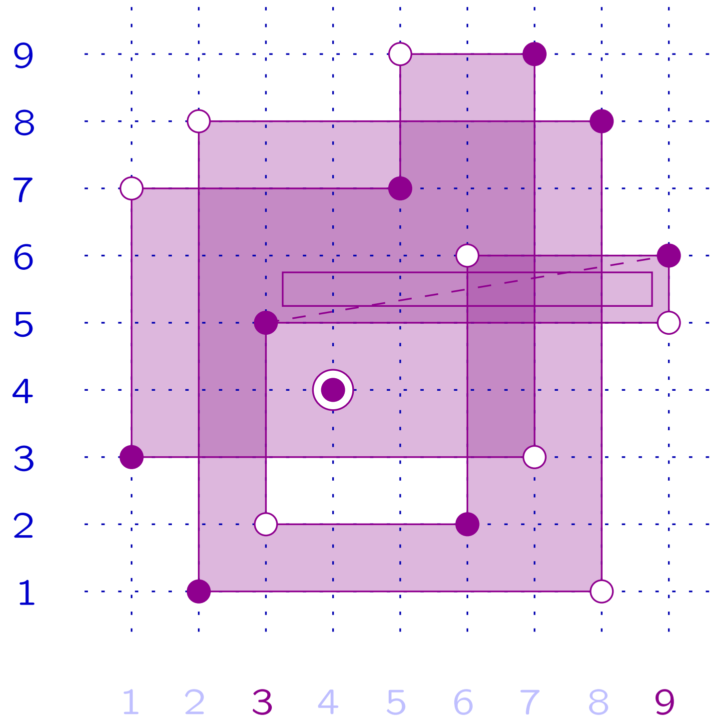
z atom of $[x, y]$ $\Leftrightarrow$ $z = x(i, j)$, with (i, j) free rise of x, good with respect to y.

Example $x = 315472986$ ($\bullet$)

$y = 782496315$ ($\circ$)

$(3, 9)$ free rise of x

good w.r.t. y



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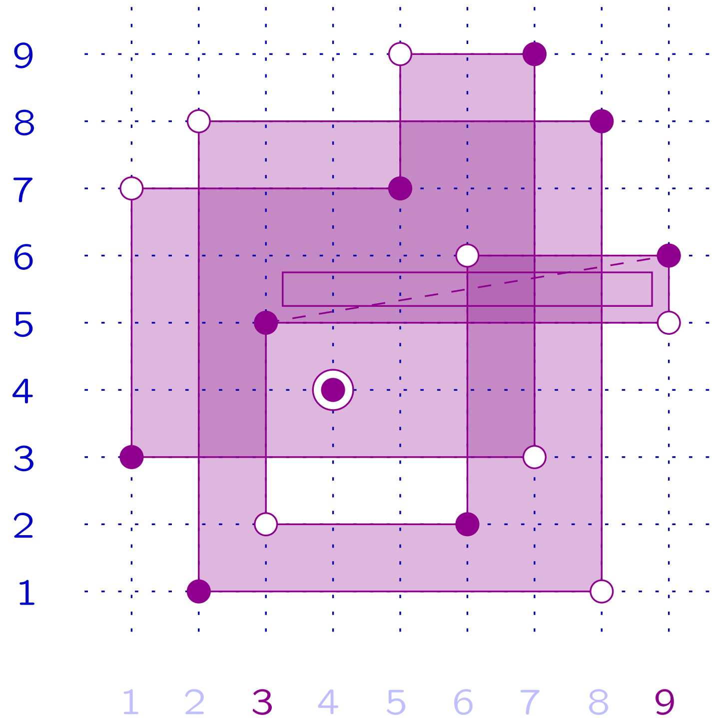
$(3, 9)$ free rise of x

good w.r.t. y

$z_1 = x(3, 9)$



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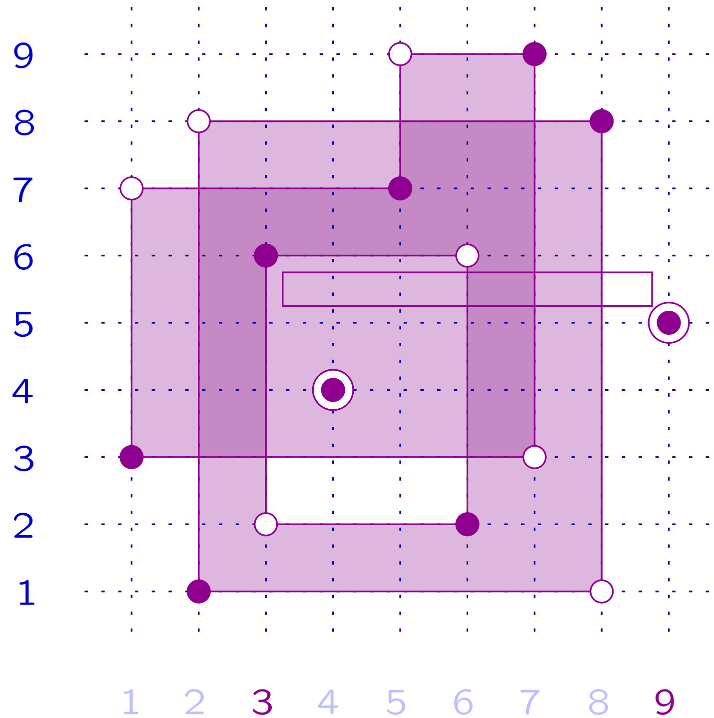

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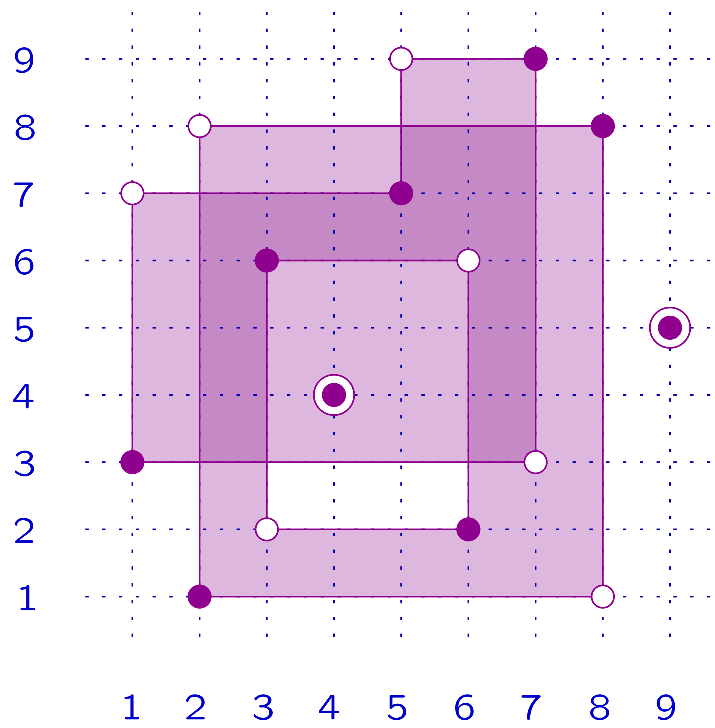
$(3, 9)$ free rise of x

good w.r.t. y

$z_1 = x(3, 9)$ $(\bullet)$

$\Downarrow$

z_1 atom of $[x, y]$



Proposition Let $x, y \in S_n$, with $x < y$. Then

w coatom of $[x, y]$ $\Leftrightarrow$ $w = y(i, j)$, with (i, j) free inversion of y, good with respect to x.

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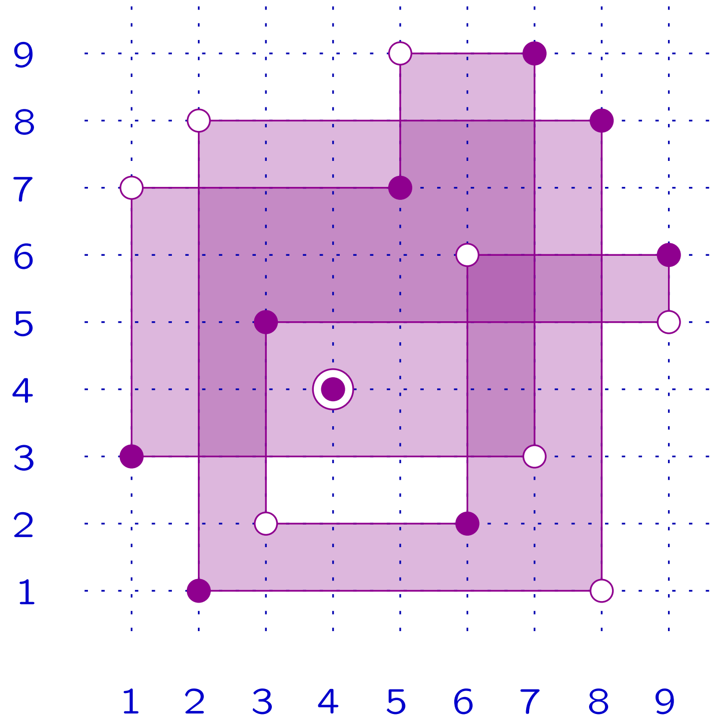

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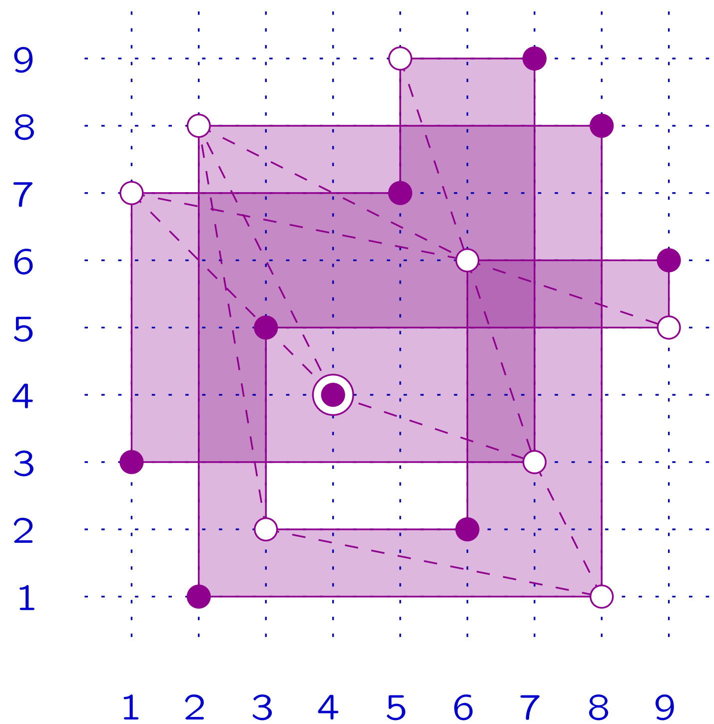
y has 11 free inversions

good w.r.t. x:



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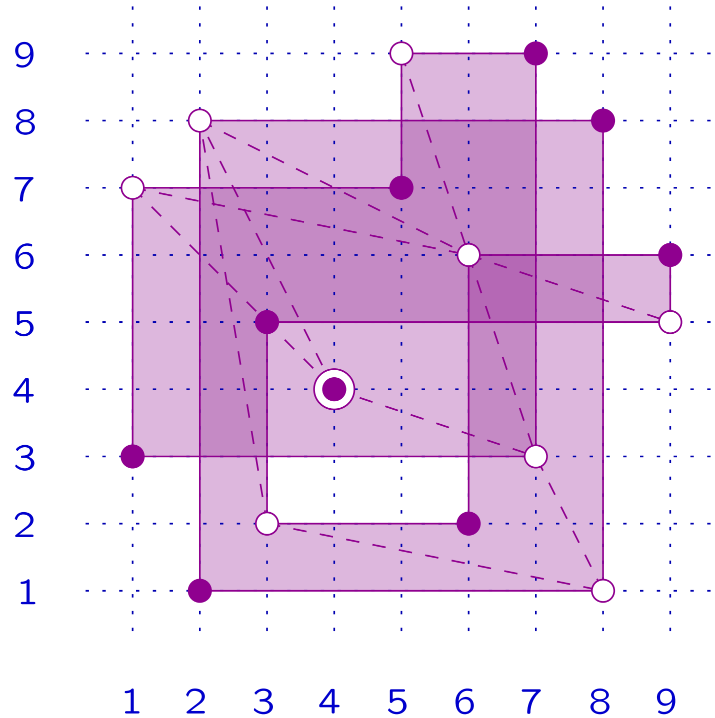
$y = 782496315$ ($\circ$)

y has 11 free inversions

good w.r.t. x

$\Downarrow$

$[x, y]$ has 11 coatoms



5. FROM THE DIAGRAM TO $\widetilde{R}_{x,y}(q)$

5.1 Symmetries

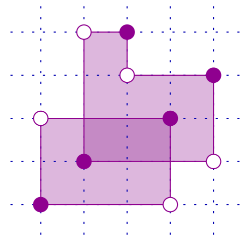
Let W be a Coxeter group.

Proposition Let $x, y \in W$, $x < y$. Then

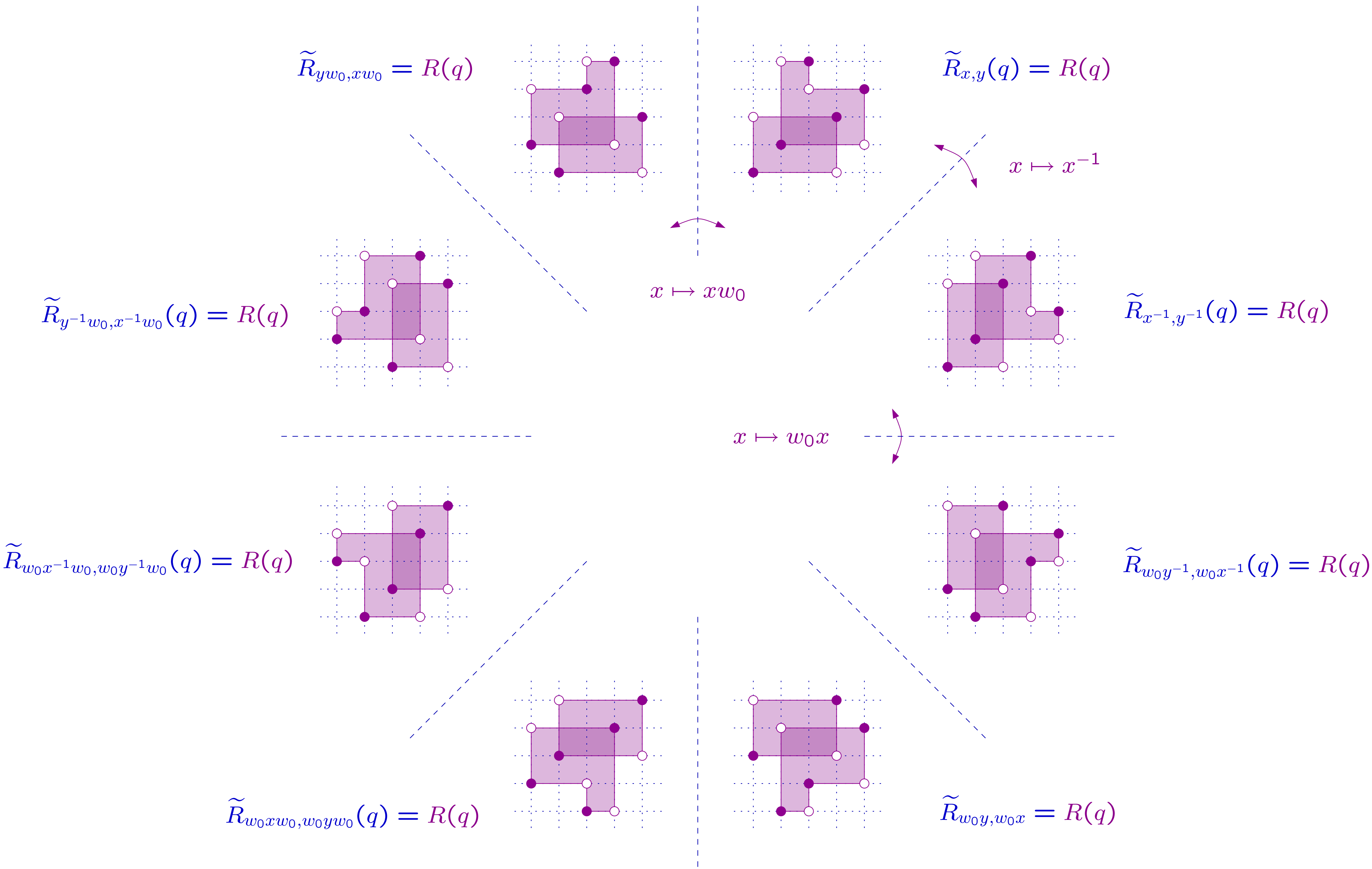
$$\widetilde{R}_{x,y}(q) \;=\; \widetilde{R}_{x^{-1},\,y^{-1}}(q).$$

If W is finite, then

$$\begin{aligned}\widetilde{R}_{x,y}(q) &= \widetilde{R}_{yw_0\,,\,xw_0}(q)\\ &= \widetilde{R}_{w_0y\,,\,w_0x}(q)\\ &= \widetilde{R}_{w_0xw_0\,,\,w_0yw_0}(q).\end{aligned}$$



$$\widetilde{R}_{x,y}(q) = R(q)$$


$\widetilde{R}_{yw_0,xw_0} = R(q)$
$\widetilde{R}_{x,y}(q) = R(q)$
$x \mapsto x^{-1}$
$x \mapsto xw_0$
$\widetilde{R}_{y^{-1}w_0,x^{-1}w_0}(q) = R(q)$
$\widetilde{R}_{x^{-1},y^{-1}}(q) = R(q)$
$x \mapsto w_0x$
$\widetilde{R}_{w_0x^{-1}w_0,w_0y^{-1}w_0}(q) = R(q)$
$\widetilde{R}_{w_0y^{-1},w_0x^{-1}}(q) = R(q)$
$\widetilde{R}_{w_0xw_0,w_0yw_0}(q) = R(q)$
$\widetilde{R}_{w_0y,w_0x} = R(q)$

5.2 Reflection ordering in S_n

In the symmetric group S_n the reflections are the transpositions:

$$T = \{(i,j) : i,j \in [n]\}.$$

Proposition [Dyer] A possible reflection ordering $\prec$ on the transpositions of S_n is the lexicographic order.

Assume this order $\prec$ fixed on T. For example, in S_4:

$$(1,2) \prec (1,3) \prec (1,4) \prec (2,3) \prec (2,4) \prec (3,4).$$

5.3 Edges of the Bruhat graph

$x \xrightarrow{(i,j)} y$ in S_n means $y = x(i,j)$, with (i,j) rise of x.

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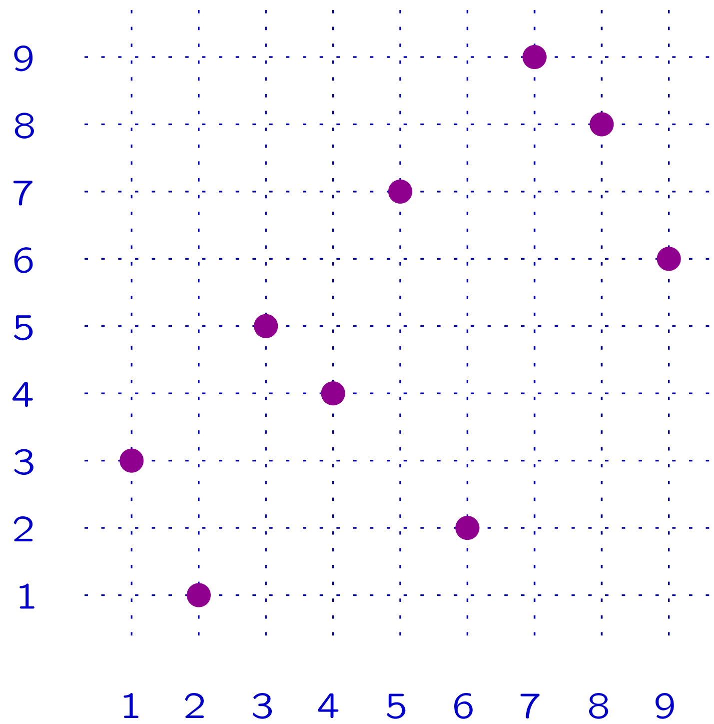
$$x \xrightarrow{(i,j)} y \text{ in } S_n \quad \text{means} \quad y = x(i,j), \quad \text{with } (i,j) \text{ rise of } x.$$

Example $x = 315472986$ (•)

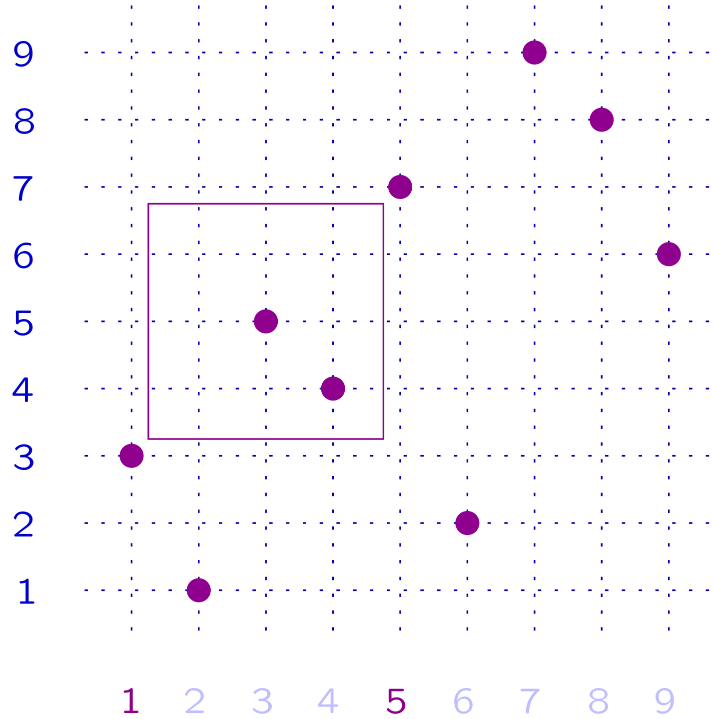


5.3 Edges of the Bruhat graph

$x \xrightarrow{(i,j)} y$ in S_n means $y = x(i,j)$, with (i,j) rise of x.

Example $x = 315472986$ (•)

$(1,5)$ rise of x



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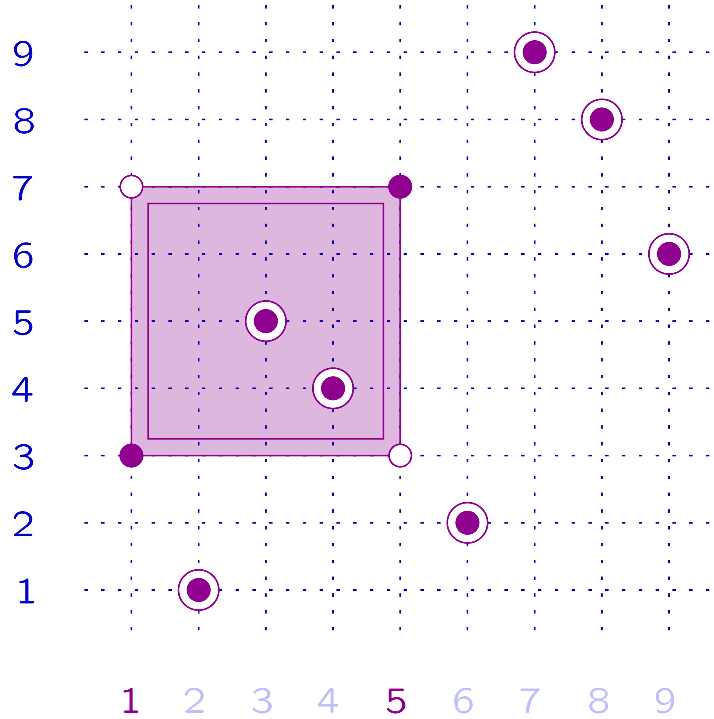

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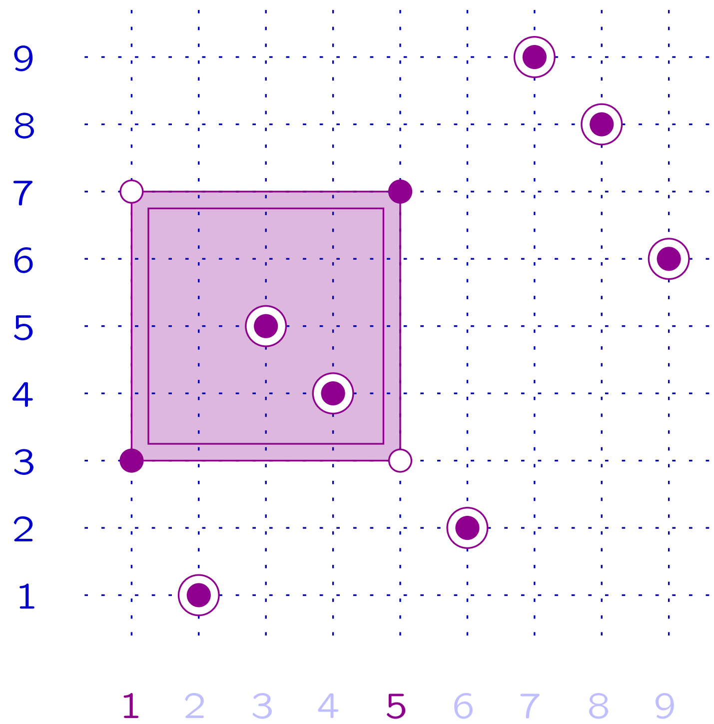
Example $x = 315472986$ ($\bullet$)

$(1,5)$ rise of x

$y = x(1,5)$ ($\circ$)

$\Downarrow$

$x \xrightarrow{(1,5)} y$



5.4 Increasing paths

Let $x, y \in S_n$, with $x < y$. An increasing path in BG from x to y is

$$x = x_0 \xrightarrow{(i_1,j_1)} x_1 \xrightarrow{(i_2,j_2)} \cdots \xrightarrow{(i_k,j_k)} x_k = y\,,$$

with $(i_1, j_1) \prec (i_2, j_2) \prec \cdots \prec (i_k, j_k)\,.$

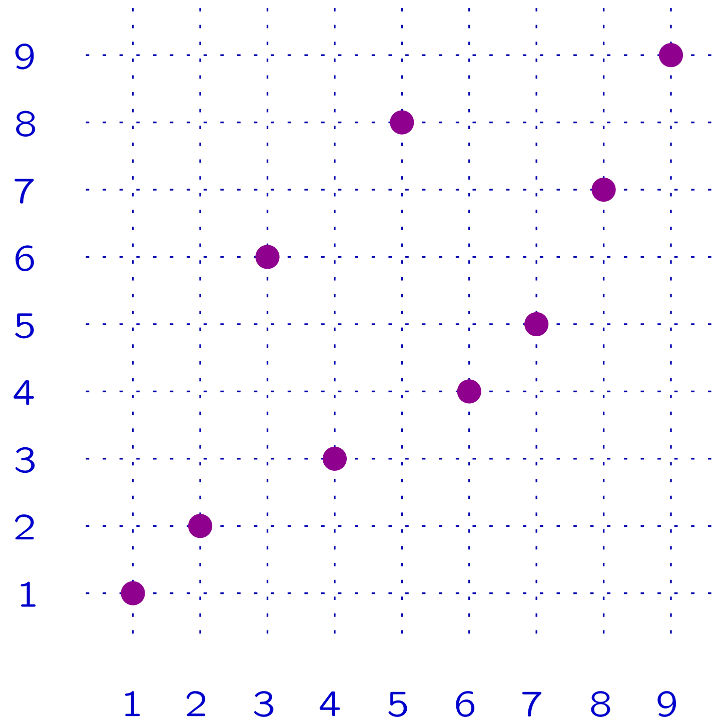
Special case: $i_1 = i_2 = \cdots = i_k = i$

$$x = x_0 \xrightarrow{(i,j_1)} x_1 \xrightarrow{(i,j_2)} \cdots \xrightarrow{(i,j_k)} x_k = y\,,$$

with $i < j_1 < j_2 < \cdots < j_k\,.$ Call it a *stair path*.

