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Calculus

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Note for the reader For some of the problems, marked with the symbol ⏭, a solution is given in Chapter 5. The coffee cup ☕ marks harder problems.

Contents

CHAPTER 1

To begin

Let us recall some basic concepts and some calculation methods that you will surely have already encountered in high school; later on, we will return to some of these ideas and examine them from a more general point of view.

1. Integer, rational and real numbers

In mathematics it is necessary to choose a starting point on whicheverybody agrees; from here, with a series of logical arguments, we obtain more and more general consequences that allow us to solve more and more complicated problems. Our starting point are the *natural numbers*:

$$1, 2, 3, 4, 5, \dots.$$

Natural numbers are so basic that you cannot define them using even simpler objects (there are no simpler objects ...). We will call *the set of natural numbers* the collection formed by these numbers; the notation used is the following:

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, \dots\}.$$

Whenever we have a collection of objects we will say that we have a *set*, the individual objects are called the *elements* of the set, and they are said to *belong* to the set. If a is an element of the A set, we write $a \in A$ (and if it doesn't belong, we write $a \notin A$), and read: *a belongs* to A, or also: *A contains a*. Sometimes zero is included in the collection; to distinguish the two cases we will use the notation

$$\mathbb{N}_0 = \{0, 1, 2, 3, 4, 5, 6, \dots\}.$$

We note that the elements of the set $\mathbb{N}$ are also in the $\mathbb{N}_0$ set (there is only one more element). When all the elements of a set A are also elements of the set B we will write $A \subseteq B$ or also $A \subset B$ and we say that *A is a subset of B*. For example we have $\mathbb{N} \subseteq \mathbb{N}_0$.

Another set that you have already encountered is the set of all *integers*, also called *relative integers*: to obtain it, just add zero and all negative integers to $\mathbb{N}$. We will use the symbol

$$\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}.$$

Thus we have $\mathbb{N} \subseteq \mathbb{N}_0 \subseteq \mathbb{Z}$. The same idea can be expressed writing

$$n \in \mathbb{N} \implies n \in \mathbb{N}_0 \implies n \in \mathbb{Z},$$

where the symbol $\implies$ means *"implies"*. The formula $A \implies B$ (read "A implies B") is a shorthand to say that if A is true, then B is also true. For practical purposes, we shall write $B \impliedby A$ with the same meaning as $A \implies B$.

Now we make a qualitative leap: we know that given two integers p and q, positive or negative, we can consider the fraction $\frac{p}{q}$, if the denominator q is different from zero (in symbols: if $q \neq 0$). We will call the set of all fractions *the set of rational numbers* $\mathbb{Q}$. If we want to use the above notation for $\mathbb{Q}$ as well, we need to modify it a bit. We want to write in a single formula that $\mathbb{Q}$ is the set of all numbers x such that x is a fraction of relative integers, with the denominator other than zero:

$$\mathbb{Q} = \left\{ x \colon x = \frac{p}{q},\ p, q \in \mathbb{Z},\ q \neq 0 \right\}$$

(and the formula is read just like this: *the set of all x such that x is equal to p/q* etc.). Of course, if p is divisible by q we obtain an integer; hence $\mathbb{Z} \subseteq \mathbb{Q}$.

You will surely remember that fractions can be expressed as decimal numbers, with an *integer part* and a *decimal part*:

$$\frac{1}{3} = 0,333333\ldots, \qquad \frac{13}{7} = 1,857142857142\ldots \qquad \frac{-28}{70} = -0,40000\ldots.$$

Indeed, when we divide two integers, the decimal expansion is a very special one: in some cases, from a certain point on we obtain a sequence of zeros (*finite* decimal expansion) and in these cases we do not write the zeros:

$$\frac{-28}{70} = -0,40000\ldots = -0,4;$$

in other cases, the decimal expansion does not stop, however there is a group of digits which always returns the same, periodically:

$$\frac{1338}{9990} = 0,1339339339\ldots = 0,1\overline{339}.$$

This repeating group is called the *period* and it is denoted by drawing a line over the group.

Nobody forbids us to consider even more general numbers, whose decimal expansion is not periodic:

$$-35,78543656895435609870076 54\ldots.$$

A "general" decimal number is called a *real number*, and the set of all real numbers (all possible decimal expansions) is denoted by $\mathbb{R}$. This definition is almost perfect: the only small trouble is that there are some decimal expansions which look different, but actually give the same number. Precisely we have:

$$0,999999\ldots = 1$$

and more generally, given a finite decimal expansion, we obtain an equivalent one with the same method:

$$65,2583 = 65,258299999\ldots.$$

However this small defect does not give any problem in the definition of real numbers.

There are very many real numbers that are not rational, for example the number

$$\sqrt{2} = 1,4142135623730950488016887242096980785696718753769480731766 79\ldots$$

(ratio between diagonal and side of a square) is real but not rational. Real numbers that are not rational (= nonperiodic decimal expansions) are said to be *irrational*.

The set $\mathbb{R}$ is very rich in properties: first of all we can perform the usual operations (sum, product), furthermore given a number x we can consider its opposite $-x$, and its inverse $\frac{1}{x}$ when $x \neq 0$; so we can make the difference $x - y$ and divide $\frac{x}{y}$ if the denominator does not vanish. Another important property is that the set $\mathbb{R}$ is *ordered*. This means that given two real numbers x and y we can always determine which of the two is larger:

$$\text{one has always: either } x \leq y \text{ ot } x \geq y.$$

The only case when both inequalities are valid is when $x = y$. Watch out: the inequality $x \leq y$ (x is less than or equal to y) is correct *both* when x is smaller than y, *and* when $x = y$. For example, it is true that $1 \leq 3$ and it is also true that $2 \leq 2$. If we want to exclude equality we use the symbol $x < y$ (or $x > y$) which is called *strict inequality*.

Since the real numbers are ordered, we can put them all "in a row": the simplest way to visualize $\mathbb{R}$ is to think of a line

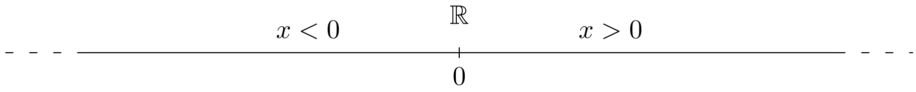


FIGURE 1.1.

This picture is called the *real line*. Inside the real line we find all the previous sets, since $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R}$. Drawing integers is easy:

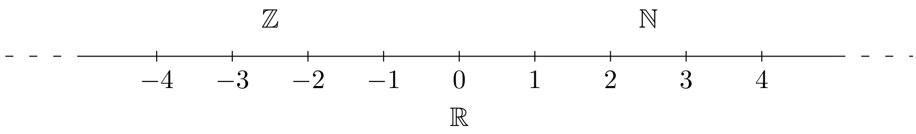


FIGURE 1.2.

but if we want to draw $\mathbb{Q}$ there is a problem: the rational numbers are not separated from each other, but they are “dense” everywhere. More precisely, however we choose two real numbers x and y, even very close to each other, between them we can find an infinite amount of rational numbers.

Given two numbers a and b, with $a < b$, we can consider the set of all real numbers between a and b; this is called an *interval* of real numbers, and a, b are called the *extrema* of the interval. In some cases it is useful to include the two extrema in the interval, in other cases not; you see easily that in total we have four possibilities: if we include both extrema we have

$$\text{the } \textit{closed interval } [a, b] = \{x \colon a \leq x \leq b\};$$

if we exclude both extrema we have

$$\text{the } \textit{open interval }]a, b[= \{x \colon a < x < b\};$$

and if we include only one of the extrema, we have the half-open intervals (also called half-closed)

$$]a, b] = \{x \colon a < x \leq b\} \qquad \text{e} \qquad [a, b[= \{x \colon a \leq x < b\}.$$

What happens if $a = b$? The closed interval $[a, a]$ contains only the point a; the set containing only the point a is also denoted by $\{a\}$. On the other hand, the open interval $]a, a[$ contains no points, because no x can verify $a < x < a$; so this is an *empty* set, often denoted by $\emptyset$.

It is very easy to visualize these sets on the real line: just consider the segment of endpoints a and b. To distinguish the various previous cases, we will draw a solid point if the point is part of the interval, and a hollow point if it is not part of it:

FIGURE 1.3.

Sometimes it is useful to consider also intervals of infinite length i.e. *half–lines*: that is, having chosen a point a, we consider all the points x that are to the right of a, or all those that are to the left. To denote these infinite intervals we use the infinity symbol ∞, and precisely we write:

$$[a, +\infty[= \{x : x \geq a\} \qquad]a, +\infty[= \{x : x > a\}$$

for the half–line at the right of a (a included or excluded), and

$$]-\infty, a] = \{x : x \leq a\} \qquad]-\infty, a[= \{x : x < a\}$$

for the half–line at the left of a.

Figure 1.4.

With the sets just defined, we can perfomr several *operations*. There are two main ones: *the union* and *the intersection*. Taking the *union* of two sets means putting together all the points that are both in the first and in the second set. For example, the union of the intervals $[1, 4]$ and $[3, 8]$ is the entire interval $[1, 8]$; the union of the intervals $]-1, 2[$ and $]-3.7]$ is the interval $]-3.7]$; the union of the intervals $[0, 1]$ and $[5, 6]$ is not an interval but is a set made up of two separate pieces. The union of two sets is indicated with $\cup$:

$$]0, 6[\cup]1, 9] =]0, 9].$$

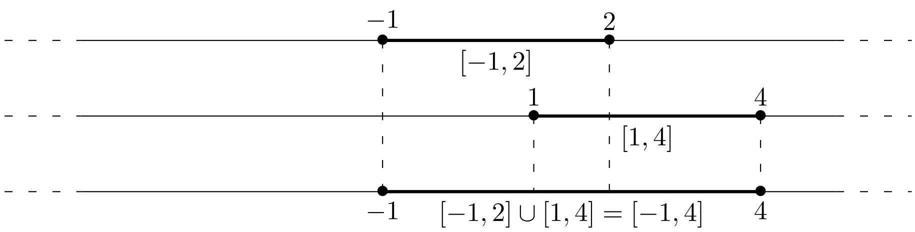


Figure 1.5.

The union of two sets A, B contains both A and B.

The second operation between sets is the intersection. Taking the *intersection* of two sets means considering only the points in common between them, i.e. the points that are both in the first and in the second set. Points which are only in one of the two sets but not the other, are discarded. The intersection of two sets is denoted by $\cap$. For instance,

$$[1, 3] \cap [2, 7] = [2, 3] \qquad [3, +\infty[\ \cap \ ]-\infty, 5[= [3, 5[.$$

Clearly, the intersection of two sets is contained in both sets.

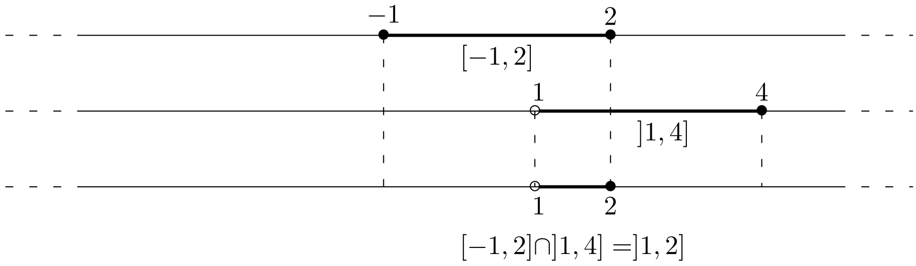


FIGURE 1.6.

Problems.

EXERCISE 1.1 (▶▶|). Can the following unions and intersections of intervals be written in a simpler form or not?

$$[1,10] \cap]4,12]; \qquad]-2,-1[\cup [0,5]; \qquad [1,2] \cup]3,4[\, .$$

If $p, q \in \mathbb{Z}$ are two integers, p is divisible by q if there is a third integer $r \in \mathbb{Z}$ such that $p = qr$. An integer is *even* if it is divisible by 2, otherwise it is *odd*. By definition a *prime number* is an integer $p > 1$ which is only divisible by ± 1 and $\pm p$.

EXERCISE 1.2. Let a, b be two integers whose product ab is divisible by 4. Which of the following statements is certainly true?

(1) a and b are both even,
(2) a and b are both divisible by 4,
(3) a is divisible by 4 or b is divisible by 4,
(4) a is even or b is even.

EXERCISE 1.3. Which of the following statements are true?

(1) If x is an irrational number, then x^2 is too.
(2) If x is an irrational number, then $x + \sqrt{2}$ is too.
(3) If x is an irrational number, then $x/2$ is rational.
(4) If x is an irrational number, then $x - \sqrt{2}$ is integer.
(5) If x is an irrational number, then $x/2$ can be rational.
(6) If x is an irrational number, then $x - \sqrt{2}$ can be integer.

2. Equalities and inequalities

In the previous section we have already observed that given two real numbers a, b, if these numbers coincide we can write this fact as an *equality* $a = b$. If a, b are distinct numbers we can instead write $a \neq b$: in other words the formula $a = b$ is true if and only if the formula $a \neq b$ is false.

We know that an equality $a = b$ remains true if the same quantity is added or subtracted to the expressions at the right and at the left of the $=$ sign: in particular, $a = b$ if and only if $a - b = 0$ (just subtract b both from the left and the right).

In a similar way, an equality remains true if the expressions to the right and left of the sign $=$ are multiplied by the same number: one must be careful since this operation is reversible only if the number by which both sides are multiplied is different from zero.

We can repeat both the previous statements and the *symmetric* and *transitive* properties in the usual "if–then" formulation:

(1) if $a = b$ and $c = d$, then $a + c = b + d$;
(2) if $a = b$ and $c = d$, then $ac = bd$;
(3) if $a = b$, then $b = a$;
(4) if $a = b$ and $b = c$, then $a = c$.

Finally, it is clear that an equality has a non trivial information content when the two parts are different ways of writing the same number, for example

$$\frac{1}{2} = \frac{2}{4}, \qquad \frac{1}{2} = 0,5, \qquad \frac{3}{9} = 0,\overline{3}, \qquad \frac{12}{99} = 0,\overline{12}\,.$$

On the other hand, an equality like $2 = 2$ is true but trivial.

We observe that if $a = b$ then $a^2 = b^2$, but the converse is generally false. For example if $a = 1$ and $b = -1$, then $a^2 = b^2$ but $a \neq b$. This is a very common mistate, so beware!

When we speak of an *inequality* we are referring instead to a formula of one of the following types:

$$a < b, \qquad a \leq b, \quad a \neq b, \quad a \geq b, \quad a > b\,.$$

For what concerns inequalities involving the symbols $<$ (strictly minor) and $\leq$ (minor or equal), the following properties are valid and are easy and immediate to understand:

(1) if $a < b$, then $a \leq b$;
(2) if $a = b$, then $a \leq b$;
(3) if $a \leq b$ and $b \leq a$, then $a = b$;
(4) if $a < b$ and $b \leq c$, then $a < c$;
(5) if $a \leq b$ and $b < c$, then $a < c$;
(6) if $a \leq b$ and $b \leq c$, then $a \leq c$.

As for the equalities, also the inequalities remain valid if the same quantity is added to both sides. For example if $a < b$ then $a + c < b + c$ for each c. More generally, if $a < b$ and $c \leq d$, then $a + c < b + d$. Indeed we have $a + c < b + c$ (add c to both sides of $a < b$), $b + c \leq b + d$ (add b to both sides of $c \leq d$) and this implies

$$a + c < b + c \leq b + d \Rightarrow a + c < b + d\,.$$

Unlike the equalities, an inequality remains valid if both members are multiplied by the same quantity only if this quantity is *strictly positive*: it is in fact well known that if we multiply by negative numbers it is necessary to "reverse" the inequality. In other words, using the "if—then" formulation, we can write;

(1) if $a \leq b$ and $c > 0$, then $ac \leq bc$;
(2) if $a < b$ and $c > 0$, then $ac < bc$;
(3) if $a \leq b$ and $c < 0$, then $ac \geq bc$;
(4) if $a < b$ and $c < 0$, then $ac > bc$.

Example 2.1. Let a, b be positive real numbers (i.e. $a > 0$ and $b > 0$). Assume that $a^2 = b^2$. Then we can prove that $a = b$.

Indeed, if $a \neq b$ we have either $a < b$ or $b < a$. If $a < b$, multiplying by $a > 0$ yields $a^2 < ab$, while multiplying by b yields $ab < b^2$ and therefore $a^2 < ab < b^2$ which is impossible since we assumed $a^2 = b^2$. In a similar way we see that $b < a$ is impossible.

The same argument used in the previous example shows that if $0 < a < b$ then $a^2 < b^2$, $a^3 < b^3$, $a^4 < b^4$ and more generally $a^n < b^n$ for all positive integer n. For example, for the cubic powers, multiplying $a < b$ by a^2, by ab and by b^2 we have the three inequalities $a^3 < a^2b < ab^2 < b^3$, which imply $a^3 < b^3$.

Example 2.2. We prove that for each pair of real numbers a, b the inequalities hold

$$2ab \leq a^2 + b^2, \qquad 4ab \leq (a + b)^2\,.$$

To this end, it is sufficient to observe that the squares of real numbers are never negative, therefore $(a-b)^2 \geq 0$ which implies $a^2+b^2-2ab \geq 0$ which is equivalent to $a^2+b^2 \geq 2ab$ and also to $a^2+b^2+2ab \geq 4ab$.

Example 2.3. We prove that for each triplet of real numbers a, b, the following inequality holds:

$$a^2+b^2+c^2 \geq ab+ac+bc\,.$$

For the proof just add the three inequalities

$$a^2+b^2 \geq 2ab, \qquad a^2+c^2 \geq 2ac, \qquad b^2+c^2 \geq 2bc,$$

which were proved in the example above, and then divide by 2.

Problems.

Exercise 1.4 (multiple choice). The sum $\frac{1}{a}+\frac{1}{b}$ of the reciprocal of two positive numbers a, b is equal to 1. Then the sum $a+b$ of the two numbers a, b is:

(1) equal to their difference;
(2) negative;
(3) equal to their product;
(4) zero;
(5) equal to 1.

Exercise 1.5. Let a be a real number such that $a^3+a=100$. Which of the following statements are true and which are false?

(1) $a>5$;
(2) $a<5$;
(3) $a>4$;
(4) $a<4$;
(5) $a^2+a>30$.

3. Practice with numbers: proportions and percentages

Proportions and percentages are very easy concepts but they are constantly used, both in all scientific fields ans also in everyday life. Therefore it is of fundamental importance to have them very clear. Let us see some examples.

A *proportion*, or proportionality ratio, is nothing more than a ratio between two quantities. To say that two pairs of numbers a, b and c, d have the same proportionality ratio means that

$$\frac{a}{b}=\frac{c}{d}\,.$$

Therefore, from the theoretical point of view this is nothing complicated. We now explain how there ratios are used.

Example 3.1. To make 720 grams of peach jam you need 1800 grams of peaches and 360 grams of sugar. How many peaches and how much sugar are needed to make 1000 grams of jam?

If we use the unknown p to indicate the grams of peaches, and the unknown z to indicate the grams of sugar, we have the proportions:

$$\frac{p}{1000}=\frac{1800}{720}, \qquad \frac{z}{1000}=\frac{360}{720},$$

which are equivalent to

$$p=1800\times\frac{1000}{720}=2500, \qquad z=1800\times\frac{360}{720}=900\,.$$

3.1. Measuring angles. The most common units of measure for angles are the *degree* and the *radian*. The convention is that a round angle measures 360 degrees or 2π radians, from which it follows, making the necessary proportions, that:

(1) a flat angle measures 180 degrees, or π radians;
(2) a right angle measures 90 degrees, or $\pi/2$ radians;
(3) the interior angles of an equilateral triangle measure 60 degrees, or $\pi/3$ radians.

More generally, to pass from measure in degrees to measure in radians, we must multiply by

$$\frac{2\pi}{360} = \frac{\pi}{180} .$$

For example, an angle of 27 degrees measures

$$27 \times \frac{\pi}{180} = \frac{3}{20}\pi \simeq 0,471 \qquad \text{radians.}$$

The symbol $\simeq$ in the previous expression indicates that we are taking the approximate decimal expansion (in this case to the third decimal place) of the number $3\pi/20$, which is irrational and therefore does not allow a finite decimal expansion.

Of course, the measure in degrees of an angle need not necessarily be an integer. For example, the interior angles of a regular heptagon measure

$$180 - \frac{360}{7} \simeq 128,572 \qquad \text{degrees.}$$

To measure smaller angles one uses the *prime*, which is the sixtieth part of a degree, and the *second*, which is the sixtieth part of a prime, that is 3600th part of a degree.

Degrees are usually indicated with a circle superscript ($^\circ$), primes with a prime ($'$) and seconds with a double prime ($''$). Thus

$$1^\circ = 60' = 3600'', \qquad 1,5^\circ = 90' = 5400'', \qquad \text{and so on.}$$

Angles are usually written in *normal form*. For instance,

$$\alpha = 20^\circ\ 12'\ 4''$$

indicates that the angle α measures 20 degrees, 12 primes and 4 seconds and this means that its amplitude is given by the sum of the three indicated quantities. So we have:

$$\alpha = \left(20 + \frac{12}{60} + \frac{4}{3600}\right)^\circ .$$

More generally, a normal form is of the type $gg^\circ\ pp'\ ss''$, where gg is a non negative integer, pp is an integer between 0 and 59, and ss is any integer belonging to the interval $[0, 60[$.

To pass from the normal form to the measure in degrees is easy: just divide the first by 60, the seconds by 3600 and then take the sum:

$$1^\circ\ 54'\ 3'' = \left(1 + \frac{54}{60} + \frac{3}{3600}\right)^\circ$$

$$2^\circ\ 14'\ 31'' = \left(2 \times 60 + 14 + \frac{31}{60}\right)'$$

$$3^\circ\ 4'\ 38'' = (3 \times 3600 + 4 \times 60 + 38)'' .$$

The transition from the measurement in degrees to the normal form is slightly more complicated. Let's see the procedure at work in a concrete example. Let's write in normal form an angle of $25,364^\circ$: the integer part are degrees, thus we have

$$25,364^\circ = 25^\circ\ pp'\ ss'' .$$

To calculate the number of primes, multiply the decimal part of the degrees by 60 and take the integer part:

$$0,364° = 0,364 \times 60 = 21,84' \quad \Rightarrow \quad pp = 21, \quad 25,364° = 25°\ 21'\ ss''\ .$$

To calculate the seconds just multiply the decimal part of primes by 60:

$$0,84' = 0,84 \times 60 = 50,4''.$$

In conclusion

$$25,364° = 25°\ 21'\ \ 50,4''\ .$$

Repeating the same procedure, in short, for the angle $65,323°$, we write

$$0,323 \times 60 = 19,38, \quad 0,38 \times 60 = 22,8\,,$$

and we obtain

$$65,323° = 65°\ 19'\ \ 22,8''\ .$$

To add (or subtract) angle amplitudes in normal form, degrees, primes and seconds must be added separately. For instance:

$$24°\ 35'\ 40''\ +\ 41°\ 11'\ 15''\ =\ 65°\ 46'\ 55''\ .$$

What can go wrong is that in this way the sum may no longer be in normal form, for example:

$$24°\ 35'\ 40''\ +\ 41°\ 51'\ 25''\ =\ 65°\ 86'\ 66''\,.$$

In this case you must first remove the excess seconds (in multiples of 60) and move them onto the primes, then remove the excess primes (always in multiples of 60) and move onto degrees. Finally, the excess degrees are removed by multiples of 360. In the previous example, since $66'' = 60'' + 6'' = 1' + 6''$ and $87' = 60' + 27' = 1° + 27'$, we obtain

$$65°\ 86'\ 66''\ =\ 65°\ 87'\ 6''\ =\ 66°\ 27'\ 6''\ ,$$

with the expression on the right in normal form.

Negative numbers may appear when computing differences of amplitudes in normal form. If the number of seconds is negative, add a multiple of $60''$ taking care to subtract the same multiple from the number of primes. For instance

$$3°\ 24'\ 12''\ -\ 1°\ 14'\ 35''\ =\ 2°\ 10'\ \ -23''\ .$$

To transform the result into normal form, add $60''$ and subtract $1'$: clearly the total remains unchanged.

$$2°\ 10'\ \ -23''\ =\ 2°\ (10' - 1')\ (60'' - 23'')\ =\ 2°\ 9'\ 37''\ .$$

In a similar way we proceed if the number of primes is negative: add a multiple of $60'$ and subtract the same multiple of $1°$:

$$3°\ 24'\ 12''\ -\ 1°\ 34'\ 5''\ =\ 2°\ \ -22'\ 7''\ =\ (2° - 1°)\ (60' - 22')\ 7''\ =\ \ 1°\ 38'\ 7''\ .$$

3.2. Percentages. A *percentage* is nothing more than a proportion multiplied by 100. For example, if a liter of grappa contains 40 centiliters of alcohol, the ratio of alcohol to the total is 0.4, while the percentage is 40% (40 *percent*). We list some practical examples of usage of percentages.

Example 3.2. During sales, a pair of shoes costs 100 Euros, while the list price was 125 Euros before the sales. What discount percentage has the merchant applied?

The discount is $125 - 100 = 25$ Euros. The proportion between the discount and the initial price is therefore $25/125 = 0.2$. To get the percentage you need to multiply by 100 thus obtaining $0,2 \times 100 = 20\%$.

Example 3.3. One hundred grams of a well–known industrial apricot snack contain 64 grams of carbohydrates, of which 37.8 is sugar. What is the percentage of sugar on the total carbohydrates?

To answer the question, just apply the definition of percentage, i.e. multiply the ratio of sugars to carbohydrates by 100:

$$\frac{37,8}{64} \times 100 = 59,06\%\,.$$

Example 3.4. 1) I have a business partner and we have decided to split the earnings always in the same way. According to our agreements, 55 % to me, 45 % to him. This month we made 3200 euros. How many euros is my share?

The problem is very easy to solve but we check it in detail. We are making a simple proportion: we have a total of 3200, which is 100 parts, and we want only 55 of those parts:

$$3200 : 100 = x : 55 \quad \text{that is} \quad \frac{3200}{100} = \frac{x}{55}$$

The answer is 55/100 of 3200:

$$x = \frac{3200}{100} \times 55 = 1760.$$

A quicker way to remember: to make 55 % of 3200 is to make the 55/100 of 3200, which is to multiply 3200 by 0.55.

2) Suppose my partner is in financial trouble and asks me to settle for only 800 euros for this month, in which we have earned 3200 euros. What percentage did I get instead of my 55 %? In addition, the following month goes very well and we earn 5000 euros. My partner tells me to take my share plus what I didn't take the previous month. How many euros do I get, and what percentage did I get out of the total of 5000?

The first question asks: what percentage of x of 3200 does 800 represent? that is

$$3200 : 100 = 800 : x \quad \text{i.e.} \quad \frac{3200}{100} = \frac{800}{x}$$

(if 3200 euros are 100 parts, 800 euros are x parts) so that

$$x = 800 \times \frac{100}{3200} = 25$$

and so I'm only getting 25 % instead of 55 %. A quicker way is to simply divide 800/3200 = 0.25 which tells us precisely that 800 euros are the 25 % of 3200 euros.

The next month, my 55% stake would be worth

$$5000 \times \frac{55}{100} = 2750.$$

Also, the month before I only pocketed 800 euros instead of my 1760, so I'm in credit of 1760−800 = 960, and this month in total I can take 960 + 2750 = 3710 euros. This sum, compared to the total of 5000 euros, represents a share of

$$\frac{3710}{5000} \times 100 = 74,2 \text{ percent}$$

(more simply, 3710/5000 = 0.742) and so I'm pocketing more than 74 percent of last month's earnings.

Example 3.5. A recent survey, carried out on a sample of 4000 people, asked to indicate various preferences regarding the color of cats. 10 % of the sample did not want to take part in the survey because they hate cats and only love dogs. The results of the interviews were as follows:

- 15% prefers black cats;
- 55% prefers white cats;
- 25% prefers grey cats;

- 5% does not know.

The next day, on the Gatto Quotidiano newspaper, the news appears "more than half of Italians prefer white cats". Is the news correct or is it the usual journalistic exaggeration?

Let's do the math: 10 % of the sample of 4000 is calculated as follows:

$$4000 : 100 = x : 10$$

and so $x = 400$ people hate cats; the interview was done only to 4000-400 = 3600 people. 55 % responded in favor of white cats: it is

$$\frac{55}{100} \times 3600 = 1980$$

people, clearly less than half of the sample. Precisely,

$$\frac{1980}{4000} \times 100 = 49,5$$

therefore only 49.5 of Italians love white cats. The next day, great hilarity on the Resto del Felino.

Example 3.6. A sporting goods store carries out the following promotion: if you buy three items, you pay the cheapest one only half price. What is the maximum percentage discount that a customer can get with this promotion?

We denote by x, y, z the prices of the three items in descending order, that is $x \geq y \geq z$. The customer then pays $x + y + \frac{z}{2}$ obtaining an absolute discount of $\frac{z}{2}$. The percentage discount is calculated using the formula:

$$\frac{\frac{z}{2}}{x+y+z} \times 100 = \frac{z}{2(x+y+z)} \times 100\,.$$

This value ranges from a minimum of 0% (when $z = 0$) to a maximum of

$$\frac{z}{2(z+z+z)} \times 100 = \frac{z}{6z} \times 100 = \frac{100}{6} = 16,66\%$$

obtained when the three items have the same price.

Problems.

Exercise 1.6. 100 grams of low-fat yogurt contain 4.7 grams of protein. How much protein is contained in a 125 gram jar of the same yogurt?

Exercise 1.7. Find the measure in radians of an angle of 20°.

Exercise 1.8 (⏭). Express the following angles in degrees and radians:

$$12^\circ\ 27'\ 30''\,, \qquad 17^\circ\ 2'\ 50''\,, \qquad 30^\circ\ 12'\ 30''\,.$$

(approximation to the third decimal place is requested and the calculator can be used).

Exercise 1.9. Put in normal form the following angles:

$$12,3^\circ, \quad 1230', \quad 34,34^\circ, \quad 10000''\,.$$

Exercise 1.10. Put in normal form the triple of the angle $130^\circ\ 50'\,, 50''$.

Exercise 1.11. On December 15, 2017, a sweater cost € 111. On January 15, 2018, the same sweater was sold for sale at a price of 100 euros. What percentage discount has the shopkeeper given to the customer?

Exercise 1.12. Consider an adult man weighing 80 kilograms and formed by 60% of water (by weight, before breakfast). Knowing that jellyfish are made up of 99% water (by weight), how many kilograms of jellyfish must our man swallow to reach 90% water after breakfast?

Exercise 1.13. In the academic year 2735/2736, 150 students enrolled in the first year of the degree course in Sciences applied to intergalactic travel. In the first exam session of the Mathematics course, 80% of the students enrolled and only 40% passed the exam. In the second session, all those who had not taken or passed the exam before showed up, and 50% of them pessed the exam. Calculate:

(i) the percentage of those who passed on the total number of members;
(ii) the percentage of those who passed in the first session on those registered;
(iii) the percentage of those who passed in the first session on the total of those who passed.

Exercise 1.14. The *brilliance* B of a set of monuments is the percentage of white surface on the total (at a certain moment T), and can be measured with special instruments; the *blackening* is the remaining percentage. The *blackening index* $N(\Delta T)$ is the amount of blackening (i.e. lost brightness) in a fixed period of time ΔT.

In two archaeological sites in Rome and Milan, between December 2003 and February 2005, the following data relating to brilliance were collected:

	Roma	Milano
Dec.2003	75	80
Dec.2004	70	50
Feb.2005	68	47

Compute:

(i) the blackening index in the two sites in 2004;

(ii) the blackening index in the first two monthsof 2005.

Exercise 1.15. A Swiss and an Abruzzese take the two ends of a 738 kilometer long tunnel at the same time. Knowing that the speed of the Swiss is 0.000025 % of the speed of the neutrino and the speed of the Abruzzese is 0.000020 % of the speed of the neutrino, determine how many kilometers the Swiss traveled before colliding with the Abruzzese.

4. Practice with numbers: powers and roots

Powers are a very useful notation that is used everywhere in mathematics, but you have to be careful. If x is a number (or more generally a numeric expression) and n is a positive integer then

$$x^n = \underbrace{x \cdot x \cdot \dots \cdot x}_{n \text{ factors}}$$

denotes the product of x with itself n times, such as:

$$x^7 = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x\,.$$

The first three basic rules of powers are:

- (product with the same base) $x^n \cdot x^m = x^{n+m}$;
- (product with the same exponent) $x^n \cdot y^n = (xy)^n$;
- (power of a power) $(x^n)^m = x^{nm}$.

For example, we have the equalities:

$$2^3 \cdot 2^7 = 2^{10}, \qquad 8^3 = (2^3)^3 = 2^9, \qquad 2^3 3^3 = 6^3,$$
$$2^{10} + 2^9 = 2 \cdot 2^9 + 2^9 = (2+1)2^9 = 3 \cdot 2^9\,.$$

If $x \neq 0$ and $n > m$, then $x^m \neq 0$ and dividing by x^m the relation $x^m \cdot x^{n-m} = x^n$ we get the formula

$$\frac{x^n}{x^m} = x^{n-m}\,. \qquad (4.1)$$

We can remove the $n > m$ limitation from this formula by setting $x^0 = 1$ and more in general

$$x^{-n} = \frac{1}{x^n}, \qquad n \in \mathbb{Z}, \quad x \neq 0\,. \tag{4.2}$$

We reiterate the importance of the condition $x \neq 0$ in order to define the powers of x with null and negative exponents: under this condition the three basic rules $x^n \cdot x^m = x^{n+m}$, $x^n \cdot y^n = (xy)^n$ and $(x^n)^m = x^{nm}$ are still valid for all $x, y \neq 0$ and $n, m \in \mathbb{Z}$. For instance

$$2^4 \cdot 4^{-3} = 2^4 \cdot (2^2)^{-3} = 2^4 \cdot 2^{-6} = 2^{-2} = \frac{1}{4}\,.$$

We observe that if $x \geq 1$ then $x^n \geq 1$ for every $n > 0$; moreover $x^2 = x \cdot x \geq x \cdot 1 = x$, $x^{n+1} = x^n \cdot x \geq x^n \cdot 1 = x^n$ and therefore there is an infinite chain of inequalities

$$1 \leq x \leq x^2 \leq x^3 \leq \cdots \leq x^n \leq x^{n+1} \leq \cdots\,.$$

It is well known that if $x > 1$ then the powers x^n grow very fast as n increases, so much so that the term *exponential growth* has entered the common lexicon to indicate rapid growth phenomena.

Now we want to go a step further and extend the possible exponents to all rational numbers, so that the basic rules of powers continue to hold. Let us first consider the case of the exponent $\pm 1/2$: we want the relations

$$(a^{\frac{1}{2}})^2 = a^{\frac{1}{2}\cdot 2} = a, \qquad (a^{-\frac{1}{2}})^2 = a^{-1} = \frac{1}{a}$$

to be true. We see that $a^{1/2}$ must be a number such that its square is a; a positive number with this property is called the *square root* of a and is denoted by $\sqrt{a}$. Since the squares of real numbers are all non–negative, a necessary condition for the definitions $a^{\frac{1}{2}}$ and $a^{-\frac{1}{2}}$ to make sense is $a > 0$.

The condition $a > 0$ is also sufficient due to the following theorem which we state without proof.

Theorem 4.1. *For every positive integer $n > 0$ and every non–negative real number $a \geq 0$ there exists a unique non-negative real number $\sqrt[n]{a} \geq 0$, called* n-th root of a, *such that*

$$\left(\sqrt[n]{a}\right)^n = a\,.$$

When $n = 2$, we write simply $\sqrt{a}$ in place of $\sqrt[2]{a}$. The Theorem 4.1 implies in particular that for every positive real number b it holds $\sqrt{b^2} = b$ and more generally $\sqrt[n]{b^n} = b$ for any integer n.

We see that the formulas

$$(a^{\frac{1}{n}})^n = a^{\frac{1}{n}\cdot n} = a, \qquad a > 0, \qquad n > 0,$$

force us to define

$$a^{\frac{1}{n}} = \sqrt[n]{a} \quad = n\text{–th root of } a,$$

and as a consequence

$$a^{\frac{n}{m}} = (a^{\frac{1}{n}})^m = (\sqrt[n]{a})^m, \qquad a, n > 0, \quad n, m \in \mathbb{Z}$$

which we can write also

$$a^{\frac{n}{m}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}.$$

Example 4.2. We have $\sqrt{2^3} = \sqrt{2^2 \cdot 2} = \sqrt{2^2} \cdot \sqrt{2} = 2\sqrt{2}$, and also:

$$\begin{aligned} 2^{\frac{3}{2}} &= (2^3)\frac{1}{2} = \sqrt{8}\,, \\ &= (2^{\frac{1}{2}})^3 = \sqrt{2}^3\,, \\ &= 2^{1+\frac{1}{2}} = 2^1 \cdot 2^{\frac{1}{2}} = 2\sqrt{2}\,. \end{aligned}$$

If a, b are positive real numbers and $s \in \mathbb{Q}$ the formula $a^s b^s = (ab)^s$ is still true. For example:

$$2^4 3^4 = 6^4, \qquad 2^3\pi^2 = 2(2\pi)^2, \qquad \sqrt{2^3 3^5} = \sqrt{2^3}\,\sqrt{3^5} = 2\cdot 3^2\sqrt{2}\,\sqrt{3}, \qquad \text{and so on.}$$

Example 4.3. We can perform the following simplifications and rationalizations:

$$\frac{2+\sqrt{20}}{2} = \frac{2+\sqrt{2^2\cdot 5}}{2} = \frac{2+2\sqrt{5}}{2} = \frac{2(1+\sqrt{5})}{2} = 1+\sqrt{5}\,;$$

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{(\sqrt{2})^2} = \frac{\sqrt{2}}{2}; \qquad \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}\sqrt{3}}{(\sqrt{2})^2} = \frac{\sqrt{6}}{2}\,;$$

$$\frac{1}{1+\sqrt{2}} = \frac{1}{1+\sqrt{2}}\cdot\frac{1-\sqrt{2}}{1-\sqrt{2}} = \frac{1-\sqrt{2}}{1^2-(\sqrt{2})^2} = \frac{1-\sqrt{2}}{1-2} = \sqrt{2}-1\,.$$

Example 4.4. If $a, b, c > 0$, we have

$$\frac{\sqrt[3]{a^4b^2c}}{\sqrt[2]{abc^3}} = \frac{(a^4b^2c)^{\frac{1}{3}}}{(abc^3)^{\frac{1}{2}}} = \frac{a^{\frac{4}{3}}b^{\frac{2}{3}}c^{\frac{1}{3}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{3}{2}}} = a^{\frac{4}{3}-\frac{1}{2}}b^{\frac{2}{3}-\frac{1}{2}}c^{\frac{1}{3}-\frac{3}{2}} = a^{\frac{5}{6}}b^{\frac{1}{6}}c^{-\frac{7}{6}}\,.$$

Problems.

Exercise 1.16. The square root of $(-3)^2$ is equal to:

$$-3, \quad 3, \quad 9, \quad 81, \quad \sqrt{3}\,?$$

Exercise 1.17. The sum $2^9 + 2^9$ is equal to:

$$2^{18}, \quad 2^{10}, \quad 4^9, \quad 4^{18}, \quad \text{or an irrational number?}$$

Exercise 1.18. Simplify the expressions:

$$(2^n + 2^{n+1})^2, \qquad (3^n + 3^{n+1})^2 - 7\cdot 9^n\,.$$

Exercise 1.19. If $8^{x+3} = 2^{x+1}$, what is the value of x?

Exercise 1.20. Simplify the fractions:

$$\frac{\sqrt{50}}{10}, \qquad \frac{8-2\sqrt{8}}{4}, \qquad \frac{1}{2-\sqrt{3}}, \qquad \frac{11-\sqrt{121}}{\sqrt{17}+\sqrt{19}}\,.$$

Exercise 1.21 (▶▶|). Which of the following numbers are integers?

$$(1+\sqrt{3})^2 - 2\sqrt{3}, \quad (1+\sqrt{2})^2 - \sqrt{2}, \quad \frac{\sqrt{200}}{\sqrt{8}}, \quad (1+\sqrt{2})(1+\sqrt[4]{2})(1-\sqrt[4]{2})\,.$$

Exercise 1.22. Simplify the following expressions:

$$\frac{\sqrt[3]{a^{12}b^6c^7}}{\sqrt{a^5b^3c}}, \qquad \frac{\sqrt[4]{a^9b^5c^2}}{\sqrt[3]{a^8b^8c^8}}, \qquad \frac{\sqrt[5]{a^2b^3c^3}}{\sqrt[3]{abc^2}}\,,$$

$$\sqrt{a^{12}b^6c^3}, \qquad \sqrt[3]{8a^{10}}, \qquad \frac{\sqrt[3]{27a^4b^3}}{\sqrt{16a^3b^5}}, \qquad \frac{a^{\frac{3}{2}}b^{\frac{9}{4}}\sqrt[3]{c^3}}{\sqrt{a^3}\sqrt[4]{b^9}}.$$

Exercise 1.23. Put each of the following sequences of numbers in order of size:

(1) 2^5, 5^3, 3^4, 10^1;

(2) 2^{500}, 5^{300}, 3^{400}, 10^{100};

(3) 2^{-10}, 10^{-2}, $\frac{1}{2000}$, $\frac{1}{20}$, $\frac{2}{1000}$;

(4) 2^{200}, 4^{30}, 8^{25}, 16^{51}.

EXERCISE 1.24. Which of the following numbers is half of $\left(\frac{1}{2}\right)^{50}$?

$$\left(\frac{1}{4}\right)^{50}, \quad \left(\frac{1}{2}\right)^{25}, \quad \left(\frac{1}{2}\right)^{49}, \quad \left(\frac{1}{2}\right)^{51}, \quad \left(\frac{1}{4}\right)^{25}, \quad \left(\frac{1}{2}\right)^{100}.$$

EXERCISE 1.25. Which of the following simplifications are right and which are wrong?

$$\frac{2^{2n}+3}{2^n+3} = 2^n, \qquad \frac{2^{2n}+2^n}{2^n} = 2^n+1, \qquad \frac{3}{5^n}(5^n+3) = 9, \qquad \frac{3^n}{2^m}\frac{2}{3} = \frac{1^n}{1^m}.$$

5. Practice with numbers: real powers, logarithms, scientific notation

More generally, the powers a^b can be defined for any real number $a > 0$ and any real number b (even irrational). The same rules always apply:

$$a^b a^c = a^{b+c}, \qquad (a^b)^c = a^{bc}, \qquad a^{-b} = \frac{1}{a^b}, \qquad 1^b = 1,$$

$$\text{and} \quad a^b \le a^c \quad \text{every time that} \quad a \ge 1 \text{ and } b \le c\,.$$

It is important to remember that in order to compute the powers a^b where the *exponent* b is any real number, the *base* a must be a strictly positive number. We have already dealt with the case of fractional powers:

$$\begin{aligned} a^{\frac{n}{m}} &= (a^{\frac{1}{n}})^m = (\sqrt[n]{a})^m, \\ &= (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}\,. \end{aligned}$$

To define the powers a^b with any real number b, it is necessary to treat separately the three cases $a > 1$, $a = 1$ and $0 < a < 1$. The case $a = 1$ is the simplest since $1^b = 1$ whatever the exponent b is.

When $a > 1$ we know that $a^s \le a^t$ for any pair of rational numbers s, t such that $s \le t$. If b is any real number, the power a^b is defined as *the unique number*[1] *such that*

$$a^s \le a^b \le a^t$$

for any pair of rational numbers s, t such that $s \le b \le t$. In a more informal way we can therefore say that the real number a^b is approximated by the fractional power a^s when the rational number s approximates b: the closer s is to b, the closer a^s is to a^b.

EXAMPLE 5.1. The first five digits of the decimal development of π are $\pi = 3,1415\ldots$. Thus we have $3,1415 \le \pi \le 3,1416$ and for the properties of the powers we have

$$2^{3,1415} \le 2^\pi \le 2^{3,1416}$$

which, as seen on fractional powers, can be written as

$$\sqrt[10000]{2^{31415}} \le 2^\pi \le \sqrt[10000]{2^{31416}} = \sqrt[2500]{2^{7854}}.$$

Bringing the development of $\pi = 3.141592653\ldots$ to 10 decimal digits we obtain a better approximation

$$2^{3,141592653} \le 2^\pi \le 2^{3,141592654}\,.$$

[1]The proof that such a number exists and is unique is analogous to the proof of Theorem 4.1 and is therefore omitted.

When $0 < a < 1$ we reason in the same way as in the case $a > 1$, keeping in mind that $a^b \geq a^c$ if $b \geq c$. For example, since $3,141 \leq \pi \leq 3,142$ one has

$$(0,41)^{3,141} \geq (0,41)^{\pi} \geq (0,41)^{3,142}.$$

Alternatively we can write $a^b = (a^{-1})^{-b}$ and observe that $a^{-1} > 1$ when $0 < a < 1$.

If a, b are positive real numbers and $s \in \mathbb{R}$ the formula still holds $a^s b^s = (ab)^s$: for example $2^\pi 3^\pi = 6^\pi$, $2^{\sqrt{2}} 3^{\sqrt{2}} = 6^{\sqrt{2}}$ and so on.

We remember the notion of decimal logarithm (we will return to logarithms later). Suppose we know that

$$10^a = x$$

(where a can be any real number). Then the number a is called the *decimal logarithm*, or *in base 10*, of the number x, and it is written

$$a = \log_{10} x = \log_{10}(x)$$

(parentheses are used only if necessary). We see that the following rules are true:

$$\log_{10}(10^a) = a \qquad \text{ossia} \qquad 10^{\log_{10} x} = x$$

For example, the decimal logarithm of $1000 = 10^3$ is 3; the decimal logarithm of 10^{47} is 47; the decimal logarithm of $1/10 = 10^{-1}$ is -1; the decimal logarithm of $\sqrt{10}$ is $1/2$. On the other hand, the logarithm of 1 is zero:

$$\log_{10}(1) = \log_{10}(10^0) = 0.$$

What if the number x can't be written as a power of 10? For example, what is $\log_{10} 3$? To answer this question you need a calculator; we can calculate how many digits of the decimal development we want, but an exact answer is impossible. The approximate value is given by $\log_{10} 3 = 0,4771212547\ldots$. The values of the logarithms of integers between 1 and 10, rounded to the third decimal place, are shown in the following table:

$\log_{10} 2 \simeq 0,301$	$\log_{10} 3 \simeq 0,477$	$\log_{10} 4 \simeq 0,602$	$\log_{10} 5 \simeq 0,699$
$\log_{10} 6 \simeq 0,778$	$\log_{10} 7 \simeq 0,845$	$\log_{10} 8 \simeq 0,903$	$\log_{10} 9 \simeq 0,954$

Rules for the powers follow immediately from the corresponding rules for the logarithm. The following rules should be remembered:

$$\log_{10}(a \cdot b) = \log_{10}(a) + \log_{10}(b), \qquad \log_{10}(a^b) = b \cdot \log_{10} a$$

that is, the logarithm of the product is the sum of the logarithms, and the logarithm of a power is the exponent times the logarithm of the base. A consequence of the previous rules is

$$\log_{10}\left(\frac{a}{b}\right) = \log_{10}(a) - \log_{10}(b)$$

(indeed $a/b = a \cdot b^{-1}$) and also

$$\log_{10}\left(\frac{1}{b}\right) = -\log_{10}(b).$$

EXAMPLE 5.2. We use the properties of the logarithm to show that the number $\log_{10} 3$ is irrational, i.e. it cannot be written as a quotient of integers. If, by contradiction, you had $\log_{10} 3 = a/b$, then $a = b \log_{10} 3 = \log_{10}(3^b)$ which is equivalent to equality $10^a = 3^b$ which is impossible for $a, b \in \mathbb{N}$ (10^a is even, 3^b is odd).

A very common use of powers is the representation of numbers in scientific notation. The *scientific notation* is a way of writing numbers that allows to compare them and understand how big (or small) they are much more quickly. A number written in scientific notation has the form

$$x = 3,12571 \cdot 10^7$$

while the traditional notation of the same number would be

$$x = 31.257.100$$

The general form is the following:

$$x = m, pqrs \ldots \cdot 10^n$$

that is, the number must be written as the product of a power of 10 (n can also be negative or null) by a decimal number $m, pqrs \ldots$ in which the integer part m at the left of the comma is an integer between 1 and 9. Then

- the exponent n is called *order of magnitude* of the number x;
- the digits to the right of the comma are called *significant digits* of the number x.

We note a very useful fact. If we calculate the decimal logarithm of $x = m, pqrs \ldots \cdot 10^n$ we see that

$$\log_{10} x = \log_{10}(m, pqrs \ldots \cdot 10^n) = \log_{10}(m, pqrs) + n$$

and the number $\log_{10}(m, pqrs)$ is between 0 and 1 since the number $m, pqrs$ is between $1 = 10^0$ and $10 = 10^1$. So we see that to calculate the order of magnitude of a number x it is sufficient to *calculate the decimal logarithm of x and take the integer part.*

Example 5.3. Let us see concretely how to proceed to write a number in scientific notation. If a number is a power of 10, this is trivial:

$$1000 = 1 \cdot 10^3, \qquad 1000000 = 1 \cdot 10^6, \qquad 0,00001 = 1 \cdot 10^{-5}, \qquad 1 = 1 \cdot 10^0.$$

Note that we can also write $1000 = 10 \cdot 10^2 = 0,1 \cdot 10^4$ but these are not scientific notations. In particular, the order of magnitude of 1000 is 3, that of $0,00001$ is -5; and indeed we have $\log_{10}(1000) = 3$, $\log_{10})0,00001 = -5$.

If we have a more complicated number like $x = 121{,}950.394$, we can write

$$x = 1,21950394 \cdot 10^5$$

so the order of magnitude is 5 (note that $5 + 1 = 6$ is the number of digits of the integer part of x). Instead,

$$0,0007432... = 7,432 \cdot 10^{-4}$$

has order of magnitude -4.

Very often not all the digits are written but only the first ones: for example

$$121.950,39 = 1,21950394 \cdot 10^5 \simeq 1,219 \cdot 10^5$$

is the expression approximated to the third significant digit.

Example 5.4. A case in which scientific notation is very convenient concerns large powers, if use decimal logarithms in the proper way. Suppose we want to write the number 8^{100} in scientific notation, knowing that

$$\log_{10} 8 = 0,903...$$

Since

$$\log_{10}(8^{100}) = 100 \cdot \log_{10} 8 = 90,3...$$

we see that the order of magnitude is 90 (this tells us that 8^{100} has 91 digits). Since $90.3 \cdots = 90 + 0.3 \ldots$, we can write

$$8^{100} = 10^{90,3\ldots} = 10^{0,3\ldots} \cdot 10^{90}$$

and we're almost done. To complete the representation in scientific notation we should calculate more precisely $10^{0,3\cdots}$, but for many problems the representation we have obtained is already sufficient.

If we have two numbers written in scientific notation, comparing them is very easy: the one with the largest order of magnitude is the largest; and if the two numbers have the same order of magnitude, the significant figures are compared. For example:

$$\begin{aligned} 1,902\cdot 10^{46} &\quad \text{è più grande di} \quad 8,342\cdot 10^{44}\\ 3,141\cdot 10^{-6} &\quad \text{è più grande di} \quad 2,992\cdot 10^{-9}\\ 4,554\cdot 10^{12} &\quad \text{è più grande di} \quad 3,109\cdot 10^{12}\end{aligned}$$

and so on. Sometimes the most efficient way to compare two numbers with each other it is precisely to write them in scientific notation.

Example 5.5. Knowing that $\log_{10} 6 = 0.77815\ldots$, tell if 6^{100} is larger than 60^{50} or not. We have immediately

$$\log_{10}(6^{100}) = 100\cdot\log_{10} 6 = 100\cdot 0,77815\cdots = 77,8\ldots$$

so that 6^{100} has order of magnitude 77. On the other hand

$$\log_{10}(60^{50}) = 50\cdot\log_{10} 60 = 50\cdot(\log_{10} 6+\log_{10} 10) = 50\cdot(0,77815\cdots+1) = 39,9\ldots$$

so that 60^{50} has order of magnitude 39 and is much smaller.

Problems.

Exercise 1.26. Simplify the following expressions:

$$\log_{10}\left(\frac{2^4 10^7}{3^{11}7^3}\right),\qquad \log_{10}\left(\sqrt[n]{a^{12}b^p c^{-1}}\right),\qquad \log_{10}(3,55\cdot 10^{-100}).$$

Exercise 1.27. Simplify the following expressions:

$$\log_{10}(a^3b^2c^5) - \log_{10}(a^3) - \log_{10}\sqrt{a^3b},\qquad \log_{10}(100) - 4\log_{10} 5$$

$$\log_{10}\left(\frac{\sqrt{abc}}{25}\right) + 2\log_{10} 10,\qquad \log_{10} 338 - 2\log_{10} 13.$$

Exercise 1.28. Knowing that $\log_{10} 8 = 0,9030...$, compute the order of magnitude of $8^{50}, 80^{20}, 800^{30}$.

Exercise 1.29. Compute the order of magnitude of 3^{100}, 9^{100}, 90^{20} and order these numbers in ascending size (use that $\log_{10} 3 \simeq 0,477$).

Exercise 1.30. Compute the order of magnitude of 8^{100} and of 80^{20}, and determine which of the two numbers is larger. (use that $\log_{10} 2 \simeq 0,301$).

Exercise 1.31. Compute the order of magnitude of 9^{50} (use that $\log_{10} 3 \simeq 0,477$).

6. Practice with algebraic operations: rational functions

To do mathematics, and more generally science, it is necessary to know how to handle with familiarity mathematical objects more complex than simple numbers. For example, it is necessary to perform algebraic operations between polynomials and rational functions without errors. Recall that a polynomial of degree n in the variable x is an expression of the type

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

with each a_i a number that can be real, rational or integer. Even a simple number a_0 can be regarded as a polynomial (of degree 0). The number 0 is therefore also a polynomial, which is called *null polynomial*. The *degree* of a polynomial $p(x)$ is the largest integer n such that the power x^n appears in $p(x)$ with coefficient $a_n \neq 0$. By convention, the degree of the null polynomial is equal to $-\infty$.

Polynomial $p(x)$	Degree $\deg(p(x))$
0	$-\infty$
$a \neq 0$	0
$ax + b,\ \ a \neq 0$	1
$ax^2 + bx + c,\ \ a \neq 0$	2
$ax^3 + bx^2 + cx + d,\ \ a \neq 0$	3

The *principle of identity* of polynomials holds: two polynomials are equal if and only if they have the same degree and the same coefficients of each power of x. In other words, the two polynomials

$$a_0 + a_1 x + \cdots + a_n x^n, \qquad b_0 + b_1 x + \cdots + b_m x^m,$$

are equal if and only if $n = m$ and $a_i = b_i$ for each index i.

It is well known that polynomials can be added and multiplied: for example

$$(x + 2x^2) + (x^2 - 1) = 3x^2 + x - 1, \qquad (x + 1) + (x^2 - 1) = x^2 + x\,,$$
$$(x + 2)(x^2 - 1) = x^3 + 2x^2 - x - 2, \qquad (x + 1)(x^2 - 1) = x^3 + x^2 - x - 1\,,$$
$$(x + a)^2 = x^2 + 2ax + a^2, \qquad (x + a)^3 = x^3 + 3ax^2 + 3a^2 x + a^3\,,$$
$$(x + a)(x - a) = x^2 - a^2, \qquad (x - a)(x^2 + ax + a^2) = x^3 - a^3\,.$$

It is useful to observe that the degree of the product of two polynomials is equal to the sum of the degrees, while the degree of the sum (or of the difference) is always less than or equal to the maximum of the degrees of the single polynomials:

$$\deg(p(x)q(x)) = \deg(p(x)) + \deg(q(x)), \quad \deg(p(x) \pm q(x)) \leq \max(\deg(p(x)), \deg(q(x)))\,.$$

For polynomials, the *product deletion rule* applies: if $p(x)$ is a polynomial other than 0 and $p(x)q(x) = p(x)r(x)$ for two polynomials $q(x), r(x)$, then $q(x) = r(x)$. In fact, if $p(x)q(x) = p(x)r(x)$ then $p(x)(q(x) - r(x)) = 0$; if it were $q(x) \neq r(x)$ then the polynomial $q(x) - r(x)$ would have degree $n \geq 0$; if $p(x)$ has degree $m \geq 0$, then the product $p(x)(q(x) - r(x))$ has degree $n + m$ and therefore it cannot be the null polynomial.

A *rational function* is an expression of the form

$$\frac{p(x)}{q(x)}, \qquad (\text{ read: } p(x) \text{ over } q(x)\ ),$$

where $p(x)$ and $q(x)$ are polynomials, with $q(x) \neq 0$. For example, the following expressions are rational functions:

$$\frac{x+2}{1}, \quad \frac{x+1}{x-1}, \quad \frac{x^2+x+1}{x^3-2}, \quad \frac{x^{134}-x^{17}}{x^{22}-3}, \quad \frac{1}{x^3+1} .$$

When the denominator is the polynomial 1, we simply write $\dfrac{p(x)}{1} = p(x)$ and therefore any polynomial can be interpreted as a rational function: for example

$$\frac{x+1}{1} = x+1, \qquad \frac{x^2+x+1}{1} = x^2+x+1 .$$

Like for numerical fractions we have the equalities $\dfrac{1}{2} = \dfrac{2}{4}$ and $\dfrac{6}{8} = \dfrac{9}{12}$, also for rational functions there are similar equalities, such as

$$\frac{x^2}{x^3} = \frac{1}{x}, \qquad \frac{x^2-3x+2}{x^2-2x+1} = \frac{(x-2)(x-1)}{(x-1)^2} = \frac{x-2}{x-1} .$$

In general the following properties are true:

(1) multiplying (or dividing) the numerator and denominator of a rational function by the same polynomial other than 0, the rational function does not change;
(2) the following rule holds:

$$\frac{p(x)}{q(x)} = \frac{r(x)}{s(x)} \quad \text{if and only if} \quad p(x)s(x) = r(x)q(x) .$$

For example we have $\dfrac{x^2-4x+4}{x} = \dfrac{x^3-5x^2+8x-4}{x^2-x}$, since

$$(x^2-4x+4)(x^2-x) = (x^3-5x^2+8x-4)x = x^4-5x^3+8x^2-4x .$$

We can therefore rewrite the product cancellation rule by saying that two polynomials $p(x)$, $q(x)$ are equal if and only if $\dfrac{p(x)}{s(x)} = \dfrac{q(x)}{s(x)}$ for some polynomial $s(x)$ which is not null.

Operations between rational functions are quite similar to those between numerical fractions:

- The sum of rational functions with *the same denominator* is obtained by adding their respective numerators; ditto for the difference. For example,

$$\frac{1}{x-1} + \frac{x}{x-1} = \frac{1+x}{x-1}, \qquad \frac{x+1}{x^2+1} - \frac{x}{x^2+1} = \frac{1}{x^2+1} .$$

- If the fractions to sum do not have the same denominator, before proceeding to the sum with the above rule, the numerator and denominator of each rational function are multiplied by an appropriate polynomial so that all the fractions have the same denominator. For example,

$$\begin{aligned}\frac{1}{x-1} + \frac{x}{x+1} &= \frac{x+1}{(x-1)(x+1)} + \frac{x(x-1)}{(x-1)(x+1)} \\ &= \frac{x+1+x(x-1)}{(x-1)(x+1)} = \frac{x^2+1}{x^2-1} .\end{aligned}$$

- The product is obtained by multiplying the numerators and denominators separately. For example,

$$\frac{1}{x-1} \cdot \frac{x}{x+1} = \frac{x}{(x-1)(x+1)} = \frac{x}{x^2-1} .$$

- The inverse of a non–zero rational function is obtained by exchanging the numerator with the denominator, and vice versa:
$$\left(\frac{x+1}{x-1}\right)^{-1} = \frac{1}{\frac{x+1}{x-1}} = \frac{x-1}{x+1}\,.$$
- To perform a division, take the product with the inverse of the divider:
$$\frac{x^3-1}{x+2} : \frac{x-1}{x^2} = \frac{x^3-1}{x+2}\cdot\frac{x^2}{x-1} = \frac{x^2(x^3-1)}{(x+2)(x-1)}\,.$$
Often the division sign : is replaced with a fraction line:
$$\frac{\frac{x^2+1}{x-2}}{\frac{x+1}{x-1}} = \frac{x^2+1}{x-2} : \frac{x+1}{x-1} = \frac{x^2+1}{x-2}\cdot\frac{x-1}{x+1} = \frac{(x^2+1)(x-1)}{(x-2)(x+1)}\,.$$

Finally we remember the following well–known identities:
$$\frac{x^2-1}{x-1} = x+1, \quad \frac{x^3-1}{x-1} = x^2+x+1, \quad \frac{x^4-1}{x-1} = x^3+x^2+x+1,$$
and more generally for any positive integer n
$$\frac{x^n-1}{x-1} = x^{n-1}+x^{n-2}+\cdots+x+1\,.$$
Similarly we have:
$$\frac{x^3+1}{x+1} = x^2-x+1, \quad \frac{x^5+1}{x+1} = x^4-x^3+x^2-x+1,$$
and more generally for any positive, odd integer n
$$\frac{x^n+1}{x+1} = x^{n-1}-x^{n-2}+\cdots-x+1\,.$$

Remark 6.1. It is well known that the polynomial x^2+1 can not be decomposed as the product of two polynomials of first degree (with real coefficients). Similarly, one can prove that for any integer $n = 2^h$ power of 2, the polynomial x^n+1 can not be decomposed as a product of polynomials of lower degree with *rational coefficients*. On the other hand, it can also be shown that each polynomial decomposes as a product of polynomials with *real coefficients* of degree 1 and 2. For example we have:
$$x^4+1 = (x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)\,,$$
$$x^6+1 = (x^2+1)(x^2+\sqrt{3}x+1)(x^2-\sqrt{3}x+1)\,.$$

Problems.

Exercise 1.32. Perform the following operations between polynomials:
$$(1+x)(1-x+x^2-x^3), \qquad (1-x)(1+x+x^2+x^3), \qquad (x+a)^2-(x-a)^2,$$
$$1+2x+\frac{(2x)^2}{2}+\frac{(2x)^3}{6}-\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}\right)^2\,.$$

Exercise 1.33. Perform the following operations between rational functions:
$$\frac{x}{1+x}+\frac{x^2-1}{x^3+1}\,, \qquad \frac{x-2}{x^2-4}-\frac{1}{x+2}\,, \qquad \frac{1+\frac{x}{x-1}}{x+\frac{2}{x-1}}+\frac{x}{x^2-1}\,,$$
$$\frac{3-x}{2x+5}-\frac{2x+4}{x-3}-\frac{x^2-3x+1}{2x^2-x-15}\,, \qquad \frac{x-1}{2x+3}-\frac{2x^2+5x-1}{4x^2-9}+\frac{6x-7}{(2x-3)^2}\,,$$

$$\frac{x^3+x^2}{1-x^2}+1-x-\frac{1-x^2}{x-1}-\frac{x^2+1}{1-x}\,,\qquad \frac{1-x}{1+x}\left(\frac{1}{1-x}-\frac{x}{1-x^2}\right)\,,$$

$$\left(x+1-\cfrac{1}{1-\cfrac{x}{1+x-\cfrac{x}{1-2x}}}+\frac{x^2}{1-3x}\right):\left(\frac{x^2-3x}{2x^2-8}+\frac{x-1}{x+2}\right)\,.$$

EXERCISE 1.34. Let a, b be rational numbers. If

$$a=\frac{1}{2}\quad\text{and}\quad\frac{a}{a+b}=\frac{1}{7}\,,$$

what is the value of b?

EXERCISE 1.35. Let a, b be rational functions. If

$$a=\frac{1}{x}\quad\text{and}\quad\frac{a}{a+b}=x\,,$$

what is the value of b?

EXERCISE 1.36 (⏭). Compute, if they exist, two real numbers a, b such that

$$\frac{1}{x^2+x}=\frac{a}{x}+\frac{b}{x+1}\,.$$

EXERCISE 1.37. Compute, if they exist, three real numbers a, b, c such that

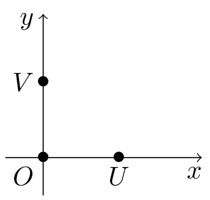
$$\frac{1}{x^3-x}=\frac{a}{x}+\frac{b}{x+1}+\frac{c}{x-1}\,.$$

7. The Cartesian plane

By *Cartesian plane* we mean the Euclidean plane with a Cartesian reference system (O, U, V): here O, U, V are three distinct points in the plane such that the two segments $\overline{OU}$ and $\overline{OV}$ are perpendicular to each other in O and have the same length.

To construct a Cartesian reference, we fix two lines perpendicular to each other in the plane (and we think the first line "horizontal", i.e. parallel to the line that joins our eyes). The two lines will be called respectively the x axis (or the axis of abscissas) and the y axis (or the axis of ordinates).

Let O be the point of intersection of the two lines, which is called the origin of the reference system, and fix a point U on the first line and a point V on the second, so that U, V have the same distance from O.



We identify the axis of abscissas with the set of real numbers so that the numbers $0, 1$ correspond to the points O and U respectively. Similarly, we identify the axis of ordinates with the set of real numbers so that the numbers $0, 1$ correspond to the points O and V respectively.

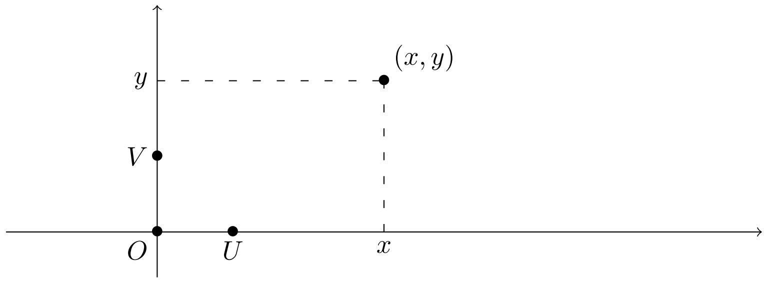


FIGURE 1.7.

Let us now consider a point P of the plane; to it we associate two real numbers x and y, which correspond to the projections of the point P on the axes of the abscissas and ordinates, respectively. In this way, to each point P of the plane we associate an ordered pair (x, y) of real numbers, which are called the *coordinates* of P; vice versa, to each ordered pair of real numbers we associate a point of the plane. The point O then corresponds to the $(0, 0)$ coordinate point, U becomes the $(1, 0)$ coordinate point, and V becomes the $(0, 1)$ coordinate point .

The distance between two points in the plane P, Q is equal to the length of the segment with endpoints P, Q. If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, the Pythagorean theorem allows you to calculate the distance between P and Q using the formula (see Figure 1.8):

$$d(P, Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\ .$$

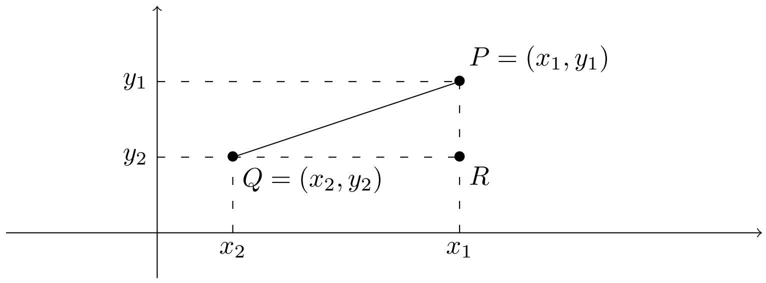


FIGURE 1.8. By the Pythagorean theorem

$$d(P, Q) = \sqrt{\overline{PR}^2 + \overline{QR}^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\ .$$

EXAMPLE 7.1. The distance between the points $P = (1, 2)$ and $Q = (3, 1)$ is equal to

$$d(P, Q) = \sqrt{(1 - 3)^2 + (2 - 1)^2} = \sqrt{(-2)^2 + 1^2} = \sqrt{5}\,.$$

The condition for a point (x, y) to be equidistant from P and Q is obtained by equating the distances:

$$\sqrt{(x - 1)^2 + (y - 2)^2} = \sqrt{(x - 3)^2 + (y - 1)^2}\,.$$

Squaring both sides we obtain

$$(x - 1)^2 + (y - 2))^2 = (x - 3)^2 + (y - 1)^2$$

which after suitable simplifications becomes $4x - 2y = 5$.

Equation of the line: first form. Recall that any straight line L on the plane not parallel to the y axis is defined by an equation of the type $y = mx + q$. This means that the point P with coordinates (x, y) belongs to the line L if and only if $y = mx + q$; in particular the point $(0, y)$ belongs to L if and only if $y = q$ and therefore $(0, q)$ coincides with the intersection of the line L with the axis y. The number m is called *slope* or also *angular coefficient* of the line.

Given two points (x_1, y_1) and (x_2, y_2) with $x_1 \neq 2$, it is easy to determine the equation $y = mx + q$ of the line passing through the two points. In fact, it is enough to solve the two equations simultaneously

$$y_1 = mx_1 + q, \qquad y_2 = mx_2 + q,$$

in the unknowns m, q. Computing the difference we get $y_1 - y_2 = m(x_1 - x_2)$, while multiplying the equations x_2 and x_1 respectively and taking the difference we obtain $y_1x_2 - y_2x_1 = q(x_2 - x_1)$. We then found the formula:

$$m = \frac{y_1 - y_2}{x_1 - x_2}, \qquad q = \frac{x_1y_2 - x_2y_1}{x_1 - x_2}\,. \tag{7.1}$$

EXAMPLE 7.2. We calculate the equation $y = mx + q$ of the line passing through the points $(1, 2)$ and $(2, 3)$. By applying Formula 7.1 we obtain

$$y = \frac{2-3}{1-2}x + \frac{3-4}{2-1} = x + 1\,.$$

To calculate in which points of the plane two straight lines L_1, L_2 of equations $y = m_1x + q_1$ and $y = m_2x + q_2$ intersect, we need to solve the linear system

$$\begin{cases} y - m_1x = q_1 \\ y - m_2x = q_2 \end{cases}.$$

It is very easy to see that if $m_1 \neq m_2$ the system admits a single solution; if $m_1 = m_2$ and $q_1 = q_2$ the two lines are coincident, while if $m_1 = m_2$ and $q_1 \neq q_2$ the system does not admit solutions, i.e. there are no intersection points between the two lines which therefore are parallel.

Equation of the line: second form. The first form studied above is simple but has the disadvantage of not including the lines parallel to the y axis, i.e. the lines of equation $x + c = 0$, for some $c \in \mathbb{R}$. We can work around this problem by writing the equation of a line in the form

$$ax + by + c = 0, \qquad a, b, c \text{ not all } = 0\,.$$

In this case we are considering all possible lines, including those parallel to the y axis, with equation $x + c = (1)x + (0)y + c = 0$, and the non parallel lines with equation

$$mx - y + q = 0 \iff y = mx + q\,.$$

Unfortunately, the advantage of incorporating all the lines is paid with the disadvantage that two distinct equations may determine the same line. In fact we have that two equations $ax + by + c = 0$ e $a'x + b'y + c' = 0$ determine the same line if and only if the triplets (a, b, c) and (a', b', c') are proportional, i.e. if and only if there is a real number $\lambda \neq 0$ such that $a' = \lambda a$, $b' = \lambda b$ e $c' = \lambda c$.

A line L of equation $ax + by + c = 0$ is not parallel to the y axis if there is only one $t \in \mathbb{R}$ such that the point $(0, t)$ belongs to L, that is if and only if the equation $bt + c = 0$ has a unique solution. This is possible if and only if $b \neq 0$. Hence every straight line L of equation $ax + by + c = 0$ can be written in the first form if and only if $b \neq 0$: more precisely

$$ax + by + c = 0 \iff y = \frac{-a}{b}x + \frac{-c}{b}$$

and the ratio $-a/b$ is equal to the slope (angular coefficient) of L .