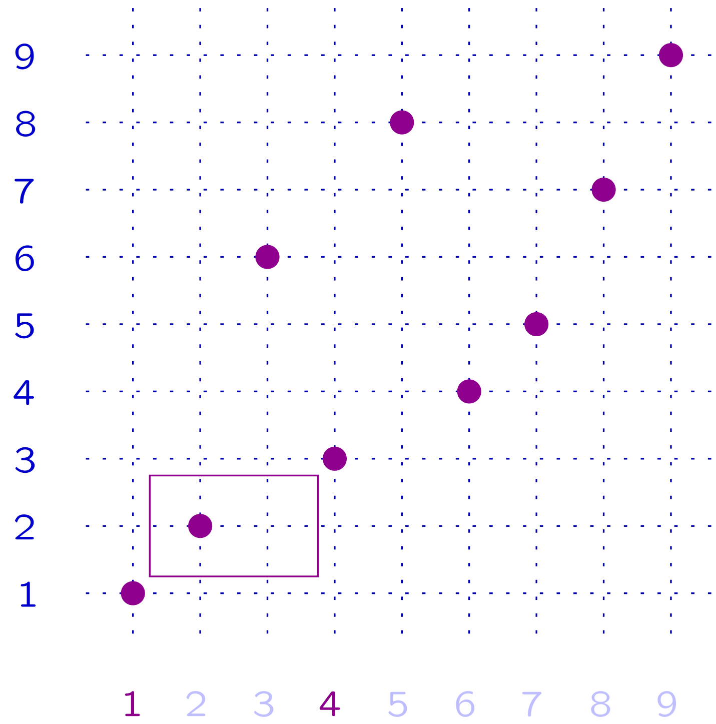
General case: an increasing path is a sequence of stair paths.

Example $x = 126384579$ (•)



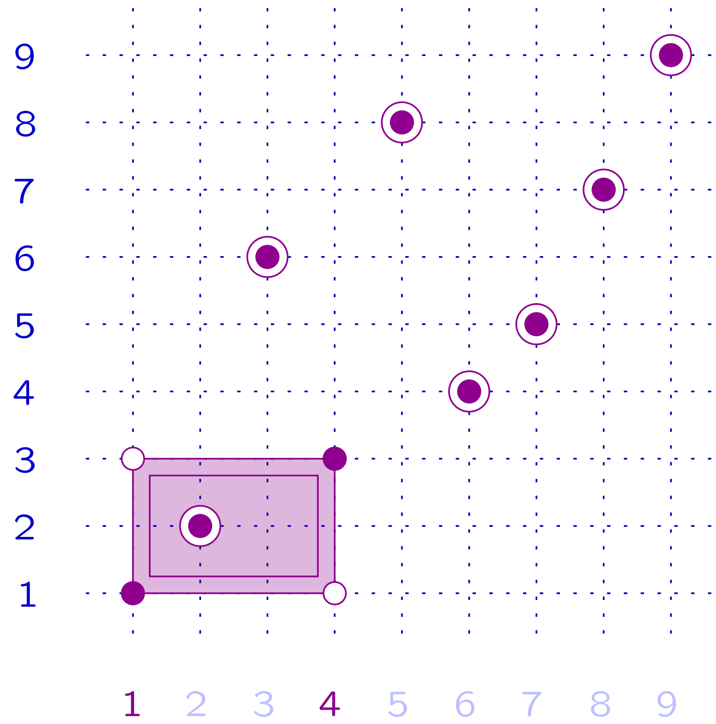
Example $x = 126384579$ (•)

$(1, 4)$ rise of x



Example $x = 126384579$ $(\bullet)$

$(1,4)$ rise of x



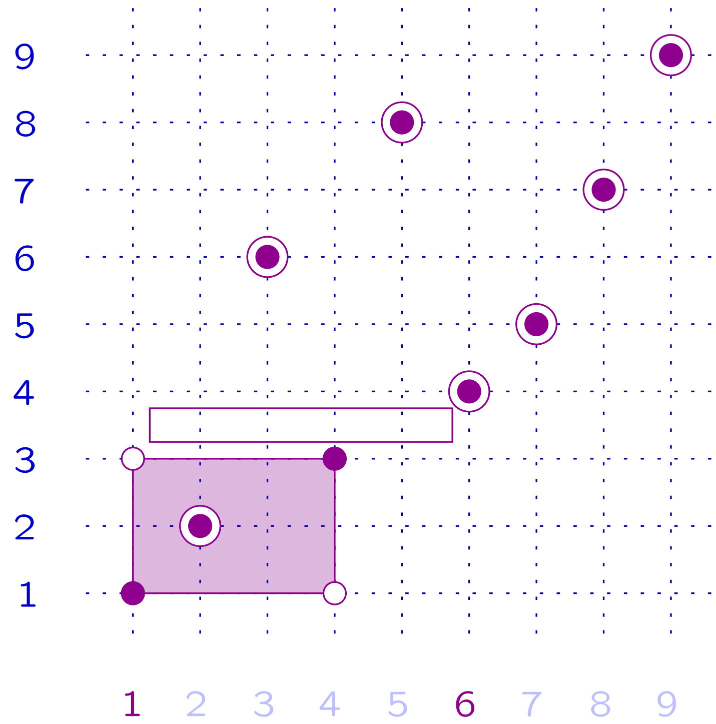
$$x \xrightarrow{(1,4)} x_1$$

$(\bullet)$ $(\circ)$

Example $x = 126384579$ $(\bullet)$

$(1,4)$ rise of x

$(1,6)$ rise of x_1



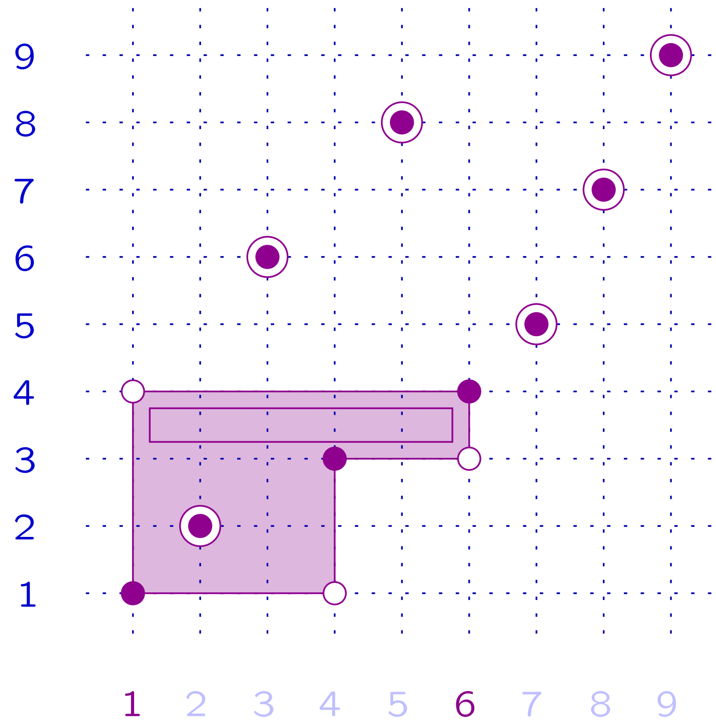
$$x \xrightarrow{(1,4)} x_1$$

$(\bullet)$ $(\circ)$

Example $x = 126384579$ $(\bullet)$

$(1, 4)$ rise of x

$(1, 6)$ rise of x_1



$$x \xrightarrow{(1,4)} x_1 \xrightarrow{(1,6)} x_2$$

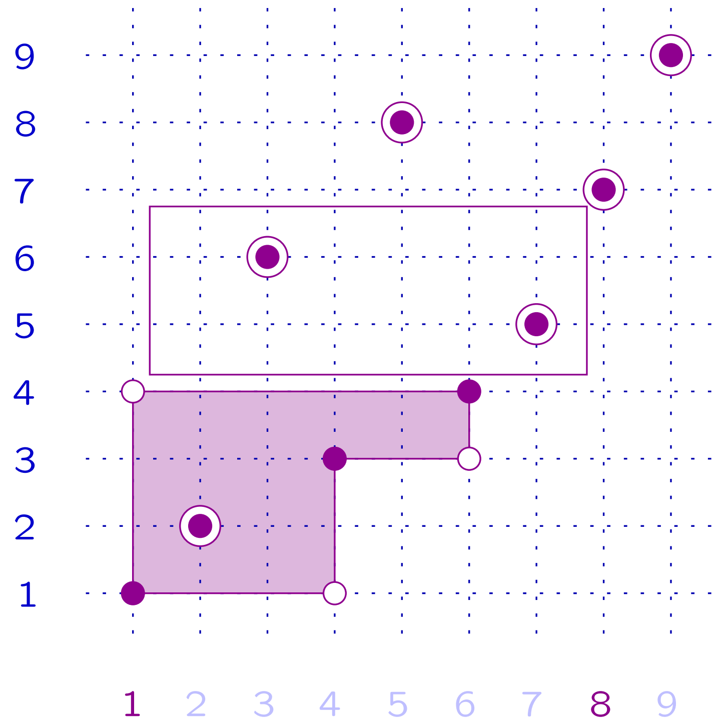
$(\bullet)$ $(\circ)$

Example $x = 126384579$ $(\bullet)$

$(1,4)$ rise of x

$(1,6)$ rise of x_1

$(1,8)$ rise of x_2



$$x \xrightarrow{(1,4)} x_1 \xrightarrow{(1,6)} x_2$$

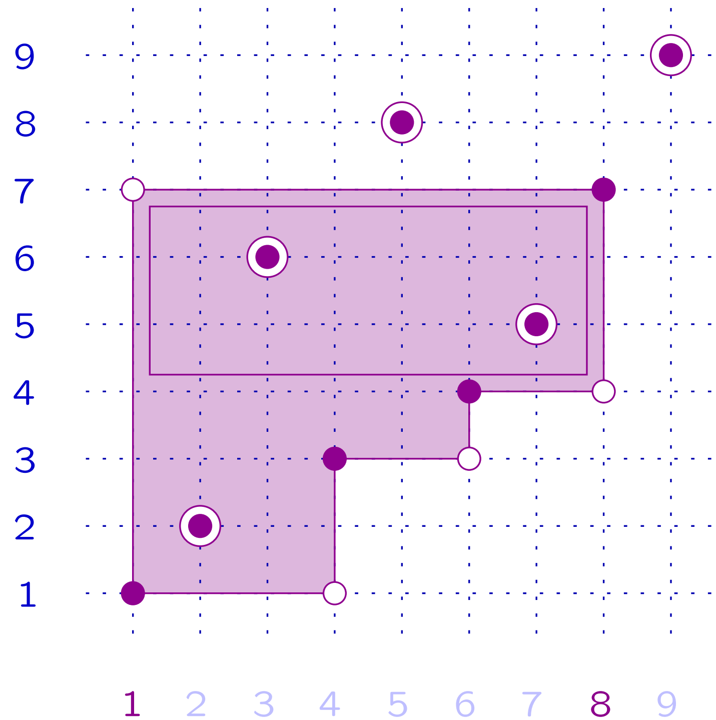
$(\bullet)$ $(\circ)$

Example $x = 126384579$ $(\bullet)$

$(1,4)$ rise of x

$(1,6)$ rise of x_1

$(1,8)$ rise of x_2



$$x \xrightarrow{(1,4)} x_1 \xrightarrow{(1,6)} x_2 \xrightarrow{(1,8)} x_3$$

$(\bullet)$ $(\circ)$

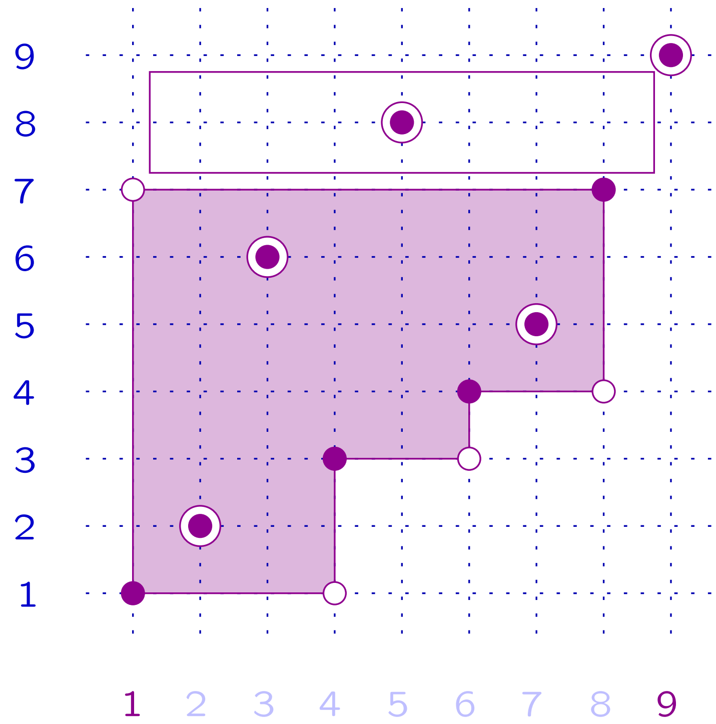
Example $x = 126384579$ $(\bullet)$

$(1,4)$ rise of x

$(1,6)$ rise of x_1

$(1,8)$ rise of x_2

$(1,9)$ rise of x_3



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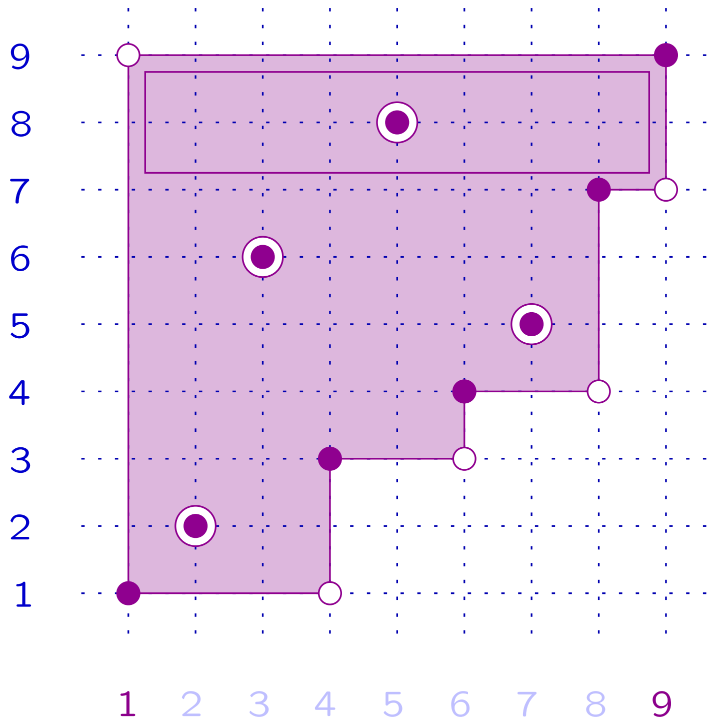
$(1,9)$ rise of x_3

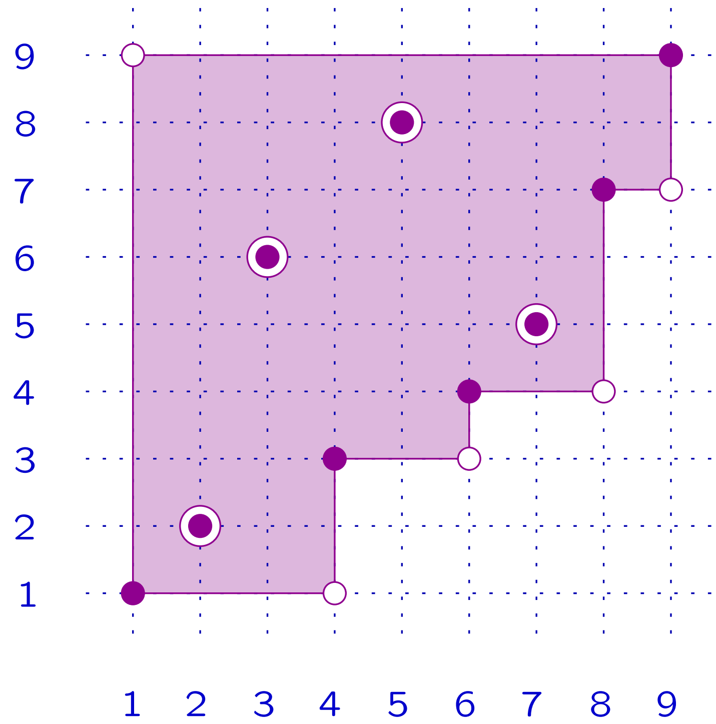


$$x \xrightarrow{(1,4)} x_1 \xrightarrow{(1,6)} x_2 \xrightarrow{(1,8)} x_3 \xrightarrow{(1,9)} y$$

$(\bullet)$ $(\circ)$

Example $x = 126384579$ $(\bullet)$

Stair diagram:



Stair path: $\underset{(\bullet)}{x} \xrightarrow{(1,4)} x_1 \xrightarrow{(1,6)} x_2 \xrightarrow{(1,8)} x_3 \xrightarrow{(1,9)} \underset{(\circ)}{y}$

Definition Let $x \in S_n$. A *stair* of x is an increasing sequence

$$s = (i, j_1, \dots, j_k) \in [n]^k$$

such that $(x(i), x(j_1), \dots, x(j_k))$ is also increasing.

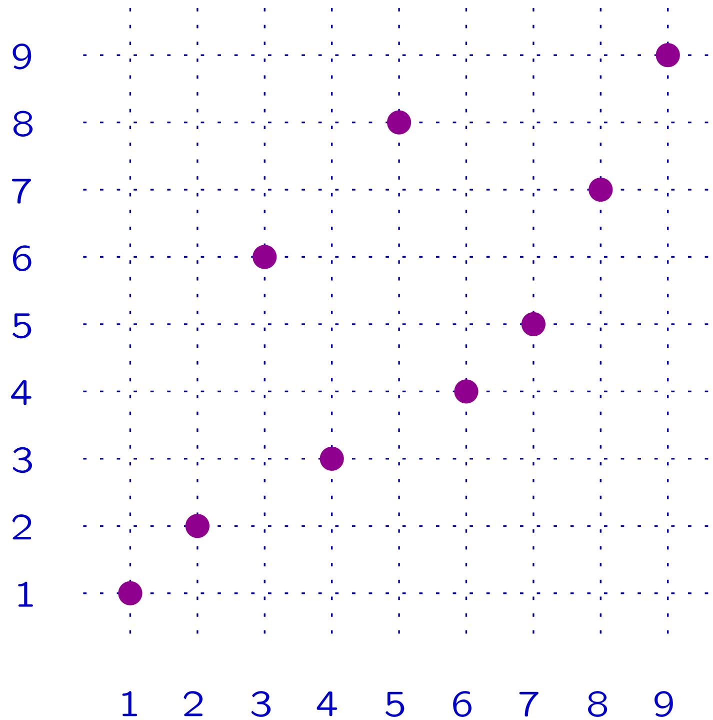
The permutation *obtained* from x by *performing* the stair s is

$$xs = x(i, j_k, \dots, j_1).$$

The *stair area* associated with s is

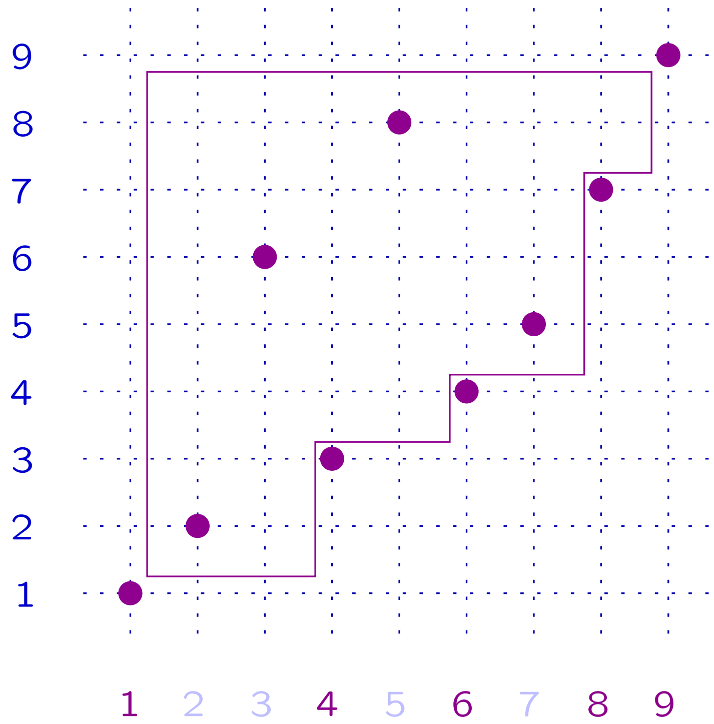
$$Stair_x(s) = \Omega(x, xs).$$

Example $x = 126384579$ $(\bullet)$



Example $x = 126384579$ $(\bullet)$

$(1, 4, 6, 8, 9)$ stair of x



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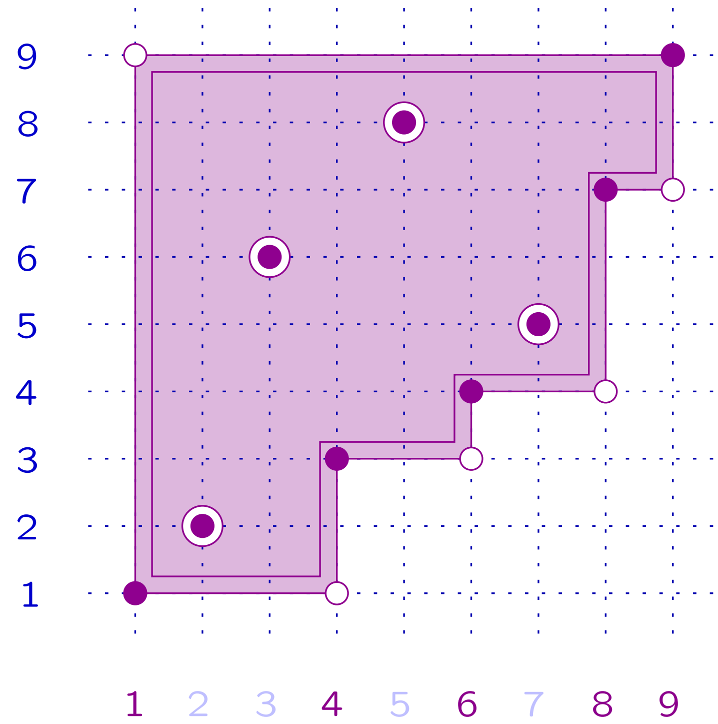
$(1,4,6,8,9)$ stair of x

$\Downarrow$

$y = x(1,9,8,6,4)$ $(\circ)$

obtained from x by

performing $(1,4,6,8,9)$



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$(1,4,6,8,9)$ stair of x

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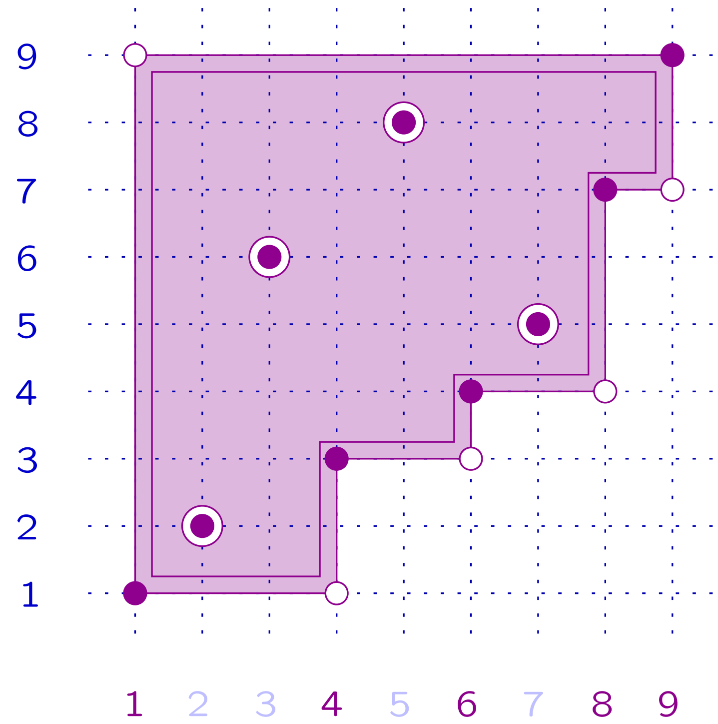

Stair path: $\underset{(\bullet)}{x} \xrightarrow{(1,4)} x_1 \xrightarrow{(1,6)} x_2 \xrightarrow{(1,8)} x_3 \xrightarrow{(1,9)} \underset{(\circ)}{y}$

Definition Let $x, y \in S_n$, $x < y$. The *difference index* of (x, y) is

$$di = \min\{k : x(k) \neq y(k)\}.$$

The *stair index* of (x, y) is $si = x^{-1}y(di)$.