Problems.

EXERCISE 1.38. Calculate the equations (in the form $y = mx + q$) of the straight lines passing through each of the following pairs of points in the plane:

(1) $(0, 0)$ and $(1, 0)$;
(2) $(1, 0)$ and $(0, 1)$;
(3) $(2, 1)$ and $(1, 1)$;
(4) $(-1, -10)$ and $(-10, -1)$.

EXERCISE 1.39. Given the points $P = (2, 3)$ and $Q = (5, 7)$ in the Cartesian plane, calculate their distance $d(P, Q)$ and the equation of the line L that contains them both. Determine the point R of intersection of L with the x axis and all the points S of the line L such that $d(S, R) = d(P, Q)$.

EXERCISE 1.40. Calculate the coordinates of the intersection point of the line L of equation $x + y + 1 = 0$ with the line M passing through the points $(0, 0)$ and $(1, 1)$.

8. Review of trigonometry

Trigonometry essentially deals with the study of sine and cosine functions, the functions derived from them (tangent, cotangent, etc.) and their applications to the geometry of triangles. Later (in Chapter 3) we will study some properties of these functions from the point of view of analysis; here we limit ourselves to recalling some elementary properties and some geometric applications.

We recall the geometric construction of the functions $\sin x$, and $\cos x$. Let us consider the unit circle in the Cartesian plane, with center at the origin O and radius 1. We call A the point $(1, 0)$ at the intersection between the unit circle and the abscissa axis.

If P is a point on the circle, we can measure the width of the angle AOP with the length of the arc of circumference $\widehat{AP}$; this value is called the measure in *radians* of the angle.

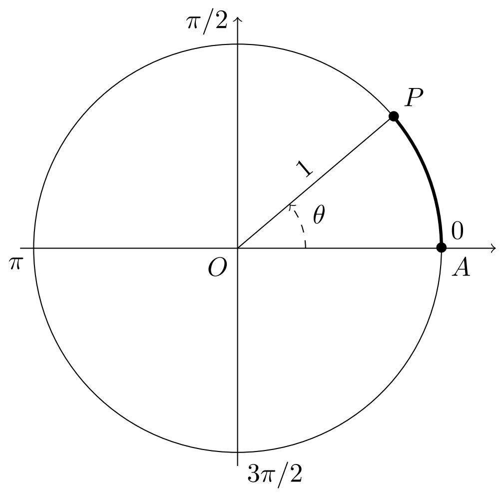


FIGURE 1.9. The measure in radians of the angle θ is equal to the length of the arc $\widehat{AP}$ in the circle of radius 1, or equivalently, to twice the area of the circular sector OAP.

For example:

$$90^\circ \text{ (straight angle)} = \frac{\pi}{2} \text{ radians} \qquad 270^\circ = \frac{3\pi}{2} \text{ radians}$$

$$180^\circ \text{ (flat angle)} = \pi \text{ radians} \qquad 360^\circ \text{ (round angle)} = 2\pi \text{ radians}$$

$$45^\circ = \frac{\pi}{4} \text{ radians} \qquad 30^\circ = \frac{\pi}{6} \text{ radians} \qquad 60^\circ = \frac{\pi}{3} \text{ radians}$$

$$120^\circ = \frac{2\pi}{3} \text{ radians} \qquad 135^\circ = \frac{3\pi}{4} \text{ radians} \qquad 150^\circ = \frac{5\pi}{6} \text{ radians}$$

REMARK 8.1. The area A of a circular sector of size θ radians in a circle of radius r is equal to $\theta r^2/2$. In fact the ratio between this area and the area πr^2 of the circle is equal to the ratio between the amplitude of the angle and the amplitude of the round angle, i.e. . Therefore

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi} \iff A = \frac{\theta r^2}{2}.$$

We can now define the sine and cosine of an angle expressed in radians: given a point P on the unit circumference, if s is the length of the arc AP, the *sine* and the *cosine* of s are exactly the ordinate and abscissa of the point P:

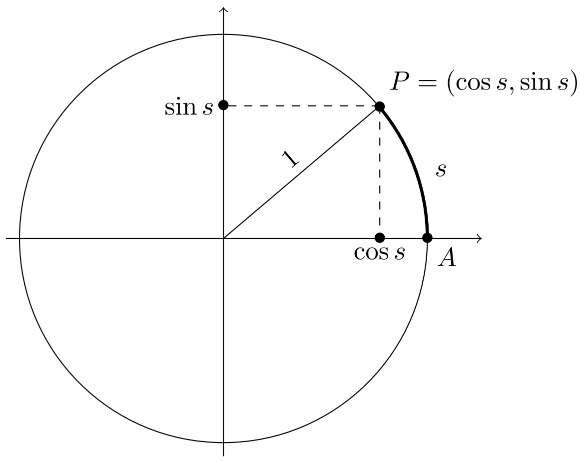


FIGURE 1.10. Cosine and sine of s are the coordinates of the image of the point $A = (1, 0)$ by rotating s radians counterclockwise.

It is easy to verify that

$$\sin(0) = 0 \qquad \cos(0) = 1$$

$$\sin\left(\frac{\pi}{6}\right) = \sin(30^\circ) = \frac{1}{2} \qquad \cos\left(\frac{\pi}{6}\right) = \cos(30^\circ) = \frac{\sqrt{3}}{2}$$

$$\sin\left(\frac{\pi}{4}\right) = \sin(45^\circ) = \frac{1}{\sqrt{2}} \qquad \cos\left(\frac{\pi}{4}\right) = \cos(45^\circ) = \frac{1}{\sqrt{2}}$$

$$\sin\left(\frac{\pi}{3}\right) = \sin(60^\circ) = \frac{\sqrt{3}}{2} \qquad \cos\left(\frac{\pi}{3}\right) = \cos(60^\circ) = \frac{1}{2}$$

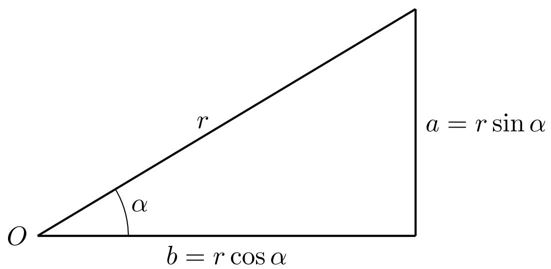
$$\sin\left(\frac{\pi}{2}\right) = \sin(90^\circ) = 1 \qquad \cos\left(\frac{\pi}{2}\right) = \cos(90^\circ) = 0$$

$$\sin\left(\frac{2\pi}{3}\right) = \sin(120°) = \frac{\sqrt{3}}{2} \qquad \cos\left(\frac{2\pi}{3}\right) = \cos(120°) = -\frac{1}{2}$$
$$\sin\left(\frac{3\pi}{4}\right) = \sin(135°) = \frac{1}{\sqrt{2}} \qquad \cos\left(\frac{3\pi}{4}\right) = \cos(135°) = -\frac{1}{\sqrt{2}}$$
$$\sin\left(\frac{5\pi}{6}\right) = \sin(150°) = \frac{1}{2} \qquad \cos\left(\frac{5\pi}{6}\right) = \cos(150°) = -\frac{\sqrt{3}}{2}$$
$$\sin(\pi) = \sin(180°) = 0 \qquad \cos(\pi) = \cos(180°) = -1$$
$$\sin\left(\frac{3\pi}{2}\right) = \sin(270°) = -1 \qquad \cos\left(\frac{3\pi}{2}\right) = \cos(270°) = 0.$$

When the point P is in the starting position A, the angle measures zero radians. If we move P counterclockwise, after one complete turn, the point returns to the position A, and then the angle is 2π radians. If we continue to move the point counterclockwise, the angle goes beyond 2π and the point P retraces the positions it passed through in the first turns. In particular, the sine and cosine values take again the same values after a complete revolution:

$$\sin(s + 2\pi) = \sin s, \qquad \cos(s + 2\pi) = \cos s\,.$$

Let us consider a right triangle with legs (catheti) a, b and hypotenuse r



and let $\alpha > 0$ be the angle between the hypotenuse and the cathetus b. Then you have $b = r\cos\alpha$, $a = r\sin\alpha$.

The sine and cosine functions satisfy numerous identities, but here we will limit ourselves to mentioning only a few: the elementary identity, which follows from the Pythagorean Theorem,

$$\sin^2 s + \cos^2 s = 1,$$

the *addition formulas*

$$\sin(a + b) = \sin a \cos b + \cos a \sin b,$$
$$\cos(a + b) = \cos a \cos b - \sin a \sin b,$$

and the *properties of symmetry*

$$\sin(-a) = -\sin a, \qquad \cos(-a) = \cos a.$$

A few more formulas follow immediately from the sum formulas and the symmetry properties: the *subtraction formulas*

$$\sin(a - b) = \sin a \cos(-b) + \cos a \sin(-b) = \sin a \cos b - \cos a \sin b,$$
$$\cos(a - b) = \cos a \cos(-b) - \sin a \sin(-b) = \cos a \cos b + \sin a \sin b\,,$$

the *duplication formulas*

$$\sin(2a) = 2\sin a \cos a, \qquad \cos(2a) = \cos^2(a) - \sin^2(a) = 2\cos^2(a) - 1 = 1 - 2\sin^2(a),$$

and the *bisection formulas*

$$\sin\left(\frac{a}{2}\right) = \pm\sqrt{\frac{1 - \cos a}{2}}, \qquad \cos\left(\frac{a}{2}\right) = \pm\sqrt{\frac{1 + \cos a}{2}}\,.$$

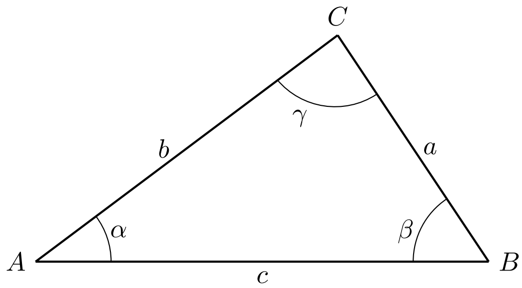
EXAMPLE 8.2. An angle x is between 0 and π radians, and we know that $\cos x = \frac{1}{4}$. We want to calculate the value of $\sin\left(x + \frac{\pi}{4}\right)$.
Since $0 < x < \pi$ we know that $\sin x \geq 0$ and therefore we have

$$\sin x = \sqrt{1 - \cos^2 x} = \sqrt{1 - \frac{1}{16}} = \frac{\sqrt{15}}{4}.$$

Since $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, from the addition formula for the sine we obtain

$$\sin\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{15}}{4}\frac{\sqrt{2}}{2} + \frac{1}{4}\frac{\sqrt{2}}{2} = \frac{\sqrt{30} + \sqrt{2}}{8}.$$

We conclude this paragraph by recalling two applications of trigonometry to the geometry of triangles. Let us consider a triangle with sides a, b, c and denote with α the angle opposite to the side a, with β the angle opposite to the side b, and with γ the angle opposite to the side c.



The first application is the *Carnot's theorem*: the following formula is valid

$$a^2 = b^2 + c^2 - 2bc\cos\alpha.$$

Note that if $\alpha = \pi/2$, i.e. if the triangle is right and a is the hypotenuse, the theorem is reduced to $a^2 = b^2 + c^2$ i.e. the Pythagorean Theorem.
The second application is the *Sine Theorem*: the following formula holds

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

Note that for a right triangle, with hypotenuse a and catheti b, c, so where the angle $\alpha = \pi/2$, we have simply

$$b = a\sin\beta, \qquad c = a\sin\gamma.$$

EXAMPLE 8.3. Two sides of a triangle are long 8 and 16, and the angle between them is $pi/3$ radians. How long is the third side?
Just apply Carnot's Theorem:

$$c^2 = 8^2 + 16^2 - 2\cdot 8\cdot 16\cdot\cos\frac{\pi}{3} = 64 + 256 - 256\cdot\frac{1}{2} = 192$$

so that

$$c = \sqrt{192} = 8\sqrt{3}.$$

EXAMPLE 8.4. During a walk (on the plains) I walk from the starting point for 1 km to the north, and then for 2 km to the south east. How many km as the crow flies I am from the starting point?
If we call A the starting point, B the point where I changed direction, and C the ending point, we know that the side AB = c is 1 Km long, the side BC = a is long 2 Km, and the

angle ABC $= \beta$ is 45 degrees i.e. $\pi/4$. The problem asks to determine the length of the third side CA $= b$. From Carnot's theorem we see that

$$b^2 = a^2 + c^2 - 2ac\cos\beta = 4 + 1 - 4\cos(\pi/4) = 5 - 4\cdot\frac{1}{\sqrt{2}} = \frac{5\sqrt{2}-4}{\sqrt{2}}$$

so that the solution is

$$b = \sqrt{\frac{5\sqrt{2}-4}{\sqrt{2}}}\,.$$

Problems.

Exercise 1.41. Knowing that $\pi/2 = 1.57\ldots$ and that $50\cdot 2\pi = 314,15\ldots$, determine if $\sin(315)$ and $\cos(315)$ (angle of 315 radians) are positive, null or negative.

Exercise 1.42. Using the cosine duplication formula

$$\cos(2\alpha) = 2\cos^2(\alpha) - 1,$$

calculate the cosine of an angle of $15^\circ = \dfrac{\pi}{12}\ rad$.

Exercise 1.43. We knoe that an angle α satisfies $\cos(2\alpha) = -\frac{3}{4}$, $\sin(2\alpha) < 0$ and $\sin(\alpha) < 0$. What is the value of $\cos(\alpha)$?

CHAPTER 2

Equations and inequalities

In most practical problems where mathematics can be applied, the solution of the problem lies in the solution of an equation or a system of equations, which, depending on the case, can be linear, algebraic, differential, integral, etc. In this chapter we recall the simpler forms of equations, for the most part already well known to the reader.

1. Equations and systems

When we write an equality, in some cases, for example $2 = 2$ or $a = a$, we are simply observing a fact that we know to be true; in other cases, we are saying something false, for example no one forbids us to write $1 = 2$, but of course this is a false statement.

An *equation* is a different kind of statement. An equation is a very concise way of describing a problem to be solved. For example, if we say

"solve in x the equation $2x - 5 = 7$"

we are actually asking:

"find all real numbers x such that double x minus five is exactly seven".

Therefore, the solutions of an equation can be many, or only one, or none (in this case we say that the equation is *impossible*). The letter indicating the quantity to be determined is also called the *unknown*; in the previous example we used the letter x to denote the unknown.

Very often, just to complicate things, several letters are used in an equation; all letters indicate real numbers, but only one is the unknown. In these cases, to make it clear what the unknown is, we say:

"solve in x the equation $ax + b = c$".

The other letters a, b, c indicate fixed real numbers, but which we do not want to choose immediately; in this way we reserve the freedom to choose them after having solved the equation, and the formulas we obtain for the solution can be applied to many different equations. Sometimes these other letters are called *parameters*.

How do you solve an equation? It depends. Some equations are very easy to solve, some are very difficult. The equations we have written so far are very easy, they are *first degree equations* that you learn to solve in high school. For example, the equation $ax+b = c$, when $a \neq 0$ (a different from 0) is solved immediately in two steps:

$$ax + b = c \quad \iff \quad ax = c - b \quad \iff \quad x = \frac{c - b}{a}.$$

The symbol $\iff$ reads "*if and only if*", or "*is equivalent to*", meaning that the two expressions on the right and left are same thing, written differently. Notice that in the first step we subtracted b from both members, in the second step we divided both members by a; the equations are always solved like this, with a series of steps that transform the equations into simpler equivalent forms.

What happens if $a = 0$? The equation becomes a bit strange because the unknown disappears. But let's try to go all the way: we have two possibilities. If the numbers b and c are different, the equation $0 \cdot x + b = c$ is clearly impossible. What if $b = c$? For example, what does it mean to solve for x the equation $0 \cdot x + 3 = 3$? Very simple: all x are ok, so the solutions are all real numbers.

Also in high school the *second degree equations* are studied:

$$ax^2 + bx + c = 0, \qquad a \neq 0\,.$$

The standard solution procedure leads to the following rule: the *discriminant* $\Delta = b^2 - 4ac$ is calculated. If $\Delta < 0$, there are no solutions in the real numbers. If $\Delta > 0$, there are two distinct solutions given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a}\,.$$

Finally, if $\Delta = 0$, the above formula is still correct, but gives only one solution $x = -\frac{b}{2a}$; it is also said that the two roots coincide. (To study the second degree equation more completely, *complex numbers* are required).

It is useful to remember that finding the solutions of a second degree equation is equivalent to decomposing the polynomial as a product of linear factors. In fact, if $a \neq 0$ and if α, β are the solutions of the equation $ax^2 + bx + c = 0$, then the formula holds

$$ax^2 + bx + c = a(x - \alpha)(x - \beta)\,. \tag{1.1}$$

REMARK 1.1. Sometimes the equation to solve is already written as a product:

$$a(x + b)(x + c) = 0, \qquad \text{with} \quad a \neq 0\,.$$

It is always possible to expand the product

$$a(x + b)(x + c) = ax^2 + a(b + c)x + abc$$

and then apply the solution formula for the equations of second degree. But it is much more convenient to observe that a product of real numbers vanishes if and only if at least one of the factors vanishes. In this case, since $a \neq 0$ we obtain that $a(x + b)(x + c) = 0$ if and only if $x + b = 0$ or $x + c = 0$. It follows that the solutions of the equation are $x = -b$ and $x = -c$.

In a completely analogous way we have that the solutions of the equation

$$\alpha(\beta x + b)(\gamma x + c) = 0, \qquad \text{with} \quad \alpha, \beta, \gamma \neq 0$$

are precisely $x = -\frac{b}{\beta}$ and $x = -\frac{c}{\gamma}$.

EXAMPLE 1.2. The solutions of the equation $(x - 1)(x + 2)(2x - 3) = 0$ are $x = 1$, $x = -2$ e $x = \frac{3}{2}$, and coincide with the values of x which make equal to zero at least one of the three factors $(x - 1)$, $(x + 2)$ or $(2x - 3)$.

EXAMPLE 1.3. We calculate the solutions of the equation $x^3 + 2x^2 + x + 2 = 0$. Recalling a bit of polynomial decomposition techniques learned in high school it is not difficult to obtain that

$$x^3 + 2x^2 + x + 2 = (x^2 + 1)(x + 2)$$

and therefore $x^3 + 2x^2 + x + 2 = 0$ if and only if $x^2 + 1 = 0$ or $x + 2 = 0$. The first factor never vanishes: in fact, since $x^2 \geq 0$ for every real number x, we have that $x^2 + 1$ is always greater than or equal to 1. The $x + 2$ factor vanishes for $x = -2$. We can conclude by saying that the equation $x^3 + 2x^2 + x + 2 = 0$ has the unique solution $x = -2$.

Almost always in concrete problems it happens that you have to solve several equations at the same time. In this case we have a more complicated problem called a *system*. We use the following notation: solving the system

$$\begin{cases} x^2 - 3x + 2 = 0 \\ 6x - 5 = 1 \end{cases}$$

means "to find all the numbers x that solve *both* the first *and* the second equation". Note that there is only one unknown x in the system. A procedure that usually works is the following: first we solve each equation of the system, separately; in this way we obtain the set of solutions of the first equation, of the second equation, and so on. At the end, the various sets are compared and we look for the values of x in common among all the sets of solutions: that is, we compute the *intersection* of the solutions of each equation. In the previous system,

the first equation has for solutions $x = 1, 2$;

the second equation has for solution $x = 1$;

these two sets of solutions have in common only the value $x = 1$, so the only solution of the system is $x = 1$.

REMARK 1.4. In many cases the procedure just described for solving systems of equations can be speeded up in the following way. We choose one of the equations and we solve it. Then, for each of the solutions we have found, we verify if is a solution of the other equations of the system. For example, let's consider the system of equations

$$\begin{cases} x^2 - 3x + 2 = 0 \\ x^7 - x^3 + x - 1 = 0\,. \end{cases}$$

The second equation is of the seventh degree and therefore very difficult to solve, while the first equation is easily solved and has as solutions $x = 1$ and $x = 2$. Calculating $x^7 - x^3 + x - 1$ for $x = 1$ and $x = 2$ we obtain respectively:

$$1^7 - 1^3 + 1 - 1 = 0, \qquad 2^7 - 2^3 + 2 - 1 = 121 \neq 0\,.$$

Thus $x = 1$ is the unique solution of the system.

In some cases the problem contains a single equation, but to solve it easily it is better to write it as a system. For instance, suppose we want to find the solutions of $\dfrac{p(x)}{q(x)} = a$, with $p(x)$, $q(x)$ polynomials and a a number. Since we cannot divide by 0, solving the previous equation is equivalent to finding the values of x such that

$$q(x) \neq 0 \qquad \text{and} \qquad p(x) - aq(x) = 0\,.$$

In practice, we first find the solutions $x_1, x_2, \ldots$ of the equation $p(x) - aq(x) = 0$, the solutions $y_1, y_2, \ldots$ of the equation $q(x) = 0$ and then we look for the elements of the set $x_1, x_2, \ldots$ which **do not** belong to the set $y_1, y_2, \ldots$.

Important. The numbers belonging to both sets $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ **must not** be considered solutions, even if in some cases they can be considered *limit solutions*, but we will talk about this later.

EXAMPLE 1.5. Let's solve the equation

$$\frac{x^2 + 1}{x^2 - 2x + 1} = 1\,. \tag{1.2}$$

The equation $x^2 - 2x + 1 = 0$ has only one solution $x = 1$ (double root), so for each $x \neq 1$ we can multiply both sides by $x^2 - 2x + 1$ and we obtain the equation

$$x^2 + 1 = x^2 - 2x + 1$$

which, after the necessary simplifications, becomes $2x = 0$ and has $x = 0$ as the only solution. Since $0 \neq 1$ we have that $x = 0$ is also the only solution of the equation (1.2).

EXAMPLE 1.6. Let's solve the equation

$$\frac{x^2 - 1}{x^2 - 2x + 1} = 3\,. \tag{1.3}$$

As in the previous example, the denominator vanishes for $x = 1$, and for each $x \neq 1$ we can multiply both sides by $x^2 - 2x + 1$ to get the equation

$$x^2 - 1 = 3(x^2 - 2x + 1)$$

which, after the necessary simplifications, becomes $2(x^2 - 3x + 2) = 0$ and has as solutions $x = 1$ and $x = 2$. Since we must discard $x = 1$, it follows that $x = 2$ is the only solution of (1.3). If we try to substitute 1 instead of x in (1.3) we find $\frac{0}{0} = 3$ which doesn't make any sense (you can not divide by 0, even 0 itself).

Problems.

Exercise 2.1 (▶▶|). For which values of the parameter k the equation

$$x^2 + kx + (k^2 - 1) = 0$$

in the unknown x has only one (double) solution.

Exercise 2.2. Solve the following equations:

$$3(x-1)(x+2) = 0, \qquad 2(x+1)(2x+1)(3x+1) = 0, \qquad (x+1)(x-1) = 1.$$

Exercise 2.3. Solve the following equations:

$$\frac{x-1}{x(x-2)} = 2, \qquad \frac{x^2}{(x-1)(x+1)} = 2, \qquad \frac{x^2+2x+1}{x^2-2x+1} = 4\,.$$

Exercise 2.4. Solve the following equations:

$$\frac{1}{x} + \frac{1}{x+1} = 2, \qquad x + \frac{1}{x} = 2, \qquad \frac{1}{x} - \frac{1}{x+1} + \frac{1}{x+2} = 0\,.$$

Exercise 2.5. Using formula (1.1) simplify the following rational functions:

$$\frac{x^2-2x+1}{x-1}\,, \qquad \frac{x-2}{x^2-3x+2}, \qquad \frac{2x^2+5x+2}{x^2+x-2}\,, \qquad \frac{x^2+10x-200}{x^3-10x^2} \cdot \frac{x^3-x}{2x+40}\,.$$

Exercise 2.6. Solve the following systems of equations:

$$\begin{cases} x^2 - 4x + 3 = 0 \\ 5x - 4 = 1 \end{cases} \qquad \begin{cases} x^2 - 4x + 4 = 0 \\ x^2 - 5x + 6 = 0 \end{cases} \qquad \begin{cases} x^2 - 6x + 8 = 0 \\ x^3 - x^2 + 6x - 16 = 0 \end{cases}$$

2. Inequalities

Like equations, *inequalities* are another synthetic way to describe a problem. An inequality looks like an equation, but instead of the sign of equality =, it appears one of the 5 signs

$$\neq, \quad <, \quad >, \quad \leq, \quad \geq\,.$$

For example, solving the following first degree inequality

$$5 - 8x > 13$$

means to find all the numbers x that satisfy this condition.

First degree inequalities are solved like first degree equations, there is only one important difference to remember: when you multiply or divide both sides of an inequality by a *negative* number, the inequality *changes direction.* For example,

$$2 \leq 5 \quad \Longleftrightarrow \quad -2 \geq -5.$$

To solve the above inequality a few steps are enough:

$$5 - 8x > 13 \iff -8x > 13 - 5 \iff -8x > 8 \iff x < \frac{8}{-8} \iff x < -1.$$

If you prefer, we can also solve like this:

$$5 - 8x > 13 \iff 5 - 13 > 8x \iff -8 > 8x \iff \frac{-8}{8} > x \iff -1 > x$$

which is exactly the same.

Usually the set of solutions is an interval, or a union of intervals; it is very useful to always draw the set of solutions on the real line, this helps to solve and sometimes to discover errors.

Second degree inequalities have the form

$$ax^2 + bx + c > 0, \quad \text{or} < 0, \geq 0, \leq 0, \neq 0.$$

These inequalities can be solved in a very simple way: in fact we only have to establish the sign of the trinomial $ax^2 + bx + c$. As one studies in high school, to understand the sign of the trinomial just imagine a parabola with the branches facing upwards when $a > 0$ (glass shape, Figure 2.1),

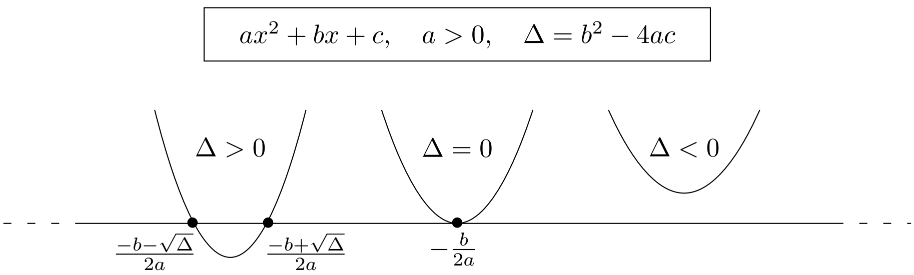


Figure 2.1.

and downwards when $a < 0$ (hat shape, Figure 2.2).

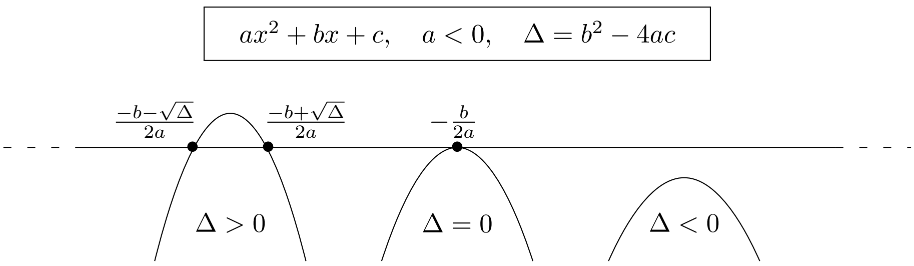


Figure 2.2.

The sign of the trinomial is the same as the sign of the parabola.

Example 2.1. For example, let's solve the inequality

$$x(x-4) \geq 3 - 2x.$$

First of all we take everything to the first member and we collect: we get

$$x^2 - 2x - 3 \geq 0.$$

$x^2 - 2x - 3$ is an upward parabola, it has two distinct roots $x = -1$ and $x = 3$, so it is positive in the area outside the roots and negative in the interval between the two roots.

In the inequality, the zone ≥ 0 is required, so the solution is given by the zone outside the roots, *roots included* (note the greater *or equal*):

$$x \leq -1 \quad \text{and} \quad x \geq 3.$$

Of course we can also consider *systems of inequalities*, or mixed systems with equations and inequalities; in these cases a drawing may be indispensable!

EXAMPLE 2.2. Solve the system

$$\begin{cases} 3(x^2-5) < (x+1)x \\ 1-x < 3x-4. \end{cases}$$

First of all we simplify and solve the two equations separately:

$$\iff \begin{cases} 2x^2 - x - 15 < 0 \\ 4x > 5 \end{cases} \iff \begin{cases} -\frac{5}{2} < x < 3 \\ x > \frac{5}{4}. \end{cases}$$

Finally, we must find the points in common between the solutions of the first and second equations (i.e. make the intersection of the two sets):

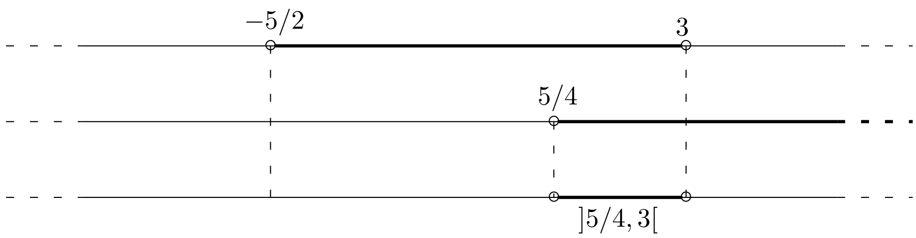


FIGURE 2.3.

and we obtain that the solutions are all numbers x such that

$$\frac{5}{4} < x < 3.$$

EXAMPLE 2.3. Solve the system

$$\begin{cases} 5x - 3 \geq -2x^2 \\ 8x - 7 < 2x + 3 \\ x(2x-5) = (x-2)(x+2). \end{cases}$$

We have immediately

$$\iff \begin{cases} 2x^2 + 5x - 3 \geq 0 \\ 6x < 10 \\ x^2 - 5x + 4 = 0 \end{cases} \iff \begin{cases} x \leq -3 \;\text{ e }\; x \geq \frac{1}{2} \\ x < \frac{5}{3} \\ x = 1 \;\text{ e }\; x = 4. \end{cases}$$

We see immediately from the diagram

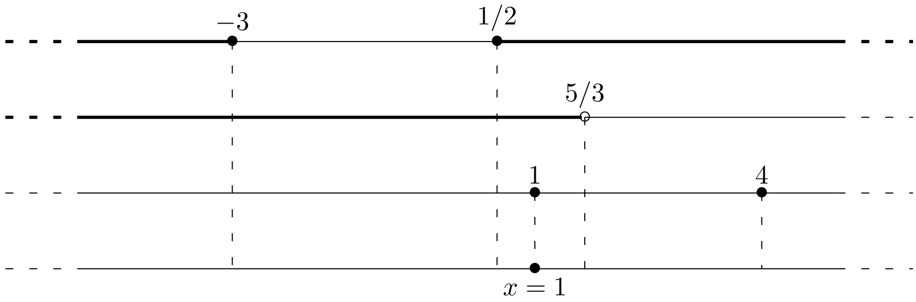


FIGURE 2.4.

that the only solution of the system is $x = 1$.

Problems.

EXERCISE 2.7. Solve the inequalities (this is not a system! These are four separate inequalities):

$$2x^2 \le 1 - x; \qquad x(2-3x) > x - 6;$$
$$2x \ge -x^2 - 1; \qquad x^2 + x < -1.$$

EXERCISE 2.8 (▶▶|). Solve the following systems:

$$a) \begin{cases} x^2 - 1 < x \\ x - 2 > 2x \end{cases} \qquad b) \begin{cases} x^2 - 2x \ge 0 \\ 2x \le x + 1 \end{cases} \qquad c) \begin{cases} x^2 - 2x - 3 = 0 \\ x^2 < 4 \end{cases}$$

EXERCISE 2.9. Solve the following systems:

$$\begin{cases} 2x - 3 > 2 \\ x + 1 < 5 \\ x - 3 < 6 \end{cases} \qquad \begin{cases} 7x + 5 > -1 \\ x^2 - 2x - 3 < 0 \\ 12 - 3x > -3 - x \end{cases} \qquad \begin{cases} 2x + 3 \ne 1 \\ x^2 + 2x - 3 \ge 0 \\ 3 - x < 15 + 2x. \end{cases}$$

3. The absolute value

Given a real number x, we want to define its *absolute value*, also called *modulus*, which is denoted by the symbol $|x|$. Let's start with some examples:

the absolute value of 3 is 3;

the absolute value of -10 is 10;

the absolute value of 0 is 0.

Is this clear? The absolute value of a positive number is equal to the number; the absolute value of a negative number is the opposite of the number. (The absolute value of zero is zero because it can be considered both positive and negative and the result is the same). Hence the result is always a positive number. Summing up:

$$|x| = \begin{cases} x & \text{if } x \ge 0, \\ -x & \text{if } x \le 0. \end{cases}$$

The same rule applies in more general cases. For example:

$$|x-2| = \begin{cases} x - 2 & \text{if } x - 2 \ge 0, \\ -x + 2 & \text{if } x - 2 \le 0 \end{cases}$$

which can also be written like this:

$$|x-2| = \begin{cases} x-2 & \text{if } x \geq 2, \\ 2-x & \text{if } x \leq 2. \end{cases}$$

When an absolute value appears in an equation (or inequality), we are actually dealing with two different equations (or inequalities), depending on the sign of the unknown x. Let's see an example.

EXAMPLE 3.1. We solve the equation

$$|x-2| = 5\,.$$

This equation is satisfied if and only if $x-2=5$ or $x-2=-5$. In the first case we find $x=7$, in the second $x=-3$.

EXAMPLE 3.2. Suppose we want to solve the equation

$$2x + |x| = 1 - x.$$

First of all we must interpret the presence of the absolute value. Since $|x|$ means two different things depending on whether it $x \geq 0$ or $x \leq 0$, let's proceed as follows: divide all possible real numbers x into two groups and study them separately. First group: all $x \geq 0$; for these x we know that $|x| = x$, so we need to solve the equation

$$2x + x = 1 - x \qquad \Longleftrightarrow \qquad 4x = 1 \qquad \Longleftrightarrow \qquad x = \frac{1}{4}.$$

We note that this equation is exactly the original one for $x \geq 0$, but for negative x it has nothing to do with the original equation! Since the new equation has a solution $x = \frac{1}{4}$ in the area under study, we can accept it: we have found a solution of the original equation. But we can no stop here: we must also study the case $x \leq 0$. For these values of x we know that $|x| = -x$, so the original equation becomes

$$2x - x = 1 - x \qquad \Longleftrightarrow \qquad 2x = 1 \qquad \Longleftrightarrow \qquad x = \frac{1}{2}.$$

Also in this case we have intepreted the absolute value, we have obtained a new equation, and we have solved it; but unfortunately the solution found is not in the right area. In fact, the new equation is equivalent to the original one only for $x \leq 0$; for $x \geq 0$ it has nothing to do with it. In conclusion, the solution is only one: $x = \frac{1}{4}$.

Let's study another example; the method is always the same! Note that if we start from a second degree equation, when we solve the absolute value we get *two different equations* of the second degree; therefore, depending on the cases, the number of solutions may vary from 0 (no solution) to 4.

EXAMPLE 3.3. Solve the equation

$$x^2 - x - 2 = |x+1|.$$

First of all we must understand the meaning of the absolute value:

$$|x+1| = \begin{cases} x+1 & \text{if } x \geq -1, \\ -x-1 & \text{if } x \leq -1. \end{cases}$$

Then we must distinguish the two zones $x \geq -1$ and $x \leq -1$ and in each of them we get a different equation to be solved only in that zone. Let's try: if $x \geq -1$ instead of $|x+1|$ we must write $x+1$ and we get the equation

$$x^2 - x - 2 = x + 1 \qquad \Longleftrightarrow \qquad x^2 - 2x - 3 = 0 \qquad \Longleftrightarrow \qquad x = -1 \text{and } x = 3.$$

Both solutions fall into the $x \geq -1$ area under examination, so we can accept both of them. We pass to the zone $x \leq -1$: here the absolute value means $|x+1| = -x-1$, and by replacing we obtain the equation

$$x^2 - x - 2 = -x - 1 \iff x^2 - 1 = 0 \iff x = +1 \text{ and } x = -1.$$

The solution -1 is acceptable, however we have already found it as the solution of the first case. Instead we have to discard the second solution $+1$ because it does not fall into the area under examination which is $x \leq -1$. Final answer of the problem: the equation has exactly two solutions $x = -1$ and $x = 3$.

Example 3.4. We solve the equation $|x^2 - 1| = |x^2 + 1| - 1$. Since $x^2 + 1$ is always greater than or equal to 0, the equation becomes

$$|x^2 - 1| = x^2 + 1 - 1 = x^2 \ .$$

If $x^2 \geq 1$ we have $x^2 - 1 \geq 0$ and the equation becomes $x^2 - 1 = x^2$, which is clearly without solutions. If $x^2 \leq 1$ we have $x^2 - 1 \leq 0$ and the equation becomes $1 - x^2 = x^2$, that is $2x^2 = 1$ which has as solutions $x = \pm 1/\sqrt{2}$.

Of course we can also consider *inequalities* containing an absolute value; the procedure is completely analogous to the previous one.

Example 3.5. Solve the inequality

$$|3 - x| < \frac{1}{2}x - 1.$$

The absolute value means

$$|3 - x| = \begin{cases} 3 - x & \text{if } x \leq 3, \\ -3 + x & \text{if } x \geq 3. \end{cases}$$

As usual, we split all x into two groups. In the zone $x \leq 3$ we substitute the absolute value $|3 - x|$ by its meaning $3 - x$ and we obtain the inequality

$$3 - x < \frac{1}{2}x - 1 \qquad \iff \qquad -\frac{3}{2}x < -4 \qquad \iff \qquad x > \frac{8}{3}.$$

Thus, examining all the $x \leq 3$, we found that the solutions are all the numbers $x > \frac{8}{3}$: that is, we obtained that all the x satisfying

$$\frac{8}{3} < x \leq 3$$

are solutions of the given inequality.

Let's move on to the second group $x \geq 3$; in this case the equation means

$$-3 + x < \frac{1}{2}x - 1 \qquad \iff \qquad \frac{1}{2}x < 2 \qquad \iff \qquad x < 4$$

and therefore all x such that

$$3 \leq x < 4$$

are also solutions.

If we put together all the solutions obtained, that is $\frac{8}{3} < x \leq 3$ and $3 \leq x < 4$, we can conclude that the solutions of the given inequality are all x satisfying

$$\frac{8}{3} < x < 4.$$

Example 3.6. Solve the inequality

$$x^2 \leq 2x + 2|x|.$$

We must distinguish the two cases $x \geq 0$ and $x \leq 0$. When $x \geq 0$ we have $|x| = x$, then the equation becomes

$$x^2 \leq 4x \iff x^2 - 4x \leq 0 \iff 0 \leq x \leq 4$$

(parabola pointing up, with roots 0 and 4). Since we're looking into the $x \geq 0$ zone, we've got the solutions

$$0 \leq x \leq 4.$$

We move to the zone $x \leq 0$ where $|x| = -x$: the equation becomes

$$x^2 \leq 2x - 2x \iff x^2 \leq 0 \iff x = 0$$

and therefore $x = 0$ is a solution, but we already knew it from the first group. In conclusion, the solutions of the inequality are the x such that

$$0 \leq x \leq 4\,.$$

Of course, systems of inequalities containing modules are solved with the same methods: first we solve *separately* each equation, and then we look for the common solutions among all the equations of the system.

Problems.

EXERCISE 2.10 (⏭). Solve the following equations:

$$a)\quad |2x-1| = |x|+4; \qquad b)\quad |x-1| = 3-x; \qquad c)\quad |x-3| = 2|x-5|;$$

EXERCISE 2.11. Solve the following equations:

$$|x-1| = 2x-3; \qquad |2x+1|+x = 1; \qquad x\cdot|x| - 2x + 1 = 0;$$
$$2x\cdot|x| - x = -2; \qquad x = |x|; \qquad x\cdot|x| - 3x + 2 = 0;$$
$$x^2 = x + |x| - 1; \qquad 2x\cdot|x| = 2|x| - 3; \qquad x = 3x(|x|-2)+3;$$
$$x\cdot|x-1| = 2x+1; \qquad (x+|x|)^2 = 3x-1; \qquad x^2+x+1 = x\cdot|x| - 6x - 5.$$
$$x\cdot|x| - 2x + 1 = 0; \qquad |(x-1)(x-2)| = x-3; \qquad 6|x| + 2|x-1| = 3|x-2|\,.$$

EXERCISE 2.12. Solve the following inequalities:

$$|x-1| \leq x+2; \qquad |2-x| > 3; \qquad x+1 \geq |x-5|;$$
$$|x+1| - x^2 \geq 0; \qquad 2x\cdot|x| - x \geq -2; \qquad 2x^2 < |x+1|;$$
$$4x^2 + 4|x| < 3; \qquad 4x\cdot|x| + 4|x| - 3 \geq 0; \qquad x\cdot|x| - |x| + 2 < 0.$$
$$|x^2+3x-4| < 2, \qquad \frac{x^2+3x-4}{x-6} \geq 0, \qquad (|x|+1)(|x|-1) \leq 0\,.$$

EXERCISE 2.13. Solve the following systems of inequalities:

$$\begin{cases} x^2 - 2x \geq 0 \\ 2x \leq |x|+1 \end{cases} \qquad \begin{cases} 2x\cdot|x| + x - 3 = 0 \\ |x| > 1 \end{cases} \qquad \begin{cases} 2|x-2| < 3x \\ x^2 + |x| < 4 \end{cases}$$

$$\begin{cases} (x-1)(x+2) > 3 \\ |x+1| < 16 \\ 2x^2 + 4 = 8 \end{cases} \qquad \begin{cases} 7|x| - 5 > -1 \\ x^2 - 2|x| - 3 < 0 \\ 12 - 3x > -3 - x \end{cases} \qquad \begin{cases} x \leq 5|x| \\ x^2 > 4 \\ 6x - 1 < x + 3. \end{cases}$$

4. Irrational equations and inequalities

In some cases the equations or inequalities to be solved contain a square root; to eliminate it it is necessary to square it, since $(\sqrt{A})^2 = A$; but you have to pay attention to two problems:

1) a square root is always a positive number (≥ 0);

2) the root argument must always be a positive number.

Let us see what happens if you don't pay attention to these facts. Consider the equation

$$\sqrt{x} = 2 - x.$$

If we square both sides regardless of the two problems mentioned above, we get

$$x = (2-x)^2 \iff x^2 - 5x + 4 = 0 \iff x = 1 \text{ and } x = 4.$$

We check if these two numbers are solutions of the original equation or not: $x = 1$ is a solution, in fact by substituting we get $1 = 1$. On the contrary, substituting $x = 4$ we get $2 = -2$, which is absurd. What is wrong?

The problem is very simple: if we start from a false relation like $2 = -2$, by squaring we can get a true relation: $4 = 4$. Thus, sometimes, by squaring we add solutions that were not there at the beginning.

We see now how the exercise can be solved correctly. We observe that in the equation $\sqrt{x} = 2 - x$ the square root is defined only if $x \geq 0$, so we must impose this condition. Then we observe that the first member is always positive, so the second member must also be positive: $2 - x \geq 0$. Under these conditions, both sides are positive and if we square we don't add solutions:

$$\sqrt{x} = 2 - x \iff \begin{cases} x \geq 0 \\ 2 - x \geq 0 \\ x = (2-x)^2 \end{cases}$$

and now we solve the system we have obtained:

$$\begin{cases} x \geq 0 \\ 2 - x \geq 0 \\ x^2 - 5x + 4 = 0 \end{cases} \iff \begin{cases} x \geq 0 \\ x \leq 2 \\ x = 1 \text{ and } x = 4 \end{cases} \iff x = 1.$$

Summing up, in general: *to eliminate a square root correctly we have to set up a system:*

$$\sqrt{A(x)} = B(x) \iff \begin{cases} A(x) \geq 0 \\ B(x) \geq 0 \\ A(x) = B(x)^2. \end{cases}$$

We study another example:

$$\sqrt{7 - 6x} = 2 - x.$$

We set up the system:

$$\begin{cases} 7 - 6x \geq 0 \\ 2 - x \geq 0 \\ 7 - 6x = (2-x)^2 \end{cases} \iff \begin{cases} x \leq \frac{7}{6} \\ x \leq 2 \\ x^2 + 2x - 3 = 0 \end{cases} \iff \begin{cases} x \leq \frac{7}{6} \\ x \leq 2 \\ x = 1 \text{ e } x = -3 \end{cases}$$

and in this case we don't have to discard any solution and we get $x = 1$ and $x = -3$.

In a similar way we can study *irrational inequalities*, i.e. which contain roots. The simplest type is the following:

$$\sqrt{A(x)} \leq B(x).$$

Here, too, one must be careful before squaring; an inequality can be squared only if we already know that both sides are positive:

$$2 \leq 3 \iff 4 \leq 9$$

but if we try to square the inequality

$$-3 \le 2$$

we get the absurd $9 \le 4$ and therefore we see that if the two members are not both positive the direction of the inequality can change.

To solve the inequality

$$\sqrt{A(x)} \le B(x)$$

we follow the steps:

1) impose that the root is defined, so $A(x)$ must be positive;

2) note that the second member is greater than a root which is always positive, so $B(x)$ must also be positive;

3) at this point we can safely take the square of both sides.

In other words, *we need to set up the system*

$$\sqrt{A(x)} \le B(x) \quad \Longleftrightarrow \quad \begin{cases} A(x) \ge 0 \\ B(x) \ge 0 \\ A(x) \le B(x)^2. \end{cases}$$

The inequality $\sqrt{A(x)} < B(x)$ can be solved in a completely analogous way. Here is an example:

$$\sqrt{8x+24} < 2x-2 \iff \begin{cases} 8x+24 \ge 0 \\ 2x-2 \ge 0 \\ 8x+24 < (2x-2)^2 \end{cases} \iff \begin{cases} x \ge -3 \\ x \ge 1 \\ x^2-4x-5 > 0 \end{cases}$$

The second degree inequality in the last system has as solutions the values of x outside the range of the roots (which are $x = -1$ and $x = 5$), i.e. all $x > 5$ and all $x < -1$. Thus we arrive at the system

$$\begin{cases} x \ge -3 \\ x \ge 1 \\ x > 5 \quad \text{or} \quad x < -1 \end{cases}$$

and it is sufficient to draw the usual diagram to discover that the solutions of the system are all the numbers $x > 5$.

Instead, to solve the inequality

$$\sqrt{A(x)} \ge B(x)$$

(or $\sqrt{A(x)} > B(x)$ which is analogous) we must distinguish two cases. If the term $B(x)$ is strictly negative, then there is nothing more to solve, because the first member is a root which is always positive. In this way we immediately get the first group of solutions:

$$I. \begin{cases} A(x) \ge 0 \\ B(x) < 0 \end{cases}$$

(we must always impose that the root is defined, i.e. A must always be positive otherwise the starting expression is not defined). If, on the other hand, B is positive, then we can safely square and we obtain the second group of solutions:

$$II. \begin{cases} A(x) \ge 0 \\ B(x) \ge 0 \\ A(x) \ge B(x)^2. \end{cases}$$

To sum up, *to solve the inequalities of the type* $\sqrt{A(x)} \geq B(x)$ *we must impose two distinct systems:*

$$\sqrt{A(x)} \geq B(x) \quad \Longleftrightarrow \quad I. \begin{cases} A(x) \geq 0 \\ B(x) < 0 \end{cases} \quad \text{plus} \quad II. \begin{cases} A(x) \geq 0 \\ B(x) \geq 0 \\ A(x) \geq B(x)^2. \end{cases}$$

The solutions will be all solutions of system I *plus* all solutions of system II.

An example: to solve the inequality

$$x - 3 \leq \sqrt{7 - 3x}$$

we must solve two systems:

$$I. \begin{cases} 7 - 3x \geq 0 \\ x - 3 < 0 \end{cases} \qquad II. \begin{cases} 7 - 3x \geq 0 \\ x - 3 \geq 0 \\ 7 - 3x \geq (x-3)^2. \end{cases}$$

Solving as before, we obtain the solutions from the first system

$$I. \begin{cases} x \leq \dfrac{7}{3} \end{cases}$$

while the second system is impossible (just compare the first two lines) and does not give other solutions. All solutions are therefore given by $x \leq \frac{7}{3}$.

Problems.

Exercise 2.14 (▶▶|). Solve the following irrational equations:

$$a)\ x + 5 = \sqrt{3 - 3x}; \qquad b)\ x + 5 = \sqrt{3x - 3}; \qquad c)\ \sqrt{2x + 5} = 3x - 3;$$

Exercise 2.15. Solve the following irrational equations:

$$2x - 1 = \sqrt{1 + x}; \qquad \sqrt{x - 3} + 5 + x = 2x - 1; \qquad x - 1 + \sqrt{2 - 6x} = 3;$$

$$\sqrt{x} = 3x - \frac{1}{4}; \qquad \sqrt{x + 1} = 2 - x; \qquad \sqrt{x - \sqrt{x}} = 2\ .$$

Exercise 2.16. Solve the following equations containing square roots and absolute values:

$$\sqrt{x^2} - |x| = 0; \qquad 2\sqrt{|x| - 1} = x; \qquad \sqrt{x^2 + 1} = |x| + \frac{1}{2}.$$

Exercise 2.17. Solve the following irrational inequalities:

$$x + 5 < \sqrt{3 - 3x}; \qquad x + 5 \leq \sqrt{3x - 3}; \qquad \sqrt{2x + 5} < 3x - 3;$$

$$x + 5 \geq \sqrt{3 - 3x}; \qquad x + 5 > \sqrt{3x - 3}; \qquad \sqrt{2x + 5} \geq 3x - 3;$$

$$2x - 1 < \sqrt{1 + x}; \qquad \sqrt{x - 3} + 5 + x \geq 2x - 1; \qquad x - 1 + \sqrt{2 - 6x} < 3.$$

Exercise 2.18. Solve the following inequalities:

$$\frac{x+1}{x-1} \geq 0; \quad \frac{x(x+2)}{x+3} < 0; \quad \frac{x|x+2|}{x+3} \leq 0; \quad \frac{|x-4|(5-6x)}{2x+9} > 0; \quad \frac{x^2(5-x)}{3x-7} < 0.$$

5. Sign and set of definition of an expression

Let us consider the following inequality:

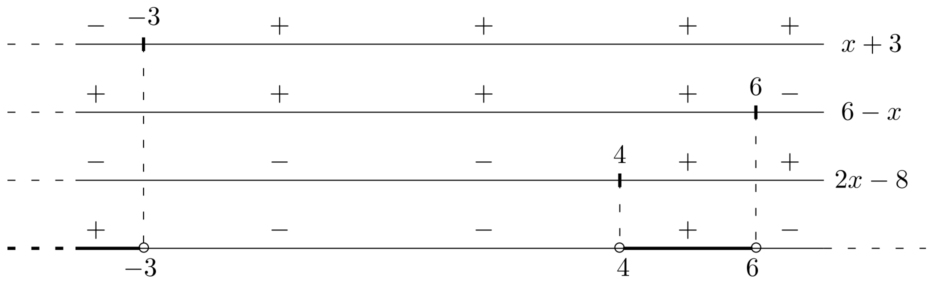
$$(x+3)(6-x)(2x-8) > 0.$$

To solve it we could proceed as in the previous examples and begin to multiply, divide, carry to the second member etc., but there is actually a much simpler way. In fact, it is sufficient to observe that the exercise simply asks: *for which values of x is the expression on the left positive?* That is, all we are interested in is the *sign* of the product $(x+3)(6-x)(2x-8)$.

Since the sign of a product is the product of the signs, it is sufficient to study separately the sign of $(x+3)$, $(6-x)$ and $(2x-8)$ and finally take the product of all signs. In practice we proceed like this: we observe that

$x+3$ has sign $+$ when $x > -3$, and has sign $-$ otherwise;
$6-x$ has sign $+$ when $x < 6$, and has sign $-$ otherwise;
$2x-8$ has sign $+$ when $x > 4$, and has sign $-$ otherwise.

Then we report these results in the following sign diagram, from which it is very easy to establish the product of the signs, shown on the lowest line:



and therefore the product is positive

$$\text{for all} \quad x < -3 \quad \text{and for all} \quad 4 < x < 6.$$

Pay attention to what happens in the points $x = -3$, $x = 4$ and $x = 6$; in these points the product vanishes, but in the inequality we ask that the product be strictly positive, so the three points must be discarded.

The same method works if the expression we are studying is a mixture of products and ratios:

$$\frac{2x-3}{(x+2)(6-x)} \leq 0$$

in fact the ratio of two signs is identical to the product of the signs. In the case of the ratios, of course, it is necessary to exclude the values of x for which the denominator vanishes! Let's solve the exercise:

$2x-3$ has sign $+$ when $x > \frac{3}{2}$, and has sign $-$ elsewhere;
$x+2$ has sign $+$ when $x > -2$, and has sign $-$ elsewhere;
$6-x$ has sign $+$ when $x < 6$, and has sign $-$ elsewhere.

We draw the usual diagram; we also observe that the inequality is with $\leq$, so besides the case of negative sign we also accept the case in which the fraction vanishes; finally we remove the points that make the denominator vanish. In conclusion we get

$$-2 < x \leq \frac{3}{2} \quad \text{and} \quad 6 < x.$$

One last useful observation: if any of the factors is an *absolute value*, then care must be taken where it vanishes; in fact, when it is not zero, it is strictly positive and therefore it is useless to insert it in the diagram, because it can not change the sign of the expression. For example to solve

$$\frac{|x-4|}{(3-x)|x-2|} \geq 0$$

there is no need to draw the diagram: in fact $|x-2|$ is always positive or null, $|x-4|$ is always positive or null, therefore the sign of the whole expression coincides with the sign of $3-x$ and the solution is: $x < 3$, $x \neq 2$ (discard $x = 2$ and $x = 3$ because they make the denominator vanish).

When writing an expression containing a variable, such as

$$\sqrt{\frac{x+2}{x-3}}$$

we mean that the variable x is an arbitrary real number, and depending on the value of x the expression takes on a different value. But you need to be careful: for some values of x the expression cannot be calculated. For example, in the previous expression a division appears, and dividing by zero is not an admissible operation; furthermore, a square root appears, and the square root of a negative number cannot be extracted (in real numbers). The set of all x for which the expression can be calculated is called *definition set*, or *set of definition* of the expression. Let us study the definition set of the previous expression: we must impose that the denominator is non–zero and that the root argument is ≥ 0, that is to say we obtain a system

$$\begin{cases} x-3 \neq 0 \\ \\ \dfrac{x+2}{x-3} \geq 0 \end{cases} \iff \begin{cases} x \neq 3 \\ \\ \dfrac{x+2}{x-3} \geq 0. \end{cases}$$

Note that in the second line we used the $\geq$ because and if the root argument is canceled there is no problem ($\sqrt{0} = 0$); the problem exists only if the argument is strictly negative. We solve the second inequality with the known methods and we obtain

$$\begin{cases} x \neq 3 \\ x \leq -2 \text{ e } x > 3 \end{cases}$$

and therefore the definition set is simply

$$x \leq -2 \text{ e } x > 3\,.$$

Using a more synthetic notation we can also write

$$D.S. = \,]-\infty, -2] \cup \,]3, +\infty[\,.$$

Problems.

EXERCISE 2.19 (▶▶|). For simplicity of notation we denote by log the logarithm in any base > 1. Determine the sign and definition set of the expression

$$\log(x-1) + \log(6-x)\,.$$

EXERCISE 2.20. Determine the definition set of the following expressions:

$$\sqrt{\frac{x+1}{x-1}}; \qquad \sqrt{\frac{x(x+2)}{x+3}}; \qquad \sqrt{x-\frac{1}{x}}; \qquad \sqrt{1-x^2}; \qquad \sqrt{x^2-3x+2};$$

$$x^2+\frac{1}{2-x}; \qquad \sqrt{\frac{x|x-2|(x-3)}{4-3x}}; \qquad \frac{1}{x^2-5x+4}; \qquad \sqrt{x|x|-2x+3};$$

$$\sqrt{10-2x-x^2}; \qquad \sqrt{|x-4|(x+2|x|-2)}; \qquad \frac{1}{x+|x|}; \qquad \sqrt{\frac{|x-2|}{|x-3|}};$$

$$\sqrt{|x+4|}; \qquad \sqrt{\frac{1}{|x+4|}}; \qquad \frac{1}{|x^2-8x+10|}; \qquad \frac{1}{\sqrt{x+1}}; \qquad \frac{1}{\sqrt{x}+1}.$$

CHAPTER 3

Functions of a real variable

First of all, some terminology. Suppose we have a set of numbers A, and that M is one of these numbers: $M \in A$. Then we say that the number M is the *maximum* of A if it is the greatest of all numbers in A: with a formula we write

$$M \geq a \qquad \forall a \in A$$

where the symbol $\forall$ means "for every". Similarly, if the number m in the set A is the smallest of all numbers in A, we say that m is the *minimum* of A:

$$m \leq a \qquad \forall a \in A.$$

Warning: not all sets have maximum or minimum. For example, the interval

$$I = [1, 2[$$

certainly has a minimum, in fact, the point $m = 1$ is the smallest of the set; but it has no maximum! In fact the point 2, which could be the maximum, does not belong to the interval but is outside. In this case we say that 2 is the *upper bound* of the interval; this means that all numbers $x < 2$ close enough to 2 are in the set I, but the point $x = 2$ itself is not in the set. Similarly, the interval

$$J =]-2, 5]$$

has maximum $M = 5$, but has no minimum: the point -2 is called the *lower bound* of J, but it is not the minimum because it is outside the interval J. Of course, if there is a maximum M then M is also the upper bound; and in the same way, if there is a minimum m, then m is also the lower bound. There is also another case in which there is no maximum or minimum: the set can be *unbounded*. For example, the half line

$$I = [2, +\infty[$$

has a minimum $m = 2$ but clearly has no maximum M; in fact, given any value M, inside I we can always find numbers larger than M. In these cases we conventionally say that the upper bound of I is $+\infty$ (plus infinity). What about the following half line?

$$J =]-\infty, 3[$$

We see that J has no maximum and no minimum; the upper bound is 3; again by convention, we say that the lower bound is $-\infty$.

To indicate the maximum and minimum of a set A we write

$$\max A, \qquad \min A;$$

while to indicate the upper and lower bound of A we write

$$\sup A, \qquad \inf A$$

and indeed one commonly says "the sup of A" instead of the upper bound, and "the inf of A" instead of the lower bound.

For the curious readers, the precise definition of inf and sup is the following. We call *majorant* of the set A any real number M with the property $M \geq x$ for all $x \in A$: the number M is "to the right" of the set A. If it is not possible to find a majorant (for example if $A = \mathbb{N}$) then we say that $\sup A = +\infty$. If, on the other hand, there exists a majorant, then we call $\sup A$ the minimum of all majorants of A. The fact that such a minimum exists is not a theorem but is part of the definition of real numbers. Similarly, a number m with the property $m \leq x$ for all $x \in A$ is called a *minorant* of A; if there are no minorants, we say that $\inf A = -\infty$, otherwise $\inf A$ is defined as the maximum of all minorants of A.

1. The concept of function

Given a set of real numbers A, a *function* f defined on A with real values is a "rule" which associates to every number x in A a value $f(x)$. For example: if we choose $A = [0, +\infty[$, the set of positive numbers, we can define on A the function square root of x

$$f(x) = \sqrt{x}$$

which associates the value $\sqrt{x} \in \mathbb{R}$ to each number $x \in A$. The set A on which the function is defined is called the *domain* of f, and to indicate that f is a function on A with real values we write

$$f\colon A \to \mathbb{R}.$$

We write also

$$x \mapsto f(x)$$

to indicate that f takes the value x into the value $f(x)$. Another example: if we choose

$$A = \{x\colon x \neq 2\}$$

that is, all real numbers except 2, on A we can define the function

$$f(x) = \frac{1}{x-2}$$

which associates to each number $x \in A$ the value $\frac{1}{x-2} \in \mathbb{R}$.

The set of all the values of a function $f : A \to \mathbb{R}$ is also called the *image* of f and is denoted by $f(A)$.

The best way to understand these concepts is to visualize them with a drawing. For this purpose we use the Cartesian plane: we already know that each pair (x, y) can be represented with a point of the plane on which we have drawn two Cartesian axes. To represent a function $f\colon A \to \mathbb{R}$, we proceed as follows: for each number $x \in A$ in the domain, we calculate the value $f(x)$, and then draw a point corresponding to the pair $(x, y) = (x, f(x))$. If we repeat this operation for all the numbers $x \in A$, eventually the points we draw will form a curve called the *graph* of the function f:

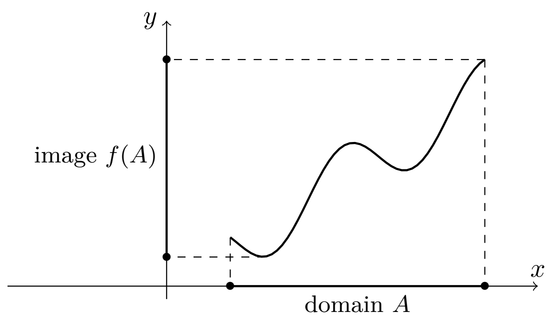


FIGURE 3.1.

We see that if we have drawn the graph of a function, it is quite easy to visualize its image $f(A)$. Often, in a synthetic way, we speak of the "graph of the function $y = f(x)$"; more precisely this means "the set of all points of the Cartesian plane of the form $(x, y) = (x, f(x))$ as x varies in A".

When two functions f and g are defined on the same set A, we can immediately define the *sum* $f + g$ and the *product* $f \cdot g$ of the two functions; of course this means $f(x) + g(x)$ and $f(x)g(x)$ respectively. If g does not vanish, we can also consider the *ratio* function $\frac{f}{g}$ i.e. $f(x)/g(x)$. For example:

$$f(x) = x^2,\ g(x) = \sqrt{x} \quad \Longrightarrow \quad f + g = x^2 + \sqrt{x}, \quad fg = x\sqrt{x}, \quad \frac{f}{g} = \frac{x^2}{\sqrt{x}} = x^{\frac{3}{2}}.$$

Moreover, given two functions f and g, we can consider their *composition* which is indicated with $g \circ f$ or also with $g(f(x))$: the composition can be defined when the values of f are contained in the domain of g, and it corresponds to "introduce the function f inside the function g". For example:

$$f(x) = x^2, \quad g(x) = x + 1 \quad \Longrightarrow \quad g(f(x)) = x^2 + 1;$$
$$f(x) = x + 1, \quad g(x) = x^2 \quad \Longrightarrow \quad g(f(x)) = (x + 1)^2;$$
$$f(x) = x^3, \quad g(x) = \sqrt{2x - 3} \quad \Longrightarrow \quad g(f(x)) = \sqrt{2x^3 - 3}.$$

This example shows among other things that the functions $g \circ f$ and $f \circ g$ are not necessarily equal and may be different.

To complete the picture, we give some basic terminology. A function is called *injective* if it never takes the same value twice: that is, f injective means

$$f(x) \neq f(y) \quad \text{for} \quad x \neq y.$$

From the graph of f, it is very easy to check if f is injective: if there is a horizontal line that cuts the graph in two distinct points, then f cannot be injective, otherwise it is.

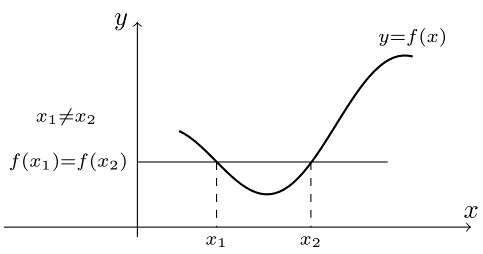


FIGURE 3.2. An example of a **non** injective function: we can find an horizontal line which cuts the graph in two distinct points.

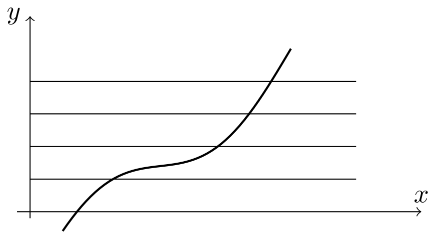


FIGURE 3.3. An example of an injective function: every horizontal line cuts the graph at most in one point.

A function is said to be *increasing* if it has the property

$$x \leq y \quad \Longrightarrow \quad f(x) \leq f(y)$$

that is, if "moving x to the right the values of the function increase". Note that if the function does not grow but remains constant, the above definition still applies; if instead we want to talk about a function that does not remain constant but actually grows, then we must define a *strictly increasing* function with the property

$$x < y \quad \Longrightarrow \quad f(x) < f(y)$$

with strict inequality. It is easy to recognize the graph of increasing functions:

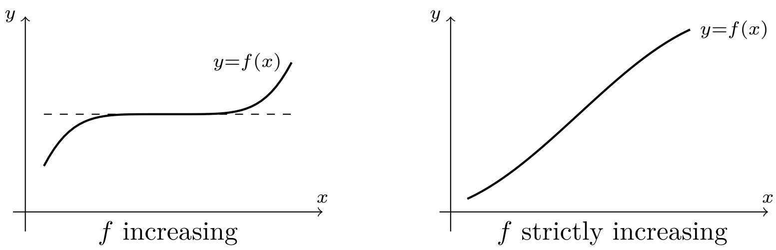


FIGURE 3.4.

Similarly we can define a *decreasing* function:

$$x \leq y \quad \Longrightarrow \quad f(x) \geq f(y)$$

and a *strictly decreasing* function:

$$x < y \quad \Longrightarrow \quad f(x) > f(y).$$

Note that a strictly increasing or decreasing function is always injective; why? (think of the graph ...)

Most functions are neither increasing nor decreasing; usually, functions are increasing on some intervals and decreasing on others. For example, the parabola $f(x) = x^2$ is decreasing for negative x that is to say on $A =]-\infty, 0]$, while it is increasing for positive x that is to say on $B = [0, +\infty[$. Then we speak of *intervals of increase or decrease* of the function f.

Finally, we introduce the concept of *inverse function.* Suppose we have an injective function $f\colon A \to \mathbb{R}$; thus f never takes the same value twice, and if we choose a value y in the image, there is only one $x \in A$ such that $f(x) = y$ (if there were any two, the function would not be injective!). But in this way we have created a new "rule", that is a new function: to each number y in the image of f we associate the unique number $x \in A$ such that $f(x) = y$. This new function is denoted by f^{-1} and we call it the *inverse function* of f: if f has domain A and image $f(A) = B$, then the inverse function has B as domain and A as image.

If we know the graph of f, to plot the graph of f^{-1} it is sufficient to reflect it with respect to the bisector of the axes: in fact we are simply exchanging the role of the ordinate axis and the abscissa axis.

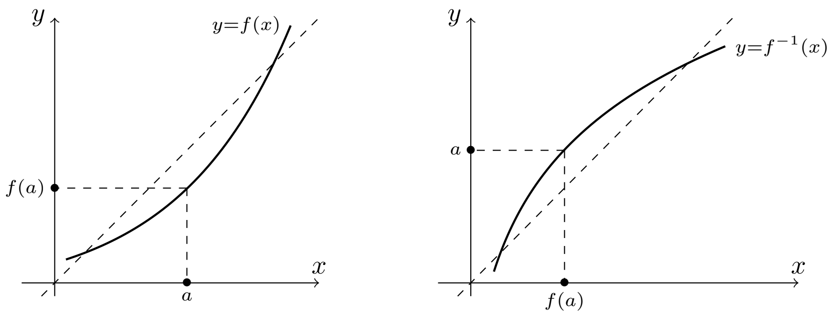


FIGURE 3.5. The graphs of f and f^{-1} can be obtained from each other by a reflection with respect to the line $x = y$.

How to compute the inverse function? let's see in practice: if $f\colon A \to \mathbb{R}$ is the function $f(x) = x^2$ on the set of positive numbers $A = [0, +\infty[$, to compute the inverse function of f we write $y = x^2$ and extract x as a function of y: we get $\sqrt{y}$, so the inverse function is the square root : $f^{-1}(x) = \sqrt{x}$ (remember to swap the role of x and y!). If we want to invert the function

$$f(x) = \frac{1}{x+2},$$

we write $y = f(x)$ and get x as a function of y:

$$y = \frac{1}{x+2} \implies x = \frac{1}{y} - 2$$

and therefore the inverse function is

$$f^{-1}(x) = \frac{1}{x} - 2.$$

Finally, we note that if we compose f with f^{-1} we are first passing from the value x to the value $f(x)$, and then we are going back, so

$$f^{-1}(f(x)) = x;$$

and in the same way

$$f(f^{-1}(x)) = x.$$

One last useful definition: given a function $f\colon \mathbb{R} \to \mathbb{R}$, we say that

f is *even* if $f(-x) = f(x)$

f is *odd* if $f(-x) = -f(x)$.

The graphs of even functions and odd functions are easily recognizable: even functions are symmetric with respect to the ordinate axis, while odd functions are symmetric with respect to the origin:

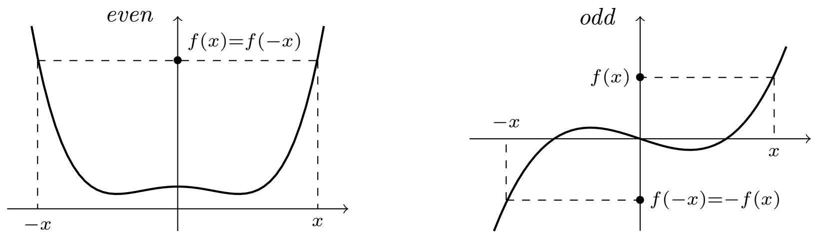


FIGURE 3.6.

Problems.

EXERCISE 3.1. Determine upper bound (sup), lower bound (inf) and, if they exist, maximum and minimum of the following sets:

$$[0,2]; \qquad]0,2]; \qquad [1,2]\cup[3,5[\,; \qquad]-\infty,3]\cup\,]4,5[\,; \qquad [-1,2]\cup\,]3,+\infty[\,.$$

EXERCISE 3.2. If $f(x)=\sqrt{x^2-3x+2}$, compute the values

$$f(0), \quad f(1), \quad f(2), \quad f(3).$$

EXERCISE 3.3 (⏭). Write the composite functions $f(g(x))$ and $g(f(x))$ for

(1) $f(x)=\sqrt{x}, \quad g(x)=x^2-3x.$
(2) $f(x)=\dfrac{1}{2x-4}, \quad g(x)=x+1.$
(3) $f(x)=\dfrac{x+1}{x-1}, \quad g(x)=2x-3.$

EXERCISE 3.4. Write the inverse function for the following functions:

$$x^2-1; \qquad \frac{3}{5x-4}; \qquad \frac{x+1}{x-1}; \qquad 6-12x; \qquad 5+\frac{2}{2-x}.$$

2. Elementary functions

Some functions are so important that they deserve a special study: they are called the *elementary functions*. They are the powers, the trigonometric functions (sine, cosine, tangent etc.), the exponential function and the logarithm.

Recall that to define a function $f\colon A\to\mathbb{R}$ in a precise way, two things must be assigned:
1) the domain (definition set) A in which x varies;
2) the rule $f(x)$ to compute the value of f at x.