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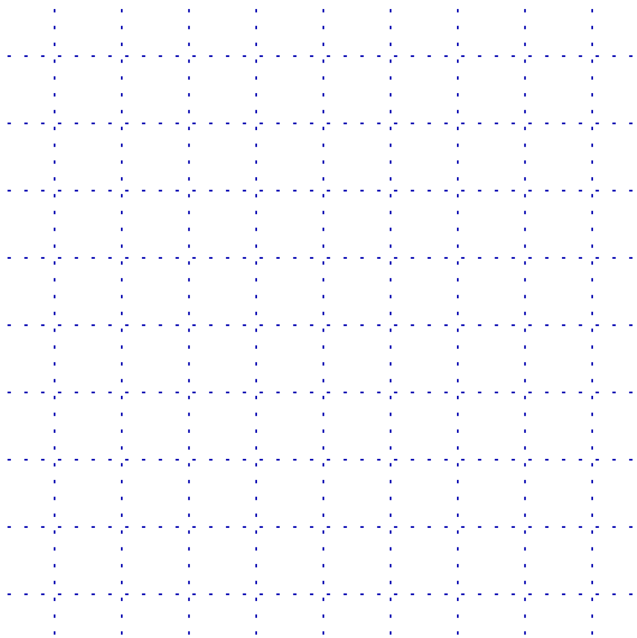
$x \; < \; y$

$(\bullet) \quad (\circ)$



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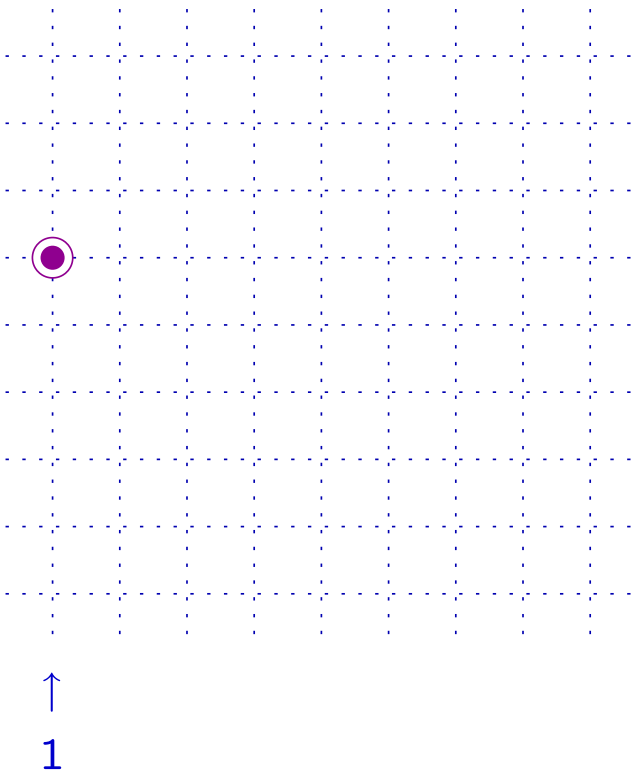

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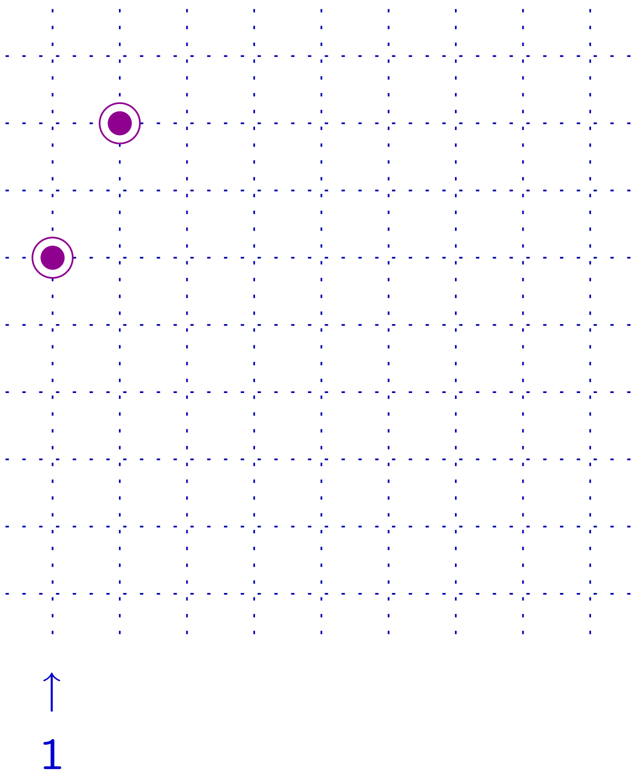

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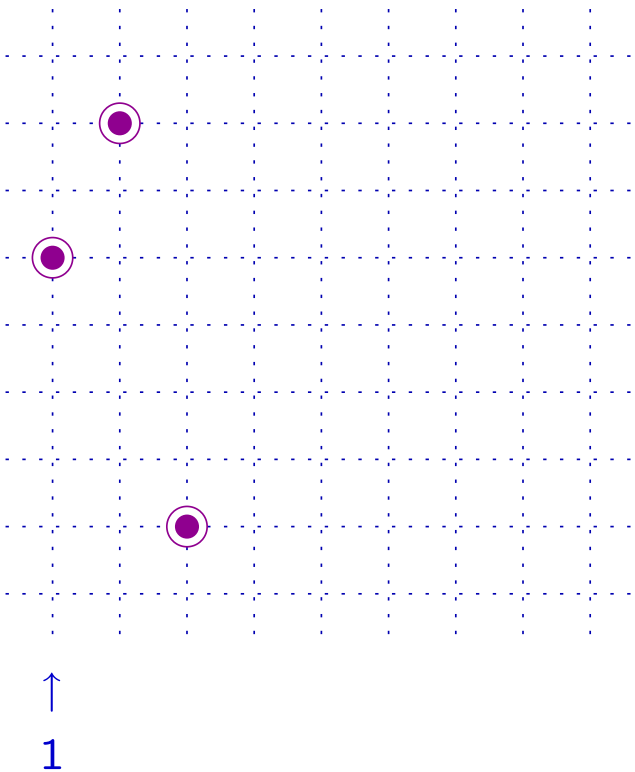

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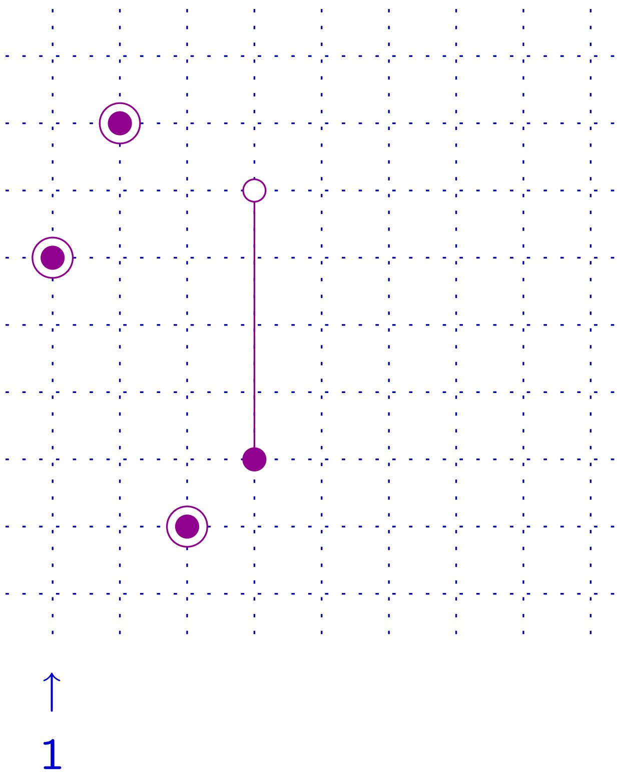

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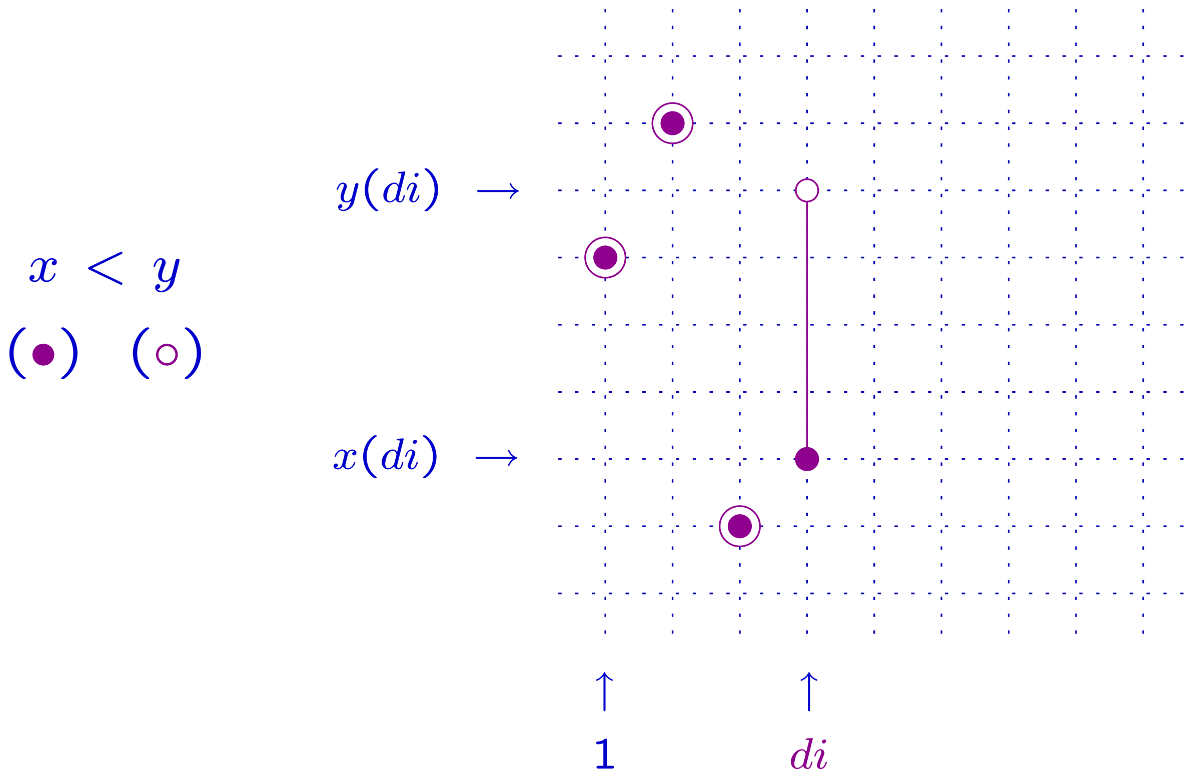

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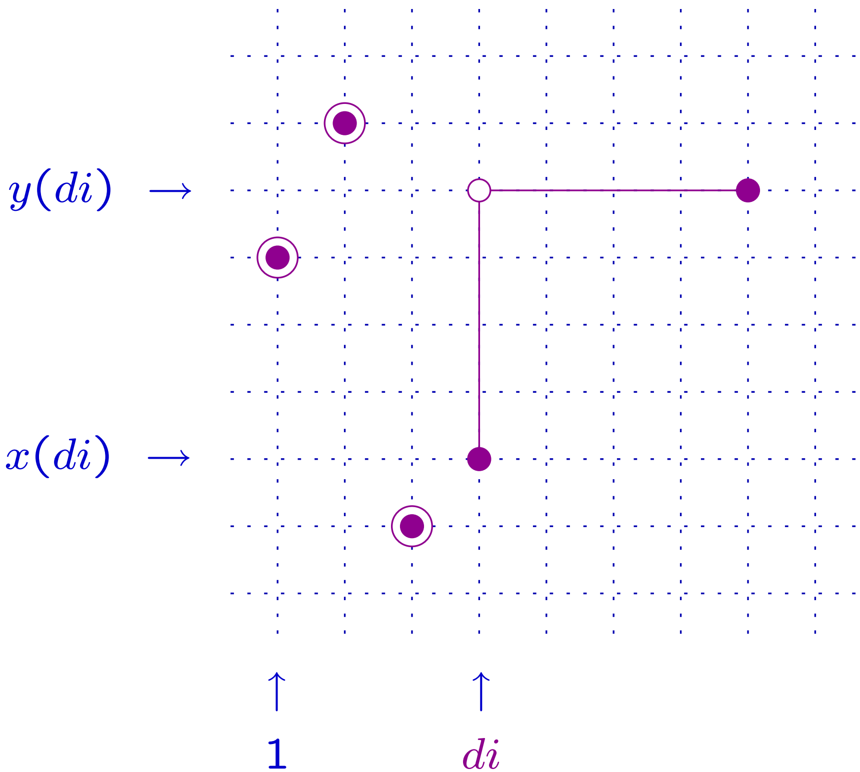
$x < y$

(•) (○)



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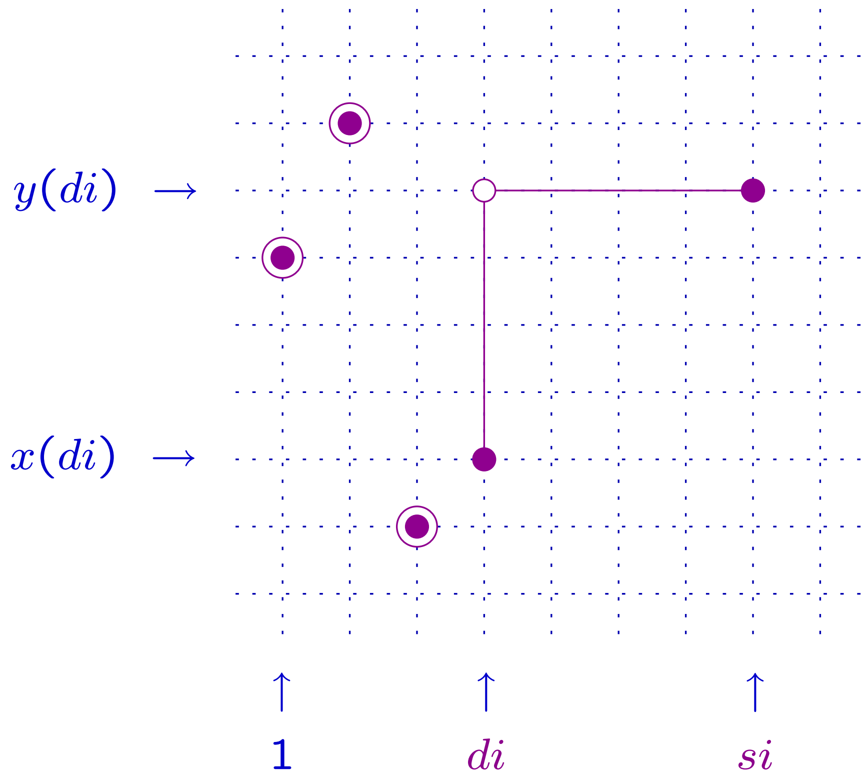
$(\bullet) \quad (\circ)$



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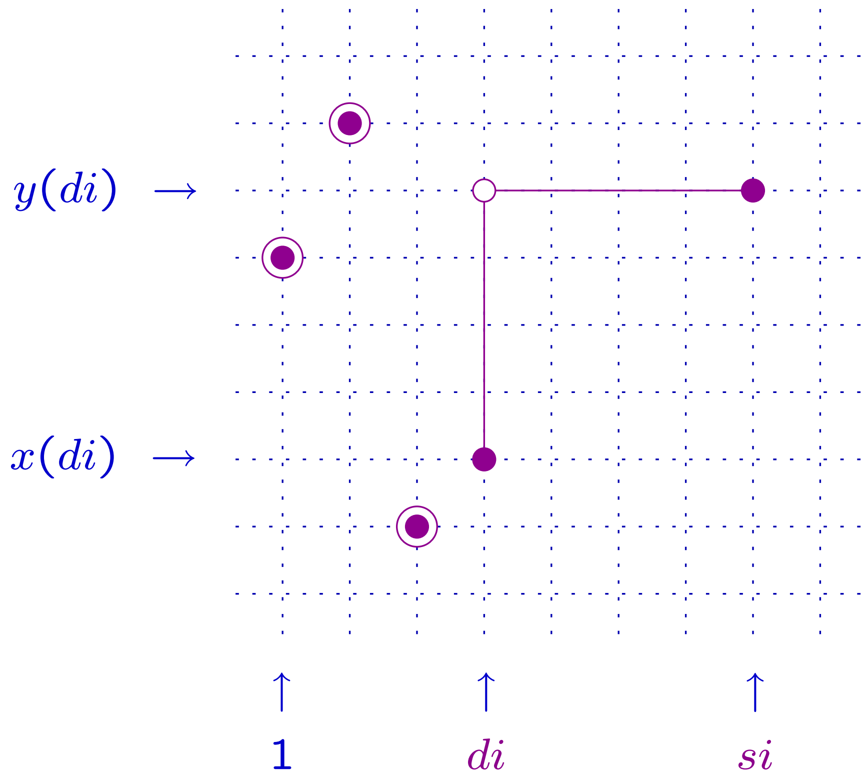

Note that

$x(di) < y(di)$

and $di < si$.

Definition Let $x, y \in S_n$, $x < y$. The *difference index* of (x, y) is

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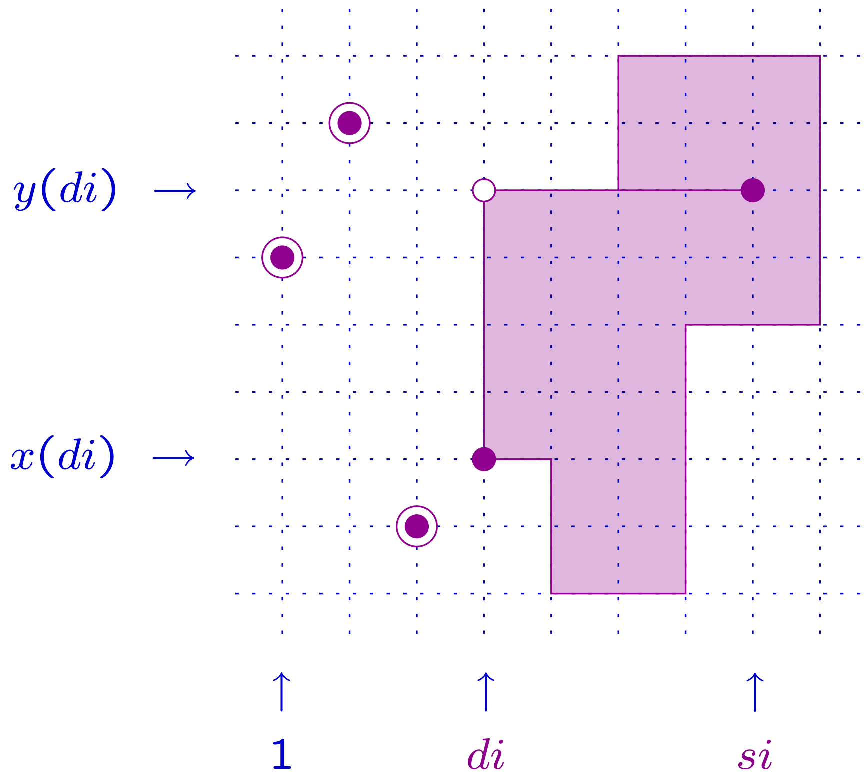
$x < y$

$(\bullet)$ $(\circ)$



Note that

$x(di) < y(di)$

and $di < si$.

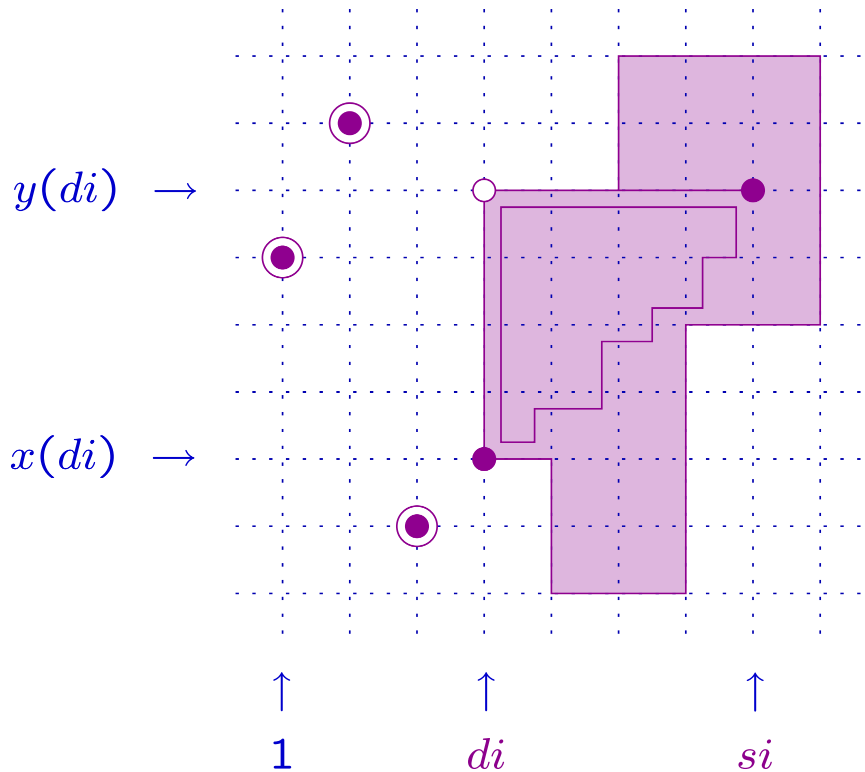
Definition Let $x, y \in S_n$, $x < y$. The *difference index* of (x, y) is

$$di = \min\{k : x(k) \neq y(k)\}.$$

The *stair index* of (x, y) is $si = x^{-1}y(di)$.

$x < y$

$(\bullet)$ $(\circ)$



Note that

$x(di) < y(di)$

and $di < si$.

Definition Let $x, y \in S_n$, with $x < y$. A stair s of x is *good* w.r.t. y if

$$Stair_x(s) \subseteq \Omega(x, y)$$

Proposition Let $x, y \in S_n$, with $x < y$. Let s be a stair of x. Then

$$xs \leq y \quad \Leftrightarrow \quad s \text{ is good w.r.t. } y.$$

Definition A stair s of x, good w.r.t. y, is an *initial stair* of (x, y) if

$$s = (di, j_1, j_2, \ldots, j_{k-1}, si)$$

Proposition An initial stair of (x, y) always exists.