LINE AND PARABOLA

A function of the type

$$f(x)=ax+b$$

with a and b fixed constants, is called a *linear function*. Clearly the definition set of the expression $ax+b$ is the entire $\mathbb{R}$. The graph of f is a straight line; the number a is called the *angular coefficient* of the line and expresses the slope of the line. When $a>0$ the line is increasing, when $a<0$ the line is decreasing. In the case $a=0$ the straight line is horizontal, and in fact the function is reduced to the *constant* function $f(x)=b$ which always assumes the same value in all points:

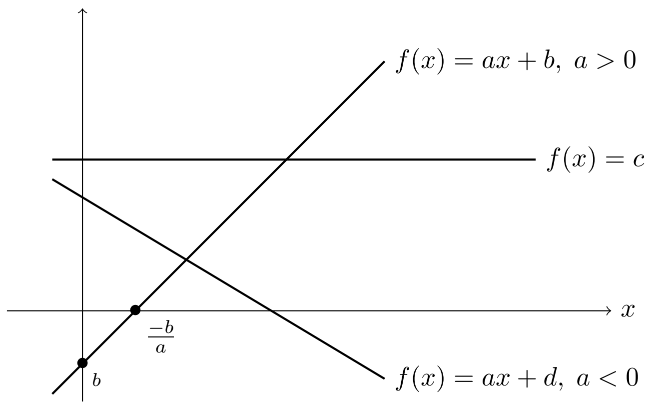


FIGURE 3.7. Graphs of linear functions are lines in the plane.

Linear functions are polynomials of degree 1 (or zero when $a = 0$). If we consider polynomials of degree 2

$$f(x) = ax^2 + bx + c$$

with $a \neq 0$, we still have functions defined on all $\mathbb{R}$, whose graph is a *parabola*. The parabola points upwards if $a > 0$, and downwards if $a < 0$. We have already discussed the zeros of the function $f(x)$ and the positivity and negativity of the parabola. We add that the intervals of increase and decrease of $f(x)$ are two: in the case $a > 0$, we have that $f(x)$ is decreasing for $x \leq -\frac{b}{2a}$ and increasing for $x \geq -\frac{b}{2a}$; instead, when $a < 0$, $f(x)$ is increasing for $x \leq -\frac{b}{2a}$ and decreasing for $x \geq -\frac{b}{2a}$.

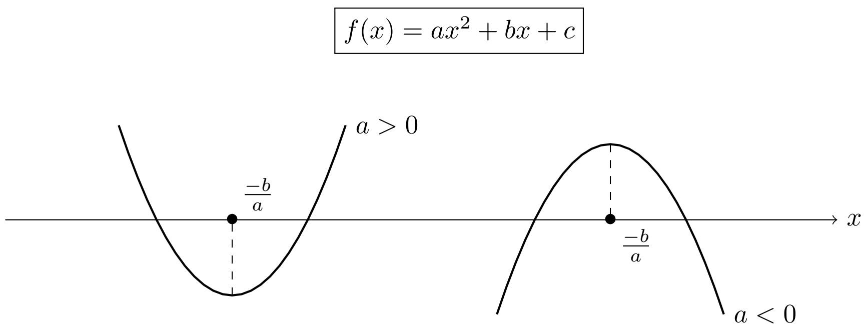


FIGURE 3.8. The graph of the quadratic function $f(x) = ax^2 + bx + c$ is a parabola with a vertical symmetry axis $x = -\frac{b}{a}$.

POWERS.

We know well that if x is a real number and $n \geq 1$ a positive integer, x^n indicates the product

$$x^n = x \cdot x \cdot \dots \cdot x \qquad n \text{ times;}$$

furthermore, for $x \neq 0$ we also set $x^0 = 1$, while the power 0^0 *is not defined.* Integer negative powers are defined simply using the rule

$$x^{-n} = \frac{1}{x^n}.$$

Also remember that if $x \geq 0$, with $\sqrt[n]{x}$ we indicate the *root n-th* of the number x that is the only positive number that raised to n gives as a result x; it is also written

$$x^{\frac{1}{n}} = \sqrt[n]{x}.$$

Note that $x^{\frac{1}{n}} = \sqrt[n]{x}$ is exactly the inverse function of x^n:

$$(x^n)^{\frac{1}{n}} = (x^{\frac{1}{n}})^n = x.$$

Now if $x > 0$ we can define any power of the type $x^{\frac{p}{q}}$ where $\frac{p}{q}$ is a rational number (that is, a fraction of two integers p and q): just define

$$x^{\frac{p}{q}} = (x^p)^{\frac{1}{q}}.$$

But we can extend this definition further, and if $x > 0$ we can define the power x^a where a is any real number; the usual rules apply

$$x^{a+b} = x^a x^b, \quad x^{ab} = (x^a)^b, \quad x^{-a} = \frac{1}{x^a}$$

and so on.

We now study the graph of these functions. The *positive integer powers* $f(x) = x^n$ are defined for each x and have a different behaviour depending on whether n is even or odd.

When n is even, we obtain an even function, i.e. symmetric with respect to the ordinate axis: for example $f(x) = x^2$ satisfies

$$f(-x) = (-x)^2 = x^2 = f(x).$$

All these functions have the same behaviour: they are decreasing for $x \leq 0$ and increasing for $x \geq 0$.

Instead the powers x^n with n odd are odd functions, i.e. symmetric with respect to the origin:

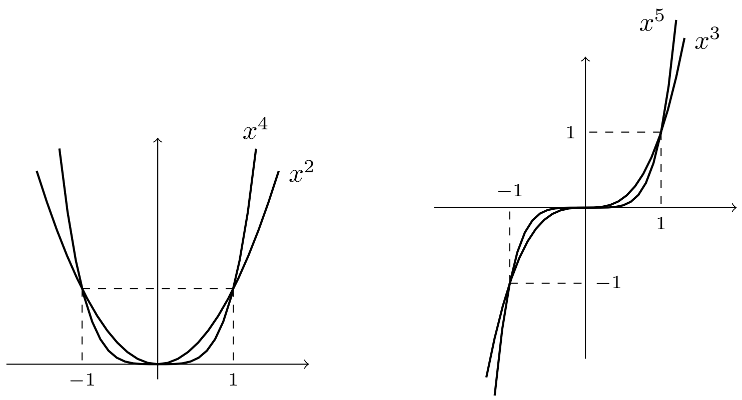


FIGURE 3.9. Graph of integer powers of x.

For example $f(x) = x^3$ satisfies

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$

All these functions have the same behaviour: they are strictly increasing on all $\mathbb{R}$.

Let us now consider the negative powers, viz

$$f(x) = x^{-n} = \frac{1}{x^n}, \quad x = 1, 2, 3, \ldots.$$

Negative powers are only defined for $x \neq 0$, and again we must distinguish n even from n odd. If n is even then $\frac{1}{x^n}$ is increasing for $x < 0$ and decreasing for $x > 0$. On the other hand, if n is odd then the function $\frac{1}{x^n}$ is increasing both for $x < 0$ and for $x > 0$.

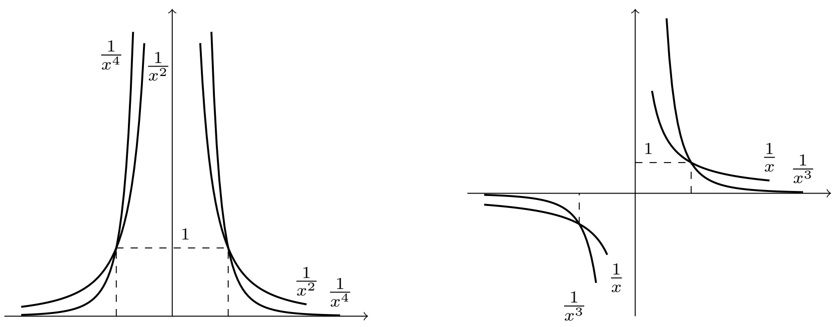


FIGURE 3.10. Graphs of negative integer powers of x.

Finally we study the graph of *real powers* x^a. Here too we have two cases: when $a > 0$, the function $f(x) = x^a$ is defined on $x \geq 0$, it is strictly increasing and positive; in particular when $a = \frac{1}{n}$ we get the *n-th roots*, and when a is integer we get back the powers; instead if $a < 0$ the function $f(x) = x^a$ is defined only for $x > 0$, it is strictly decreasing and positive:

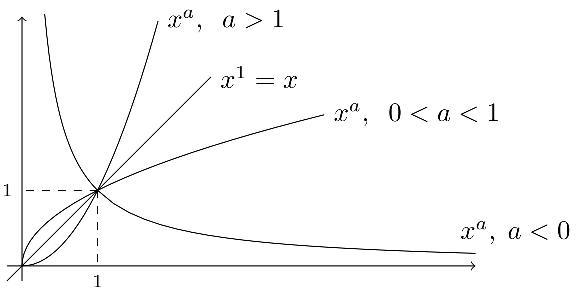


FIGURE 3.11. Graphs of real powers of x.

THE EXPONENTIAL FUNCTIONS

If in the real power a^b we keep the base $a > 0$ constant and vary the exponent, we get the *exponential functions*

$$f(x) = a^x.$$

These functions are defined on all $\mathbb{R}$, and their behavior is very simple: they are always strictly positive (they never vanish), moreover

- when $a > 1$ the exponential function is strictly increasing;
- when $0 < a < 1$ the exponential function is strictly decreasing;
- and of course, when $a = 1$ we obtain the constant function $f(x) = 1^x = 1$.

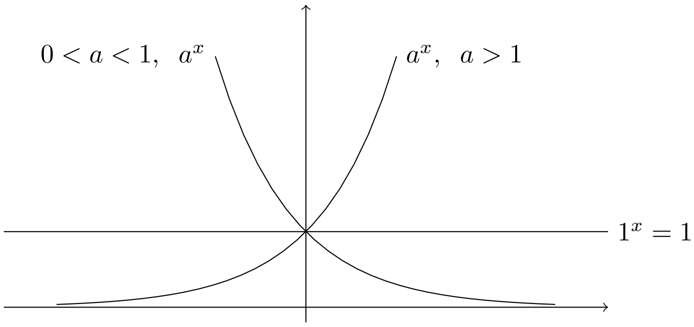


FIGURE 3.12. Graphs of exponential functions.

Note that all exponential functions have value 1 for $x = 0$, in fact $a^0 = 1$. If we now draw the line $g(x) = 1 + x$, we notice that it too passes through the point $(0, 1)$ where all the previous curves cross. It is possible to prove that only one of the exponential curves lies entirely above this line (and is actually tangent to it, while all the others cut it in two points): this happens when the base has the value

$$e = 2,718281828459045\ldots$$

which is called the *Napier's constant*. In other words

the constant e is the unique real number such that $e^x \geq 1 + x$ for all x.

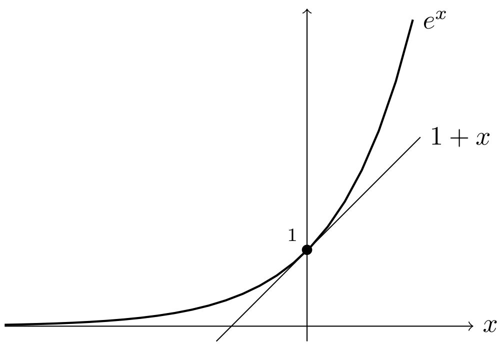


FIGURE 3.13. A visualization of the fundamental property of the number e.

One can prove that e is not a rational number[1] and therefore its decimal expansion is infinite and non persiodic. The corresponding exponential function

$$f(x) = e^x$$

is very important, so much so that it is usually simply called *the exponential.*

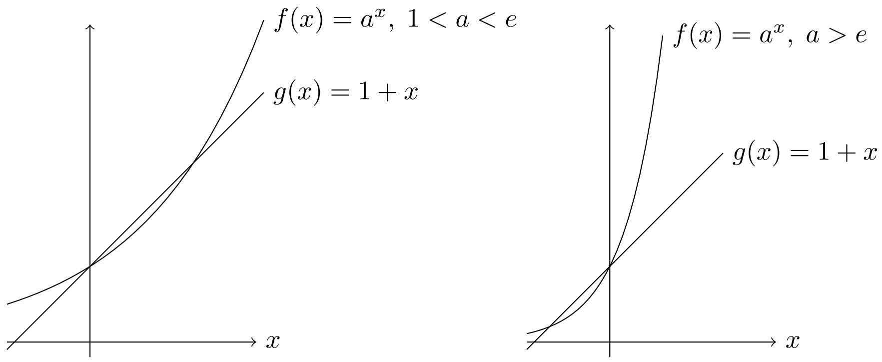


FIGURE 3.14. The graph of a^x when $a \neq e$.

THE LOGARITHM.
As we have seen, the exponential function $f(x) = e^x$ is strictly increasing, therefore injective, and we can consider the corresponding inverse function. The graph is obtained simply by reflecting the graph of e^x with respect to the bisector of the axes:

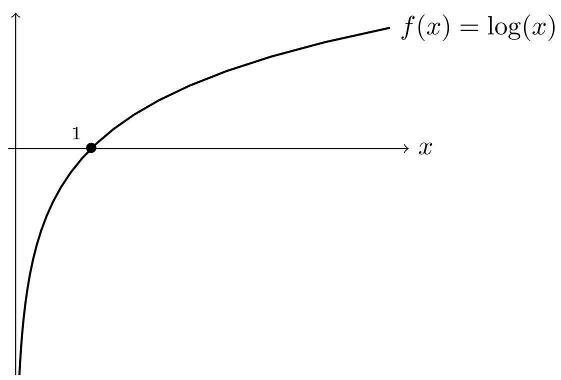


FIGURE 3.15. Graph of the function $\log(x)$

[1]It is easy to find rational numbers which approximate e. Indeed, if n is a positive integer, raising to the n–th power the relations $e^{\frac{1}{n}} \geq 1 + \frac{1}{n}$ and $e^{-\frac{1}{n+1}} \geq 1 - \frac{1}{n+1}$ we obtain the inequalities

$$\left(1 + \frac{1}{n}\right)^n \leq e \leq \left(1 + \frac{1}{n}\right)^{n+1}$$

which for large n give good approximations of e: for example, using a calculator, we find that

$$\left(1 + \frac{1}{10^4}\right)^{10^4} \simeq 2,7181, \qquad \left(1 + \frac{1}{10^4}\right)^{10^4+1} \simeq 2,7185\ .$$

For a better aproximation of Napier's constant see Problem **??**.

The function thus obtained is called *(natural) logarithm* and is indicated with $\log x$. Since $\log x$ is the inverse of e^x, we have the properties

$$e^y = x \iff y = \log x$$

$$e^{\log x} = x \quad \forall x > 0, \qquad \log(e^x) = x \quad \forall x$$

which we will use often. From the properties of the exponential we immediately obtain the following properties of the logarithm function:

$$\log(ab) = \log a + \log b, \qquad \log\left(\frac{a}{b}\right) = \log a - \log b$$

$$\log(a^b) = b \log a \qquad \log 1 = 0, \qquad \log e = 1.$$

Also, the definition set of $\log x$ (which coincides with the image of e^x) is the positive half line $x > 0$:

$$\log x \text{ is defined only for } x > 0,$$

and the function $\log x$ is strictly increasing.

THE TRIGONOMETRIC FUNCTIONS.

We resume the study of the trigonometric functions $\sin x$, $\cos x$ and $\tan x$ already begun in the first chapter. Now we study these functions from the point of view of analysis. Let us consider the unit circle in the Cartesian plane, with center at the origin O and radius 1. We call A the point $(1, 0)$ at the intersection between the circle and the abscissa axis.

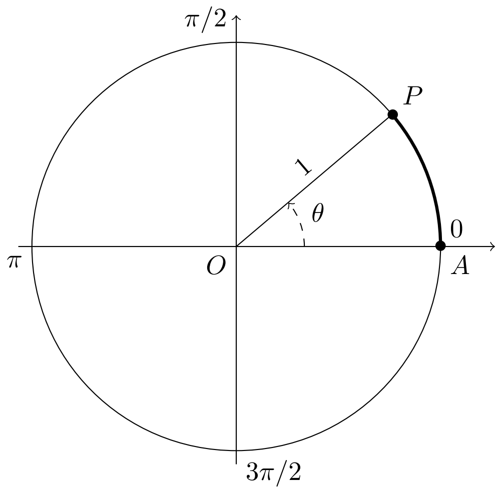


FIGURE 3.16. The measure in radians of an angle θ is equal to the length of the arc of circle $\widehat{AP}$.

Now, if P is a point on the circle, we can measure the width of the angle AOP using the length of the arc of circle AP. For example, if AOP is an angle of 90°, the arc will be long $\frac{\pi}{2}$; if the angle is 180° the arc will be long π, and so on. We say then that we are measuring angles in *radians*: a right angle measures $\frac{\pi}{2}$ radians and so on.

Then, if P is any point on the unit circle, and if s is the arc length AP, the *sine* and *cosine* of s are exactly the ordinate and abscissa of the point P:

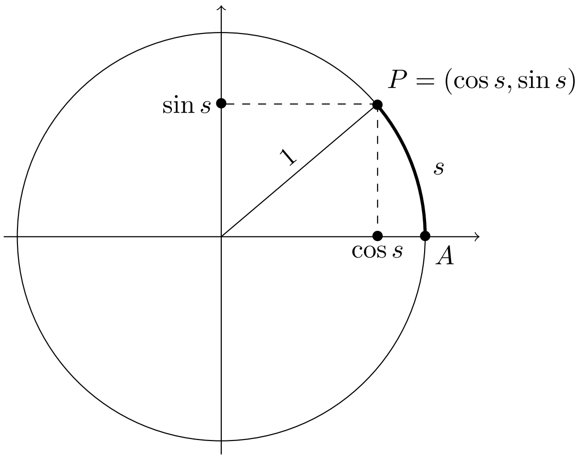


FIGURE 3.17. Cosine and sine of s are the coordinates of the image of the point $A = (1, 0)$ after a counterclockwise rotation of s radians.

From the Pythagorean Theorem we immediately obtain the relation

$$\sin^2 s + \cos^2 s = 1.$$

It is easy to verify that

$$\begin{aligned} \sin 0 &= 0, & \cos 0 &= 1 \\ \sin\left(\frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}}, & \cos\left(\frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}} \\ \sin\left(\frac{\pi}{2}\right) &= 1, & \cos\left(\frac{\pi}{2}\right) &= 0 \\ \sin \pi &= 0, & \cos \pi &= -1 \\ \sin\left(3\frac{\pi}{2}\right) &= -1, & \sin\left(3\frac{\pi}{2}\right) &= 0. \end{aligned}$$

Moreover we have also

$$\begin{aligned} \sin\left(\frac{\pi}{6}\right) &= \frac{\sqrt{3}}{2}, & \sin\left(\frac{\pi}{6}\right) &= \frac{1}{2} \\ \sin\left(\frac{\pi}{3}\right) &= \frac{1}{2}, & \sin\left(\frac{\pi}{3}\right) &= \frac{\sqrt{3}}{2}. \end{aligned}$$

When the point P is in the starting position A, the angle measures zero radians. If we move P counterclockwise, after one complete revolution the point returns to the position A, and then the angle is 2π radians. If we continue to move the point counterclockwise, the angle passes 2π and the point P retraces the positions it passed through in the first revolution. In particular, sine and cosine take on the same values after a complete revolution:

$$\sin(s + 2\pi) = \sin s, \qquad \cos(s + 2\pi) = \cos s.$$

Functions with this property are called *periodic*, and precisely a function such that

$$f(x + T) = f(x)$$

for all x is called *periodic of period* T, or T *-periodic*. So sine and cosine are 2π -periodic functions.

The graph of the two functions is the following:

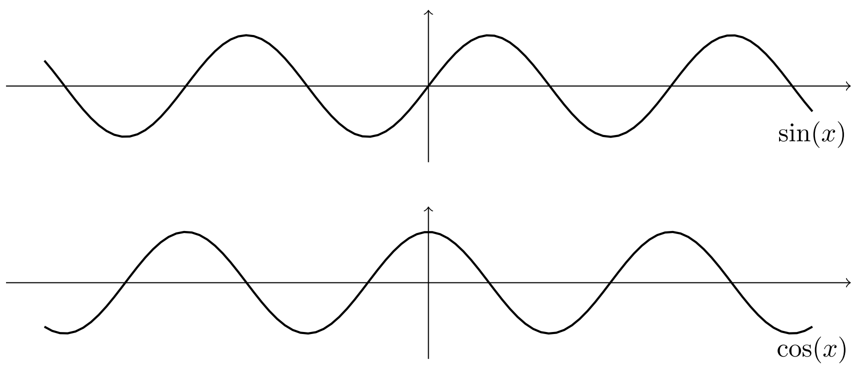


FIGURE 3.18. Graph of the functions $\sin(x)$ and $\cos(x)$.

Let us recall some other important properties: the sine and cosine functions are always in the range between -1 and $+1$:

$$-1 \leq \sin x \leq 1, \qquad -1 \leq \cos x \leq 1.$$

Also the *addition formulas* are valid

$$\sin(a+b) = \sin a \cos b + \cos a \sin b, \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

and from the graph of the two functions it is easy to understand that

$$\sin\left(x + \frac{\pi}{2}\right) = \cos x, \qquad \cos\left(x + \frac{\pi}{2}\right) = -\sin x.$$

The sine function is *odd*, while the cosine function is *even*:

$$\sin(-x) = -\sin x, \qquad \cos(-x) = x.$$

The zeroes of the sine are the points of the type $\pi, 2\pi, 3\pi, \dots$:

$$\sin x = 0 \iff x = k\pi,\ k \in \mathbb{Z}$$

while the zeros of the cosine are the points of the type $\frac{\pi}{2}, \frac{\pi}{2} + \pi, \frac{\pi}{2} + 2\pi, \dots$:

$$\cos x = 0 \iff x = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}.$$

We recall also that the *tangent* function is given by

$$\tan x = \frac{\sin x}{\cos x}$$

and of course it is not defined when the denominator vanishes, that is

the domain of $\tan x$ is given by $x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$.

The function $\tan x$ is periodic with period π and it is odd. Its graph is showed in Figure 3.19:

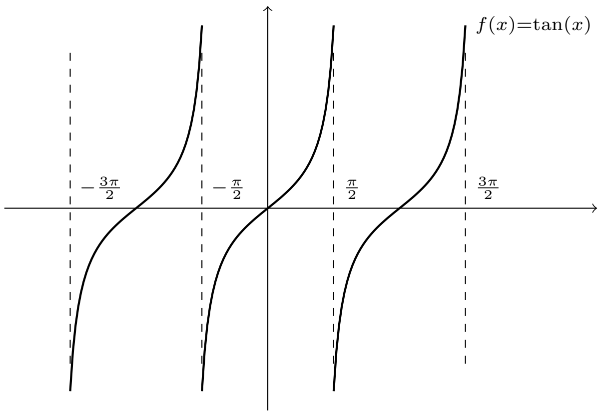


FIGURE 3.19. Graph of the tangent function.

It is easy to give a geometric description of the tangent. We take a point P on the unit circle as in Fig. 3.20, and consider the right triangle OPB; we know that OB is the cosine of the arc $s = AP$, while PB is the sine of s.

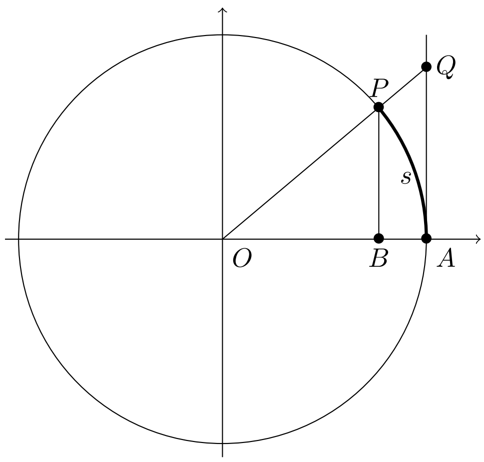


FIGURE 3.20. The radius of the circle is $OA = 1$. By Talete's Theorem we have

$$AQ = \frac{AQ}{AO} = \frac{BP}{BO} = \frac{\sin s}{\cos s} = \tan s.$$

By the similarity of the triangles OPB and OQA we immediately see that AQ represents the tangent of s.

We also see that if P is in the first quadrant, the area of the circle section OAP is equal to $s/2$, that of the triangle OAP is equal to $(\sin s)/2$, while that of the triangle OAQ is

$(\tan s)/2$; it follows that for $0 \leq s < \pi/2$ the inequalities hold

$$\text{segment } PB \leq \text{ arc } PA \leq \text{ segment } QA \tag{2.1}$$

that is to say

$$0 \leq \sin s \leq s \leq \tan s \qquad \text{for } \ 0 \leq s < \frac{\pi}{2}\,. \tag{2.2}$$

Since $\sin s$ and $\tan s$ are odd functions, the inequality holds in reverse for $-s$:

$$\tan s \leq s \leq \sin s \leq 0 \qquad \text{for } \ -\frac{\pi}{2} \leq s \leq 0\,.$$

We can summarize these inequalities in one:

$$|\sin s| \leq |s| \leq |\tan s| \qquad \text{per } \ |s| \leq \frac{\pi}{2}\,.$$

To conclude, we observe that the sine, cosine and tangent functions are not injective, therefore the inverse function cannot be defined; but if we consider only an increasing (or decreasing) stretch of these functions, then we get injective functions and we can invert them. For example, the function

$$f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}, \qquad f(x) = \sin$$

is increasing, its inverse function is called *arcsine* and is denoted by $\arcsin x$; while the function

$$f : [0, \pi] \to \mathbb{R}, \qquad f(x) = \cos x$$

is decreasing, its inverse function is called *arcosine* and is denoted by $\arccos x$:

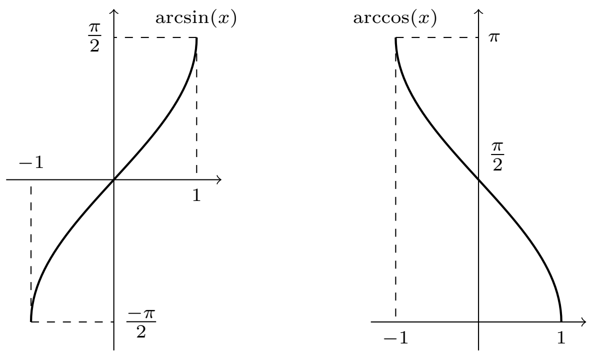


FIGURE 3.21. Graph os the arcsine and arcosine functions.

Finally, the function

$$f : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\to \mathbb{R}, \qquad f(x) = \tan x$$

Is increasing, its inverse function is calleda *arctangent* and is denoted by $\arctan x$:

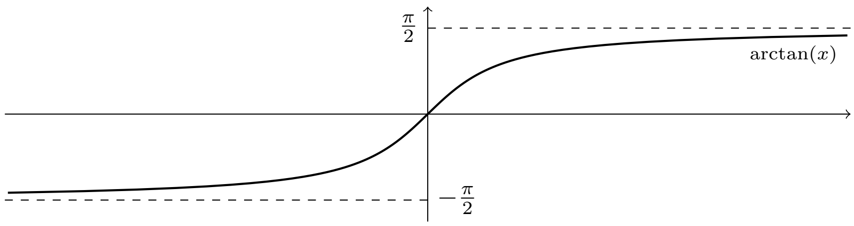


FIGURE 3.22. Graph of the arctangent function.

We note that $\arctan x$ is defined on all $\mathbb{R}$, is strictly increasing, and its values are always between $-\pi/2$ and $\pi/2$.

PIECEWISE DEFINED FUNCTIONS

Often it will be useful to consider piecewise defined functions, i.e. using different expressions on different intervals: for example, we can define $f\colon \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} -x & \text{if } x \leq 0, \\ x & \text{if } x \geq 0. \end{cases}$$

Do you recognize it? this is simply the *absolute value* (modulo) function, whose graph is the following:

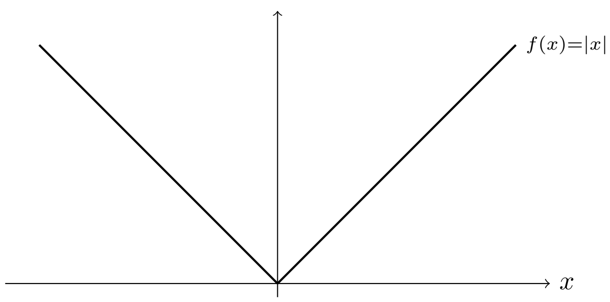


FIGURE 3.23. Graph of the function $|x|$.

Another example is the function sign of x:

$$\operatorname{sgn}(x) = \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0. \end{cases}$$

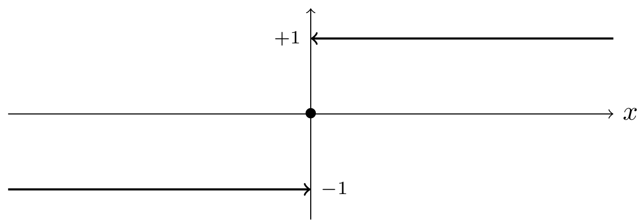
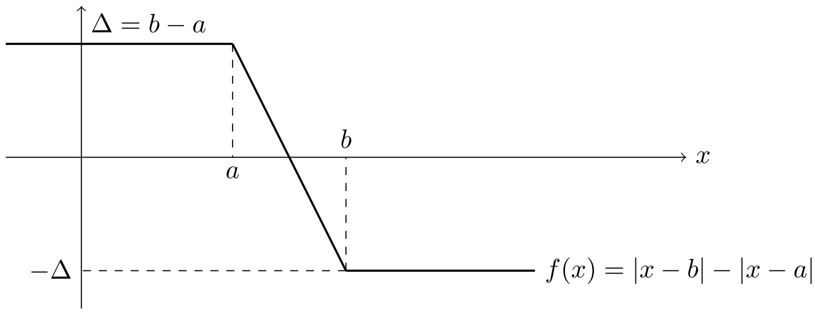


FIGURE 3.24. Graph of the function sgn(x).

EXAMPLE 2.1. Let a, b be two real numbers, with $a < b$, and let's study the function $f(x) = |x - b| - |x - a|$. This is clearly a piecewise linear function, and in fact, writing $\Delta = b - a$ we have:

$$f(x) = \begin{cases} (b-x) - (a-x) = \Delta & \text{if } x \leq a, \\ (b-x) - (x-a) = b + a - 2x = \Delta - 2(x-a) & \text{if } a \leq x \leq b, \\ (x-b) - (x-a) = -\Delta & \text{if } b \leq x \ . \end{cases}$$

Its graph is the following:



3. Limits of functions

We now introduce the concept of *limit of a function at a point*. The idea is the following: if we have defined a function f on A we know how to calculate the values $f(x)$ corresponding to the numbers $x \in A$. Now, imagine that we make the point x vary and move it towards a fixed point x_0, and we follow the corresponding values $f(x)$. If we are lucky, when x approaches x_0 the values $f(x)$ also approach a value of L; then we say that L is the limit of f at the point x_0. Note that we don't care about the value of f exactly at that point: we are just studying the behavior of the values $f(x)$ as x gets close to x_0. Of course we can move x towards x_0 from the right or from the left, or from both sides.

Here are the precise definitions:

DEFINITION 3.1. Let $f\colon]x_0, b[\to \mathbb{R}$ be a function. We say that f *has limit* L *in* x_0 *from the right* (or f *tends to* L *from the right in* x_0) if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{for} \quad x_0 < x < x_0 + \delta.$$

We write

$$L = \lim_{x \to x_0^+} f(x),$$

or also

$$f(x) \to L \quad \text{as} \quad x \to x_0^+.$$

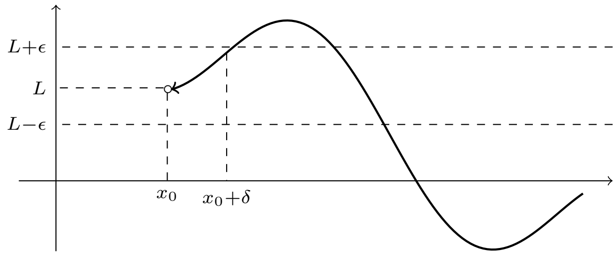


FIGURE 3.25. Limit as $x \to x_0^+$.

DEFINITION 3.2. Let $f\colon]a, x_0[\to \mathbb{R}$ be a function. We say that f *has limit* L *from the left in* x_0 (or f *tends to* L *from the left in* x_0) if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{for} \quad x_0 - \delta < x < x_0.$$

we write

$$L = \lim_{x \to x_0^-} f(x),$$

or also

$$f \to L \quad \text{as} \quad x \to x_0^-.$$

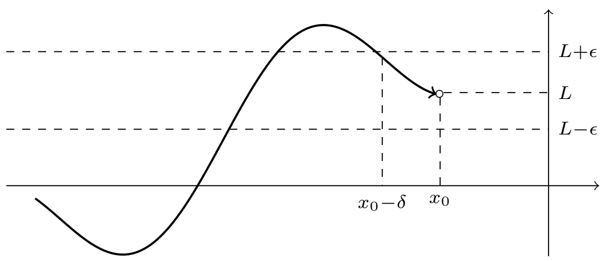


FIGURE 3.26. Limit as $x \to x_0^-$.

DEFINITION 3.3. We say that f *has limit* L *in* x_0 (or f *tends to* L *in* x_0) if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{for} \quad 0 < |x - x_0| < \delta.$$

We write

$$L = \lim_{x \to x_0} f(x),$$

or also

$$f \to L \quad \text{as} \quad x \to x_0.$$

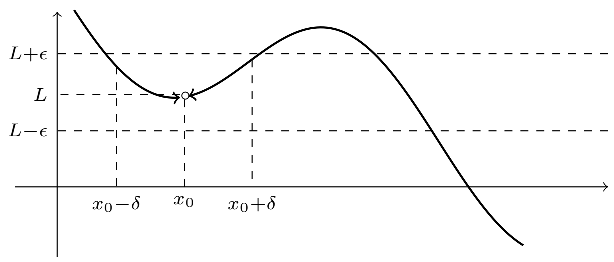


FIGURE 3.27. Limit for $x \to x_0$.

REMARK 3.4. Various cases can occur: a function f can have no limits at a point x_0; it can have a limit from the right but not from the left, and vice versa; or it can have limits both from the right and from the left. In the latter case, the two limits can be the same or different; when they are equal, then also the limit of f in x_0 exists.

In other words: saying that f tends to L in x_0 is the same as saying that f tends to L both from the right and from the left!

REMARK 3.5. Warning: in the previous definitions we are not interested in knowing the value of f at the point x_0 where we calculate the limit; we are only interested in the values $f(x)$ for x close to x_0. Let's see a very simple example: the function

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

has value 0 in all points close to the origin, but in the origin its value is 1. So the right and left limits of $f(x)$ for $x \to 0$ are equal to 0, and therefore also the limit of $f(x)$ in 0 is equal to 0:

$$\lim_{x \to 0} f(x) = 0.$$

However, the value of f in 0 is $f(0) = 1$.

EXAMPLE 3.6. The function sign of x is defined as follows:

$$f(x) = \operatorname{sgn}(x) = \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0. \end{cases}$$

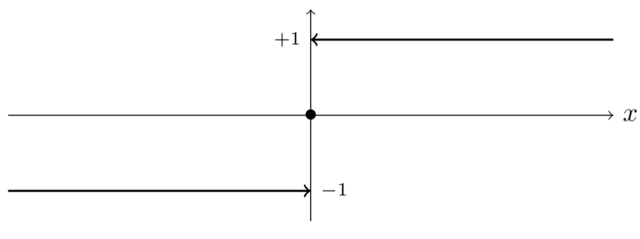


FIGURE 3.28.