5.5 The stair method

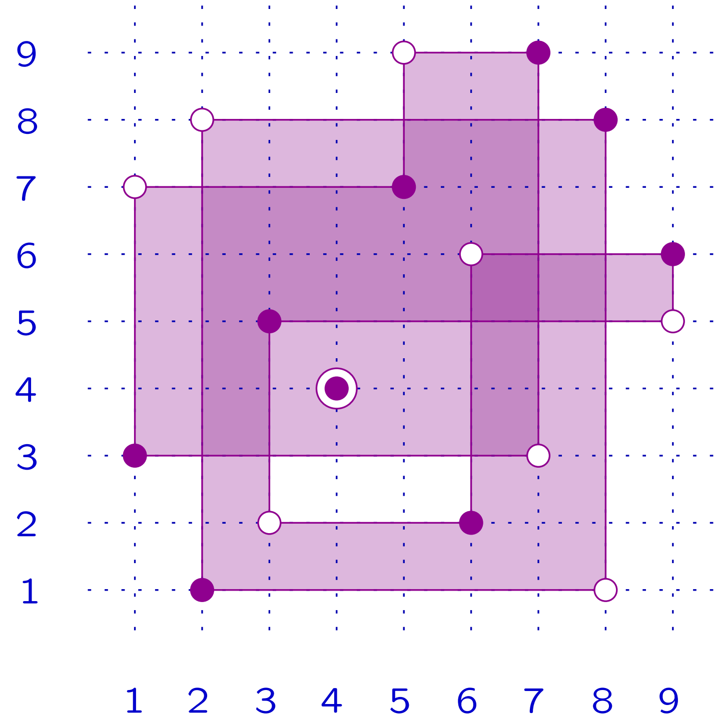
General algorithm: given $x, y \in S_n$, with $x < y$

1. choose an initial stair s of (x, y);
2. call x_1 the permutation obtained from x by performing s;
3. recursively apply the procedure on (x_1, y).

Proposition Let $x, y \in S_n$, with $x < y$. The stair method allows to generate all possible increasing paths in BG from x to y.

So, in particular, it allows to compute $\widetilde{R}_{x,y}(q)$.

Example $x = 315472986$ $(\bullet)$ $y = 782496315$ $(\circ)$

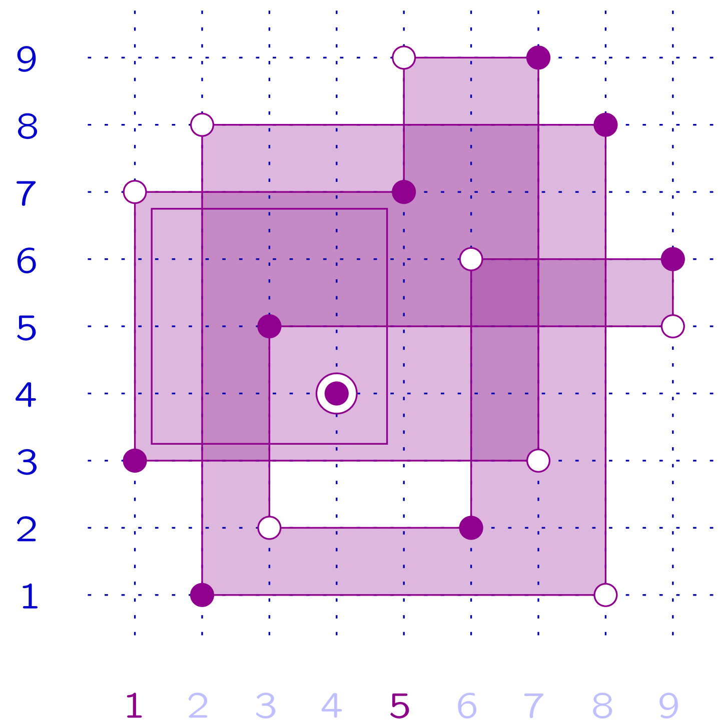


$$x < y$$
$$(\bullet) \qquad (\circ)$$

Example $x = 315472986$ $(\bullet)$ $y = 782496315$ $(\circ)$

Initial stairs of (x, y):

$(1, 5)$



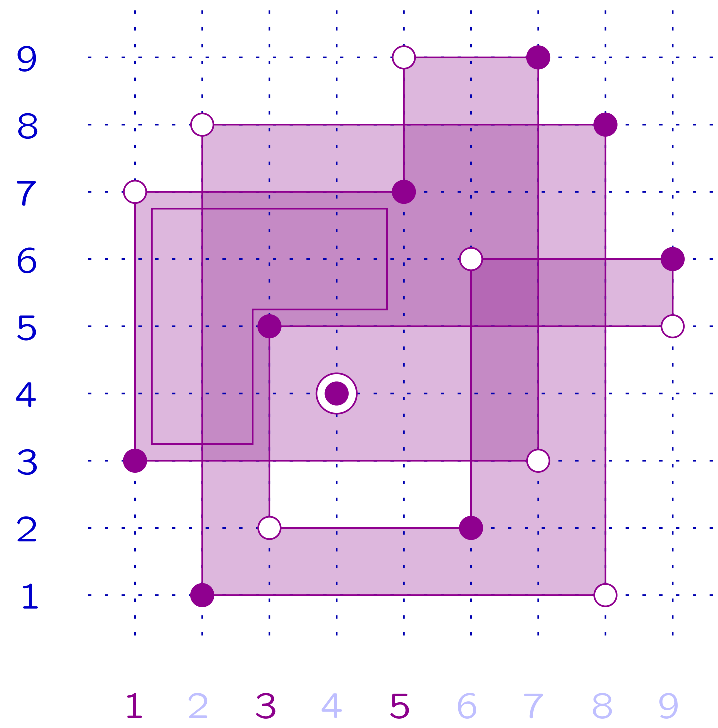
$x < y$

$(\bullet)$ $(\circ)$

Example $x = 315472986$ $(\bullet)$ $y = 782496315$ $(\circ)$

Initial stairs of (x, y):

$(1,5)$, $(1,3,5)$



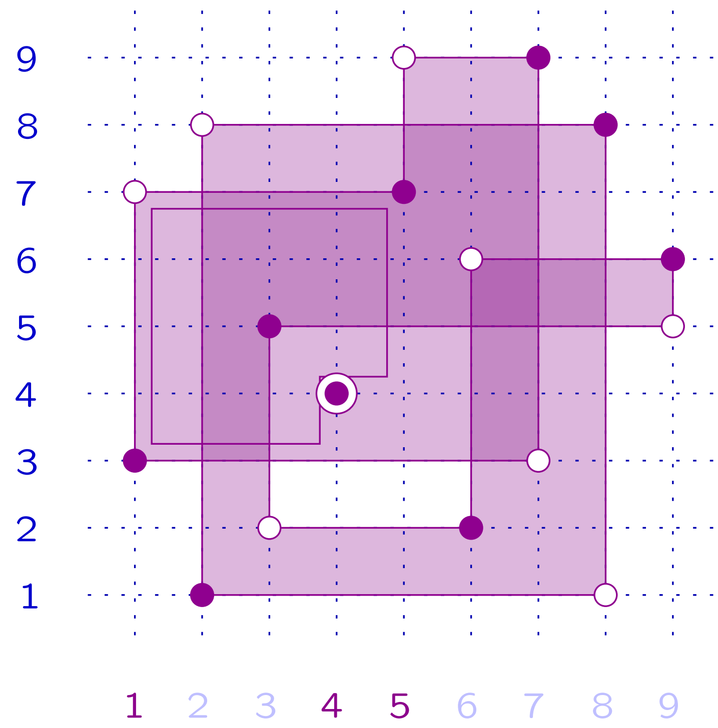
$x < y$

$(\bullet)$ $(\circ)$

Example $x = 315472986$ ($\bullet$) $y = 782496315$ ($\circ$)

Initial stairs of (x, y):

$(1, 5)$, $(1, 3, 5)$ and $(1, 4, 5)$.

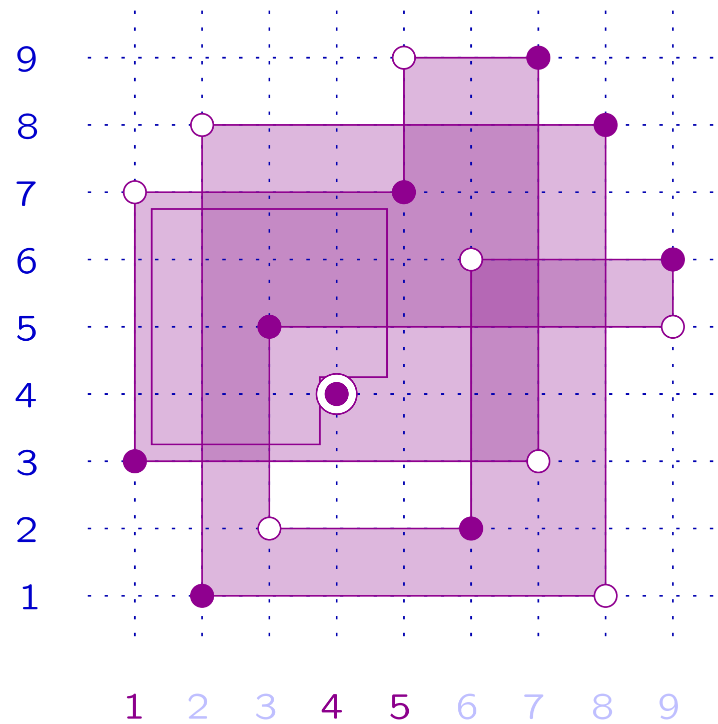


x $<$ y

($\bullet$) ($\circ$)

Example $x = 315472986$ $(\bullet)$ $y = 782496315$ $(\circ)$

$(1, 4, 5)$ initial stair of (x, y)



$$x \quad < \quad y$$
$$(\bullet) \qquad (\circ)$$

Example $x = 315472986$ $(\bullet)$ $y = 782496315$ $(\circ)$

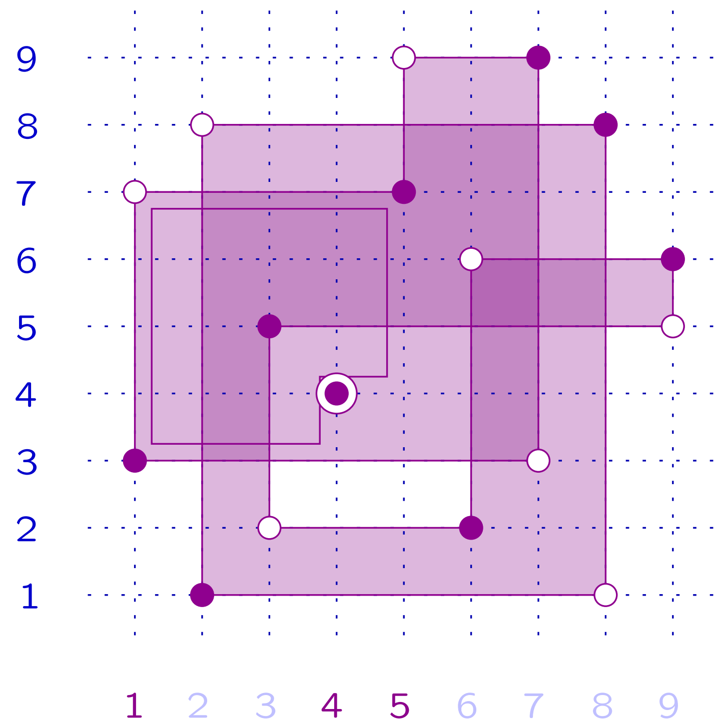
$(1,4,5)$ initial stair of (x,y)

$\Downarrow$

$x_1 = x(1,5,4)$

obtained from x by

performing $(1,4,5)$



x $<$ y

$(\bullet)$ $(\circ)$

Example $x = 315472986$ $(\bullet)$ $y = 782496315$ $(\circ)$

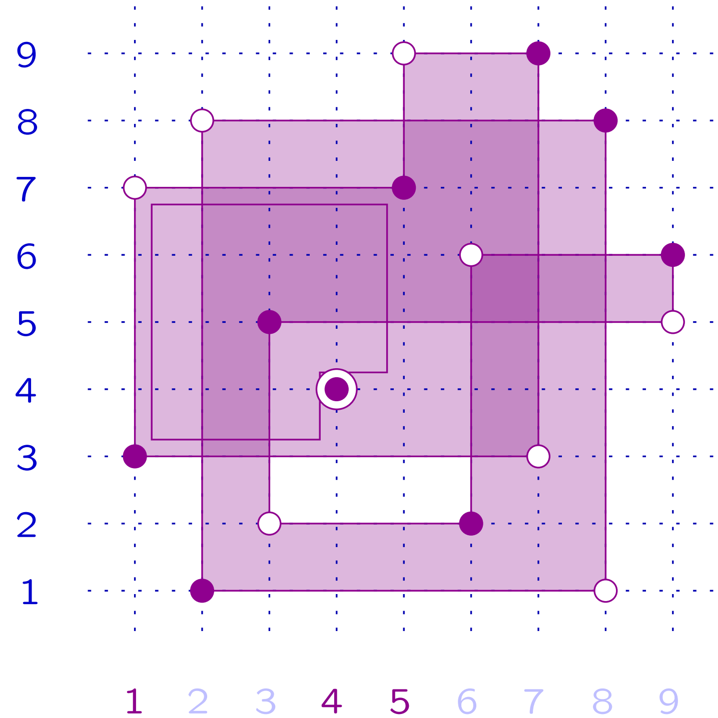
$(1,4,5)$ initial stair of (x,y)

$\Downarrow$

$x_1 = x(1,5,4)$

obtained from x by

performing $(1,4,5)$



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \qquad y$$

$(\bullet)$ $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$

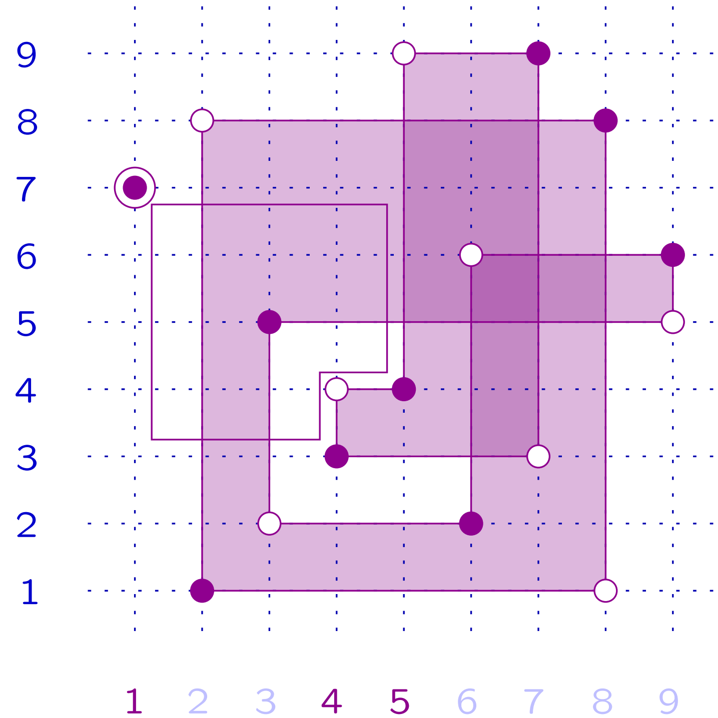
$(1,4,5)$ initial stair of (x,y)

$\Downarrow$

$x_1 = x(1,5,4)$

obtained from x by

performing $(1,4,5)$



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \qquad y$$

$(\bullet)$ $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$

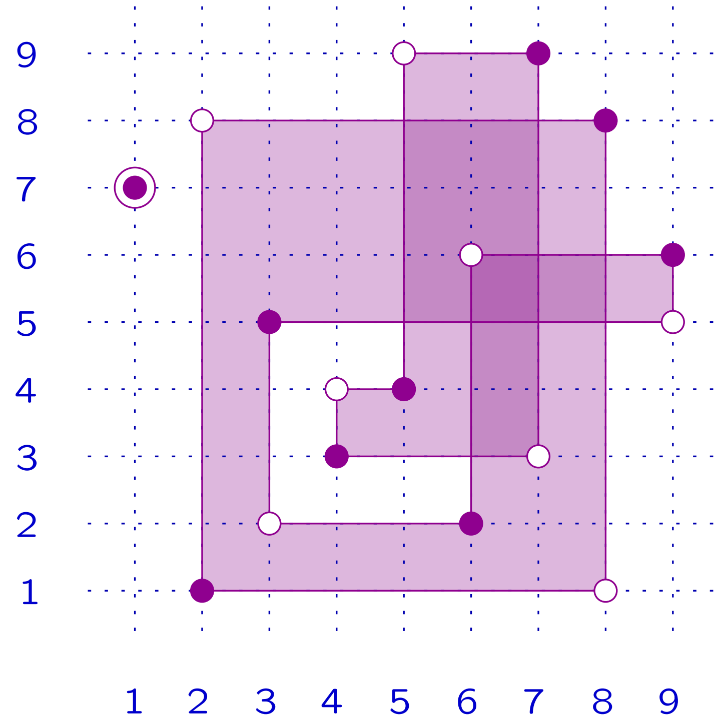
$(1,4,5)$ initial stair of (x,y)

$\Downarrow$

$x_1 = x(1,5,4)$

obtained from x by

performing $(1,4,5)$

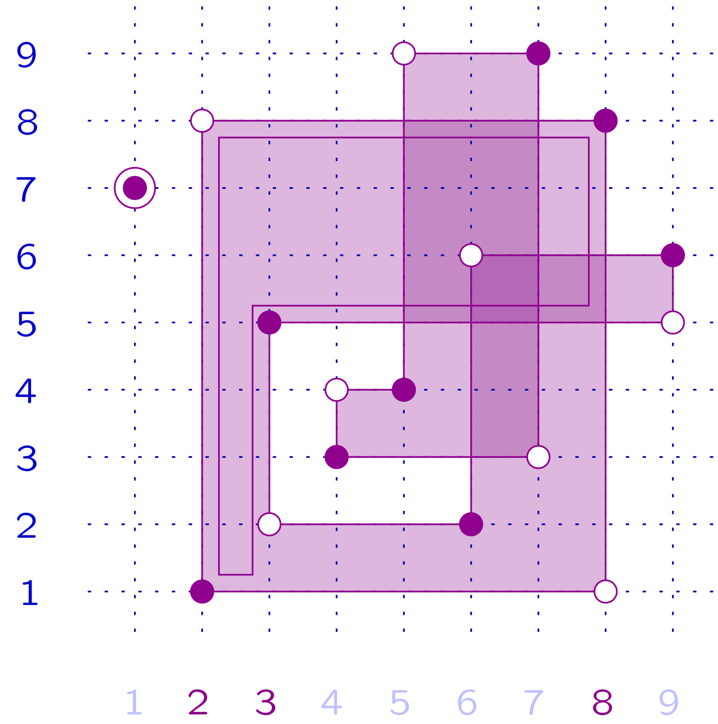


$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \quad < \quad y$$

$(\bullet)$ $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$

$(2, 3, 8)$ initial stair of (x_1, y)



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \quad < \quad y$$

$(\bullet)$ $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$

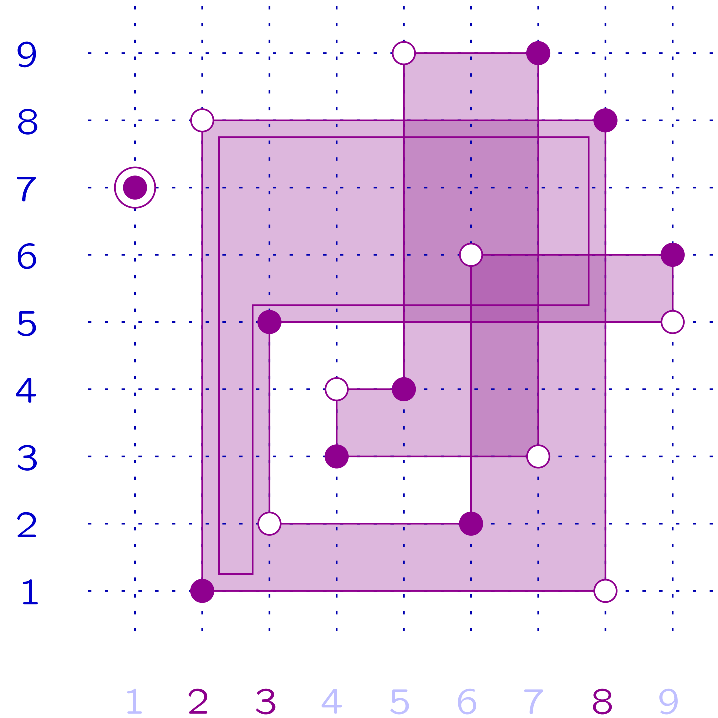
$(2,3,8)$ initial stair of (x_1, y)

$\Downarrow$

$x_2 = x_1(2,8,3)$

obtained from x_1 by

performing $(2,3,8)$



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \quad < \quad y$$

$(\bullet)$ $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$

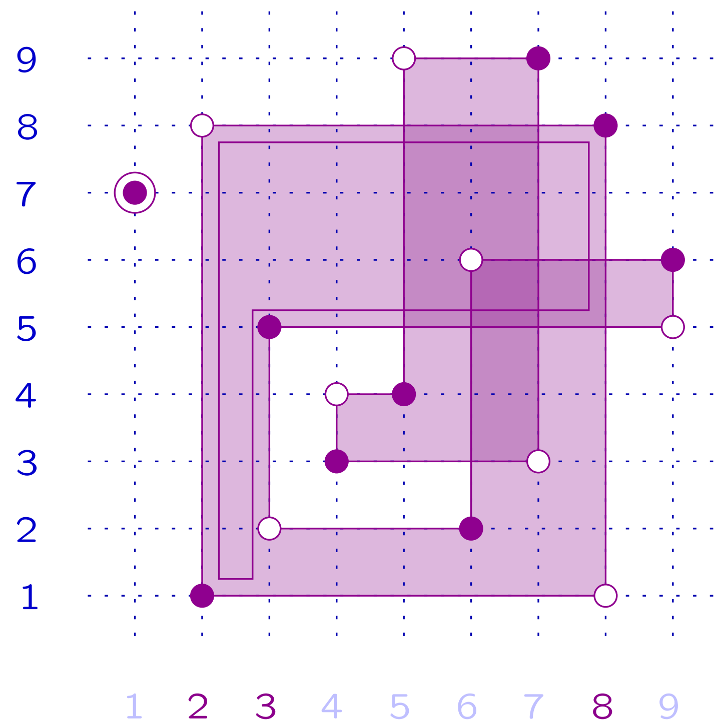
$(2,3,8)$ initial stair of (x_1, y)

$\Downarrow$

$x_2 = x_1(2,8,3)$

obtained from x_1 by

performing $(2,3,8)$



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} \underset{(\bullet)}{x_1} \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \qquad \underset{(\circ)}{y}$$

Example $x = 315472986$ $y = 782496315$ $(\circ)$

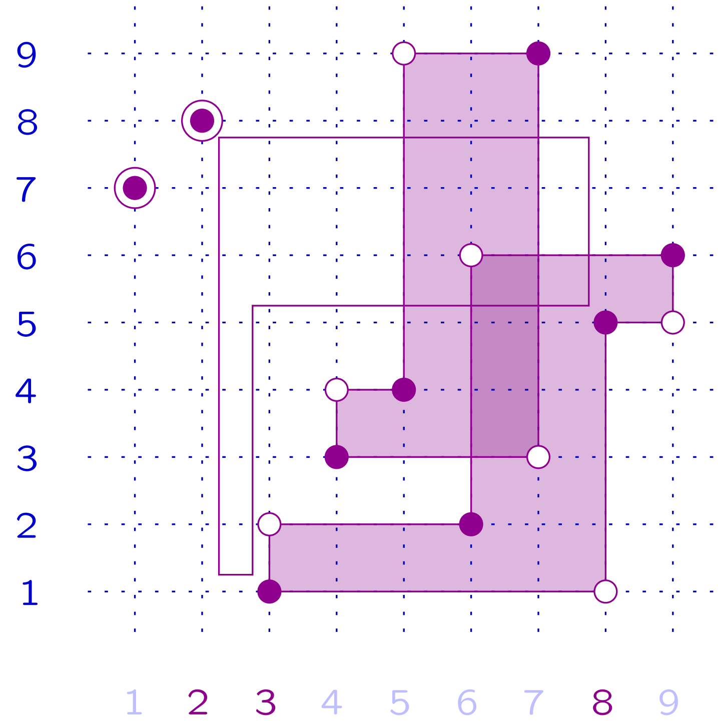
$(2,3,8)$ initial stair of (x_1, y)

$\Downarrow$

$x_2 = x_1(2,8,3)$

obtained from x_1 by

performing $(2,3,8)$



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \qquad y$$

$(\bullet)$ $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$

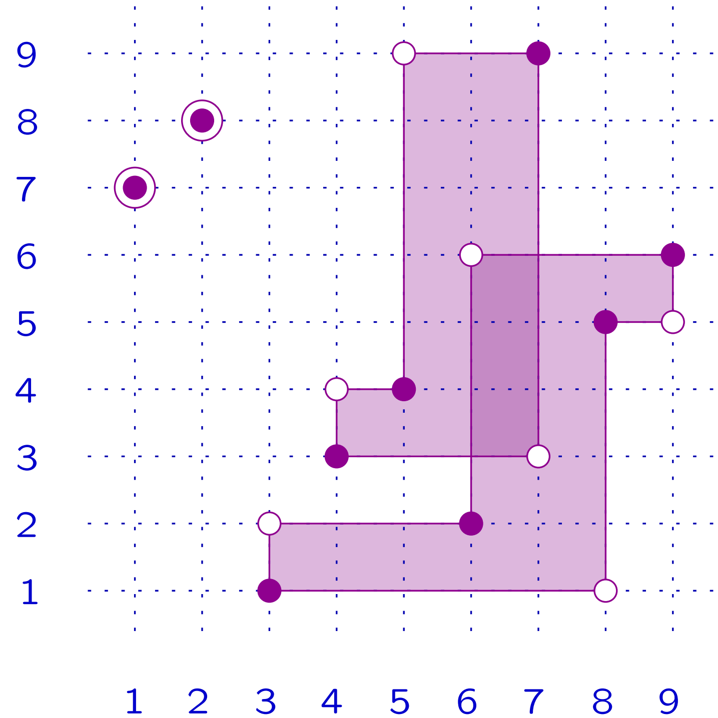
$(2,3,8)$ initial stair of (x_1, y)