Then we have:

$$\lim_{x\to 0^+} f(x) = 1, \qquad \lim_{x\to 0^-} f(x) = -1,$$

therefore the right and left limits in 0 exist but they are different from each other. We conclude that

$$\text{the } \lim_{x\to 0} f(x) \text{ does not exists.}$$

Example 3.7. Let us try to calculate the limit of the function $f(x) = x^2$ at the point $x_0 = 3$ (if it exists!)

$$\lim_{x\to 3} x^2 = L = ?$$

We want to show that the limit exists and is exactly 9. Recalling the definition, we must show that: for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x^2 - 9| < \epsilon \ \text{ for } \ 0 < |x-3| < \delta.$$

Given ϵ, how should we choose δ? First of all, we take it small: if $\delta < 1$ we have

$$|x-3| < \delta \implies |x-3| < 1 \implies 2 < x < 4 \implies |x+3| < 7.$$

But this is still not enough; let's take δ even smaller, for example $\delta = \epsilon/8$; then we can write

$$|x^2 - 9| = |x-3| \cdot |x+3| < \frac{\epsilon}{8} \cdot 7 < \epsilon$$

and now we have succeeded in proving the thesis.

Note that in this case

$$\lim_{x\to 3} f(x) = 9 = f(3)$$

that is, the limit is exactly equal to the value of the function at that point. This is the most common case: for most functions there is no need to do complicated reasoning to calculate the limit at a point, but it is sufficient to calculate the value of the function at that point.

Example 3.8. With the same reasoning just made we verify that in general

$$\lim_{x\to x_0} x^n = x_0^n$$

for each power $n \geq 0$, and indeed

$$\lim_{x\to x_0} P(x) = P(x_0)$$

for every polynomial $P(x)$: for example,

$$\lim_{x\to 2} (x^3 - 6x - 4) = 2^3 - 6\cdot 2 - 4 = -8.$$

A case that does not fall within the previous definitions but is very interesting is that of *vertical asymptotes*: for example the function

$$f(x) = \frac{1}{x}$$

has no limit for $x \to 0$, not from the right neither from the left, as you can immediately guess from its graph that we already know. In these cases we say that the function *tends to infinity* at that point, or that it has *infinite limit*. The symbol used for infinity is ∞. Let's see the precise definition:

Definition 3.9. We say that f *tends to* $+\infty$ *as* $x \to x_0^+$, and we write $\lim_{x\to x_0^+} f(x) = +\infty$, if for every M there exists δ such that

$$f(x) > M \ \text{ for } \ x_0 < x < x_0 + \delta.$$

The definitions of $\lim_{x\to x_0^-} f(x) = +\infty$, $\quad \lim_{x\to x_0} f(x) = +\infty$ are similar (replace with $x_0 - \delta < x < x_0$ and $0 < |x - x_0| < \delta$ respectively).

We say that f *tends to* $-\infty$ when the previous condition holds with $f(x) < M$ in place of $f(x) > M$. In all these cases we say also that the function *has a vertical asymptote in the point* $x = x_0$.

Exercise 3.5. Verify that

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty, \qquad \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

and therefore the function has no limit as $x \to 0$ (the limits from the right and from the left are different).

In a similar way, verify that

$$\lim_{x\to 0^+} \frac{1}{x^2} = \lim_{x\to 0^-} \frac{1}{x^2} = \lim_{x\to 0} \frac{1}{x^2} = +\infty$$

To conclude with the definitions of limit, there is a last case that we have not yet considered: it is often interesting to study the behavior of a function for very large values of x; in some cases the function tends to a value L (*horizontal asymptote*), in others the function becomes very large, and in other cases the behavior is not clear. We give the precise definitions for this situation too:

Definition 3.10. We say that f *tends to* L *as* $x \to +\infty$, and we write $\lim_{x\to+\infty} f(x) = L$, if for every ϵ there exists K such that

$$|f(x) - L| < \epsilon \quad \text{for} \quad x > K.$$

(The definition of $\lim_{x\to-\infty} f(x) = L$ is similar, just replace $x > K$ with $x < K$.) In these cases we say that the function *has an horizontal asymptote* $y = L$.

We say that f *tends to* $+\infty$ *as* $x \to +\infty$, and we write $\lim_{x\to+\infty} f(x) = L$, if for every M there exists K such that

$$f(x) > M \quad \text{for} \quad x > K.$$

(The definition of $\lim_{x\to-\infty} f(x) = +\infty$ is similar, just replace $x > K$ with $x < K$. Also the definitions of $\lim_{x\to+\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$ are similar, just replace $x > M$ with $x < M$.)

Example 3.11. Let us consider some elementary limits which immediately follow from the properties of elementary functions and from the previous definitions. First of all the powers: for $n > 0$ integer we always have

$$\lim_{x\to+\infty} x^n = +\infty$$

while of course

$$\lim_{x\to-\infty} x^n = +\infty \text{ for even } n, \quad \lim_{x\to-\infty} x^n = -\infty \text{ for odd } n.$$

For negative powers the limit is zero:

$$\lim_{x\to+\infty} \frac{1}{x^n} = \lim_{x\to-\infty} \frac{1}{x^n} = 0 \text{ for any integer } n > 0$$

therefore the abscissa axis is a horizontal asymptote. We also note that we always have for $n > 0$ integer

$$\lim_{x\to 0^+} \frac{1}{x^n} = +\infty$$

while it is necessary to distinguish

$$\lim_{x\to 0^-} \frac{1}{x^n} = +\infty \text{ for even } n, \quad \lim_{x\to 0^-} \frac{1}{x^n} = -\infty \text{ for odd } n.$$

What about the powers x^a with a real? The situation is simpler since in this case the function x^a is defined only for $x > 0$. We have only two interesting limits: $x \to +\infty$ and $x \to 0^+$. Clearly, we must distinguish the case $a > 0$, in which the function x^a is increasing:

$$a > 0 \implies \lim_{x\to+\infty} x^a = +\infty, \quad \lim_{x\to 0^+} x^a = 0$$

and the case $a < 0$, in which the function x^a is decreasing:

$$a < 0 \implies \lim_{x\to+\infty} x^a = 0, \quad \lim_{x\to 0^+} x^a = +\infty.$$

(To visualize all these properties just examine the graphs of the elementary functions in the previous paragraph).

The exponential function has a very clear behavior:

$$\lim_{x\to+\infty} e^x = +\infty, \qquad \lim_{x\to-\infty} e^x = 0.$$

Even for the logarithm function, which is defined only for $x > 0$, we have simply

$$\lim_{x\to+\infty} \log x = +\infty, \qquad \lim_{x\to 0^+} \log x = -\infty.$$

The properties of e^x also extend to all a^x with $a > 1$:

$$a > 1 \implies \lim_{x\to+\infty} a^x = +\infty, \qquad \lim_{x\to-\infty} a^x = 0.$$

Instead, the properties are reversed for the decreasing case $0 < a < 1$:

$$0 < a < 1 \implies \lim_{x\to+\infty} a^x = 0, \qquad \lim_{x\to-\infty} a^x = +\infty.$$

For the trigonometric functions the situation is more complicated. For example, if we try to calculate the limit for $x \to +\infty$ of $\sin x$, we immediately discover that the limit *does not exist*: in fact if there were a limit, the values of $f(x) = \sin x$ should get closer and closer to the value L of this limit, while we know that the function continues to oscillate between $+1$ and -1. Similar for $\cos x$, and for the limits as $x \to -\infty$.

One last remark: the $\tan x$ function has vertical asymptotes in the points $\frac{\pi}{2} + k\pi$, for example

$$\lim_{x\to\frac{\pi}{2}^-} \tan x = +\infty, \qquad \lim_{x\to-\frac{\pi}{2}^+} \tan x = -\infty$$

and this behavior is repeated periodically.

Example 3.12. And how are the limits of elementary functions calculated in all other points? For example, what is the limit

$$\lim_{x\to x_0} \sin x$$

at a given point x_0? The answer is very simple:

$$\lim_{x\to x_0} \sin x = \sin x_0.$$

That is, *to calculate the limit of* $\sin x$ *at a point* $x_0 \in \mathbb{R}$ *just calculate the value of* $\sin x$ *at that point.* We will not prove this property; we limit ourselves to observe that the same property holds for all elementary functions, and at all points x_0 where the functions are defined:

$$\lim_{x\to 3} e^x = e^3; \qquad \lim_{x\to 1} \tan x = \tan 1;$$

$$\lim_{x\to -2} \cos x = \cos(-2); \qquad \lim_{x\to 0^+} \sqrt{x} = \sqrt{0} = 0.$$

But of course, the limits

$$\lim_{x\to 0^-} \sqrt{x}, \qquad \lim_{x\to -6} \sqrt{x}$$

cannot be computed because we would leave the domain of the function $\sqrt{x}$.

Problems.

Exercise 3.6 (▶▶|). Of the following functions, only one has a limit on $x \to +\infty$; indicate which one.

$$\sqrt{1-x}, \qquad \sin(x), \qquad \log(1+\cos(x)), \qquad \frac{\sin(x)}{\log(x)}, \qquad \frac{1}{\cos(\log(x))} .$$

4. Properties of limits

To go beyond elementary functions and compute limits of more complicated functions, we need some more properties. First of all we can combine the limits we already know: for example we have

$$\lim_{x\to 3}(e^x + \log x) = e^3 + \log 3$$

that is, if we have to calculate the limit of a sum, just add the limits and so on. More precisely:

Proposition 4.1. *Assume that $f \to L_1$ e $g \to L_2$ as $x \to x_0$ (or $x \to x_0^+$, $x \to x_0^-$, $x \to +\infty$, $x \to -\infty$). Then*

$$f + g \to L_1 + L_2, \quad f \cdot g \to L_1 \cdot L_2, \quad \textit{and} \quad \frac{f}{g} \to \frac{L_1}{L_2} \textit{ se } L_2 \neq 0$$

as $x \to x_0$ (or $x \to x_0^+$, $x \to x_0^-$, $x \to +\infty$, $x \to -\infty$).

Proof. We prove the first property concerning the sum $f + g$: we know that $f \to L_1$ and $g \to L_2$, so for each ϵ we can find δ such that

$$|f(x) - L_1| < \epsilon, \quad |g(x) - L_2| < \epsilon \quad \text{as } 0 < |x - x_0| < \delta$$

and these inequalities can also be written like this:

$$L_1 - \epsilon < f(x) < L_1 + \epsilon, \quad L_2 - \epsilon < g(x) < L_2 + \epsilon$$

By adding the two inequalities we obtain

$$L_1 + L_2 - 2\epsilon < f(x) + g(x) < L_1 + L_2 + 2\epsilon \quad \text{as } 0 < |x - x_0| < \delta$$

and this means exactly $f_1 + f_2 \to L_1 + L_2$ per $x \to x_0$. The other properties can be proved in a similar way. □

Example 4.2. The preceding property immediately implies that for every polynomial $P(x)$ we have

$$\lim_{x\to x_0} P(x) = P(x_0).$$

For example,

$$\lim_{x\to 2}(2x^3 - 3x - 4) = 2\lim_{x\to 2}(x^3) - 3\lim_{x\to 2}(x) - 4 = 6.$$

Furthermore, we can easily compute many limits of functions built from elementary functions:

$$\lim_{x\to 1}\frac{\sin x + e^x}{x^2 - 5} = \frac{\sin 1 - e^1}{1^2 - 5} = \frac{e - \sin 1}{4}.$$

Remark 4.3. The above properties can be applied when L_1 and L_2 are real numbers. But it is easy to verify that many properties also extend to the case of infinite limits. Let's examine the possible cases. We start with the sum:

1) if $f \to +\infty$ and $g \to +\infty$ then $f + g \to +\infty$;
2) if $f \to -\infty$ and $g \to -\infty$ then $f + g \to -\infty$;
3) if $f \to L$ and $g \to \pm\infty$ then $f + g \to \pm\infty$.

We did not consider the case $+\infty - \infty$: in this case a general rule cannot be given because the result may be different depending on the functions f and g, and in these cases we say that

$$+\infty - \infty \quad \text{is an } \textit{indeterminate} \text{ limit}$$

or also that it is an *indeterminate form.* To understand this situation, a trivial example is enough: the following two limits are both of the type $+\infty - \infty$, but the result is very different:

$$\lim_{x\to+\infty}(2x) - \lim_{x\to+\infty} x = \lim_{x\to+\infty} x = +\infty, \qquad \lim_{x\to+\infty} x - \lim_{x\to+\infty}(2x) = -\lim_{x\to+\infty} x = -\infty.$$

For products of functions, we have:
1) if $f \to +\infty$ and $g \to +\infty$ then $f \cdot g \to +\infty$;
2) if $f \to -\infty$ and $g \to -\infty$ then $f \cdot g \to +\infty$;
3) if $f \to +\infty$ and $g \to -\infty$ then $f \cdot g \to + - \infty$;
4) if $f \to L > 0$ and $g \to \pm\infty$ then $f \cdot g \to \pm\infty$;
5) if $f \to L < 0$ and $g \to \pm\infty$ then $f \cdot g \to \mp\infty$.

Also for the product we find that the case

$$\infty \cdot 0 \quad \text{is an } \textit{indeterminate} \text{ limit.}$$

Finally, for the ratio of functions we have
1) if $f \to L$ and $g \to \pm\infty$ then $\frac{f}{g} \to 0$;
2) if $f \to \pm\infty$ and $g \to L > 0$ then $\frac{f}{g} \to \pm\infty$;
3) if $f \to \pm\infty$ and $g \to 0^+$ then $\frac{f}{g} \to \pm\infty$;
4) if $f \to \pm\infty$ and $g \to L < 0$ then $\frac{f}{g} \to \mp\infty$;
5) if $f \to \pm\infty$ and $g \to 0^-$ then $\frac{f}{g} \to \mp\infty$;

in properties 3) and 5) we used the notation

$$g \to 0^+ \iff g \to 0 \ \text{ and } \ g(x) > 0$$

and

$$g \to 0^- \iff g \to 0 \ \text{ and } \ g(x) < 0.$$

In the case of a ratio we note that

$$\frac{\infty}{\infty} \ \text{ and } \ \frac{0}{0} \quad \text{are } \textit{indeterminate} \text{ limits.}$$

The study of indeterminate limits will be resumed in the next chapter (using de l'Hôpital's Theorem).

Another very useful property concerns the composition of functions:

PROPOSITION 4.4. *Assume that $f(x) \neq a$ for every $x \neq x_0$,*

$$\lim_{x\to x_0} f(x) = a, \qquad \lim_{y\to a} g(y) = L.$$

Then, if it is possible to compose the two functions, we have

$$\lim_{x\to x_0} g(f(x)) = \lim_{y\to a} g(y) = L.$$

Analogous properties hold in cases $x \to x_0^{\pm}$, $x \to \pm\infty$, and when a or L are $\pm\infty$.

[Omitted.].

EXAMPLE 4.5. Let us see how the previous property applies: to compute

$$\lim_{x\to 2} \sin(e^x)$$

just set $y = f(x) = e^x$ and observe that

$$y = e^x \to e^2 \quad \text{as} \quad x \to 2$$

therefore
$$\lim_{x\to 2}\sin(e^x) = \lim_{y\to e^2}\sin(y) = \sin(e^2).$$
Another example:
$$\lim_{x\to+\infty}\sin\left(\frac{1}{x}\right) = \lim_{y\to 0}\sin(y) = 0$$
where we have set $y = f(x) = 1/x$.

Before turning to the calculation of limits, we give two more useful properties:

PROPOSITION 4.6 (Theorem of the two Carabinieri). *If $f(x)$ and $h(x)$ tend to the same limit L as $x \to x_0$ and the function $g(x)$ is between them, that is to say*
$$f(x) \leq g(x) \leq h(x),$$
then we have also $g \to L$ as $x \to x_0$. (Analogous properties hold for right, left and infinite limits).

PROOF. From the hypothesis we know that: for every ϵ there exists δ such that
$$L-\epsilon < f(x) < L+\epsilon, \quad L-\epsilon < h(x) < L+\epsilon$$
for $0 < |x - x_0| < \delta$; so we also have
$$L-\epsilon < f(x) \leq g(x) \leq h(x) < L+\epsilon$$
and in particular
$$L-\epsilon < g(x) < L+\epsilon$$
for $0 < |x - x_0| < \delta$, and this is precisely the thesis. □

For example, let us compute the limit of $\sqrt{x^2 + x\sin(x)}$ for $x \to +\infty$. Since we are only interested in the behavior of the function for large values of x it is not restrictive to suppose $x \geq 2$ and therefore,
$$0 \leq x \leq \frac{x^2}{2}, \qquad -\frac{x^2}{2} \leq -x \leq x\sin(x) \leq x \leq \frac{x^2}{2},$$
$$\frac{x}{\sqrt{2}} = \sqrt{\frac{x^2}{2}} \leq \sqrt{x^2 + x\sin(x)} \leq \sqrt{\frac{3x^2}{2}} = \frac{\sqrt{3}}{\sqrt{2}}x, \qquad x \geq 2.$$
The functions $\frac{1}{\sqrt{2}}x$ and $\frac{\sqrt{3}}{\sqrt{2}}x$ both tend to $+\infty$ and applying the Carabeneer's Theorem:
$$\lim_{x\to+\infty}\sqrt{x^2 + x\sin(x)} = +\infty\ .$$

PROPOSITION 4.7 (Permanence of the sign). *If a function f is positive, that is $f(x) \geq 0$, and tends to a limit L for $x \to x_0$, then $L \geq 0$ as well. In other words: the limit of a positive function is positive (and, in a similar way, the limit of a negative function is negative).*

PROOF. Proof by contradiction: if it were $L < 0$, we choose $\epsilon = \frac{|L|}{2}$ and try to apply the definition of limit: it must exists δ such that
$$L-\epsilon < f(x) < L+\epsilon$$
for $0 < |x - x_0| < \delta$. But the second inequality implies that
$$f(x) < L + \epsilon = L + \frac{|L|}{2} < 0$$
and this is absurd because we know that the function is positive. □

5. Calculation of limits

Let us quickly review the fundamental limits that follow immediately from the definitions of the elementary functions. It is better to study them carefully by referring to the graphs seen previously. For powers with a positive exponent we have

$$\lim_{x\to+\infty} x^2 = \lim_{x\to+\infty} x^3 = \lim_{x\to+\infty} x^n = +\infty, \qquad n = 1, 2, 3, \dots$$

and of course also

$$\lim_{x\to+\infty} \sqrt{x} = \lim_{x\to+\infty} \sqrt[3]{x} = \lim_{x\to+\infty} \sqrt[n]{x} = +\infty, \qquad n = 1, 2, 3, \dots$$

Also for even powers

$$\lim_{x\to-\infty} x^2 = \lim_{x\to-\infty} x^4 = \lim_{x\to-\infty} x^6 = \cdots = +\infty,$$

while for the odd powers

$$\lim_{x\to-\infty} x = \lim_{x\to-\infty} x^3 = \lim_{x\to-\infty} x^5 = \cdots = -\infty.$$

We pass to the negative powers: we have immediately for every n

$$\lim_{x\to\pm\infty} x^{-n} = 0$$

and also

$$\lim_{x\to 0_+} x^{-n} = +\infty.$$

Moreover, for even powers

$$\lim_{x\to 0^-} x^{-2} = \lim_{x\to 0^-} x^{-4} = \cdots = +\infty,$$

while for the odd powers

$$\lim_{x\to 0^-} x^{-3} = \lim_{x\to 0^-} x^{-5} = \cdots = -\infty.$$

For the exponential we have

$$\lim_{x\to+\infty} e^x = +\infty, \qquad \lim_{x\to-\infty} e^x = 0.$$

This is the same behavior as for all exponential functions with a base greater than one:

$$a > 1 \implies \lim_{x\to+\infty} a^x = +\infty, \qquad \lim_{x\to-\infty} a^x = 0$$

while if the base is less than 1 the behavior is reversed:

$$0 < a < 1 \implies \lim_{x\to+\infty} a^x = 0, \qquad \lim_{x\to-\infty} a^x = +\infty.$$

The function $\log x$ is defined only for $x > 0$ and we have

$$\lim_{x\to+\infty} \log x = +\infty, \qquad \lim_{x\to 0^+} \log x = -\infty.$$

Finally, two trigonometric functions: the $\tan x$ function has vertical asymptotes for $x = \pm\frac{\pi}{2}$ (and by periodicity, also for all $x = k\pi + \frac{\pi}{2}$) :

$$\lim_{x\to\frac{\pi}{2}^-} \tan x = +\infty, \qquad \lim_{x\to\frac{\pi}{2}^+} \tan x = -\infty$$

and in a similar way

$$\lim_{x\to-\frac{\pi}{2}^+} \tan x = -\infty, \qquad \lim_{x\to-\frac{\pi}{2}^-} \tan x = \infty.$$

For the inverse function $\arctan x$ we have

$$\lim_{x\to+\infty} \arctan x = \frac{\pi}{2}, \qquad \lim_{x\to-\infty} \arctan x = -\frac{\pi}{2}.$$

Remark 5.1. Warning: the limits of a function do not always exist! For example, the limit

$$\lim_{x\to+\infty} \sin x$$

does not exist: the $\sin x$ function continues to oscillate between $+1$ and -1 as x increases, without approaching any value L (if it tended towards a limit L, the function would have a horizontal asymptote). Similarly *the following limits do not exist*:

$$\lim_{x\to+\infty} \cos x, \qquad \lim_{x\to-\infty} \sin x, \qquad \lim_{x\to-\infty} \cos x.$$

Example 5.2. Many limits can calculated immediately using the rules seen so far. For example, let us try to calculate the limits

$$\lim_{x\to 3} \frac{2}{x^3+2}, \qquad \lim_{x\to+\infty} \frac{2}{x^3+2}, \qquad \lim_{x\to-\infty} \frac{2}{x^3+2}.$$

For the first one, thanks to the proposition on operations between limits, it is sufficient to calculate the value of the function at the point:

$$\lim_{x\to 3} \frac{2}{x^3+2} = \frac{2}{3^3+2} = \frac{2}{29}.$$

For the second it is enough to observe that the denominator

$$\lim_{x\to+\infty} (x^3+2) = +\infty$$

tends to $+\infty$, and therefore we have immediately

$$\lim_{x\to+\infty} \frac{2}{x^3+2} = 0.$$

Finally, given that

$$\lim_{x\to-\infty} (x^3+2) = -\infty$$

we also have

$$\lim_{x\to-\infty} \frac{2}{x^3+2} = 0.$$

Example 5.3. Similarly, to calculate

$$\lim_{x\to+\infty} \sin\left(\frac{1}{x}\right)$$

it is sufficient to set $y = \frac{1}{x}$ and to note that

$$x \to +\infty \implies y = \frac{1}{x} \to 0^+$$

where $y \to 0^+$ means: $y \to 0$ from the right, that is to say $y > 0$. Then the limit is trasformed into

$$\lim_{x\to+\infty} \sin\left(\frac{1}{x}\right) = \lim_{y\to 0^+} \sin y = 0.$$

Example 5.4. Given the polynomial

$$f(x) = -4x^3 + 2x^2 + 5$$

calculate the limits

$$\lim_{x\to 2} f(x), \qquad \lim_{x\to -1} f(x), \qquad \lim_{x\to+\infty} f(x), \qquad \lim_{x\to-\infty} f(x).$$

The first two are very simple: the limit is equal to the value of f at the point where the limit is taken. In fact, using the known results for limit operations,

$$\lim_{x\to 2}(-4x^3+2x^2+5) = -4\cdot\lim_{x\to 2} x^3 + 2\cdot\lim_{x\to 2} x^2 + 5 = -4\cdot 2^3 + 2\cdot 2^2 + 5 = 19.$$

Similarly

$$\lim_{x\to -1}(-4x^3+2x^2+5) = -4\cdot(-1)^3 + 2\cdot(-1)^2 + 5 = 11.$$

The third limit is indeterminate, in fact $-4x^3 \to -\infty$ while $2x^2 \to +\infty$; but we can rewrite the function as follows:

$$\lim_{x\to+\infty}(-4x^3+2x^2+5) = \lim_{x\to+\infty} x^3\left(-4+\frac{2}{x}+\frac{5}{x^3}\right)$$

and now we see that

$$x^3 \to +\infty, \qquad \left(-4+\frac{2}{x}+\frac{5}{x^3}\right) \to -4$$

and therefore for the known rules

$$\lim_{x\to+\infty}(-4x^3+2x^2+5) = \lim_{x\to+\infty} x^3\left(-4+\frac{2}{x}+\frac{5}{x^3}\right) = -\infty$$

(we got a form like $(+\infty)\cdot(-4)$ which is no longer indeterminate!). The last limit is calculated in the same way:

$$\lim_{x\to-\infty}(-4x^3+2x^2+5) = \lim_{x\to-\infty} x^3\left(-4+\frac{2}{x}+\frac{5}{x^3}\right) = +\infty$$

(here we get a form like $(-\infty)\cdot(-4)$).

Example 5.5. We calculate the limit of the ratio of two polynomials:

$$\lim_{x\to+\infty}\frac{2x^3-6x^2+x-1}{x^3+2x^2+5}.$$

Note that both the numerator and denominator tend to $+\infty$, so we have an indeterminate form. But the two infinites are "of the same order", so the limit is finite: to see this we factor out the maximum power of x in the numerator and denominator and simplify:

$$\frac{2x^3-6x^2+x-1}{-3x^3+2x^2+5} = \frac{x^3\left(2-\frac{6}{x}+\frac{1}{x^2}-\frac{1}{x^3}\right)}{x^3\left(-3+\frac{2}{x}+\frac{5}{x^3}\right)} = \frac{2-\frac{6}{x}+\frac{1}{x^2}-\frac{1}{x^3}}{-3+\frac{2}{x}+\frac{5}{x^3}} \to -\frac{2}{3}$$

as $x \to +\infty$.

If, on the other hand, the two polynomials have different degrees, the relationship will be dominated by the higher degree. For example:

$$\lim_{x\to+\infty}\frac{x^4+x^7}{3x^5-x^6} = \lim_{x\to+\infty}\frac{x^7\left(\frac{1}{x^3}+1\right)}{x^6\left(\frac{3}{x}-1\right)} = \lim_{x\to+\infty} x\cdot\frac{\frac{1}{x^3}+1}{\frac{3}{x}-1} = -\infty$$

because the numerator has a higher degree. As a rule of thumb we can write:

$$\frac{x^4+x^7}{3x^5-x^6} \sim \frac{x^7}{-x^6} = -x \quad \text{as} \quad x \to \pm\infty$$

so that

$$\lim_{x\to+\infty}\frac{x^4+x^7}{3x^5-x^6} = -\infty, \qquad \lim_{x\to-\infty}\frac{x^4+x^7}{3x^5-x^6} = +\infty.$$

Similarly, we can say that

$$\frac{-3x^4+5x^2}{8x^5+x^2} \sim \frac{-3x^4}{8x^5} = -\frac{3}{8x} \quad \text{as} \quad x \to \pm\infty$$

that is to say

$$\lim_{x\to+\infty}\frac{-3x^4+5x^2}{8x^5+x^2} = \lim_{x\to-\infty}\frac{-3x^4+5x^2}{8x^5+x^2} = 0.$$

Example 5.6. Warning: the previous rule applies for $x \to \pm\infty$ (the highest degree dominates when x is large!).

In the other points the limit is calculated according to the usual rules. To calculate the limit of a polynomial ratio for $x \to x_0$, if the denominator does not vanish in x_0 just calculate the function at $x = x_0$. If, on the other hand, the denominator vanishes in x_0 (and the numerator does not vanish), the ratio tends to infinity, and it is enough to understand the sign of the expression to establish if the limit is $+\infty$ o $-\infty$. For example:

$$\lim_{x\to 2^+} \frac{1}{x-2} = +\infty, \qquad \lim_{x\to 2^-} \frac{1}{x-2} = -\infty, \qquad \lim_{x\to 7} \frac{1}{x-2} = \frac{1}{5}.$$

$$\lim_{x\to 3^+} \frac{1-2x}{x^2-9} = +\infty, \qquad \lim_{x\to 3^-} \frac{1-2x}{x^2-9} = -\infty, \qquad \lim_{x\to 0} \frac{1-2x}{x^2-9} = -\frac{1}{9}.$$

If both the numerator and the denominator vanish at x_0, we have an indeterminate form $\frac{0}{0}$; if we know the roots of the polnomials we can simplify the fraction, otherwise we can apply the Theorem of de l'Hôpital which we will study in the next chapter.

Example 5.7. One of the most useful methods is the *change of variables* method, that is, the application of Proposition 4.4. For example the limit

$$\lim_{x\to+\infty} e^{2x}$$

setting $y = 2x$ and observing that

$$x \to +\infty \iff y \to +\infty$$

becomes

$$\lim_{x\to+\infty} e^{2x} = \lim_{y\to+\infty} e^{y} = +\infty.$$

Similarly:

$$\lim_{x\to+\infty} e^{-5x} : \quad y = -5x \to -\infty \implies \lim_{x\to+\infty} e^{-5x} = \lim_{y\to-\infty} e^{y} = 0$$

$$\lim_{x\to 0^+} e^{\frac{1}{x}} : \quad y = \frac{1}{x} \to +\infty \implies \lim_{x\to 0^+} e^{\frac{1}{x}} = \lim_{y\to+\infty} e^{y} = +\infty$$

$$\lim_{x\to 0^-} e^{\frac{1}{x}} : \quad y = \frac{1}{x} \to -\infty \implies \lim_{x\to 0^-} e^{\frac{1}{x}} = \lim_{y\to-\infty} e^{y} = 0$$

$$\lim_{x\to 0^-} e^{-\frac{3}{x}} : \quad y = -\frac{3}{x} \to +\infty \implies \lim_{x\to 0^-} e^{-\frac{3}{x}} = \lim_{y\to+\infty} e^{y} = +\infty$$

A few more examples concerning powers:

$$\lim_{x\to+\infty} (1/2)^x = 0$$

because the base $\frac{1}{2}$ is less than 1. Similarly

$$\lim_{x\to+\infty} (2^x - 3^x) = \lim_{x\to+\infty} 3^x \left(\left(\frac{2}{3}\right)^x - 1 \right) = -\infty$$

since $(2/3)^x \to 0$.

Example 5.8. The ratio

$$\frac{e^x}{x}$$

as $x \to +\infty$ is an indeterminate form of the type $\dfrac{\infty}{\infty}$. But we can say that

$$\lim_{x\to+\infty} \frac{e^x}{x} = +\infty\,.$$

More generally, for any positive number $a > 0$ we have

$$\lim_{x\to+\infty} \frac{e^{ax}}{x} = +\infty$$

and it is not difficult to prove this fact: it is enough to remember the inequality $e^x \geq 1 + x$ and write

$$e^{ax} = \left(e^{\frac{ax}{2}}\right)^2 \geq \left(1 + \frac{ax}{2}\right)^2$$

and therefore

$$\frac{e^{ax}}{x} \geq \frac{1}{x}\left(1 + \frac{ax}{2}\right)^2 = \frac{1}{x} + a + \frac{a^2 x}{4}$$

from which we get right away

$$\lim_{x \to +\infty} \frac{e^{ax}}{x} \geq \lim_{x \to +\infty} \left(\frac{1}{x} + a + \frac{a^2 x}{4}\right) = +\infty$$

(there is an even simpler method based on de l'Hôpital's Theorem, which we will study in the next chapter). We observe that this limit can be interpreted like this: when $x \to +\infty$,

e^x tends to $+\infty$ *much faster* than x.

The same property holds in general when we have the ratio of an exponential to a power (and x is large, i.e. $x \to \pm\infty$); the dominant quantity is always the exponential:

$$\lim_{x \to +\infty} \frac{e^x}{x^3} = \lim_{x \to +\infty} \frac{e^x}{\sqrt{x}} = \lim_{x \to +\infty} \frac{e^x}{x^{100000}} = +\infty.$$

In fact, for every $b > 0$ we can write

$$\lim_{x \to +\infty} \frac{e^x}{x^b} = \lim_{x \to +\infty} \left(\frac{e^{x/b}}{x}\right)^b = +\infty^b = +\infty\ .$$

Note that instead

$$\lim_{x \to -\infty} \frac{e^x}{x} = 0$$

is not an indeterminate form!

If we take the reciprocal of the previous functions we get

$$\lim_{x \to +\infty} \frac{x}{e^x} = 0, \qquad \lim_{x \to +\infty} \frac{\sqrt{x}}{e^x} = 0, \qquad \lim_{x \to +\infty} \frac{x^{100000}}{e^x} = 0.$$

The following limit is not indeterminate:

$$\lim_{x \to +\infty} xe^x = +\infty.$$

On the other side,

$$\lim_{x \to -\infty} xe^x$$

Is indeterminate of the form $\infty \cdot 0$. We can solve this immediately by changing the variable $y = -x$ noting that $y \to +\infty$:

$$\lim_{x \to -\infty} xe^x = \lim_{y \to +\infty} -ye^{-y} = -\lim_{y \to +\infty} \frac{y}{e^y} = 0.$$

The following limit is similar

$$\lim_{x \to +\infty} \frac{\log x}{x}.$$

Using the change of variables

$$y = \log x \implies y \to +\infty$$

(remember that $y = \log x \iff x = e^y$) we get

$$\lim_{x \to +\infty} \frac{\log x}{x} = \lim_{y \to +\infty} \frac{y}{e^y} = 0.$$

Also this limit can be interpreted as follows:

x tends to $+\infty$ *much faster* than $\log x$.

As before, we get in a similar way the limits

$$\lim_{x\to+\infty} \frac{x}{\log x} = +\infty, \qquad \lim_{x\to+\infty} \frac{\log x}{\sqrt{x}} = 0.$$

Another limit similar to the previous ones:

$$\lim_{x\to 0^+} x\log x.$$

If we set

$$y = \log x \implies y \to -\infty$$

and we observe that $e^y = x$, we have immediately

$$\lim_{x\to 0^+} x\log x = \lim_{y\to-\infty} ye^y = 0$$

(already proved).

Some limits to conclude:

$$\lim_{x\to+\infty}(e^x + x) = \lim_{x\to+\infty} e^x\left(1 + \frac{x}{e^x}\right) = +\infty$$

$$\lim_{x\to-\infty}(e^x + x) = -\infty$$

$$\lim_{x\to+\infty}(e^x - x) = \lim_{x\to+\infty} e^x\left(1 - \frac{x}{e^x}\right) = +\infty$$

$$\lim_{x\to-\infty}(e^x - x) = +\infty$$

$$\lim_{x\to+\infty} \frac{e^x + x}{e^x - x} = \lim_{x\to+\infty} \frac{e^x\left(1 + \frac{x}{e^x}\right)}{e^x\left(1 - \frac{x}{e^x}\right)} = \lim_{x\to+\infty} \frac{1 + \frac{x}{e^x}}{1 - \frac{x}{e^x}} = 1.$$

Example 5.9. Of course, not all limits can be calculated with the previous methods; in many cases ad hoc arguments are needed. For example the limit

$$\lim_{x\to+\infty} \frac{\sin x}{x}$$

exists and is equal to zero even if it is not included in the methods seen so far. The simplest way to prove this fact is to use the Theorem of the two Carabinieri: we observe that the values of the sine function are always between $+1$ and -1, thus for $x > 0$ we can write

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x};$$

but the first and third functions tend to zero for $x \to +\infty$, so the Theorem of the two Carabinieri implies that

$$\lim_{x\to+\infty} \frac{\sin x}{x} = 0.$$

Example 5.10. The rationalization of algebraic expressions can sometimes be useful in calculating limits, as in the following two examples:

$$\lim_{x\to 0} \frac{\sqrt{1+x} - 1}{x}, \qquad \lim_{x\to 0} \frac{x}{\sqrt{4-x} - 2}.$$

In the first case, multiplying the numerator and denominator by $\sqrt{1+x} + 1$ we obtain

$$\lim_{x\to 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x\to 0} \frac{1 + x - 1}{x(\sqrt{1+x} + 1)} = \lim_{x\to 0} \frac{1}{\sqrt{1+x} + 1} = \frac{1}{2}\,.$$

In the second case, multiplying the numerator and denominator by $\sqrt{4-x} + 2$ we obtain

$$\lim_{x\to 0} \frac{x}{\sqrt{4-x} - 2} = \lim_{x\to 0} \frac{x(\sqrt{4-x} + 2)}{-x} = -4\,.$$

Problems.

EXERCISE 3.7 (▶▶|). Calculate the following limits:

$$\lim_{x\to\pm\infty}\frac{x^4-3x^2}{8x^3+x^2},\qquad \lim_{x\to\pm\infty}\frac{2x^4+x^5}{3x^5-x^2},\qquad \lim_{x\to\pm\infty}\frac{x^4-x^5}{x^4-x^6}.$$

EXERCISE 3.8. Calculate the following limits:

$$\lim_{x\to\pm\infty} \text{ and } \lim_{x\to 2^\pm} \text{ of } \frac{2+5x}{2-x}$$

$$\lim_{x\to\pm\infty} \text{ and } \lim_{x\to 1^\pm} \text{ of } \frac{2x+3}{x^4-1}$$

$$\lim_{x\to\pm\infty} \text{ and } \lim_{x\to 3^\pm} \text{ of } \frac{x^4-3x^8}{x^5(2x^3-18x)}$$

$$\lim_{x\to\pm\infty} \text{ and } \lim_{x\to 2^\pm} \text{ of } \frac{4}{x^3-x^2-4}$$

$$\lim_{x\to\pm\infty} \text{ and } \lim_{x\to 1^\pm} \text{ of } \frac{(x-1)(x+5)(2x+5)}{(2x-2)(7-3x)}$$

$$\lim_{x\to\pm\infty} \text{ and } \lim_{x\to -1^\pm} \text{ of } \frac{-2x^4+x^2+3}{7x^3+x^6+6}$$

$$\lim_{x\to\pm\infty} \text{ and } \lim_{x\to 0^\pm} \text{ of } \frac{x^3+6x^2}{5x^2+x^3}.$$

EXERCISE 3.9. Using the procedure in Example 5.10 to calculate the following limits:

$$\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x},\qquad \lim_{x\to 0}\frac{x}{\sqrt{2+x}-\sqrt{2}},$$

$$\lim_{x\to 4}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2},\qquad \lim_{x\to 0}\frac{\sqrt{1+x+x^2}-1}{x},\qquad \lim_{x\to 0}\frac{x}{\sqrt{2+x}-\sqrt{2}}.$$

EXERCISE 3.10. Calculate the limits of the following functions for $x\to\pm\infty$, $x\to 0^\pm$ (if possibile!):

$$xe^{-x},\quad \frac{e^{x/2}}{x},\quad e^{-x^2},\quad e^{-\frac{1}{x^2}},\quad xe^{\frac{1}{x}},\quad xe^{-\frac{1}{x}},$$

$$\frac{e^x-1}{x},\quad \frac{e^x+x^2}{x^3-e^{-x}},\quad \frac{e^{\sqrt{x}}}{x^5},\quad \frac{e^{-x}}{x},\quad \frac{1}{x}e^{\frac{1}{x}},$$

$$\frac{e^x}{x^2},\quad \frac{e^{\sqrt{x}}}{\sqrt{x}},\quad \sqrt{e^x},\quad \log(2x)-\log x,\quad e^{-x}-x^3$$

$$x^x,\quad e^x-e^{3x},\quad x^2e^x-xe^{3x},\quad x2^x-e^x,$$

$$x^2\log\sqrt{x},\quad x\log(x^2),\quad e^{-\frac{1}{x^3}},\quad \log\frac{1}{x},\quad \frac{\log(e^x)}{e^x},$$

$$\frac{1}{1+e^{1/x}},\quad \frac{1}{x}e^{-\frac{1}{x}},\quad \log(e^x-1),\quad \log(e^{-x}-1),\quad x^2\log x,$$

$$\sqrt{x}\log x,\quad \sqrt{x}\log(x^{10}),\quad \frac{\sqrt{x}}{\log x},\quad \sqrt{\log(x^2)}$$

Exercise 3.11. For the following functions, determine the set of definition, which is always a union of intervals, and then compute the limits at the ends of these intervals:

$$\log(x^2-x),\qquad |x|,\qquad \frac{x}{2x-3},\qquad \log\frac{1}{2-x},\qquad e^{\frac{1}{x}},\qquad xe^{-\frac{1}{x}},$$

$$e^{\frac{x}{x+1}},\qquad e^{\frac{1}{x-2}},\qquad \frac{e^x}{e^x-1},\qquad \sqrt{x+1},\qquad \sqrt{\frac{1}{1-x}},$$

$$\sqrt{\frac{x}{x+1}},\qquad \sqrt{\frac{x+1}{x-1}},\qquad \sqrt{1-x^2},\qquad \frac{1}{\sqrt{1-x^2}},$$

$$\frac{1}{\sqrt{2+x^2}},\qquad \frac{x^2}{2-x},\qquad \frac{x+1}{x^2-1},\qquad \frac{e^x}{x-2},\qquad e^{2x^2-x^3}$$

$$\frac{x+1}{e^x},\qquad \frac{x}{e^{\frac{2}{x}}},\qquad \frac{x^2}{e^{\frac{1}{x}}},\qquad \log(1-e^x),\qquad \log(8-2x^2),$$

$$x\log\frac{1}{x},\qquad x\cdot\log\frac{1}{x-2},\qquad \frac{e^x+1}{e^x-1},\qquad \frac{e^{-x}-1}{e^x},\qquad \frac{e^{2x}}{e^x+1},$$

$$e^{2x-x^2},\qquad \left(1+\frac{1}{x}\right)e^x,\qquad \frac{1}{e^{x-x^2}},\qquad \log(2x^2+x),$$

$$(x+1)\log(x+1),\qquad \log(1+\sqrt{x-1}),\qquad \log\left(\frac{2}{2x+3}\right),\qquad \frac{1}{\log x},$$

$$e^{\frac{1}{x}}+e^{-\frac{1}{x}},\qquad e^{\frac{1+x}{1-x}},\qquad \frac{\log x}{\sqrt{x}},\qquad (x-1)e^{\frac{1}{x}},\qquad \frac{1}{|x|}.$$

Exercise 3.12 (⏭). Using the Theorem of the two Carabinieri, calculate the following limits:

$$\lim_{x\to+\infty}\frac{x+\sin x}{e^x},\qquad \lim_{x\to-\infty}\frac{e^{\sin x}}{x^2},\qquad \lim_{x\to 0} x\cdot\sin\left(\frac{1}{x}\right).$$

6. Special limits

Let us now turn to some limits that require special attention: these are the limits

$$\lim_{x\to 0}\frac{\sin x}{x}=1\qquad \text{e}\qquad \lim_{x\to 0}\frac{e^x-1}{x}=1\,.$$

Conside the first of the two limits. Remember the inequality

$$0\le\sin x\le x\le\tan x,\qquad \text{for}\ \ 0\le x<\frac{\pi}{2}\,.$$

From the inequalities

$$0\le\sin x\le x$$

dividing by $x>0$ we get

$$0\le\frac{\sin x}{x}\le 1.$$

Moreover, from the inequality

$$x\le\tan x=\frac{\sin x}{\cos x},$$

dividing by $x>0$ and multiplying by $\cos x$ (note that $\cos x>0$ in the considered area) we obtain

$$\cos x\le\frac{\sin x}{x}$$

and then by writing it all together we proved that

$$\cos x\le\frac{\sin x}{x}\le 1\qquad \text{for}\ \ 0<x<\frac{\pi}{2}.$$

By the change of variable $x \to -x$, since $\cos(-x) = \cos x$ and $\frac{\sin(-x)}{-x} = \frac{\sin x}{x}$, we obtain that the previous inequality is also true for $-\frac{\pi}{2} \le x < 0$. Therefore

$$\cos x \le \frac{\sin x}{x} \le 1 \qquad \text{for } \; 0 < x \le \frac{\pi}{2} \; \text{ and for } \; -\frac{\pi}{2} < x < 0.$$

But $\cos x \to 1$ for $x \to 0$, therefore applying the Theorem of the two Carabinieri we obtain

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

We study now the second limit: we want to show that

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

Recall that the exponential e^x verifies the property

$$e^x \ge 1 + x \qquad \text{for every } x.$$

Hence, putting $-x$ in place of x we obtain without effort that

$$e^{-x} \ge 1 - x \qquad \text{for every } x.$$

If $x < 1$, then multiplying both sides by the positive function $\dfrac{e^x}{1-x}$ gives

$$\frac{1}{1-x} \ge e^x \qquad \text{for every } x < 1$$

and then, by subtracting 1

$$\frac{x}{1-x} = \frac{1}{1-x} - 1 \ge e^x - 1 \ge x \qquad \text{for every } x < 1.$$

If $x > 0$, dividing both members by x we get

$$\frac{1}{1-x} \ge \frac{e^x - 1}{x} \ge 1 \qquad \text{for } \; 1 > x > 0.$$

If instead we divide by $x < 0$ we get

$$\frac{1}{1-x} \le \frac{e^x - 1}{x} \le 1 \qquad \text{per } \; x < 0.$$

So by the Theorem of the two Carabinieri we have

$$\lim_{x\to 0^+} \frac{e^x - 1}{x} = 1, \qquad \lim_{x\to 0^-} \frac{e^x - 1}{x} = 1$$

and this is equivalent to saying that

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

Example 6.1. Calculate the limit

$$\lim_{x\to +\infty} x \cdot \sin\left(\frac{1}{x}\right).$$

It is sufficient to set $y = 1/x$ and note that

$$x \to +\infty \implies y = \frac{1}{x} \to 0$$

to obtain

$$\lim_{x\to +\infty} x \cdot \sin\left(\frac{1}{x}\right) = \lim_{y\to 0} \frac{\sin y}{y} = 1.$$

Example 6.2. Calculate the limit

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}.$$

The trick is to multiply the numerator and denominator by $1+\cos x$; we obtain

$$\lim_{x\to 0}\frac{1-(\cos x)^2}{x^2(1+\cos x)}=\lim_{x\to 0}\frac{(\sin x)^2}{x^2(1+\cos x)}=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2\lim_{x\to 0}\frac{1}{1+\cos x}=1^2\frac{1}{2}=\frac{1}{2}.$$

Example 6.3. Calculate the limits

$$\lim_{x\to 0}\frac{e^{-x}-1}{x},\qquad \lim_{x\to 0}\frac{e^{2x}-1}{x}.$$

The first limit, with the substitution $y=-x$ becomes

$$\lim_{x\to 0}\frac{e^{-x}-1}{x}=\lim_{y\to 0}\frac{e^{y}-1}{-y}=-\lim_{y\to 0}\frac{e^{y}-1}{y}=-1.$$

The second limit, with the substitution $y=2x$ becomes

$$\lim_{x\to 0}\frac{e^{2x}-1}{x}=\lim_{y\to 0}\frac{e^{y}-1}{y/2}=2\lim_{y\to 0}\frac{e^{y}-1}{y}=2.$$

The same procedure shows that for any real number a we have

$$\lim_{x\to 0}\frac{e^{ax}-1}{x}=a\,.$$

Example 6.4. Calculate the limits

$$\lim_{x\to 0}\frac{e^{x}+x-1}{x},\qquad \lim_{x\to 0}\frac{e^{x}-e^{-x}}{x}\,.$$

The first limit can be written as

$$\lim_{x\to 0}\frac{e^{x}+x-1}{x}=\lim_{x\to 0}\frac{e^{x}-1}{x}+1=1+1=2\,.$$

In the second limit, we add and subtract 1 to the numerator,

$$\frac{e^{x}-e^{-x}}{x}=\frac{e^{x}-1+1-e^{-x}}{x}=\frac{e^{x}-1}{x}-\frac{e^{-x}-1}{x}$$

and we get

$$\lim_{x\to 0}\frac{e^{x}-e^{-x}}{x}=\lim_{x\to 0}\frac{e^{x}-1}{x}-\lim_{x\to 0}\frac{e^{-x}-1}{x}=1-(-1)=2.$$

Problems.

Exercise 3.13 (⏭). Calculate the limits

$$\lim_{x\to 0}\frac{e^{\pi x}-1}{x},\qquad \lim_{x\to 0}\frac{2^{x}-1}{x},\qquad \lim_{x\to 0}\frac{\sin(ax)}{x},\quad a\in\mathbb{R}.$$

Exercise 3.14. Calculate the limits of the following functions for $x\to 0$.

$$\frac{e^{x}-1-x}{x},\qquad \frac{e^{x}+e^{-x}-2}{x},\qquad \frac{e^{x}-1}{1-e^{-x}},\qquad \frac{\sin x}{x\cos x},\qquad \frac{\sin^2 x}{x^2},\qquad \frac{\sin(x^2)}{\sin^2 x}.$$

Exercise 3.15. Using the suggested substitutions and the special limits, calculate the following limits:

$$\lim_{x\to 0}\frac{\log(1+x)}{x}\quad (x=e^{y}-1);\qquad \lim_{x\to\pm\infty}x\log\left(1+\frac{1}{x}\right)\quad (x=\frac{1}{y});$$

$$\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^{x}\text{(take logarithms)};\qquad \lim_{x\to\pm\infty}\left(1+\frac{a}{x}\right)^{x}\quad (x=ay);$$

$$\lim_{x\to 2}\frac{x^2+3x-10}{2x^2-5x+2}\quad (x=y+2);\qquad \lim_{x\to\pi}\frac{\sin^2(x)}{1+\cos(x)}\quad (y=x+\pi);$$

Exercise 3.16. Calculate the following limits:

$$\lim_{x\to-\infty}\frac{x^2-3x}{x+2\sqrt{-x}},\qquad \lim_{x\to+\infty}\frac{x^{12}+x^4}{e^x};\qquad \lim_{x\to+\infty}2^{-e^x};$$

$$\lim_{x\to+\infty}\frac{x^x}{e^x};\qquad \lim_{x\to0^+}\frac{\sqrt{1-\cos(x)}}{x};\qquad \lim_{x\to0}\frac{x^3}{\sin^2(x)};$$

7. Continuous functions

From the intuitive point of view, a function $f\colon I\to\mathbb{R}$, defined on an interval I, is continuous if it is possible to draw the graph without removing the pencil from the sheet or, if you prefer, the chalk from the blackboard. The rigorous notion of continuity is given by the following definition.

Definition 7.1. Let $f\colon I\to\mathbb{R}$ be a function defined on an interval I and let x_0 be a point of I. The function is said to *continuous at* x_0 if the f limit for $x\to x_0$ exists and is equal to the value of the function at that point:

$$\lim_{x\to x_0}f(x)=f(x_0).$$

The function is said to be *continuous on* I if it is continuous at all points of I.

Example 7.2. In the previous sections we noted that all elementary functions verify the property just defined: therefore the elementary functions are all continuous on their domain.

Furthermore, the sum, the product and the ratio (if the denominator is $\neq 0$) of continuous functions are still continuous functions: this follows immediately from Proposition 4.1. For example, to see that $f+g$ is continuous just type

$$\lim_{x\to x_0}(f(x)+g(x))=\lim_{x\to x_0}f(x)+\lim_{x\to x_0}g(x)=f(x_0)+g(x_0).$$

If instead we use the Proposition 4.4 we obtain that the composition of continuous functions is also continuous: more precisely, if $f(x)$ is continuous in x_0, $g(y)$ is continuous in $y_0=f(x_0)$, and it is possible to compose the two functions, then setting $y=f(x)$ we obtain

$$\lim_{x\to x_0}g(f(x))=\lim_{y\to y_0}g(y)=g(y_0)=g(f(x_0))$$

that is, $g(f(x))$ is continuous in x_0.

Example 7.3. The functions

$$\sin(x^2-e^x+2x),\quad \log(\sqrt{x}),\quad e^{\sin(x^2)+x-3},\quad xe^{-\frac{1}{x}}$$

are continuous on their set of definition, since they are obtained through the sum, product, ratio and composition of continuous functions.

Example 7.4. An example of a function that **is not continuous** is given by the integer part $\lfloor x\rfloor$ of a real number, defined as the largest integer less than or equal to x. For example,

$$\lfloor 0\rfloor=0,\quad \lfloor 1\rfloor=1,\quad \lfloor 0,2\rfloor=0,\quad \lfloor -0,2\rfloor=-1,\quad \lfloor\pi\rfloor=3\quad \lfloor-\pi\rfloor=-4\ .$$

To verify that it is not continuous, it is sufficient to note that

$$\lim_{x\to1^-}\lfloor x\rfloor=0\neq\lfloor 1\rfloor\ .$$

We now study some very important properties of continuous functions:

Proposition 7.5 (Permanence of the sign). *Let $f\colon I\to\mathbb{R}$ be a function on the open interval I and let x_0 be a point of I. Suppose that f is continuous at x_0 and that $f(x_0)>0$. Then $f(x)>0$ for x close to x_0: that is, there exists $\delta>0$ such that $f(x)>0$ for $|x-x_0|<\delta$. (Similar property if $f(x_0)<0$).*

Proof. We apply the definition of limit to the function f: since f is continuous in x_0, its limit in x_0 is precisely $L = f(x_0)$, so from the definition of limit we know that for every ϵ there exists δ such that

$$f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon$$

for $0 < |x - x_0| < \delta$. If we now choose

$$\epsilon = \frac{f(x_0)}{2}$$

(note that $\epsilon > 0$), we get

$$\frac{f(x_0)}{2} < f(x) < f(x_0) + \frac{f(x_0)}{2}$$

for $0 < |x - x_0| < \delta$; the first of these inequalities tells us that f is strictly positive. □

Theorem 7.6 (Theorem of zeroes of continuous functions). *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Suppose that $f(a) \leq 0$ e $f(b) \geq 0$. Then there exists a point c between a and b in which the function f vanishes: $f(c) = 0$. The same result holds if $f(a) \geq 0$ and $f(b) \leq 0$.*

Explanation. The rigorous proof is quite complicated and therefore we omit it; we give instead an intuitive explanation which is very clear. The plane $\mathbb{R}^2$ is divided by the abscissa axis into two half–planes, the upper one $\{(x,y) : y > 0\}$ and the lower one $\{(x,y) : y < 0\}$. Now try to draw the graph of f: you have to connect the point $(a, f(a))$, located in the lower half plane, with the point $(b, f(b))$, located in the upper half plane. Since f is continuous you cannot lift the pencil from the paper and at some point you have to cross the abscissa axis. □

Example 7.7. We use the theorem of zeroes to prove that the polynomial of third degree $x^3 + x - 1$ has a root in the open interval $]0,1[$. To this end we observe that the continuous function

$$f\colon [0,1] \to \mathbb{R}, \qquad f(x) = x^3 + x - 1,$$

satisfies the assumptions of Theorem 7.6 since $f(0) = -1$ e $f(1) = 1$. Therefore, there exists a point c between 0 and 1 where the f function is equal to zero.

Theorem 7.8 (Weierstrass' Theorem). *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function defined on a closed bounded interval. Then there exist two points x_1 and $x_2 \in [a,b]$ such that, calling m the value of f in x_1 and M the value of f into x_2, we have*

$$f(x_1) = m \leq f(x) \leq M = f(x_2) \quad \textit{for every} \quad x \in [a,b]\ .$$

The proof requires the introduction of a long series of preliminary results on the properties of real numbers that go beyond the objectives of these notes and therefore is omitted.

The two points x_1 and x_2 found in the previous theorem are called the *point of absolute minimum* and *point of absolute maximum* of f on $[a,b]$ respectively; while m and M are called the *absolute minimum value* and *absolute maximum value* of f on $[a,b]$ respectively.

From Weierstrass' Theorem we know that a continuous function on a closed and bounded interval always has a minimum m and a maximum M on the interval. Now we show that the function also assumes all intermediate values between m and M:

Theorem 7.9 (Intermediate value theorem). *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Let m be the minimum and M the maximum of f on $[a,b]$. Also, let μ be an intermediate number between m and M:*

$$m \leq \mu \leq M.$$

Then there exists a point c between a and b in which the value of the function f is exactly μ, that is $f(c) = \mu$.

PROOF. Weierstrass' Theorem ensures that the function admits a minimum point where $f(x_1) = m$ and a maximum point where $f(x_2) = M$. Thus, if $\mu = m$ or $\mu = M$ there is nothing to prove.

If instead $m < \ mu < M$, we consider the function

$$g(x) = f(x) - \mu.$$

This function is continuous; moreover one has

$$g(x_1) = f(x_1) - \mu = m - \mu < 0, \qquad g(x_2) = f(x_2) - \mu = M - \mu > 0,$$

and therefore by the Theorem of zeros there exists a point c between x_1 and x_2 in which g vanishes:

$$g(c) = f(c) - \mu = 0$$

and from here the result follows immediately. □

Problems.

EXERCISE 3.17. Say which of the following functions satisfy the assumptions of the theorem of zeroes of continuous functions on the interval $[-1, 1]$.

$$x^2 - x - 1, \quad \frac{1}{x}, \quad x^2 + x + 1, \quad e^x - \frac{1}{2}, \quad \log(x^2), \quad \sqrt{2}\sin(x) - 1 \; .$$

8. Sequences and series

8.1. Sequences. A *sequence* is simply a function $a : \mathbb{N} \to \mathbb{R}$. Then the values of a should be written, using the usual notation, $a(1)$, $a(2)$, ..., $a(n)$, ... In practice it is much more convenient to write simply

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

and we will do so in the following. When using this notation, n is called the *index* of the sequence. Also the set of values $(a_n) = (a_q, a_2, a_3, \ldots)$ will be called a *sequence.* Pay attention to the fact that the order of the elements a_n of the sequence is important, and if you exchange some (or infinite) a_n, you get a different sequence.

Let's see some simple examples:

1) *The sequence of even numbers*: $a_n = 2n$. Writing it explicitly one has

$$(a_n) = (2, 4, 6, 8, 10, \ldots).$$

2) *The sequence of squares:* $a_n = n^2$. We have

$$(a_n) = (1, 4, 9, 16, 25, 36, \ldots).$$

3) *The sequence of the inverse of the integers*: $a_n = \frac{1}{n}$. We have

$$(a_n) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right).$$

4) *The sequence of powers of 2*: $a_n = 2^n$. We have

$$(a_n) = (2, 4, 8, 16, 32, 64, \ldots).$$

5) *The constant sequence which is always 0*: $a_n = 0$. We have

$$(a_n) = (0, 0, 0, 0, 0, \ldots).$$

Of course we can consider other constant sequences: the sequence that is always 1 i.e. $(a_n) = (1, 1, 1, 1, \ldots)$, the sequence that is always π i.e. $(a_n) = (\pi, \pi, \pi, \ldots)$, and so on.

6) *The oscillating sequence*: $a_n = (-1)^n$, that is

$$(a_n) = (-1, +1, -1, +1, -1, +1, -1, +1, \ldots).$$

7) *The sequence of factorials*: $a_n = n!$. This symbol simply indicates the product

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$$

for $n \geq 1$. Conventionally, we set $0! = 1$. It is easy to calculate the first values of this sequence:

$$(a_n) = (1, 2, 6, 24, 120, 720, 5040, 40320, 362880, \dots)$$

and we see that the factorials $n!$ grow very fast as n increases.

DEFINITION 8.1. A sequence (a_n) is said to be *bounded above* if there exists a number M such that $a_n \leq M$ for all indices n. We say that the sequence is *bounded below* if there exists a number m such that $m \leq a_n$ for all indexes n. We say that the sequence is *bounded* if it is bounded both above and below.

A sequence (a_n) is said to be *increasing* if for all the indices n we have $a_n \leq a_{n+1}$; that is, if you have $a_1 \leq a_2 \leq a_3 \leq \dots$.

A sequence (a_n) is said to be *decreasing* if for all indexes n we have $a_n \geq a_{n+1}$; that is, if you have $a_1 \geq a_2 \geq a_3 \geq \dots$.

Increasing and decreasing sequences are also called *monotone sequences*.

For example we see immediately that:

- The sequence $a_n = 2n$ of even numbers satisfies $a_n \geq 2$ for all indices, so it is bounded below by $m = 2$, but is not bounded above and therefore is not bounded; moreover it is an increasing sequence.
- The sequence of squares, that of powers of 2 and that of factorials are bounded below (what is m?) but not above and therefore they are not bounded. They are all increasing.
- The sequence $a_n = \frac{1}{n}$ is bounded above by $M = 1$ and is also bounded below by $m = 0$: in fact, for each index n we have $0 < \frac{1}{n} \leq 1$. It follows that the sequence is bounded. Furthermore, the sequence is decreasing because $\frac{1}{n} \geq \frac{1}{n+1}$ for each n.
- The constant sequence $a_n = 0$ for each n is bounded both above and below (we can take $m = M = 0$) and therefore is bounded. Also, it is both increasing and decreasing! We note that if a sequence is is both increasing and decreasing, it must be a constant sequence: in fact by definition we have $a_n \leq a_{n+1}$ and also $a_n \geq a_{n+1}$ for all n, therefore $a_n = a_{n+1}$ for all n.
- The oscillating sequence $a_n = (-1)^n$ satisfies $-1 \leq a_n \leq +1$ for every n, so it is bounded. It is not increasing nor decreasing.

A term that is used very frequently when talking about sequences is *eventually*: we say that the values of a sequence (a_n) verify a property *eventually* if the property is true starting from a certain index (that is, it may not be true for the first few terms, but starting from a certain index onwards it is always true). For example, the sequence $a_n = 2^n$ is eventually greater than 10: in fact we have $a_1 = 2 < 10$, $a_2 = 4 < 10$, $a_3 = 8 < 10$ and therefore the first three terms are less than 10, but starting from a_4 onwards all the terms of the sequence are greater than 10: $a_4 = 16 > 10$, $a_5 = 32 > 10$ and so on.

The following is the fundamental definition of the theory:

DEFINITION 8.2. A sequence (a_n) is said to be *convergent* if there exists a real number L such that, for every $\epsilon > 0$, the sequence is eventually between $L - \epsilon$ and $L + \epsilon$. We also say that a_n *converges to* L or *tends to* L, and L is called the *limit* of the sequence. In this case we write $\lim_{n\to+\infty} a_n = L$ or simply $a_n \to L$.

So "a_n converges to L" means: for every $\epsilon > 0$ there exists an index n_ϵ starting from which we have

$$L - \epsilon < a_n < L + \epsilon, \qquad n \geq n_\epsilon.$$

For example, we verify that the sequence $a_n = \frac{1}{n}$ tends to 0: indeed, fixed $\epsilon > 0$ just take any index $n_\epsilon > \frac{1}{\epsilon}$ and then for each $ngen_\epsilon$ we have $\frac{1}{n} \le \frac{1}{n_\epsilon} < \frac{1}{\epsilon}$, so that

$$0 - \epsilon < \frac{1}{n} < 0 + \frac{1}{\epsilon} \quad \text{for every} \quad n \ge n_\epsilon.$$

We see that we can take $L = 0$ in the definition and get that

$$\lim_{n\to+\infty} \frac{1}{n} = 0.$$

The constant sequence $a_n = 0$ also converges to 0. In general, if a sequence is constant with $a_n = c$, then it converges to the limit c. (Easy verification: exercise!).

Instead, the oscillating sequence $a_n = (-1)^n$ does not converge to any limit. Indeed, if we had $a_n \to L$, choosing $\epsilon = \frac{1}{2}$ we would have

$$L - \frac{1}{2} < a_n < L + \frac{1}{2} \quad \text{for} \quad n \ge n_\epsilon$$

But then it should be true that

$$a_{n+1} - a_n < \left(L + \frac{1}{2}\right) - \left(L - \frac{1}{2}\right) = 1 \quad \text{for} \quad n \ge n_\epsilon$$

while for all odd n we have $a_{n+1} - a_n = +1 - (-1) = 2$.

As for the functions, the concept of limits $\pm\infty$ is also defined for the sequences:

Definition 8.3. We say that a sequence a_n *tends to* $+\infty$, or that it *diverges to* $+\infty$, if for each number M the sequence is eventually greater than M. We say that a_n *tends to* $-\infty$, or that it *diverges to* $-\infty$, if for each number m the sequence is eventually less than m. In these cases we write $\lim_{n\to+\infty} a_n = \pm\infty$ or also simply $a_n \to \pm\infty$.

For example, the three sequences $a_n = 2n$, $a_n = n^2$, $a_n = 2^n$ and $a_n = n!$ tend to $+\infty$.

Remark 8.4 (Reduction to limits of functions). A very effective method for calculating the limits of sequences is given by the following observation: if there exists a function $f(x)$ such that $f(n) = a_n$ for all indices n , and we know how to calculate the limit of $f(x)$ for $x \to +\infty$, then we have

$$\lim_{n\to+\infty} a_n = \lim_{x\to+\infty} f(x).$$

In fact, the definition for the limit of a sequence is a special case of that for the limit of a function. Of course, the same property is valid in the case of limts of the type $\pm\infty$.

Note also that given any sequence a_n, we can always find a function $f : [1, +\infty) \to \mathbb{R}$ with the above property by defining $f(x) = a_n$ for $n \le x < n + 1$, $n = 1, 2, 3, \ldots$.

The previous observation allows us to immediately deduce from the theorems on limits of functions similar results on the limits of sequences:

Theorem 8.5. *Suppose $a_n \to L$ and $b_n \to M$. Then $a_n + b_n \to L + M$, $a_n \cdot b_n \to L \cdot M$ and (if divisions can be done) $a_n/b_n \to L/M$.*

Theorem 8.6. *Suppose that $a_n \to L$, $b_n \to L$ and that you have $a_n \le c_n \le b_n$ eventually. Then we also have $c_n \to L$.*

Remark 8.7. Note that if a sequence converges, its limit is *unique*. In fact, suppose we have $a_n \to L$ and also $a_n \to M$. Then $a_n - a_n \to L - L'$, but $a_n - a_n = 0$ so that $L - L' = 0$.

Recall that given a nonempty set $A \subseteq \mathbb{R}$ and a number M, the number M is said to be a *majorant* of A if for all $x \in A$ we have $x \le M$; similarly, a *minorant* of A is a number m such that $m \le x$ for every $x \in A$. Then, the *supremum* of A is the smallest of its majorants, while the *infimum* of A is the largest of its minorants. These two numbers are indicated respectively with

$$\sup A \quad \text{and} \quad \inf A.$$

If the set A has no majorant (i.e. it is not bounded above) we say that $\sup A = +\infty$; similarly, if it has no minorant (i.e. it is not bounded below) we say that $\inf A = -\infty$.

In particular, for a sequence (a_n), we denote by $\sup a_n$ and $\inf a_n$ respectively the supremum and the infimum of the set of all the values of the sequence.

We can state (without proof) a very useful result on monotone sequences:

THEOREM 8.8. *Let (a_n) be an increasing sequence. Then we have only two possibilities:*

(1) the sequence is not bounded above. In this case, $a_n \to +\infty$.
(2) the sequence is bounded above. In this case, $a_n \to L$ where $L = \sup a_n$.

A similar result holds if (a_n) is decreasing: either the sequence is not bounded below, in which case $a_n \to -\infty$, or it is bounded below, in which case $a_n \to L$ where $L = \inf a_n$.

Remark 8.4 allows us to effortlessly compute many limits of sequences, using the known properties of the limits of functions. Let us see some examples.

(1) $a_n = c^n$, where c is a fixed number. If we set $f(x) = c^x$, we have $a_n = f(n)$ and remembering the properties of exponential functions we have immediately: $c^n \to +\infty$ if $c > 1$, $c^n \to 0$ if $-1 < c < 1$, $c^n \to 1$ if $c = 1$. On the other hand, if $c = -1$ we obtain the oscillating sequence $a_n = (-1)^n$ which does not converge, and of course it does not converge also in the case$c < -1$.
(2) $a_n = n^p$, where p is a fixed number. Putting $f(x) = x^p$ we have $a_n = f(n)$. This implies right away that $n^p \to +\infty$ if $p > 0$, $n^0 \to 1$ because in this case the sequence is constant, and $n^p \to 0$ if $p < 0$.
(3) If a_n is given by the ratio of two polynomials, we can apply the same calculation rules found in the case of a real variable. For instance,
$$\lim_{n\to+\infty} \frac{3n^2-2n-5}{7n^2+n-2} = \lim_{x\to+\infty} \frac{3x^2-2x-5}{7x^2+x-2} = \frac{3}{7}.$$
(4) Other notable examples:
$$\lim_{n\to+\infty} \frac{e^n}{n} = \lim_{x\to+\infty} \frac{e^x}{x} = +\infty,$$
$$\lim_{n\to+\infty} \left(\frac{n}{e^n} - \frac{n}{n^2+1} + 2\right) = \lim_{x\to+\infty} \left(\frac{x}{e^x} - \frac{x}{x^2+1} + 2\right) = 0 - 0 + 2 = 2.$$

8.2. Series. A *series* is an "infinite sum" of real numbers
$$\sum_{j=1}^{\infty} a_j = a_1 + \cdots + a_n + \ldots.$$
Sometimes it is convenient to start indexes from $j = 0$, or from another integer number.

It should be clear to everyone that this "'definition" has some issues. For example, what is the meaning of the following infinite sum
$$+1 - 1 + 1 - 1 + 1 - 1 + \ldots$$
whose values are alternatively $+1$ and 0? To define the sum of a series in a meaningful way, we proceed as follows: we call *N–th partial sum* of the series the number
$$s_N = \sum_{j=1}^{N} a_j.$$
Then we say that

- the series $\sum a_j$ *converges*, if the sequence of the partial sums s_N converges to a (finite) number , which is thencalled the *sum* of the series
- the series *diverges to* $+\infty$ (or to $-\infty$) if $s_N \to +\infty$ (or $s_N \to -\infty$)
- the series *does not converge* if the sequence s_N does not converge
- finally, we say that the series *converges absolutely* if $\sum |a_j|$ converges.

A particularly simple case to deal with is that of series with *positive terms*: this means simply that $\sum a_j$ satisfies $a_j \geq 0$ for each j. Indeed, in this case s_N is an increasing sequence. Recalling Theorem 8.8, we see that we have only two possibilities:

(1) either s_N is bounded above by a constant M, so that the series converges with sum $\leq M$

(2) or s_N is not bounded above, so that the series diverges to $+\infty$.

Let us consider some examples.

(1) $a_j = 0$ for every j. Clearly $s_N = 0$ for every N and therefore the series converges with sum 0.

(2) $a_j = 1$ for all j. Since $s_N = N \to +\infty$, we see that this series diverges to $+\infty$.

(3) $a_j = (-1)^{j+1}$ which produces the series $+1-1+1-1+\cdots+$ considered above. In this case the values of s_N oscillate between $+1$ and 0, clearly the partial sums do not converge, and therefore we can say that the series $\sum(-1)^{j+1}$ *does not converge* (and does not diverge)

(4) $a_j = q^j$, where $q \neq 1$ is a given number and j starts from 0. This is called the *geometric series* of *ratio* q. In this case we can can write

$$s_N = 1 + q + \cdots + q^N = \frac{1-q^{N+1}}{1-q}.$$

To verify this, just multiply both members by $1-q$ and simplify. We see immediately that if $-1 < q < 1$ the power q^{N+1} tends to 0, and therefore the geometric series converges with sum $\frac{1}{1-q}$; if $q \geq 1$ the series diverges to $+\infty$; and finally if $q \leq -1$ the series has an oscillating behavior and does not converge nor diverges.

There exist many criteria that can be used to prove convergence or divergence of a series, especially in the case of a series with positive terms.

Proposition 8.9. *Let $\sum a_j$, $\sum b_j$ be two series with positive terms, such that $a_j \geq b_j$ for $j > j_0$. Then we have:*

(i) if $\sum a_j$ converges then $\sum b_j$ converges

(ii) if $\sum b_j$ diverges then $\sum a_j$ diverges too (obviously, to $+\infty$).

Proof. Suppose that $\sum a_j$ converges. Let s_N, r_N be the partial sums of these series, and let $A = a_1 + \cdots + a_{j_0}$ and $B = b_1 + \cdots + b_{j_0}$. Then for $j > j_0$ we have

$$s_N = A + a_{j_0+1} + \cdots + a_N \geq A - B + B + b_{j_0+1} + \cdots + b_N = A - B + r_N.$$

Since s_N converges, it follows that s_N is bounded above, thus r_N lis also bounded above and we conclude that the second series converges. The proof of (ii) is similar. □

We give now a slightly more sophisticated criterion:

Proposition 8.10. *Let $\sum a_j$ be a series with the properties $a_j > 0$ and $\frac{a_{j+1}}{a_j} \to c$ as $j \to \infty$.*

(i) If $c < 1$ then the series converges

(ii) If $c > 1$ then the series diverges to $+\infty$.