$\Downarrow$

$x_2 = x_1(2,8,3)$

obtained from x_1 by

performing $(2,3,8)$

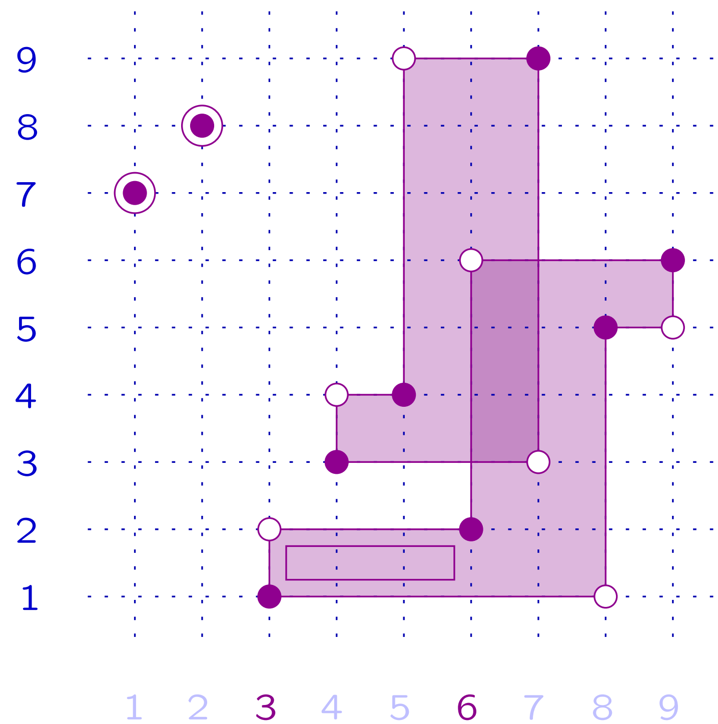


$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \quad < \quad y$$

$(\bullet)$ $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$

$(3,6)$ initial stair of (x_2, y)



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 < y$$

$(\bullet)$ $(\circ)$

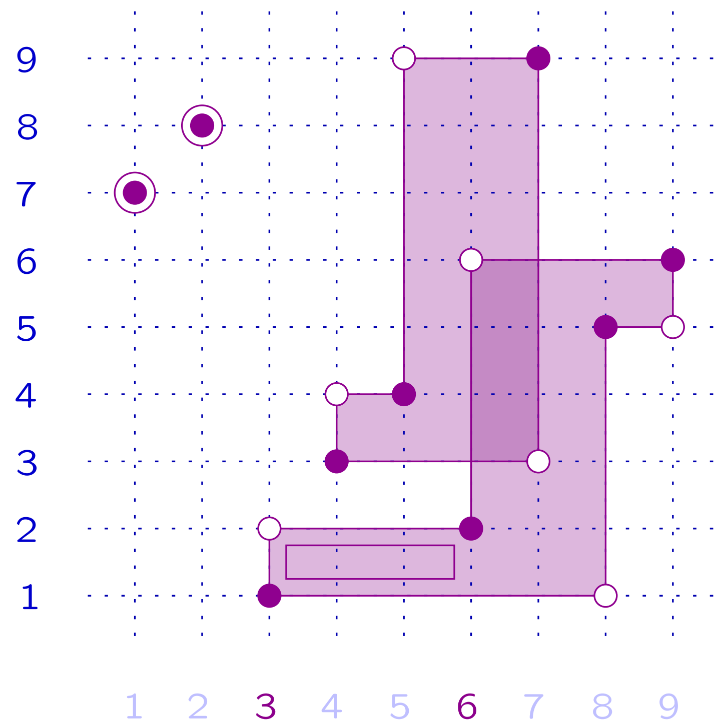
Example $x = 315472986$ $y = 782496315$ $(\circ)$

(3, 6) initial stair of (x_2, y)

$\Downarrow$

$x_3 = x_2(3,6)$

obtained from x_2 by performing $(3,6)$



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \quad < \quad y$$

$(\bullet)$ $(\circ)$

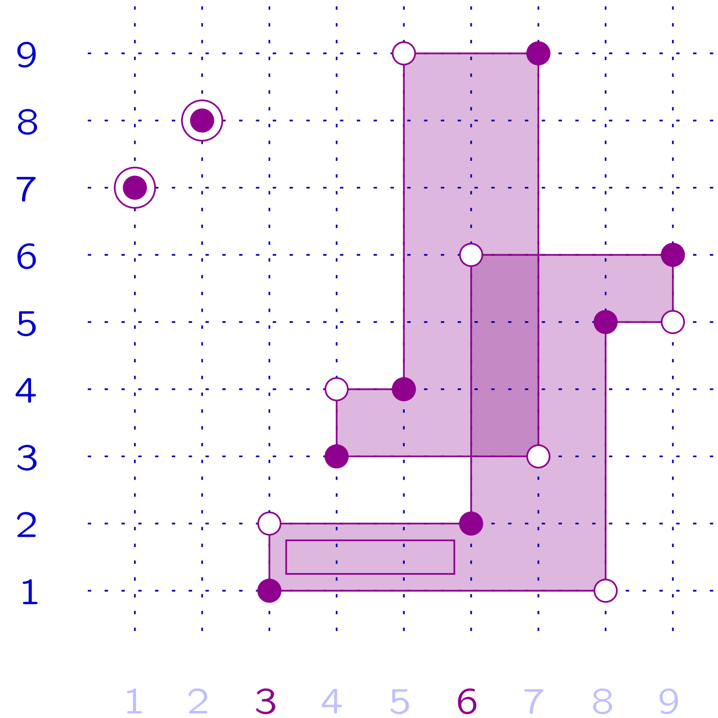
Example $x = 315472986$ $y = 782496315$ $(\circ)$

(3, 6) initial stair of (x_2, y)

$\Downarrow$

$x_3 = x_2(3, 6)$

obtained from x_2 by performing (3, 6)



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \qquad y$$

$(\bullet)$ $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$

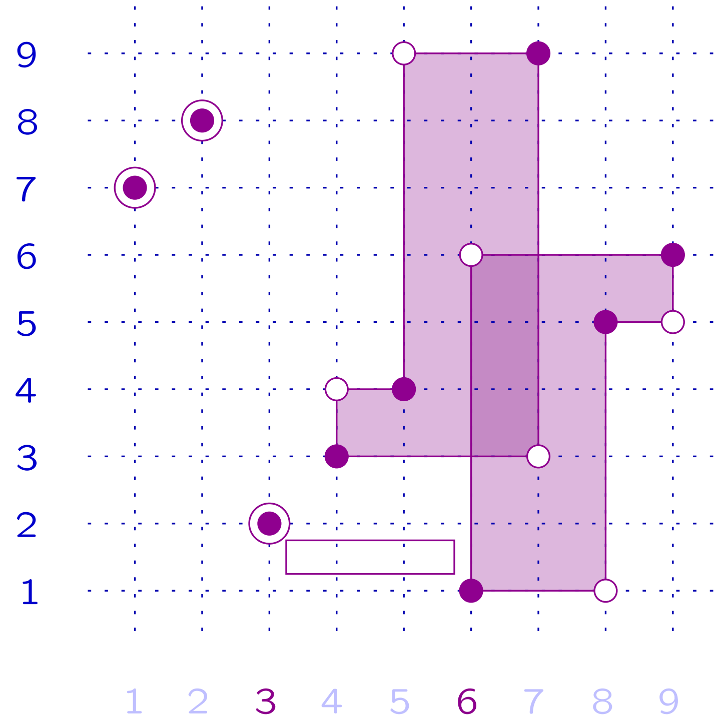
(3, 6) initial stair of (x_2, y)

$\Downarrow$

$x_3 = x_2(3, 6)$

obtained from x_2 by

performing $(3, 6)$



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \qquad y$$

$(\bullet)$ $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$

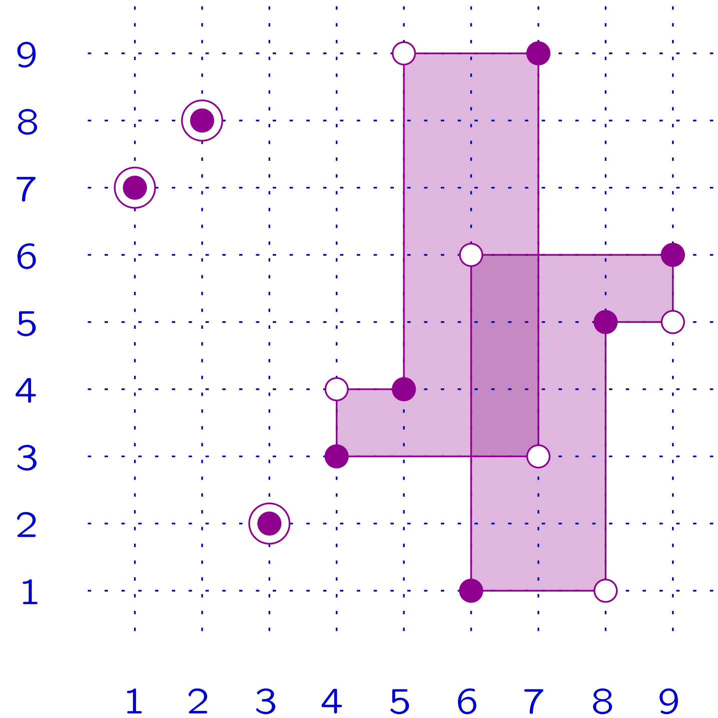
$(3,6)$ initial stair of (x_2, y)

$\Downarrow$

$x_3 = x_2(3,6)$

obtained from x_2 by

performing $(3,6)$

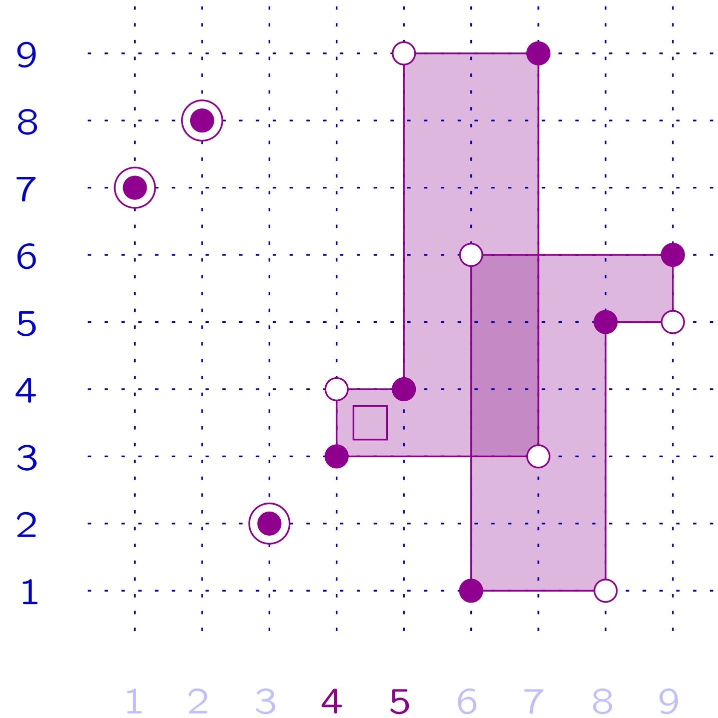


$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \quad < \quad y$$

$(\bullet)$ $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$

$(4,5)$ initial stair of (x_3, y)



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 < y$$

$(\bullet)$ $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$

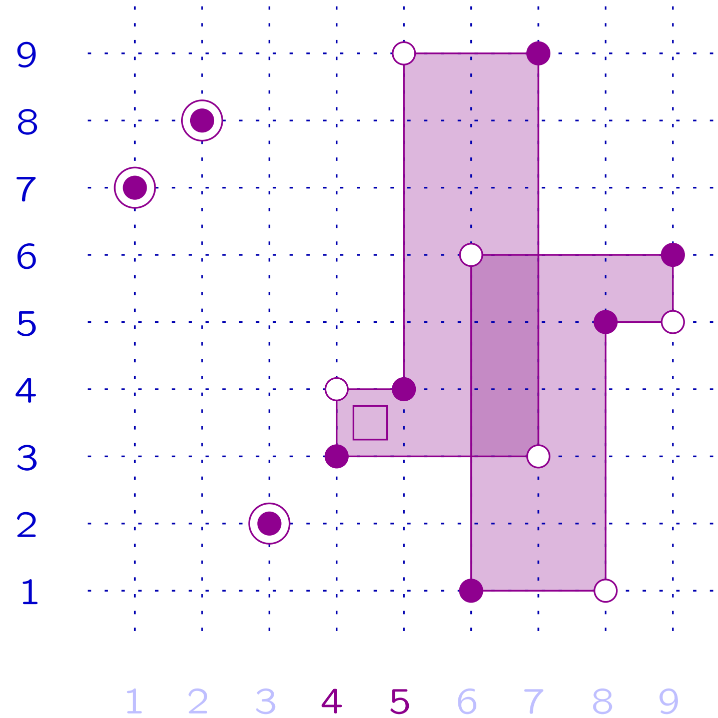
$(4,5)$ initial stair of (x_3, y)

$\Downarrow$

$x_4 = x_3(4,5)$

obtained from x_3 by

performing $(4,5)$



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \quad < \quad y$$

$(\bullet)$ $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$

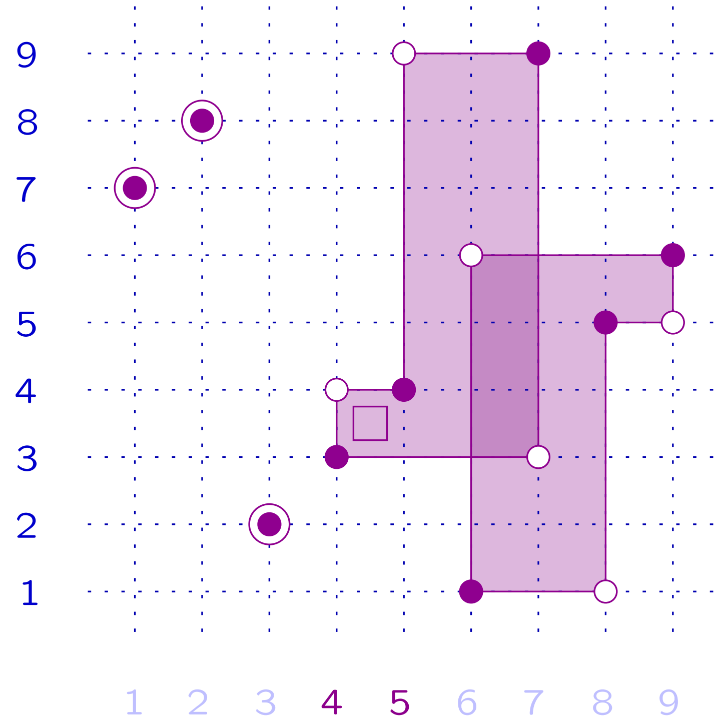
$(4,5)$ initial stair of (x_3, y)

$\Downarrow$

$x_4 = x_3(4,5)$

obtained from x_3 by

performing $(4,5)$



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \qquad y$$

$(\bullet)$ $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$

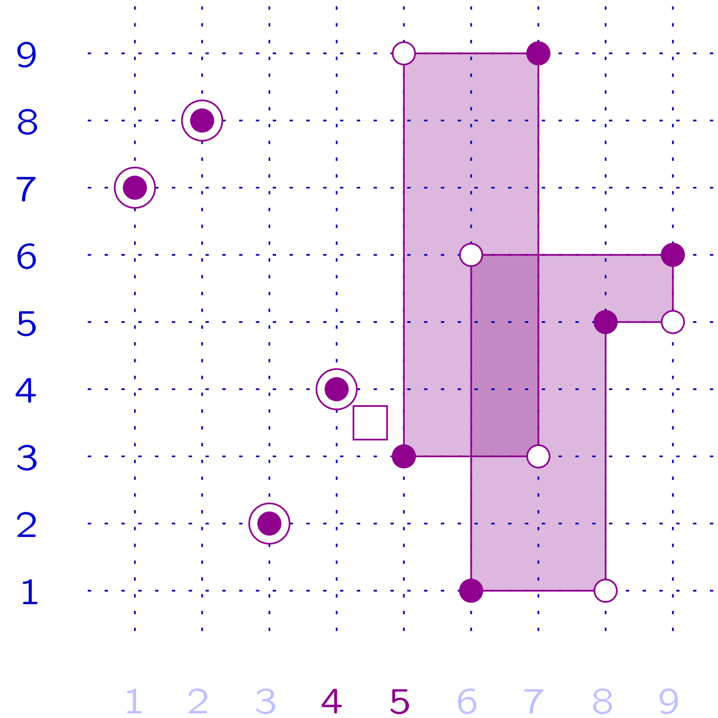
$(4,5)$ initial stair of (x_3, y)

$\Downarrow$

$x_4 = x_3(4,5)$

obtained from x_3 by

performing $(4,5)$



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \qquad y$$

$(\bullet)$ $(\circ)$

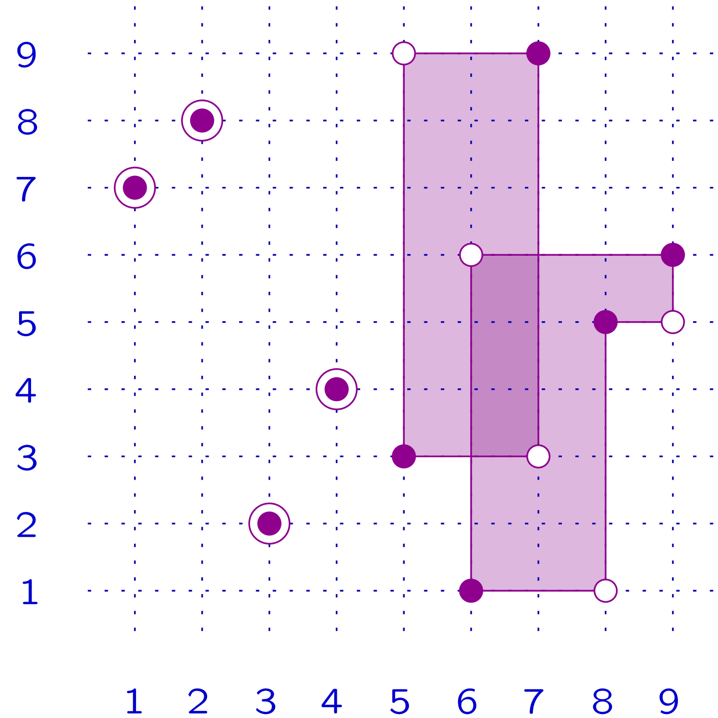
Example $x = 315472986$ $y = 782496315$ $(\circ)$

$(4,5)$ initial stair of (x_3, y)

$\Downarrow$

$x_4 = x_3(4,5)$

obtained from x_3 by performing $(4,5)$

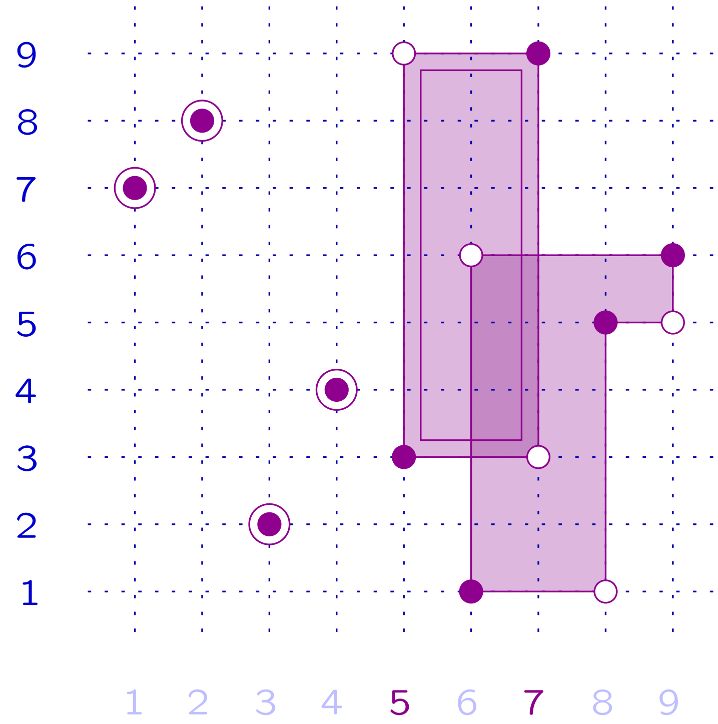


$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 < y$$

$(\bullet)$ $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$

$(5,7)$ initial stair of (x_4, y)



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \quad < \quad y$$

$(\bullet)$ $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$

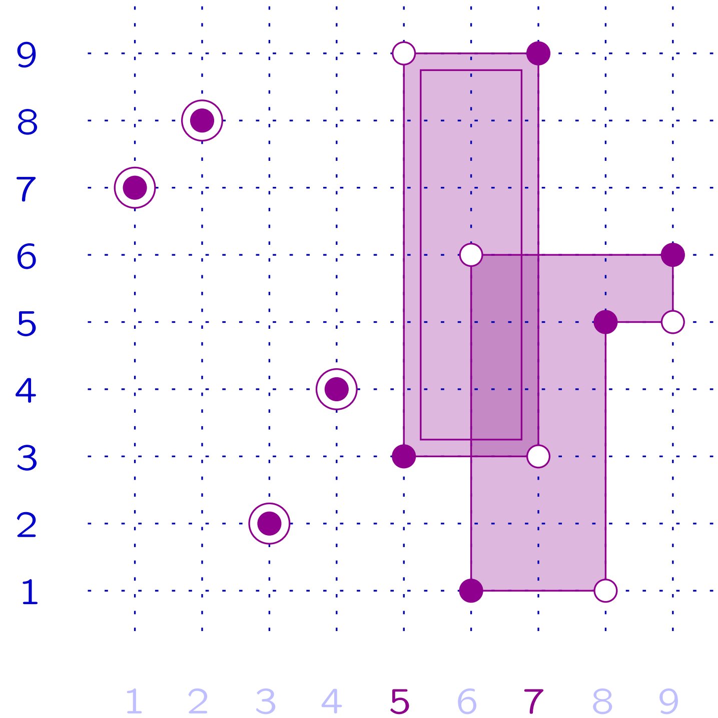
$(5,7)$ initial stair of (x_4, y)

$\Downarrow$

$x_5 = x_4(5,7)$

obtained from x_4 by

performing $(5,7)$



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 < y$$

$(\bullet)$ $(\circ)$

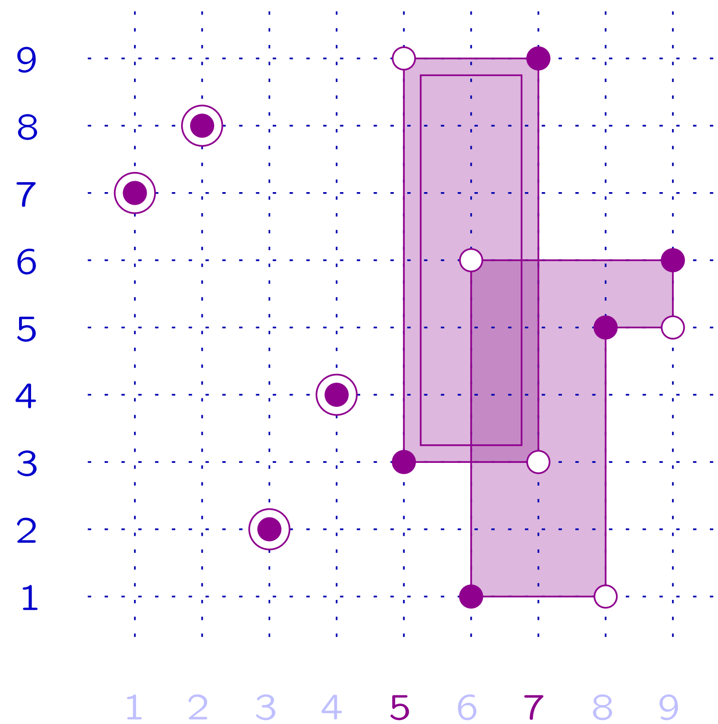
Example $x = 315472986$ $y = 782496315$ $(\circ)$

$(5,7)$ initial stair of (x_4, y)

$\Downarrow$

$x_5 = x_4(5,7)$

obtained from x_4 by performing $(5,7)$



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \xrightarrow{(5,7)} x_5 \qquad y$$

x_4 $(\bullet)$ y $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$

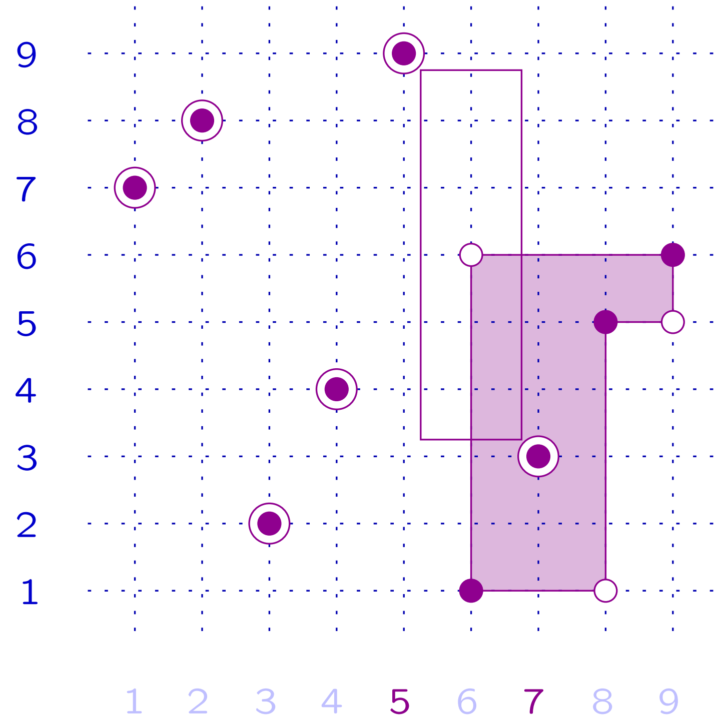
$(5,7)$ initial stair of (x_4, y)

$\Downarrow$

$x_5 = x_4(5,7)$

obtained from x_4 by

performing $(5,7)$



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \xrightarrow{(5,7)} x_5 \qquad y$$

$(\bullet)$ $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$

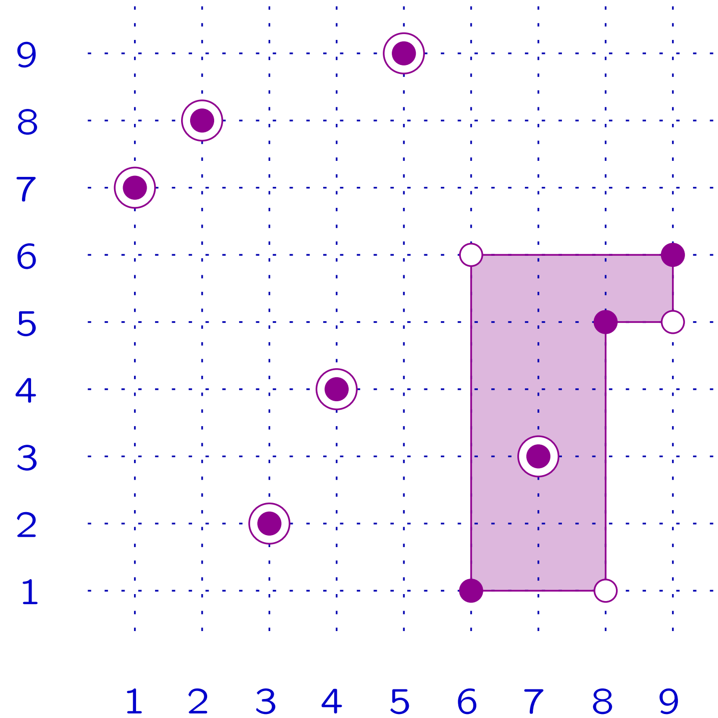
$(5,7)$ initial stair of (x_4, y)

$\Downarrow$

$x_5 = x_4(5,7)$

obtained from x_4 by

performing $(5,7)$



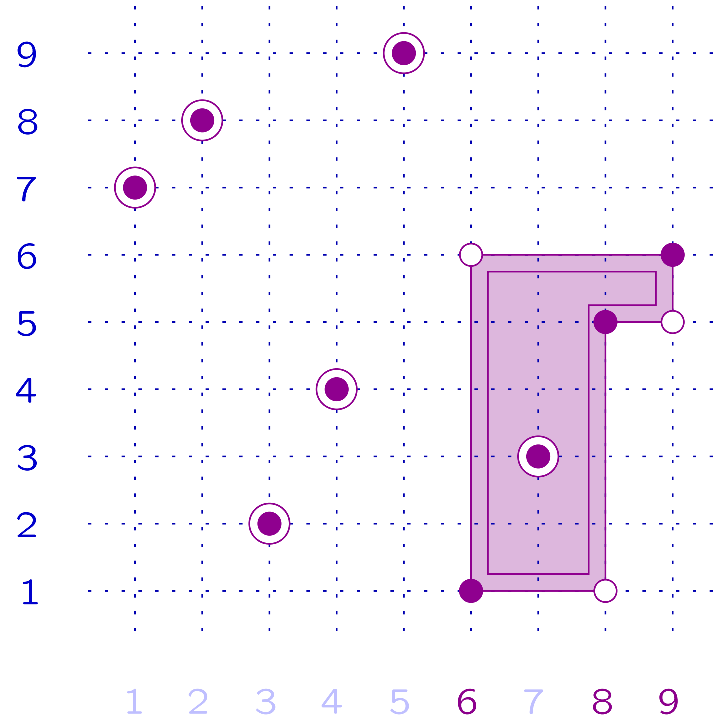
$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \xrightarrow{(5,7)} x_5 \quad < \quad y$$

$(\bullet)$ $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$

Initial stairs of (x_5, y):

$(6, 8, 9)$

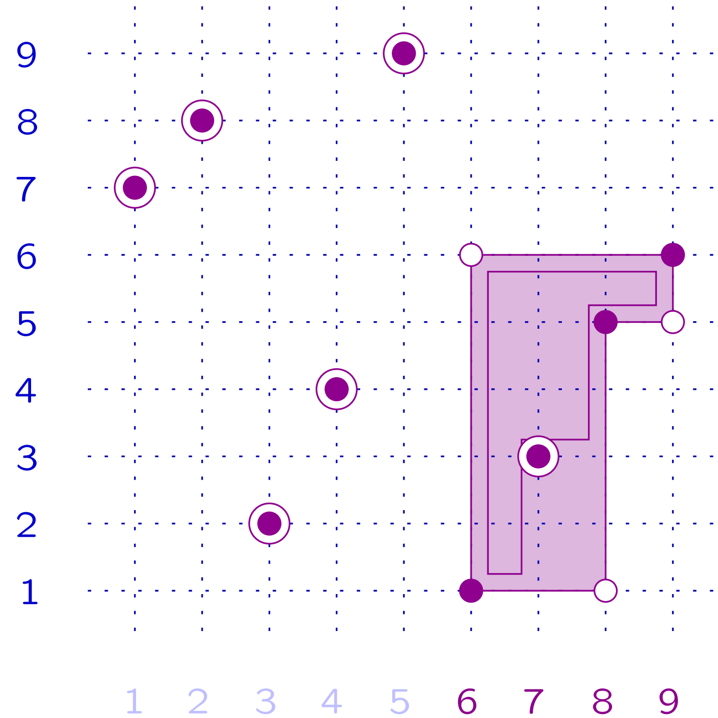


$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \xrightarrow{(5,7)} x_5 < y$$

$(\bullet)$ $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$

Initial stairs of (x_5, y):
$(6, 8, 9)$ and $(6, 7, 8, 9)$.

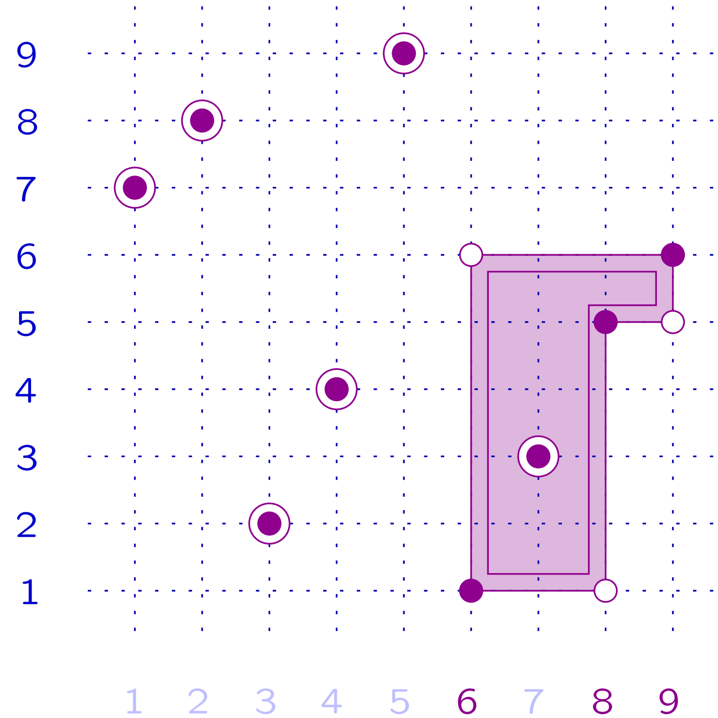


$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \xrightarrow{(5,7)} x_5 \quad < \quad y$$

$(\bullet)$ $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$

$(6, 8, 9)$ initial stair of (x_5, y)



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \xrightarrow{(5,7)} x_5 \quad < \quad y$$

$(\bullet)$ $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$

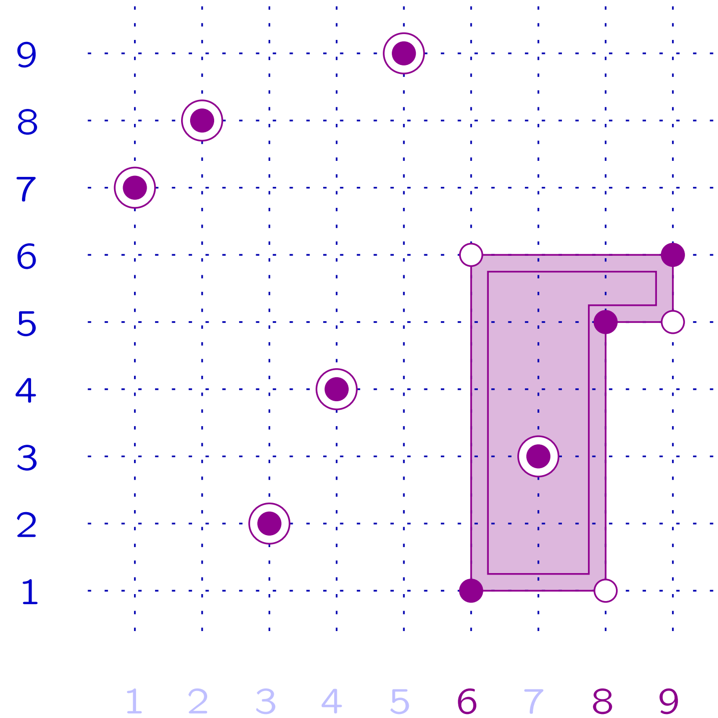
$(6,8,9)$ initial stair of (x_5, y)

$\Downarrow$

$y = x_5(6,9,8)$

obtained from x_5 by

performing $(6,8,9)$



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \xrightarrow{(5,7)} \underset{(\bullet)}{x_5} \quad < \quad \underset{(\circ)}{y}$$

Example $x = 315472986$ $y = 782496315$ $(\circ)$

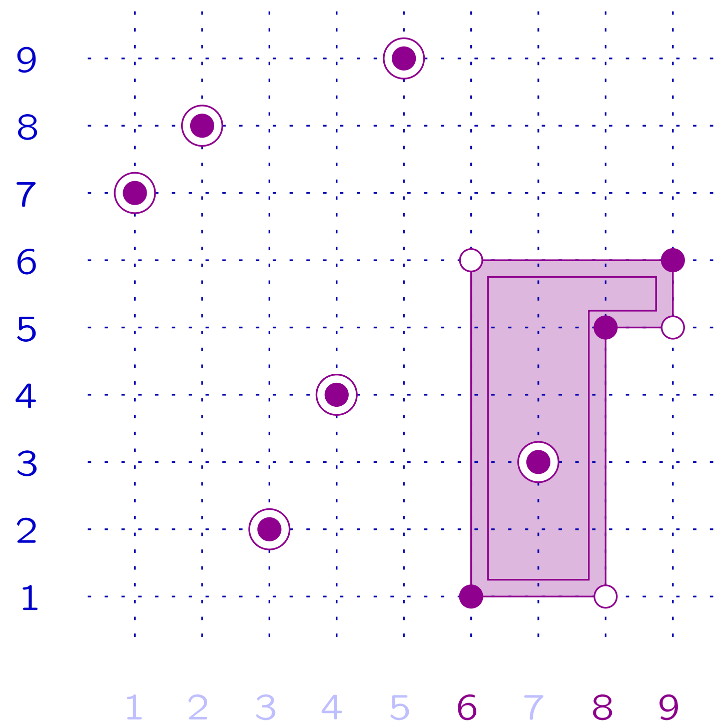
$(6,8,9)$ initial stair of (x_5, y)

$\Downarrow$

$y = x_5(6,9,8)$

obtained from x_5 by

performing $(6,8,9)$



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \xrightarrow{(5,7)} \underset{(\bullet)}{x_5} \xrightarrow{(6,8)} \bullet \xrightarrow{(6,9)} \underset{(\circ)}{y}$$

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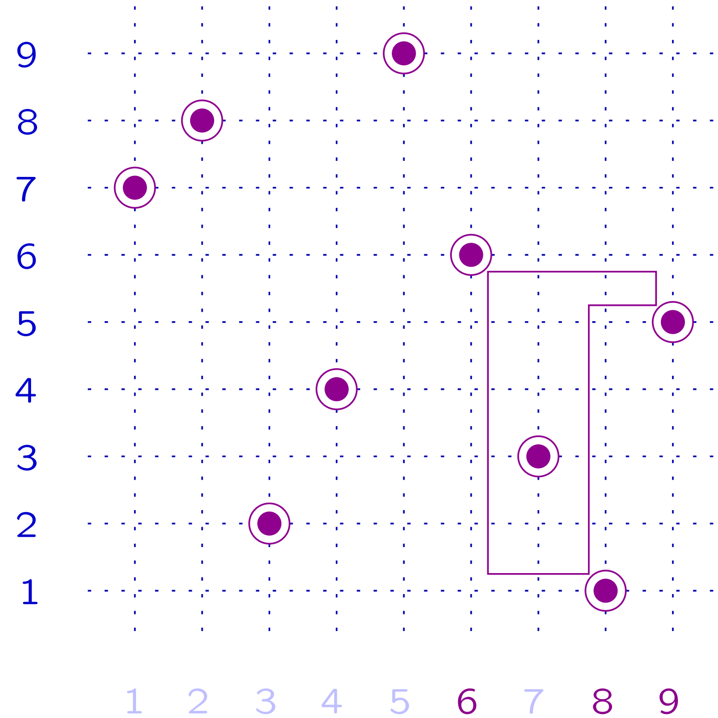

$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \xrightarrow{(5,7)} x_5 \xrightarrow{(6,8)} \bullet \xrightarrow{(6,9)} y$$

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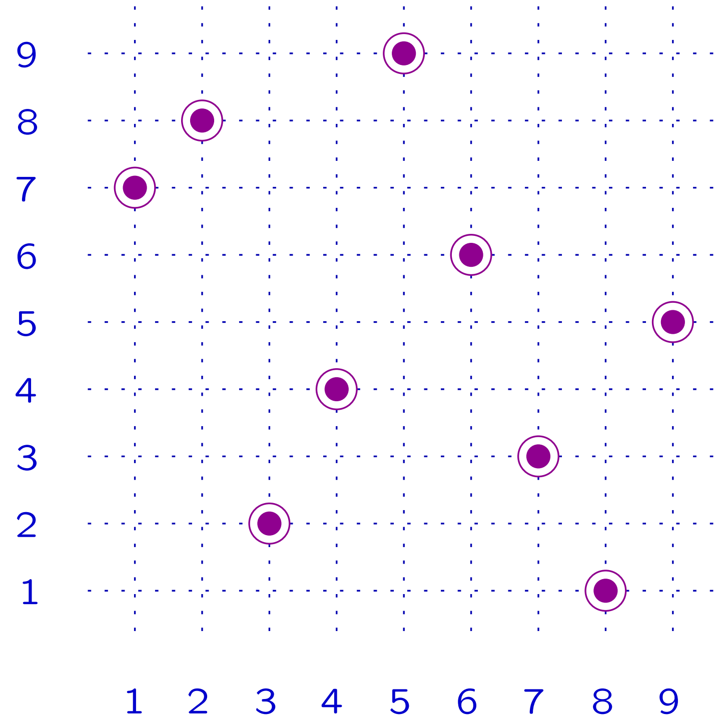
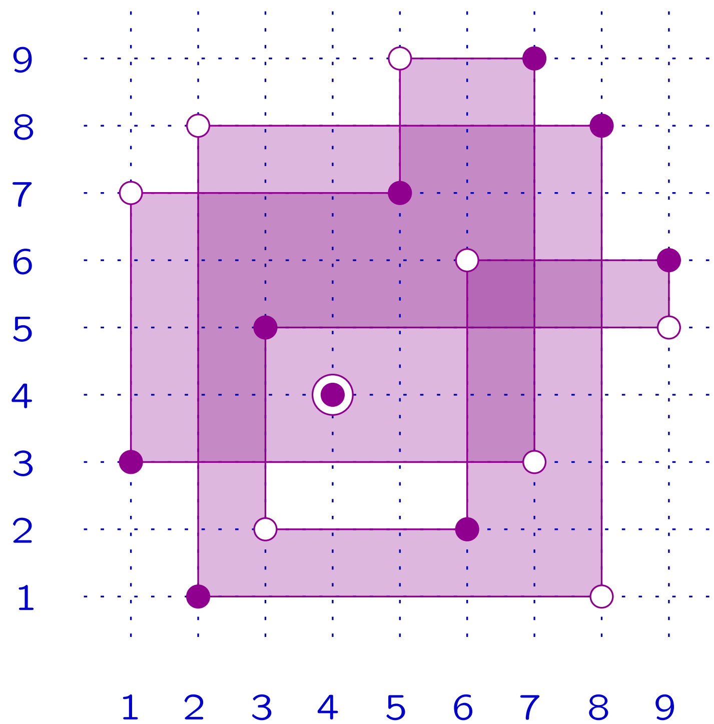


$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \xrightarrow{(5,7)} x_5 \xrightarrow{(6,8)} \bullet \xrightarrow{(6,9)} y$$

$(\bullet)$

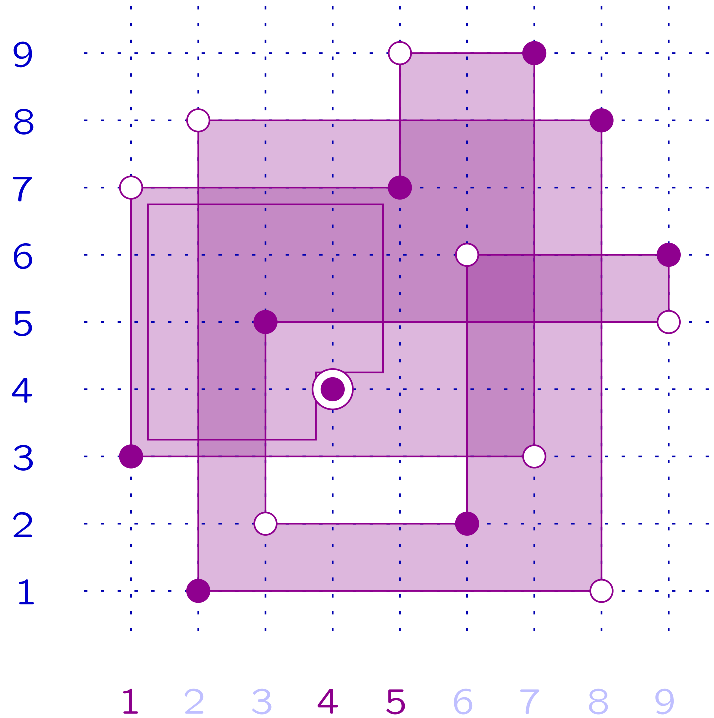
Example $x = 315472986$ $(\bullet)$ $y = 782496315$ $(\circ)$



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \xrightarrow{(5,7)} x_5 \xrightarrow{(6,8)} \bullet \xrightarrow{(6,9)} y$$

$(\bullet)$ $(\circ)$

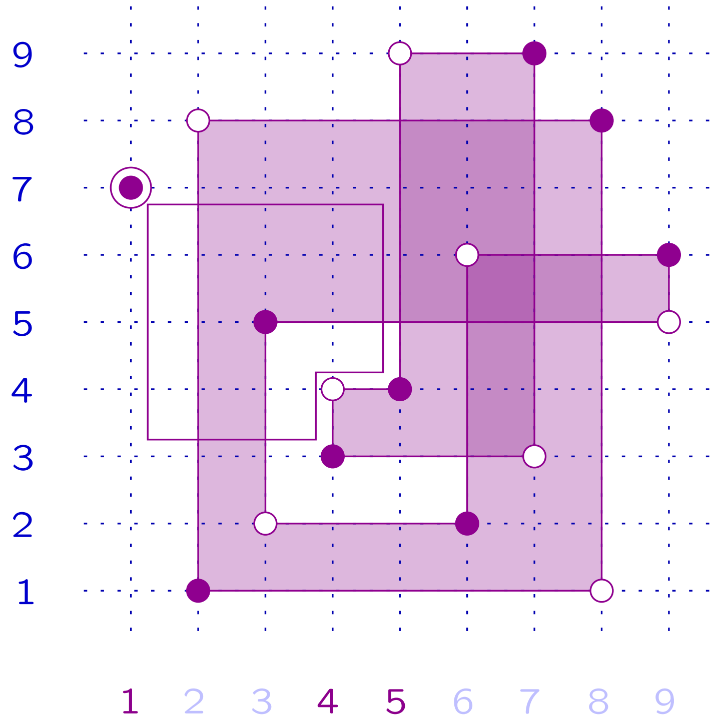
Example $x = 315472986$ (•) $y = 782496315$ (○)



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \xrightarrow{(5,7)} x_5 \xrightarrow{(6,8)} \bullet \xrightarrow{(6,9)} y$$

(•) (○)

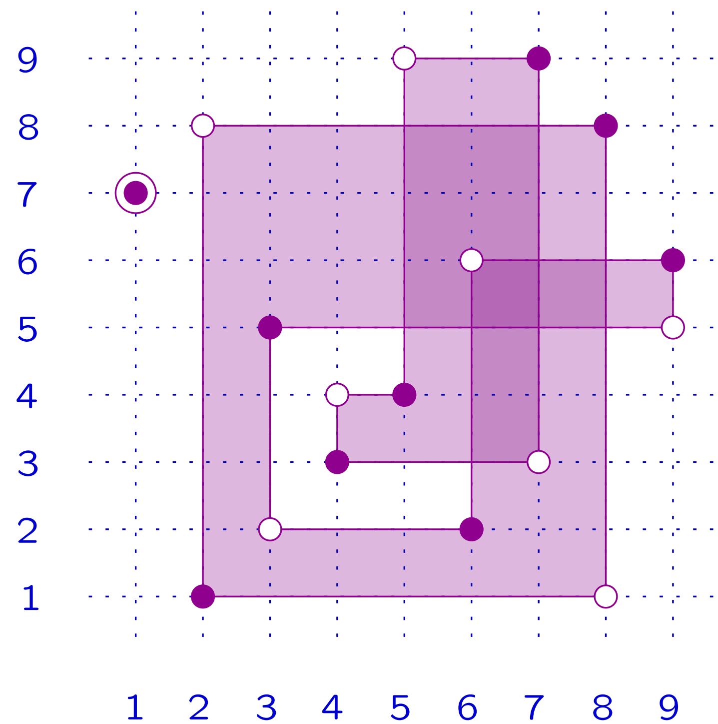
Example $x = 315472986$ $y = 782496315$ $(\circ)$



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \xrightarrow{(5,7)} x_5 \xrightarrow{(6,8)} \bullet \xrightarrow{(6,9)} y$$

$(\bullet)$ $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} \underset{(\bullet)}{x_1} \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \xrightarrow{(5,7)} x_5 \xrightarrow{(6,8)} \bullet \xrightarrow{(6,9)} \underset{(\circ)}{y}$$

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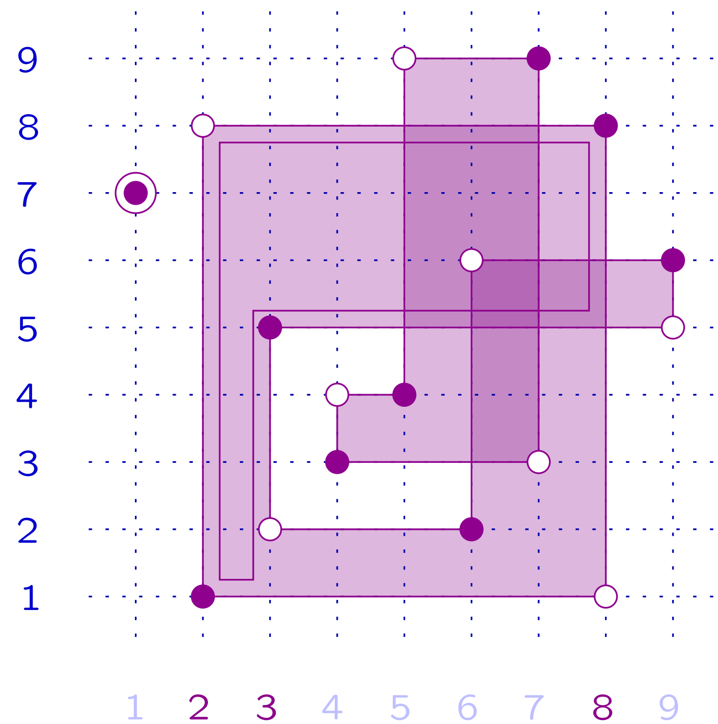

$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \xrightarrow{(5,7)} x_5 \xrightarrow{(6,8)} \bullet \xrightarrow{(6,9)} y$$

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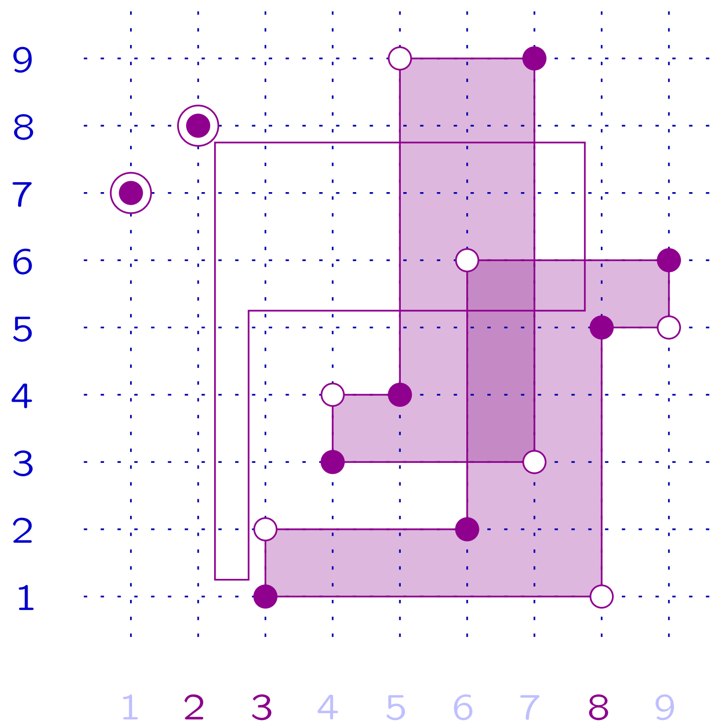

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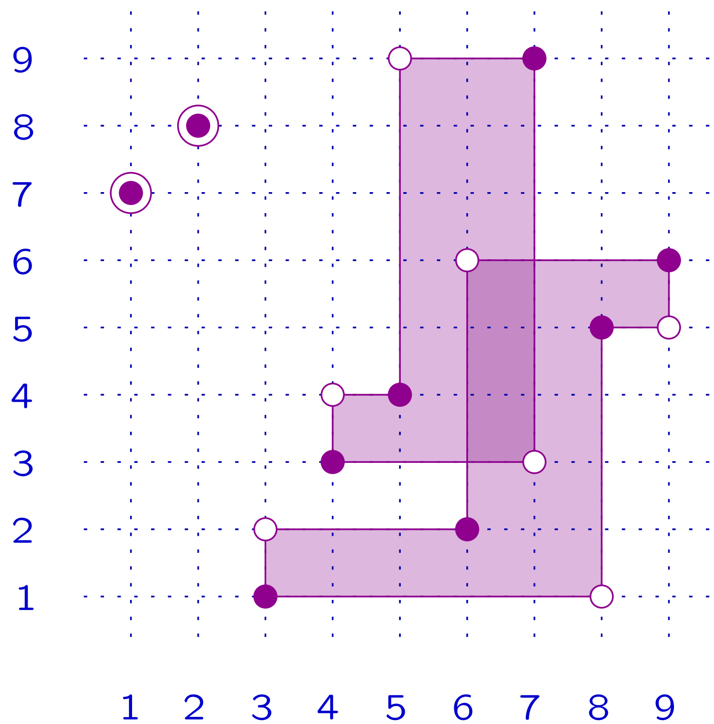

$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \xrightarrow{(5,7)} x_5 \xrightarrow{(6,8)} \bullet \xrightarrow{(6,9)} y$$

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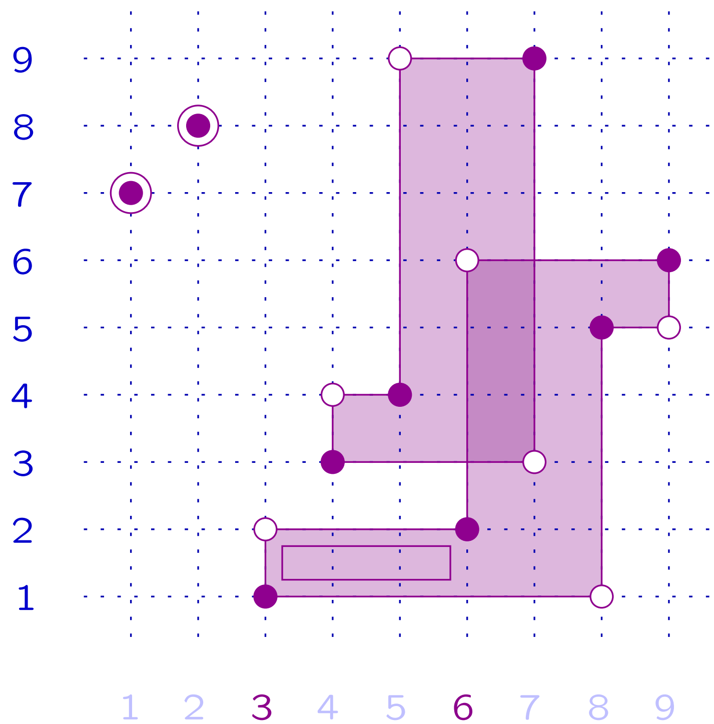

$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} \underset{(\bullet)}{x_2} \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \xrightarrow{(5,7)} x_5 \xrightarrow{(6,8)} \bullet \xrightarrow{(6,9)} \underset{(\circ)}{y}$$

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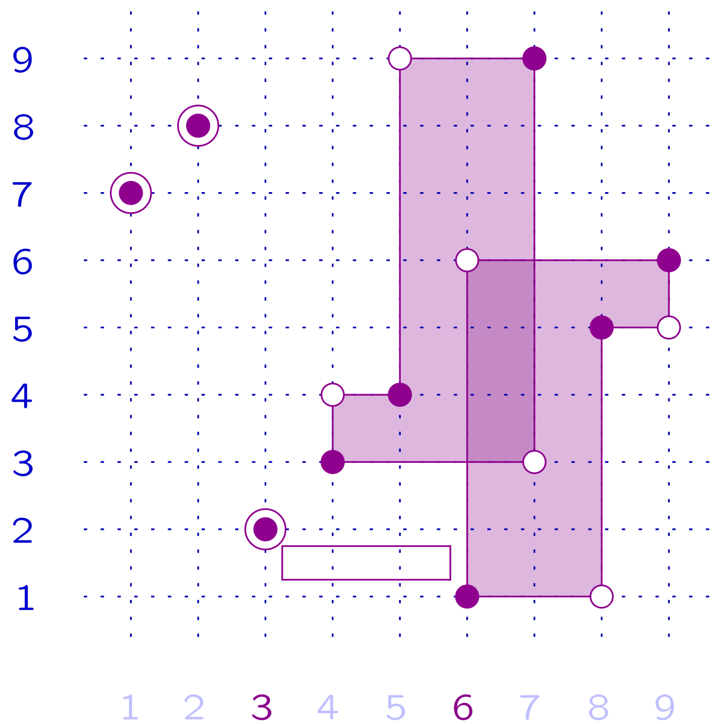
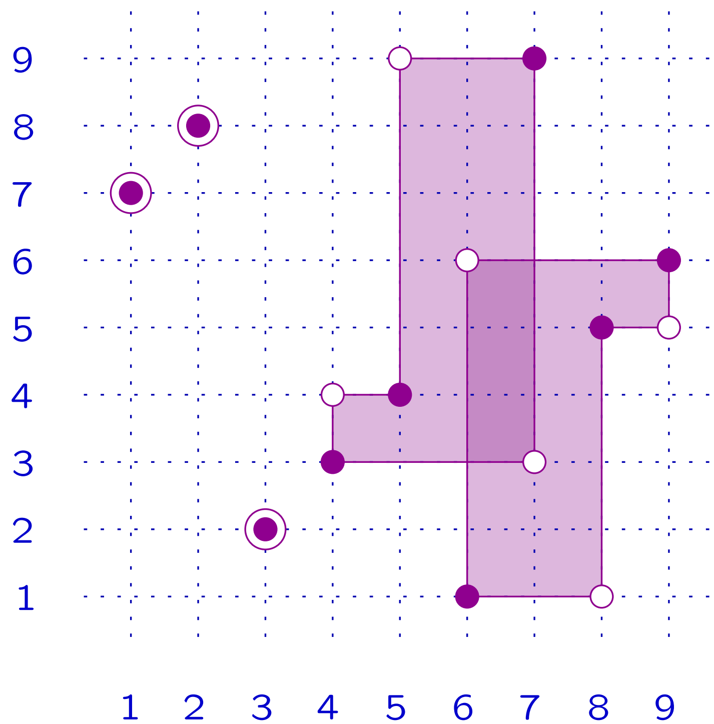


$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \xrightarrow{(5,7)} x_5 \xrightarrow{(6,8)} \bullet \xrightarrow{(6,9)} y$$

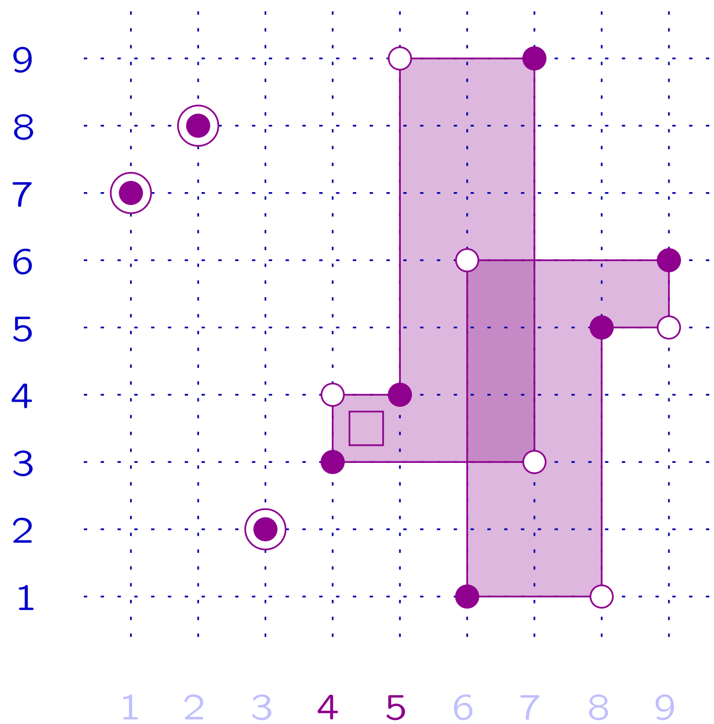
$(\bullet)$ $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$



$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} \underset{(\bullet)}{x_3} \xrightarrow{(4,5)} x_4 \xrightarrow{(5,7)} x_5 \xrightarrow{(6,8)} \bullet \xrightarrow{(6,9)} \underset{(\circ)}{y}$$

Example $x = 315472986$ $y = 782496315$ $(\circ)$



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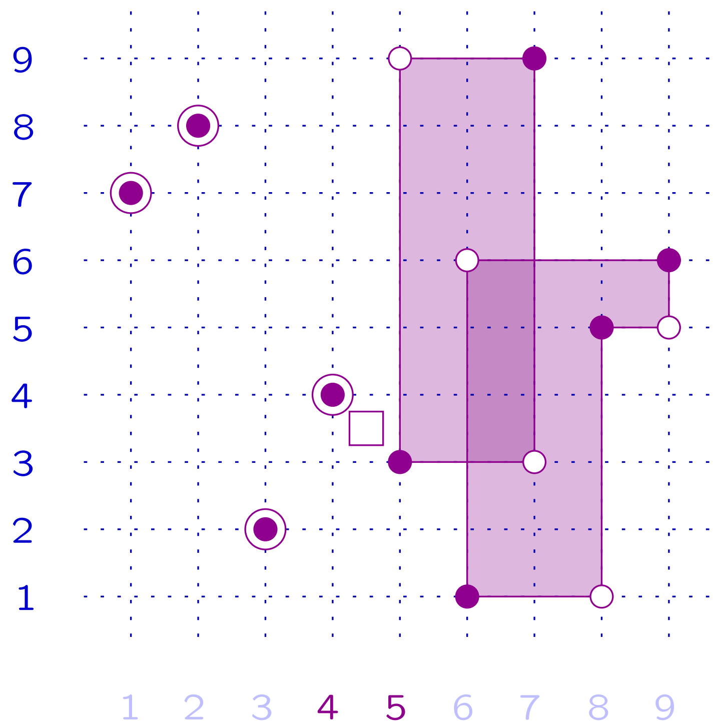

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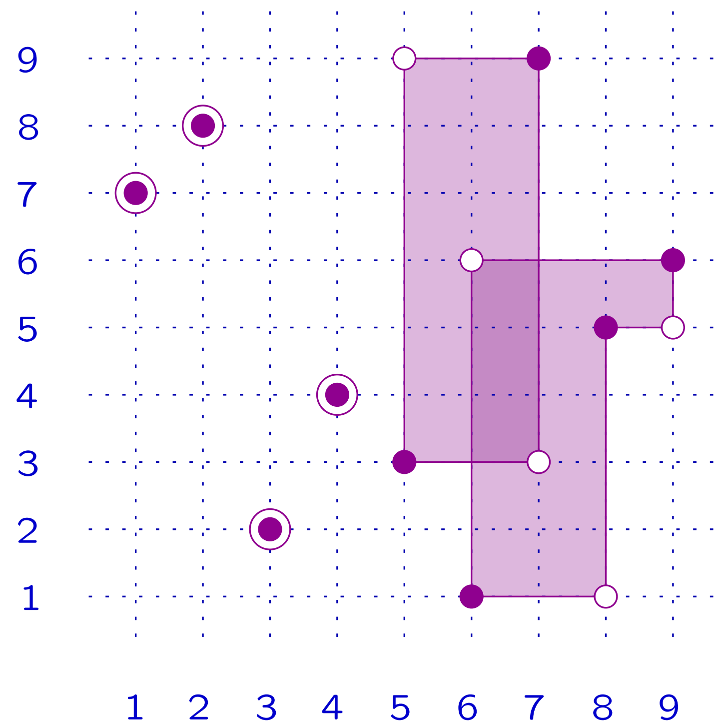
x_4: $(\bullet)$ y: $(\circ)$

Example $x = 315472986$ $y = 782496315$ $(\circ)$



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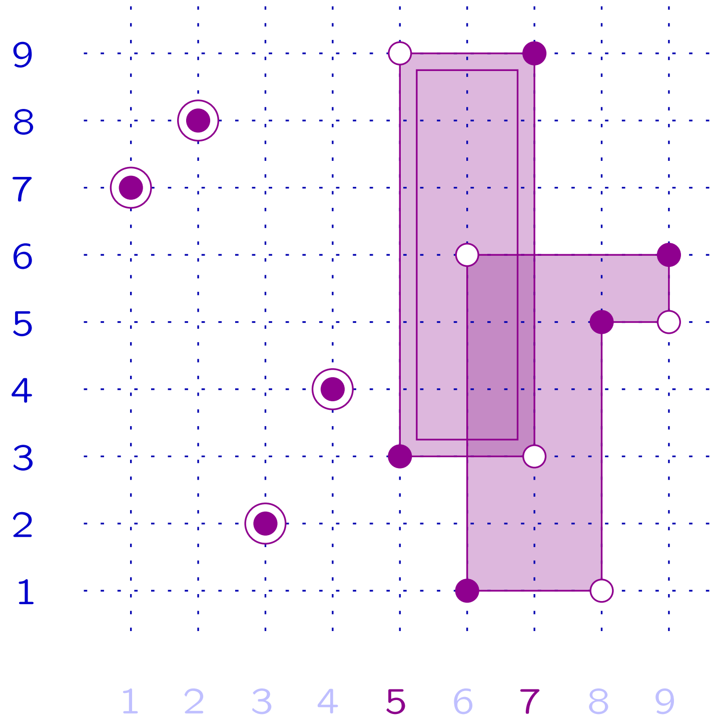

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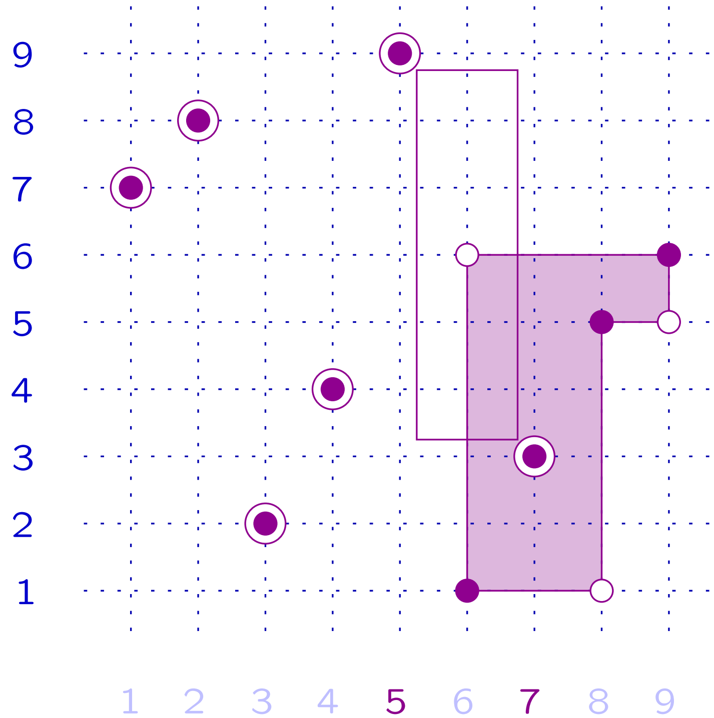

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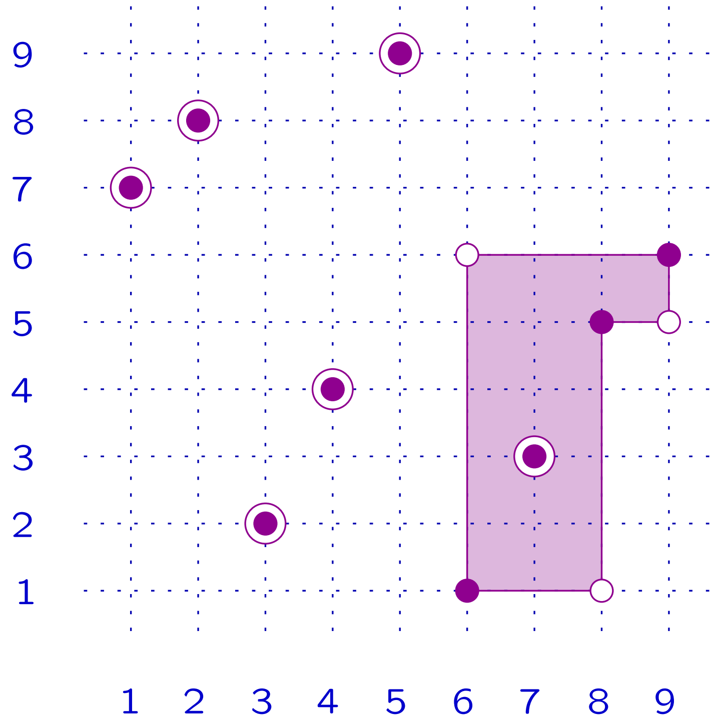

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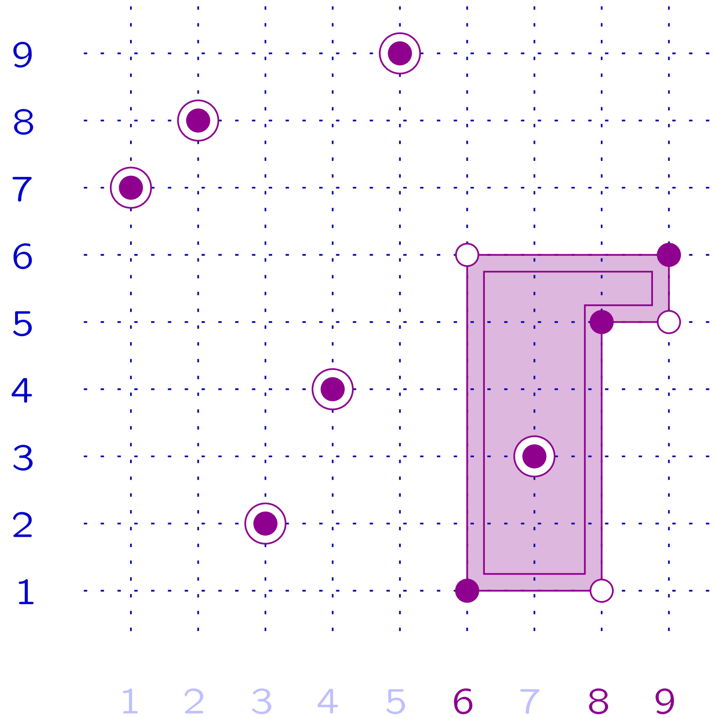

$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \xrightarrow{(5,7)} x_5 \xrightarrow{(6,8)} \bullet \xrightarrow{(6,9)} y$$

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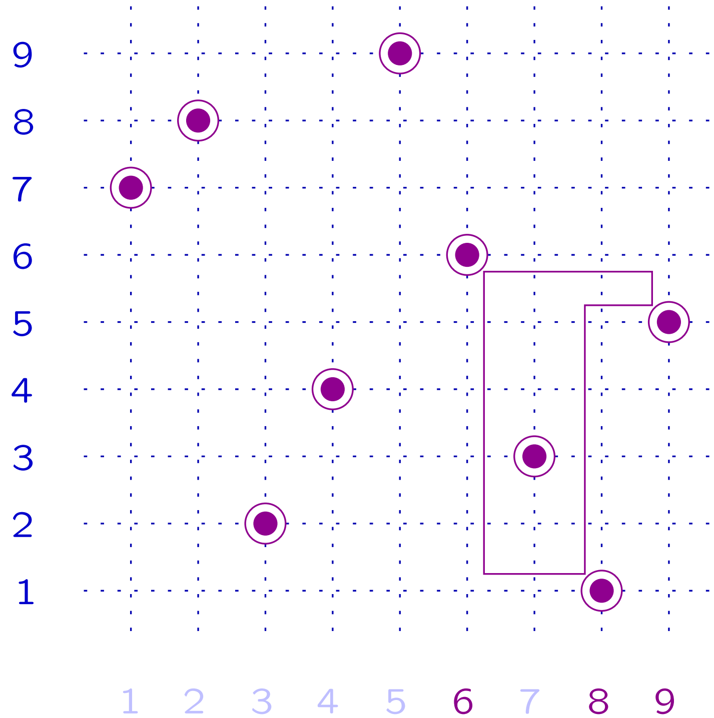

$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \xrightarrow{(5,7)} x_5 \xrightarrow{(6,8)} \bullet \xrightarrow{(6,9)} y$$

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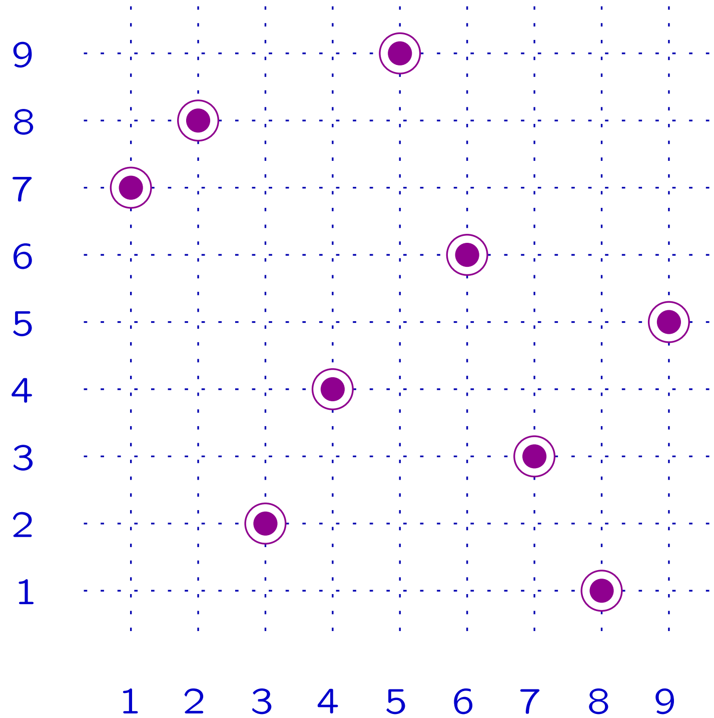

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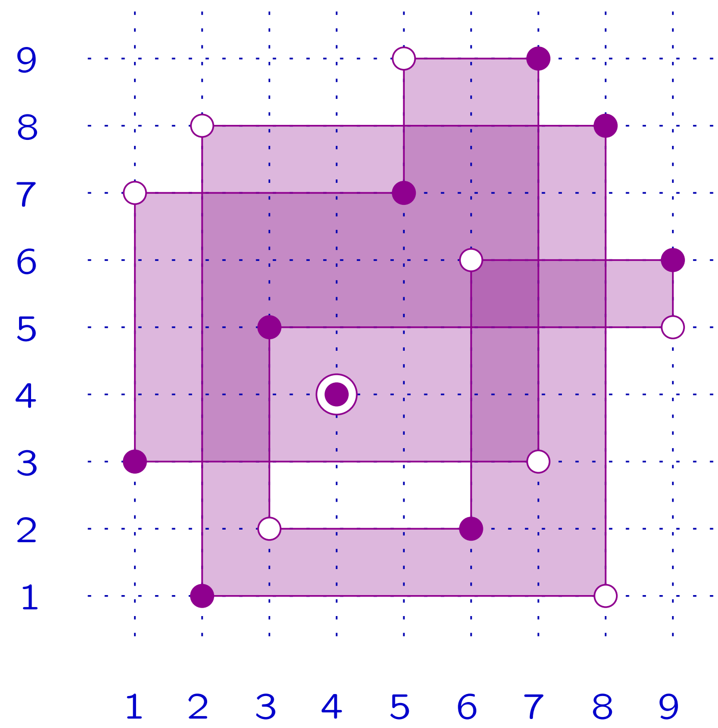
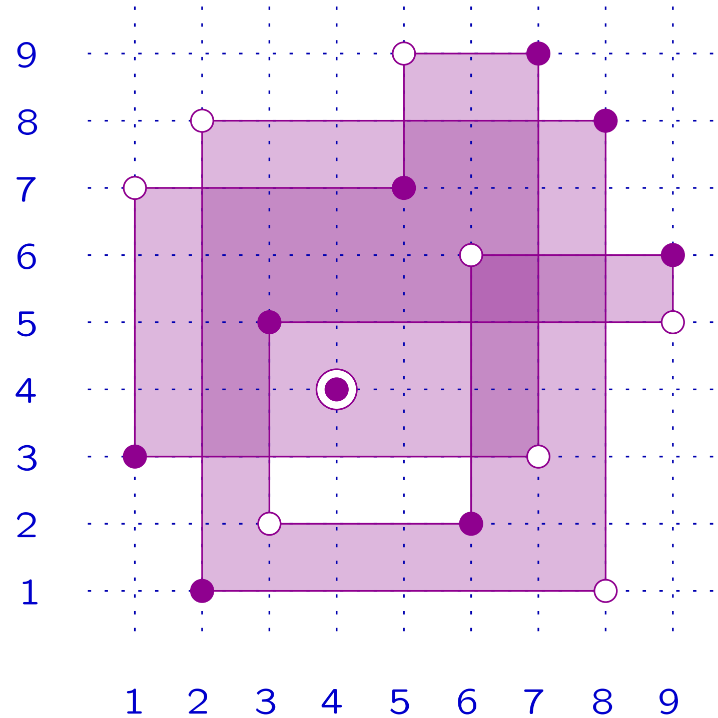
Example $x = 315472986$ ($\bullet$) $y = 782496315$ ($\circ$)



Example $x = 315472986$ ($\bullet$) $y = 782496315$ ($\circ$)

The stair method allows to generate all increasing paths in BG from x to y:



Example $x = 315472986$ ($\bullet$) $y = 782496315$ ($\circ$)

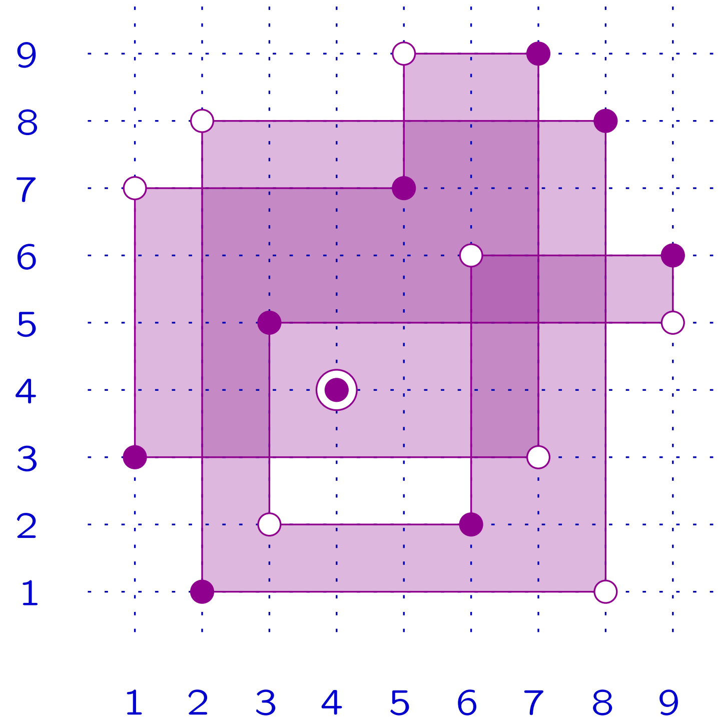
The stair method allows to generate all increasing paths in BG from x to y:

1 has length 13

4 have length 11

4 have length 9

1 has length 7



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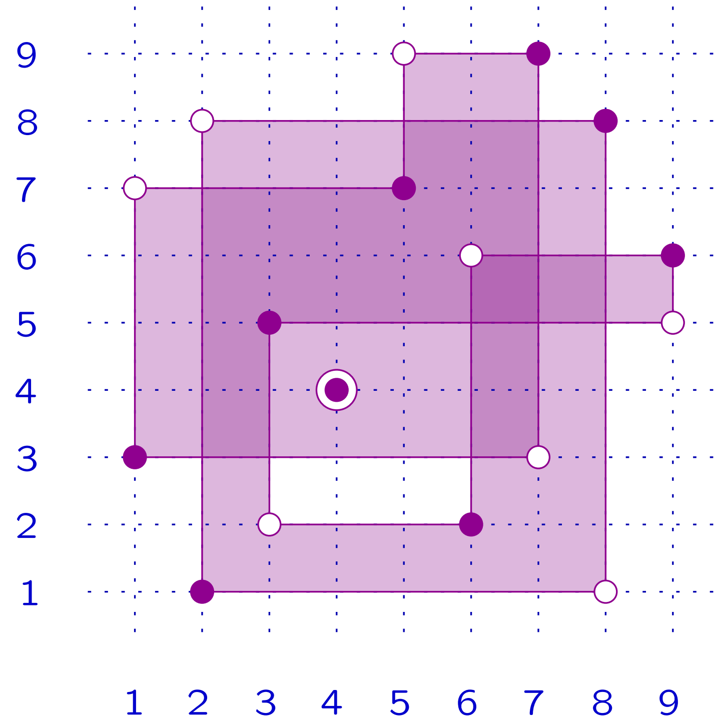


$$\Rightarrow \quad \widetilde{R}_{x,y}(q) = q^{13} + 4q^{11} + 4q^{9} + q^{7}$$

5.6 Special cases

Definition Let $x, y \in S_n$, with $x < y$. We say that

1. (x, y) has the 01-*multiplicity property* if

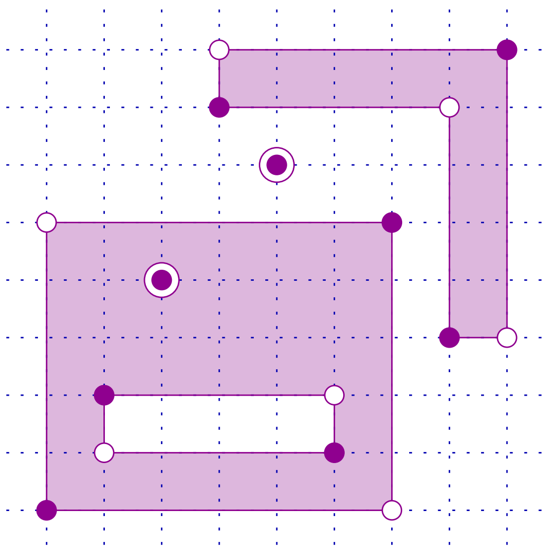
$$(x, y)[h, k] \in \{0, 1\} \quad \forall (h, k) \in \mathbf{R}^2.$$

2. (x, y) is *simple* if it has the 01-multiplicity property and

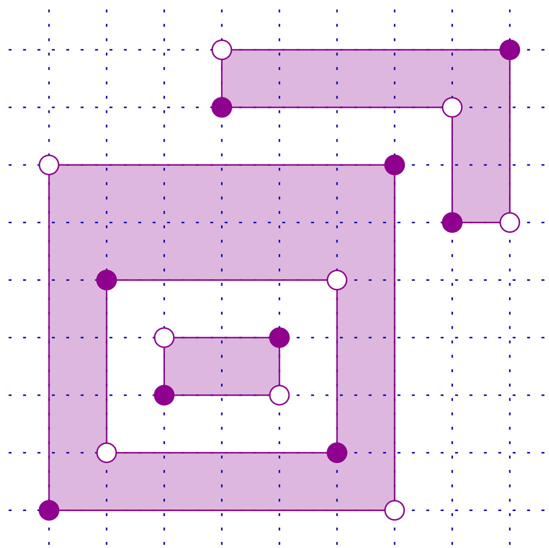
$$Fix(x, y) = \{i \in [n] : x(i) = y(i)\} = \varnothing.$$

3. (x, y) is a *permutaomino* if it is simple and $\Omega(x, y)$ is connected.

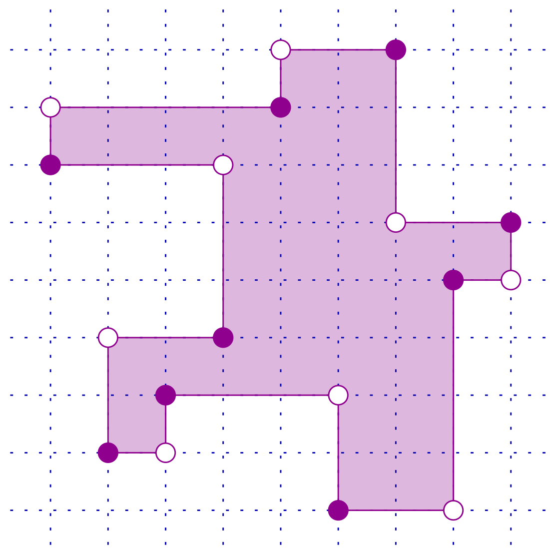
Example



01-multiplicity property



simple pair



permutaomino

Definition Let $x, y \in S_n$, with $x < y$. Let $i \in Fix(x, y)$.

The *fixed point multiplicity* of i is

$$fpm(i) = (x, y)[i, x(i)].$$

The *fixed point multiplicity* of (x, y) is

$$fpm(x, y) = \sum_{i \in Fix(x,y)} fpm(i).$$

Proposition Let $x, y \in S_n$, with $x < y$.

1. If (x, y) has the 01-multiplicity property, then

$$\widetilde{R}_{x,y}(q) = (q^2 + 1)^{fpm(x,y)} q^{a\ell(x,y)},$$

   thus

$$a\ell(x, y) = \ell(x, y) - 2fpm(x, y).$$

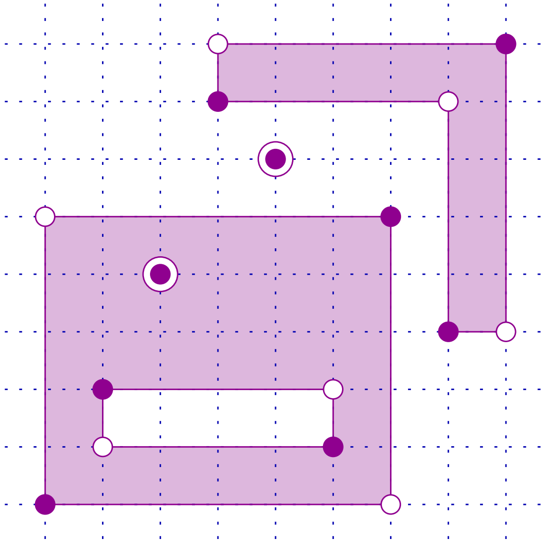
2. In particular, if (x, y) is simple, then

$$\widetilde{R}_{x,y}(q) = q^{\ell(x,y)},$$

3. and if (x, y) is a permutaomino, then

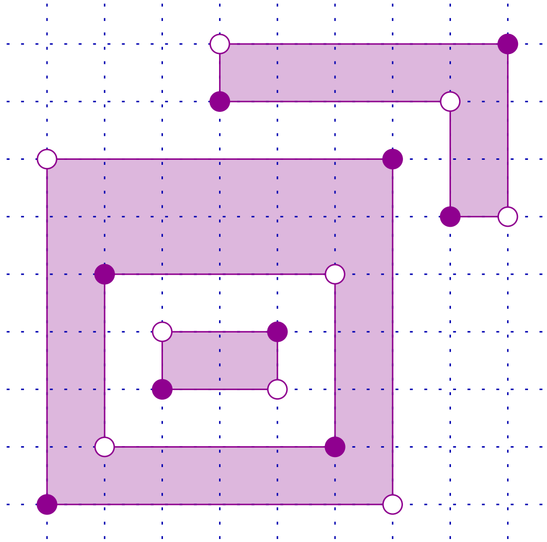
$$\widetilde{R}_{x,y}(q) = q^{(n-1)}.$$

Example



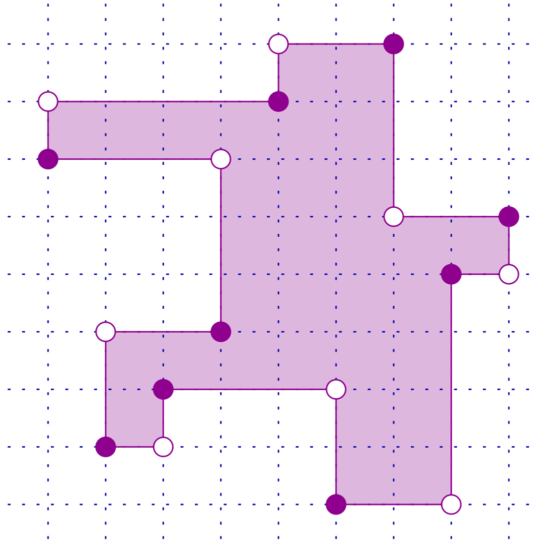
01-multiplicity property

$$\widetilde{R}_{x,y}(q) = (q^2+1)q^6$$



simple pair

$$\widetilde{R}_{x,y}(q) = q^7$$



permutaomino

$$\widetilde{R}_{x,y}(q) = q^8$$

6. PROOF SKETCH

Theorem Let $x, y \in S_n$, for some n, with $x < y$ and $\ell(x,y) = 5$. Set $a = a(x,y)$, $c = c(x,y)$ and $cap = cap(x,y)$. Then

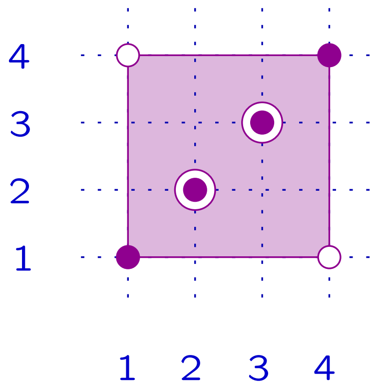
$$\widetilde{R}_{x,y}(q) = \begin{cases} q^5 + 2q^3 + q, & \text{if } \{a,c\} = \{3,4\}, \\ q^5 + 2q^3, & \text{if } a = c = 3, \\ q^5 + q^3, & \text{if } cap \in \{4,5\} \text{ but } [x,y] \ncong \mathcal{B}_5, \\ q^5, & \text{if } cap \in \{6,7\} \text{ or } [x,y] \cong \mathcal{B}_5. \end{cases}$$

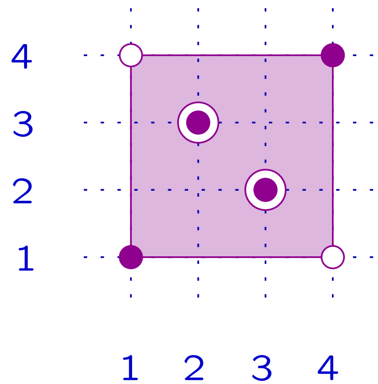
Proof sketch. Suppose known the poset structure of $[x,y]$.

By Dyer's result, it allows to determine $a\ell(x,y) \in \{1,3,5\}$.

If $a\ell(x,y) = 5$, then $\widetilde{R}_{x,y} = q^5$ is determined. In this case it is known that $[x,y]$ is a lattice and this implies either $cap(x,y) \geq 6$, or $[x,y] \cong \mathcal{B}_5$.

If $a\ell(x,y) = 1$, then (x,y) is an edge of BG. Two possible diagrams:





By the stair method: $\widetilde{R}_{x,y}(q) = q^5 + 2q^3 + q$.

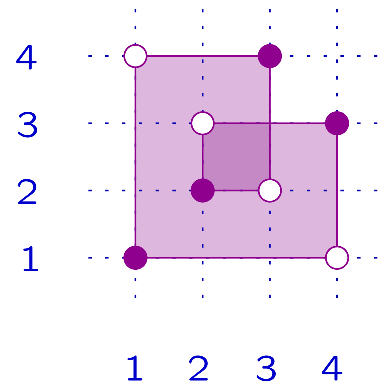
By the interpretation of atoms and coatoms:

$$\{a(x,y), c(x,y)\} = \{3, 4\}.$$

Finally, if $a\ell(x,y) = 3$, then $\widetilde{R}_{x,y}(q) = q^5 + bq^3$, for some $b \in \mathbf{N}$.

The only possibility in S_4 (up to symmetries) is the following:

the *heart* diagram:

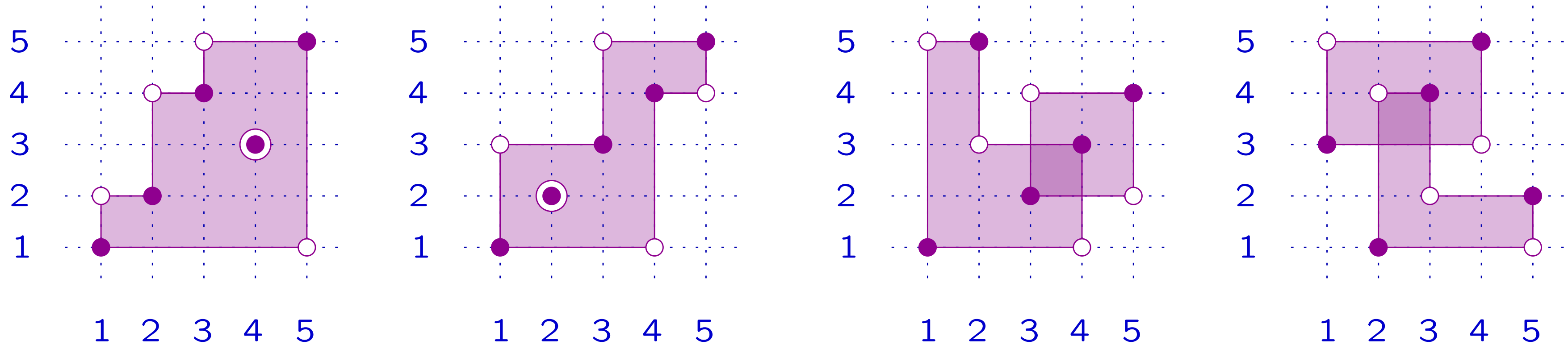


By the stair method: $\widetilde{R}_{x,y}(q) = q^5 + 2q^3$.

By the interpretation of atoms and coatoms:

$$a(x,y) = c(x,y) = 3.$$

All other cases can be easily listed. A few examples:



By the stair method: $\widetilde{R}_{x,y}(q) = q^5 + q^3$.

By the interpretation of atoms and coatoms:

$$cap(x, y) \in \{4, 5\}.$$

The boolean algebra $\mathcal{B}_5$ never occurs. □

7. EXPLICIT FORMULAS

Let $x, y \in W$, with $x < y$. For $k \in [\ell(x,y)]$ odd, set

$$be_k(x,y) = |\{(z,w) : x \leq z \rightarrow w \leq y,\ \ell(z,w) = k\}|.$$

Theorem Let $x, y \in S_n$, with $x < y$ and $\ell(x,y) = 5$. Then

$$\widetilde{R}_{x,y}(q) = q^5 + \left\lfloor \frac{be_3}{3} \right\rfloor q^3 + be_5\, q.$$

Let $x, y \in W$, with $x < y$. Set $F_i(x,y) = \{z \in [x,y] : \ell(x,z) = i\}$ and

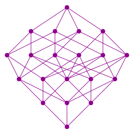
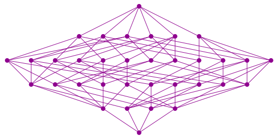
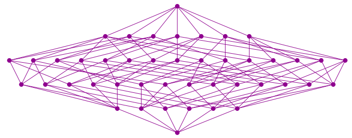
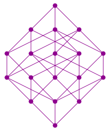
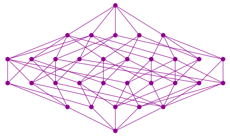
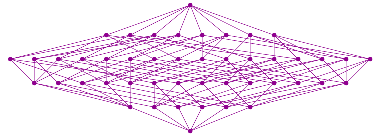
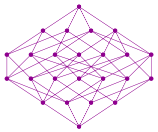
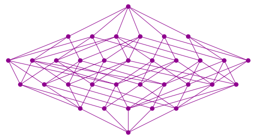
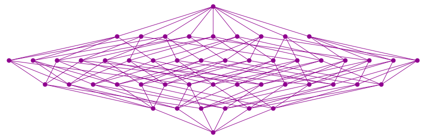
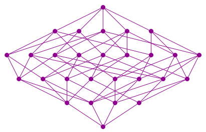
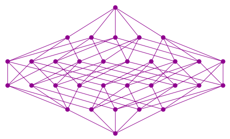
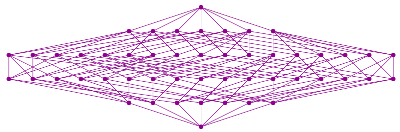
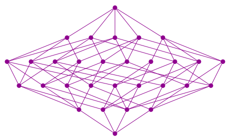
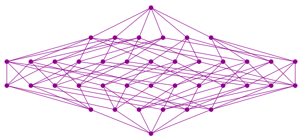
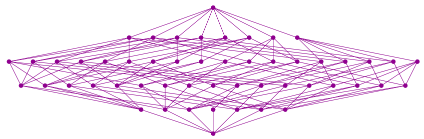
$$
\begin{aligned}
f_{i,j}(x,y) &= |\{(z,w) \in F_i(x,y) \times F_j(x,y) : z < w\}|, \\
be_{i,j}(x,y) &= |\{(z,w) \in F_i(x,y) \times F_j(x,y) : z \to w\}|.
\end{aligned}
$$

For $a, b \in \mathbf{N}$, set $\quad a \bmod b = a - b \left\lfloor \frac{a}{b} \right\rfloor$.

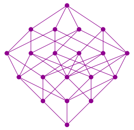
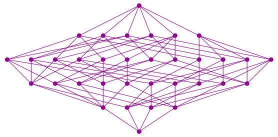
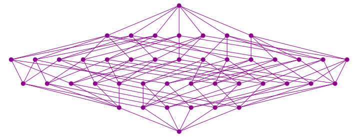
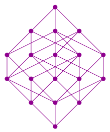
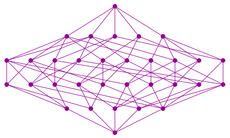
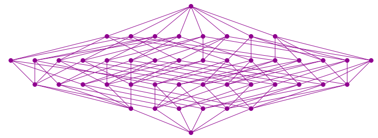
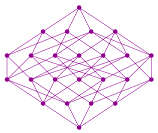
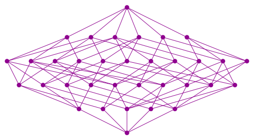
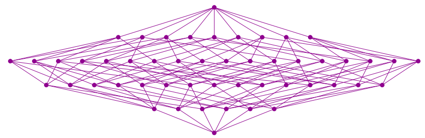
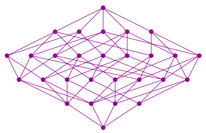
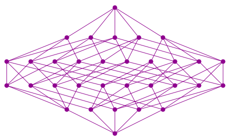
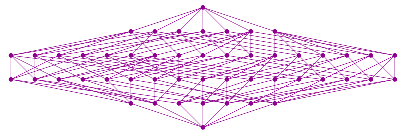
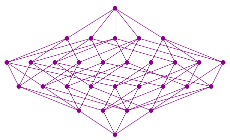
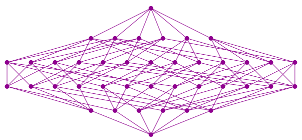
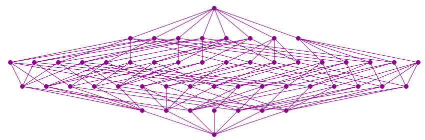
Theorem Let $x, y \in S_n$, with $x < y$ and $\ell(x,y) = 5$. Then

$$
\begin{aligned}
P_{x,y}(q) &= 1 + \left(c + \left\lfloor \frac{be_3}{3} \right\rfloor - 5 \right) q \\
&\quad + \left(10 - 3a - 3c + f_{1,4} + be_3 \bmod 3 - \frac{be_{1,4}}{2} + be_5 \right) q^2.
\end{aligned}
$$

$[x, y]$	$\widetilde{R}_{x,y}(q)$

	$q^5 + 2q^3 + q$		$q^5 + q^3$		q^5
	$q^5 + 2q^3$		$q^5 + q^3$		q^5
	$q^5 + q^3$		$q^5 + q^3$		q^5
	$q^5 + q^3$		q^5		q^5
	$q^5 + q^3$		q^5		q^5

$[x, y]$	$P_{x,y}(q)$ $P_{w_0y,w_0x}(q)$

	$1+q$ 1		$1+2q+q^2$ $1+q^2$		$1+2q$ $1+q$
	1		$1+q$ $1+q$		$1+3q$ $1+q$
	1		$1+2q$ $1+q$		$1+4q+q^2$ $1+q+q^2$
	$1+q$ 1		1		$1+2q$
	$1+q$ 1		$1+q$		$1+3q+q^2$ $1+2q+q^2$