Proof. Suppose $c < 1$ and fiz $\epsilon > 0$ such that $q = c + \epsilon < 1$. Since $\frac{a_{j+1}}{a_j} \to c$, we have $0 \leq \frac{a_{j+1}}{a_j} \leq q$ for j larger than some index j_ϵ, and therefore

$$a_{j+1} \leq q a_j \quad \text{for} \quad j \geq j_\epsilon.$$

This implies

$$a_{j+1} \leq a_j q \leq a_{j-1} q^2 \leq \cdots \leq a_{j_\epsilon} q^{j-j_\epsilon} = C q^j \quad \text{where} \quad C = a_{j_\epsilon} q^{-j_\epsilon}.$$

The series $\sum C q^j$ converges since $q < 1$, and applying the previous criterion we obtain (i). The proof of (ii) is analogous. □

A similar criterion, which is proved with a similar argument, is the following:

PROPOSITION 8.11. *Let* $\sum a_j$ *be a series with the properties* $a_j \geq 0$ *and* $\sqrt[j]{a_j} \to c$ *as* $j \to \infty$.

(i) If $c < 1$ *then the series converges*

(ii) If $c > 1$ *then the series diverges to* $+\infty$.

CHAPTER 4

Derivation

1. The derivative

The concept of derivative is very important and very natural. To have a concrete example, think of the motion of a car: if $f(t)$ is the function that expresses how far you have traveled up to a certain time t, then the speed at which you are riding is precisely the derivative of f (i.e. the speedometer shows the derivative of the odometer...).
In other words, the derivative of a function expresses the speed at which that function increases or decreases as the point x changes. Let us study the precise definition of this concept.

Definition 1.1. Let $f\colon I \to \mathbb{R}$ be a function on the open interval I and let x_0 be a point of I. For any $h \neq 0$ small enough, the *incremental ratio* of f at x_0 is the ratio

$$\frac{f(x_0+h)-f(x_0)}{h}.$$

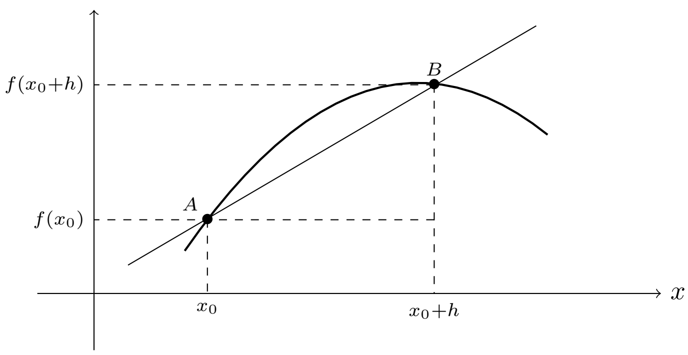


Figure 4.1. The incremental ratio.

The incremental ratio has a precise geometric meaning: it expresses the slope (= the angular coefficient) of the straight line passing through the points $A=(x_0,f(x_0))$ and $B=(x_0+h,f(x_0+h))$. If we now consider ever smaller values of h, it is not difficult to guess that the line AB gets closer and closer to the tangent line to the graph of f at the point A. The strict definition is very simple:

Definition 1.2. f is said to be *differentiable* in x_0 if the limit of the incremental ratio as $h \to 0$ exists and is finite:

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = f'(x_0).$$

This limit is called the *derivative* of f in x_0 and is denoted by $f'(x_0)$. f is said to be *differentiable* if it is differentiable at all points of its domain; the function $f'(x)$ is also called the *derivative (function)* of f.

To say that the limit exists and is finite means in particular that $f'(x_0)$ is a real number: in other words, if

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \pm\infty\,,$$

then the function f is *not* differentiable in x_0.

REMARK 1.3. The geometric meaning of the derivative is clear: when we let h tend to zero, and therefore we move the point B along the curve to the point A, the line AB gets closer and closer to the tangent line to the graph of f in A. Hence it is natural to interpret the value of $f'(x_0)$ as the slope of the tangent line to the graph at the point A. If we want to completely determine the tangent line, just observe that it must have the form $y = ax + b$; we have already noted that $a = f'(x_0)$, moreover the line must pass through $A = (x_0, f(x_0))$ and therefore

$$f(x_0) = f'(x_0)x_0 + b \implies b = f(x_0) - f'(x_0)x_0$$

and in conclusion the equation of the tangent line at the point A is

$$y = f'(x_0)(x - x_0) + f(x_0).$$

EXAMPLE 1.4. Let us study some cases in which the computation of the derivative is immediate.

If $f(x) = C$ is a constant function, the incremental ratio

$$\frac{f(x_0+h)-f(x_0)}{h} = \frac{CC}{h} \equiv 0$$

is always zero, so the limit is also zero; in conclusion, a constant function is differentiable and its derivative is the null function:

$$f(x) \equiv C = cost. \implies f'(x) \equiv 0 \quad \text{(ossia } C' = 0\text{)}.$$

If $f(x) = ax + b$, we have immediately

$$\frac{f(x_0+h)-f(x_0)}{h} = \frac{a(x_0+h)+b-(ax_0+b)}{h} \equiv a$$

and therefore the derivative of $f(x) = ax+b$ is constant and equal to the angular coefficient a:

$$(ax + b)' \equiv a.$$

If $f(x) = x^2$, then

$$\frac{f(x_0+h)-f(x_0)}{h} = \frac{(x_0+h)^2-(x_0)^2}{h} = 2x_0 + h$$

and sending h to zero you get

$$f'(x_0) = 2x_0$$

that is simply

$$(x^2)' = 2x.$$

If $f(x) = x^3$, then

$$\frac{f(x_0+h)-f(x_0)}{h} = \frac{(x_0+h)^3-(x_0)^3}{h} = 3x_0^2 + 3x_0h + h^2$$

and taking the limit as h tends to zero we obtain

$$f'(x_0) = 3x_0^2, \qquad \text{so that} \qquad (x^3)' = 3x^2\,.$$

With a similar calculation we obtain that for every integer $n \geq 1$

$$(x^n)' = nx^{n-1}.$$

Example 1.5. We compute the derivative of the function $f(x) = \sqrt{x}$ at a point $x_0 > 0$:

$$\begin{aligned} f'(x_0) &= \lim_{h\to 0} \frac{\sqrt{x_0+h} - \sqrt{x_0}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x_0+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x_0+h} + \sqrt{x_0}}{\sqrt{x_0+h} + \sqrt{x_0}} \\ &= \lim_{h\to 0} \frac{x_0+h-x_0}{h(\sqrt{x_0+h} + \sqrt{x_0})} = \frac{1}{2\sqrt{x_0}} \, . \end{aligned}$$

We can therefore write simply $(\sqrt{x})' = \dfrac{1}{2\sqrt{x}}$.

Example 1.6. It is not difficult to calculate the derivative of elementary functions. For example, consider the exponential function $f(x) = e^x$. We have

$$\frac{f(x_0+h) - f(x_0)}{h} = \frac{e^{x_0+h} - e^{x_0}}{h} = \frac{e^h - 1}{h} e^{x_0}$$

and so if we send h to zero we get immediately

$$f'(x_0) = e^{x_0}$$

that is, we obtain the simple rule

$$(e^x)' = e^x.$$

Let us now consider $f(x) = \sin x$; the incremental ratio is equal to

$$\frac{f(x_0+h) - f(x_0)}{h} = \frac{\sin(x_0+h) - \sin(x_0)}{h}$$

and recalling the addition formula

$$\sin(x_0+h) = \sin(x_0)\cos h + \cos(x_0)\sin h$$

the ratio can be written

$$\frac{f(x_0+h) - f(x_0)}{h} = \frac{\cos h - 1}{h}\sin(x_0) + \frac{\sin h}{h}\cos(x_0).$$

We want to calculate the limit as $h \to 0$. We know that $\lim_{h\to 0} \frac{\sin h}{h} = 1$; and we can use the formula

$$1 - \cos h = 2\left(\sin \frac{h}{2}\right)^2$$

to obtain

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = -\lim_{h\to 0} \frac{2}{h}\left(\sin\frac{h}{2}\right)^2 = -\lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} \sin\frac{h}{2} = 0.$$

In conclusion, the incremental ratio tends to $\cos(x_0)$ and we obtain the rule

$$(\sin x)' = \cos x.$$

In a similarly way we prove that

$$(\cos x)' = -\sin x.$$

Proposition 1.7. *If f is differentiable in x_0 then f is continuous in x_0.*

Proof. It it sufficient to start from the following identity:

$$f(x_0+h) = h \cdot \frac{f(x_0+h) - f(x_0)}{h} + f(x_0).$$

If we compute the limit for $h \to 0$ we get then

$$\lim_{h\to 0} f(x_0+h) = 0 \cdot f'(x_0) + f(x_0) = f(x_0)$$

and setting $x = x_0 + h$ we obtain

$$\lim_{x \to x_0} f(x) = f(x_0)$$

that is, f is continuous in x_0. □

EXAMPLE 1.8. Not all continuous functions are differentiable! for example the function $f(x) = |x|$ is not differentiable at the point $x_0 = 0$. To verify this, we write the incremental ratio in $x_0 = 0$:

$$\frac{f(x_0 + h) - f(x_0)}{h} = \frac{|h| - |0|}{h} = \frac{|h|}{h}.$$

Clearly, the limit as $h \to 0$ of this ratio does not exist: in fact the limit from the right is equal to

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1,$$

while the limit from the left is equal to

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1.$$

Note, however, that $|x|$ is differentiable both on the interval $x > 0$, where $|x| = x$ and therefore the derivative is equal to $+1$, and on the interval $x < 0$, where $|x| = -x$ and therefore the derivative is equal to -1.

Another example of a continuous but non differentiable function is the function defined as $g(x) = \dfrac{x}{\sqrt{|x|}}$ for $x \neq 0$, and $g(0) = 0$. We leave it to the reader the simple verification that

$$\lim_{x \to 0} g(x) = g(0), \qquad \lim_{h \to 0} \frac{g(0 + h) - g(0)}{h} = +\infty \ .$$

The following properties will enable us to differentiate all functions obtained as combinations of elementary functions:

PROPOSITION 1.9 (Rules of differentiation). *Let f and g be two differentiable functions. Then the following rules of differentiation hold:*

$$(f + g)' = f' + g', \qquad (fg)' = f'g + fg'$$

and, where g is non-zero,

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}, \qquad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

PROOF. Let us check only the first rule (the others can be proved in a similar way, albeit with some longer computations): the incremental ratio of $f + g$ at a point x_0 is equal to

$$\frac{f(x_0 + h) + g(x_0 + h) - f(x_0) - g(x_0)}{h} = \frac{f(x_0 + h) - f(x_0)}{h} + \frac{g(x_0 + h) - g(x_0)}{h}$$

and calculating the limit as $h \to 0$ we get immediately

$$\lim_{h \to 0} \frac{f(x_0 + h) + g(x_0 + h) - f(x_0) - g(x_0)}{h} = f'(x_0) + g'(x_0).$$

□

EXAMPLE 1.10. Compute the derivative of the functions $\frac{1}{x}, \frac{1}{x^2}, \ldots, \frac{1}{x^n}, \ldots$, with $n > 0$. Applying the previous rules we have

$$\left(\frac{1}{x}\right)' = -\frac{x'}{x^2} = -\frac{1}{x^2}, \qquad \left(\frac{1}{x^2}\right)' = -\frac{(x^2)'}{x^4} = \frac{-2}{x^3}, \qquad \left(\frac{1}{x^n}\right)' = -\frac{(x^n)'}{x^{2n}} = \frac{-n}{x^{n+1}}.$$

Example 1.11. The functions $\dfrac{e^x - e^{-x}}{2}$ and $\dfrac{e^x + e^{-x}}{2}$ are respectively called hyperbolic sine and hyperbolic cosine and are each derivative of the other: in fact we have

$$(e^{-x})' = \left(\frac{1}{e^x}\right)' = \frac{-e^x}{(e^x)^2} = -e^{-x},$$

so that

$$\left(\frac{e^x - e^{-x}}{2}\right)' = \frac{(e^x)' - (e^{-x})'}{2} = \frac{e^x + e^{-x}}{2}\ .$$
$$\left(\frac{e^x + e^{-x}}{2}\right)' = \frac{(e^x)' + (e^{-x})'}{2} = \frac{e^x - e^{-x}}{2}\ .$$

Proposition 1.12 (Derivative of the inverse function). *Let f be a differentiable and injective function, and let g be its inverse function. Then g is also differentiable and we have the differentiation rule*

$$g'(y) = \frac{1}{f'(x)}, \quad \textit{where we have set} \quad y = f(x).$$

[Omitted.]

Proposition 1.13 (Derivative of the composite function). *If f and g are differentiable, and it is possible to compose them, then also the composite function $h(x) = g(f(x))$ is differentiable and the following rule holds:*

$$h'(x) = g'(f(x)) \cdot f'(x).$$

[Omitted.]

Example 1.14. We are now able to differentiate almost any function. We study a few important examples:

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cdot \cos x - \sin x \cdot (\cos x)'}{(\cos x)^2} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

using the rule for the derivative of a ratio. Note that the previous rule can also be written

$$(\tan x)' = \frac{1}{\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \tan^2 x + 1.$$

To differentiate $\log x$ we can use the rule for the derivative of an inverse function, since $g(y) = \log y$ is the inverse of $y = e^x$: setting $y = e^x$, we have

$$(\log y)' = \frac{1}{(e^x)'} = \frac{1}{e^x} \equiv \frac{1}{y}$$

which can be written more comfortably

$$(\log x)' = \frac{1}{x}.$$

With the same method, the function $\arctan y$, inverse of $y = \tan x$, has for derivative

$$(\arctan y)' = \frac{1}{(\tan x)'} == \frac{1}{\tan^2 x + 1} \equiv \frac{1}{y^2 + 1}$$

and then we obtain the rule

$$(\arctan x)' = \frac{1}{x^2 + 1}.$$

Also the function $\arcsin y$, which is the inverse of $y = \sin x$, has a very simple derivative:

$$(\arcsin y)' = \frac{1}{(\sin x)'} = \frac{1}{\cos x} = \frac{1}{\sqrt{1 - \sin^2 x}} \equiv \frac{1}{\sqrt{1 - y^2}},$$

therefore the rule is

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}.$$

The rule

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

is proved in the same way.

Finally, let us deal with powers: to differentiate the function $f(x) = x^a$, $x > 0$ with a arbitrary real number, it is sufficient to write

$$f(x) = x^a = (e^{\log x})^a = e^{a \log x}$$

and then apply the rule for the derivative of a composite function:

$$f'(x) = (x^a)' = (e^{a\log x})' = e^{a\log x} \cdot (a \log x)' = e^{a \log x}\frac{a}{x} = x^a \frac{a}{x},$$

that is, after simplifying,

$$(x^a)' = ax^{a-1}$$

exactly as in the case of integer powers. The case of square roots is particularly interesting; in fact it can be written $\sqrt{x} = x^{\frac{1}{2}}$ so that

$$(\sqrt{x})' = (x^{\frac{1}{2}})' = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}\ .$$

On the other hand, for the exponential function $f(x) = a^x$, $a > 0$, we can write

$$f'(x) = (a^x)' = (e^{x\log a})' = e^{x \log a}(x \log a)' = e^{x\log a}\log a$$

and we obtain the rule

$$(a^x)' = a^x \log a.$$

As a further application of the derivation rule for the composite function we calculate the derivative of functions of the type $e^{f(x)}$, $\log(f(x))$ and $\sqrt{f(x)}$: in the first case, $e^{f(x)}$ is composite of e^y and $y = f(x)$ and applying Proposition 1.13 we obtain

$$(e^{f(x)})' = e^{f(x)}f'(x),$$

while in the second and third cases we get

$$(\log(f(x))' = \frac{f'(x)}{f(x)}, \qquad \sqrt{f(x)}' = \frac{f'(x)}{2\sqrt{f(x)}}\ .$$

For example, we have:

$$(e^{x^2})' = e^{x^2}(x^2)' = 2xe^{x^2}, \qquad (e^{\sin(x)})' = e^{\sin(x)}(\sin(x)') = \cos(x)e^{\sin(x)},$$

$$\sqrt{x+1}' = \frac{1}{2\sqrt{x+1}}, \qquad \log(\cos(x))' = \frac{\cos(x)'}{\cos(x)} = \frac{-\sin(x)}{\cos(x)} = -\tan(x)\ ,$$

$$\log(x^2+x)' = \frac{(x^2+x)'}{x^2+x} = \frac{2x+1}{x^2+x}, \qquad \sqrt{\log(x)}' = \frac{\log(x)'}{2\sqrt{\log(x)}} = \frac{1}{2x\sqrt{\log(x)}}\ .$$

Example 1.15.

$$\sqrt{x^2-1}' = \frac{2x}{2\sqrt{x^2-1}} = \frac{x}{\sqrt{x^2-1}}, \qquad \sqrt{x^2+1}' = \frac{x}{\sqrt{x^2+1}}\ ,$$

$$\log(x+\sqrt{x^2-1})' = \frac{1+\dfrac{2x}{2\sqrt{x^2-1}}}{x+\sqrt{x^2-1}} = \frac{\dfrac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}} = \frac{1}{\sqrt{x^2-1}}\ ,$$

$$\log(x+\sqrt{x^2+1})' = \frac{1+\dfrac{2x}{2\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} = \frac{\dfrac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}\ .$$

We summarize in the following tables the elementary rules of derivation:

$f(x)$	$f'(x)$
costanti	0
x^a	ax^{a-1}
$\log x$	$\dfrac{1}{x}$
e^x	e^x
a^x	$\log a \cdot a^x$
$e^{g(x)}$	$g'(x)e^{g(x)}$
$\log(g(x))$	$\dfrac{g'(x)}{g(x)}$

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\dfrac{1}{\cos^2 x}$
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\dfrac{1}{x^2+1}$
$\sin(g(x))$	$g'(x)\cos(g(x))$

Problem.

EXERCISE 4.1 (▶▶|). Calculate the derivative (with respect to x) of the following functions:

$$6x^3+2x^2-2x+1 \qquad 8x^7-4x^2 \qquad \frac{1}{x}-\frac{1}{x+1} \qquad \frac{x+1}{x-1} \qquad e^{2x}$$

$$\frac{1}{x} \qquad \sqrt{x} \qquad \sqrt{2-3x} \qquad 3^x \qquad x^x \qquad \log(\cos x) \qquad \arctan(5-3x^2).$$

EXERCISE 4.2. Calculate the derivative (with respect to x) of the following functions:

$$\frac{\pi}{x}-\log(2) \qquad \log(3) \qquad \frac{1-\tan(x)}{x} \qquad x\sin(x)-\cos(x)$$

$$\frac{x}{1-\cos(x)} \qquad \frac{1+e^x}{1-e^x} \qquad \frac{\cos(x)}{e^x} \qquad \sqrt{x}\,\cos(x)$$

$$\tan(ax) \qquad e^{-bx} \qquad \tan\left(\frac{2}{x}\right) \qquad \sin(5x) \qquad \sin x\cos x \qquad (\sin x)^2.$$

EXERCISE 4.3. Calculate the derivative (with respect to x) of the following functions:

$$\frac{1}{x^2} \qquad \frac{1}{x^n} \qquad \sqrt[3]{x} \qquad x^{\frac{1}{2}} \qquad x^{\frac{2}{3}} \qquad x^{-\frac{1}{2}} \qquad \frac{1}{\sqrt{x}} \qquad \sqrt[3]{\frac{1}{x}} \qquad \sqrt{x-1}$$

$$\sqrt[3]{2-3x} \qquad \sqrt{\frac{x+1}{x-1}} \qquad \sqrt[5]{x^2} \qquad xe^x-1 \qquad (xe^x-1)^2 \qquad \sin^2 x \qquad 2^x$$

$$(\sin x)^x \qquad \frac{x^2-1}{2-3x^2} \qquad \arcsin\sqrt{x} \qquad \frac{1+2e^x}{x+6} \qquad \tan\left(\frac{1}{\cos x}\right)$$

$$\log\frac{1}{\sqrt{x}} \qquad \log\frac{1+x}{1-x} \qquad \arctan\frac{1}{x} \qquad x\log x \qquad (3-4x)\log(2x+5)$$

$$x+xe^{1/x} \qquad \frac{1}{x}e^x \qquad e^{\frac{1}{x-1}} \qquad x^2\log(2-6x) \qquad \frac{e^{1/x}-1}{e^{1/x}+1}$$

$$\sqrt{x^3+\sin(x)} \qquad \log(\sqrt{x^2+1}-x) \qquad \log(2x+\sqrt{4x^2+1}).$$

2. Maxima and minima

One of the main applications of derivatives is in the study of the maxima and minima of functions. First of all, we recall the precise terminology on maxima and minima, which can be relative (i.e. local) or absolute (i.e. global).

DEFINITION 2.1. Let $f\colon I \to \mathbb{R}$ be a function defined on an interval I. We say that M is the *maximum* of f on I, and we write

$$M = \max_I f$$

if there exists a point $x_0 \in I$ where the value of the function is M, and in no other point of I the value of f is larger than M:

$$f(x_0) = M \geq f(x) \quad \forall x \in I.$$

the point x_0 is called *absolute maximum point* of f on I.

Instead, x_0 is said to be a *relative maximum point (or local maximum)* of f if there exists δ such that $f(x_0) \geq f(x)$ for x such that $|x - x_0| < \delta$ (i.e. if $f(x_0)$ is greater than the values of f at points close to x_0). The definitions of absolute and relative minimum point are analogous.

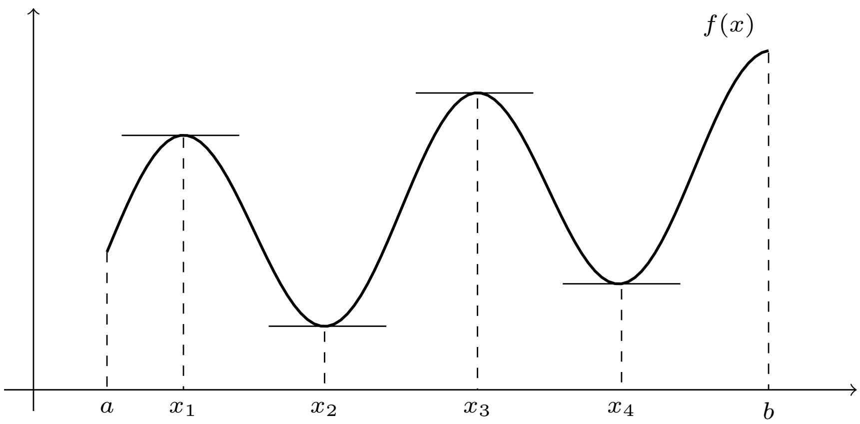


FIGURE 4.2. Points of maximum and minimum of the function $f\colon [a,b] \to \mathbb{R}$. Absolute maximum b; absolute minimum x_2; relative maxima x_1, x_3, b; relative minima a, x_2, x_4. Note that in the points x_1, x_2, x_3, x_4, which are interior to the interval $[a,b]$, the tangent line to the graph is horizontal.

Of course, an absolute maximum point is also a point of relative maximum, even if the converse is not true in general.

The result that relates maxima, minima and derivatives of f is the following. Recall that if $[a,b]$ is a closed interval, a point x_0 is said *interior* to the interval if $a < x_0 < b$, while the two points a and b are the *extrema* of the interval.

THEOREM 2.2. *Assume $f\colon [a,b] \to \mathbb{R}$ is differentiable at a point x_0* interior *to the interval $[a,b]$. If x_0 is a point of relative maximum or minimum for f, then $f'(x_0) = 0$.*

PROOF. Let us consider the case of a point of maximum (the other case is similar). Then if h is small enough we have $f(x_0) \geq f(x_0 + h)$, so that the difference

$$f(x_0 + h) - f(x_0) \leq 0$$

is negative (if h is small enough). Next, consider the incremental ratio: if $h > 0$ we have

$$\frac{f(x_0 + h) - f(x_0)}{h} = \frac{(-)}{(+)} \leq 0$$

and therefore, applying the theorem of the permanence of the sign, we obtain

$$\lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \leq 0.$$

If instead we consider the $h < 0$, we have

$$\frac{f(x_0 + h) - f(x_0)}{h} = \frac{(-)}{(-)} \geq 0$$

and therefore in the same way we have

$$\lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} \geq 0.$$

In conclusion

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = 0.$$

□

REMARK 2.3. The previous theorem does not claim that if the derivative vanishes at a poin then this point is a maximum or minimum: for example, the function $f(x) = x^3$ has derivative equal to zero in the origin but it is always strictly increasing.

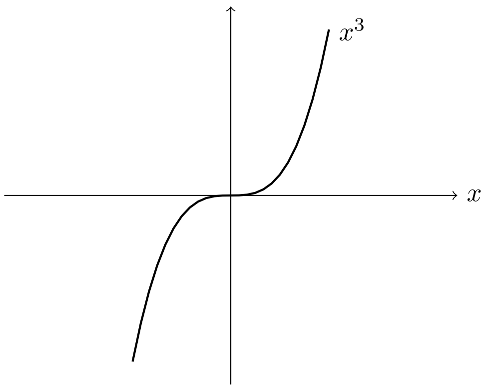


FIGURE 4.3. The derivative of x^3 vanishes at 0 but the function has no maxima and no minima.

The theorem states something different: if there is an interior point of maximum or minimum, then the derivative must vanish in that point. This gives a hint on where the maxima and minima might be: indeed for a differentiable function they can be found exclusively

1) at the extrema of the interval, or
2) at the interior points in which the derivative is zero.

After finding all the points where the derivative vanishes, we have to examine them one by one to understand if they are points of maximum or minimum, or not; and one shouldn't forget to study what happens at the extrema of the interval.

3. The Basic Theorems of Calculus

Recall that a continuous function on a closed bounded interval satisfies Weierstrass' Theorem; therefore we can always find a point where the value of f is greater than all other values, and a point where the value of f is smalled that all other values. These two points are usually distinct; if they coincide then the function must be flat, i.e. a constant.

If we add the hypothesis that f is differentiable we get several interesting results:

THEOREM 3.1 (Rolle's Theorem). *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$, differentiable on $]a,b[$. If $f(a) = f(b)$ then there is a point $c \in\,]a,b[$ where the derivative is zero: $f'(c) = 0$.*

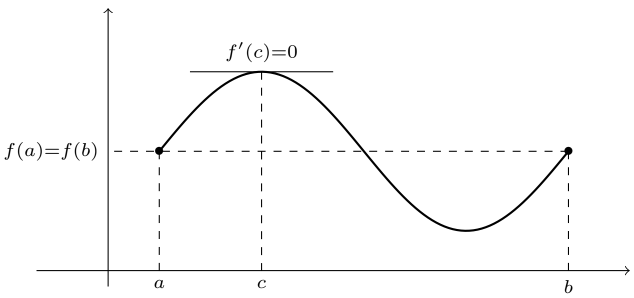


FIGURE 4.4.

PROOF. By Weierstrass' Theorem the function admits absolute maximum and minimum on $[a,b]$; let x_0 be the point of maximum and x_1 the point of minimum.

If one of these two points is *interior* to the interval $[a,b]$, then we know that in that point the derivative vanishes, and therefore the proof is over.

If, on the other hand, neither of these two points is interior, then both these points are at the extrema of the interval; but the value of f at the extrema is the same, therefore it follows that $f(a) = f(b) = f(x_0) = f(x_1)$. This means that the maximum and the minimum of f coincide i.e. the function is constant, and therefore the derivative is zero at *all* points. □

THEOREM 3.2 (Cauchy's Theorem). *Let $f, g\colon [a,b] \to \mathbb{R}$ be continuous functions on $[a,b]$ which are differentiable on $]a,b[$. Then there exists a point $c \in]a,b[$ where*

$$f'(c) \cdot [g(b) - g(a)] = g'(c) \cdot [f(b) - f(a)].$$

PROOF. Consider the function

$$F(x) = f(x) \cdot [g(b) - g(a)] - g(x) \cdot [f(b) - f(a)].$$

The function F is continuous on $[a,b]$, differentiable on $]a,b[$ and its derivative is

$$F'(x) = f'(x) \cdot [g(b) - g(a)] - g'(x) \cdot [f(b) - f(a)].$$

Also we immediately see that

$$F(a) = F(b) = f(a)g(b) - f(b)g(a).$$

So we can apply Rolle's Theorem and we get that it exists a point $c \in]a,b[$ where $F'(c) = 0$, that is

$$F'(c) = f'(c) \cdot [g(b) - g(a)] - g'(c) \cdot [f(b) - f(a)] = 0$$

hence the thesis. □

THEOREM 3.3 (Lagrange, or Mean Value Theorem). *Let* $f\colon [a,b] \to \mathbb{R}$ *be a continuous function on* $[a,b]$ *which is differentiable on* $]a,b[$. *Then there exists a point* $c \in\,]a,b[$ *where*

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

PROOF. It is sufficient to apply the previous theorem to the functions f and $g(x)=x$; the assumptions are satisfied, thus applying Cauchy's Theorem we get a point c where

$$f'(c)\cdot[g(b)-g(a)] = g'(c)\cdot[f(b)-f(a)];$$

but $g(b)=b$, $g(a)=a$, and clearly $g'(x)=(x)'=1$ at each point, so that

$$f'(c)\cdot[b-a] = f(b)-f(a)$$

and this implies the thesis. □

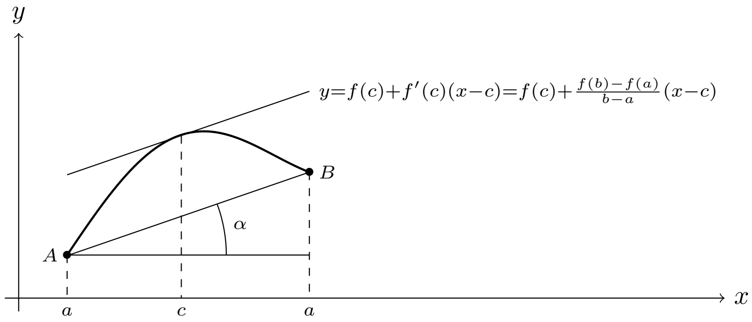


FIGURE 4.5. Mean Value Theorem: the tangent line in c is parallel to the segment AB: $f'(c)=\tan(\alpha)$.

REMARK 3.4. Recall that $f'(c)$ can be interpreted as the slope (the angular coefficient) of the tangent to the graph at $x=c$; instead, the second member is the incremental ratio between the points $x=a$ and $x=b$, which expresses the inclination of the straight line passing through the points $A=(a,f(a))$ and $B=(b,f(b))$. Lagrange's Theorem states simply that there is a point where the tangent to the graph has the same slope as the straight line through A and B.

Lagrange's Theorem has some consequences of the utmost importance:

COROLLARY 3.5. *Let* $f\colon [a,b]\to\mathbb{R}$ *be a continuous function on* $[a,b]$ *which is differentiable on* $]a,b[$. *Then*

$$f \quad \textit{is constant} \iff f'(x)=0 \quad \forall x \in]a,b[.$$

PROOF. If the function is constant, we already know that its derivative is zero at all points.

Conversely, suppose that the derivative of f is zero at all points. We choose two arbitrary points $x_1 < x_2$ within the interval, and we apply Lagrange's Theorem on the interval $[x_1,x_2]$: we obtain that there exists a point $c \in]x_1,x_2[$ such that

$$f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}.$$

Since $f'(c)=0$ for any c, we get that the ratio at the second member is equal to zero, hence $f(x_1)-f(x_2)=0$ and therefore $f(x_1)=f(x_2)$. But then we have proved that the values of f at any two points x_1,x_2 are the same, and this means that f is constant. □

Corollary 3.6. *Let* $f: [a,b] \to \mathbb{R}$ *be a continuous function on* $[a,b]$ *which is differentiable on* $]a,b[$. *Then*
(i) f *is increasing* $\iff f'(x) \geq 0 \ \forall x \in \ ]a,b[$ *;*
(ii) f *is decreasing* $\iff f'(x) \leq 0 \ \forall x \in \ ]a,b[$ *.*

Proof. We will prove only case (i), the second is completely analogous.
If f is increasing, the numerator and denominator of the incremental ratio have always the same sign: when $h > 0$

$$\frac{f(x_0 + h) - f(x_0)}{h} = \frac{(+)}{(+)} \geq 0$$

and when $h < 0$

$$\frac{f(x_0 + h) - f(x_0)}{h} = \frac{(-)}{(-)} \geq 0.$$

Thus the incremental ratio is positive, and hence also the limit of the incremental ratio, i.e. the derivative, must be positive.
Conversely, suppose that the derivative is always positive. Proceeding as in the previous corollary, we choose two arbitrary points $x_1 < x_2$ in the interval, and we apply Lagrange's Theorem on $[x_1, x_2]$: we obtain that there exists a point $c \in]x_1, x_2[$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since we know that $f'(c) \geq 0$ for any point, we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq 0$$

and multiplying by $x_2 - x_1 > 0$, we obtain

$$f(x_2) - f(x_1) \geq 0 \implies f(x_2) \geq f(x_1).$$

Therefore we have shown that for any two points x_1 and x_2,

$$x_1 < x_2 \implies f(x_1) \leq f(x_2)$$

which means precisely that f is increasing. □

Example 3.7. We compute the minimum of the function $f(x) = e^x + e^{-x}$. The function is always positive and $\lim_{x \to \pm\infty} f(x) = +\infty$. The derivative is equal to $e^x - e^{-x}$ and vanishes when $e^x = e^{-x}$, i.e. when $x = -x$. Therefore the derivative vanishes only for $x = 0$ and at this point we have $f(0) = 2$.
We note that from the relations $e^x \geq 1 + x$ and $e^{-x} \geq 1 - x$ it follows immediately that

$$e^x + e^{-x} \geq (1 + x) + (1 - x) = 2 \quad \text{for every} \quad x \ .$$

Problems.

Exercise 4.4 (▶▶|). Say which of the following functions satisfy the hypotheses of Rolle's theorem in the interval $[-1, 1]$.

$$x^2 - 3, \quad x^3 + x^2 + 4, \quad x^2 - |x|, \quad x^2 + \frac{1}{x^2}, \quad \cos x + \sin(x^2), \quad \tan(\pi x).$$

Exercise 4.5. Calculate the absolute minimum of the function $2^x + 2^{-x}$.

4. De l'Hôpital's Theorem

The use of derivatives allows in many cases to calculate with ease some indeterminate limits. For example, the following result can be used to calculate the limits of the form $\frac{0}{0}$:

THEOREM 4.1 (De l'Hôpital's theorem). *Let* $f, g\colon [a,b] \to \mathbb{R}$ *be two continuous functions on* $[a,b]$ *which are differentiable on* $]a,b[$. *Let* $x_0 \in\,]a,b[$ *be such that* $f(x_0) = g(x_0) = 0$, *and assume that* $g'(x) \neq 0$ *for* $x \neq x_0$. *If the limit* $\lim_{x\to x_0} \dfrac{f'(x)}{g'(x)}$ *exists, then also the limit* $\lim_{x\to x_0} \dfrac{f(x)}{g(x)}$ *exists, and the two limits coincide:*

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

PROOF. We apply Cauchy's Theorem to the two functions f and g on the interval of extrema x_0 and x; it follows that there is an intermediate point z such that

$$f'(z) \cdot [g(x) - g(x_0)] = g'(z) \cdot [f(x) - f(x_0)].$$

Since $f(x_0) = g(x_0) = 0$ the relation simplifies:

$$f'(z)g(x) = g'(z)f(x)$$

and dividing by $g'(z) \neq 0$ we get

$$\frac{f(x)}{g(x)} = \frac{f'(z)}{g'(z)}.$$

If we now calculate the limit as $x \to x_0$, and observe that also $z \to x_0$ since it is between x and x_0, we get

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{z\to x_0} \frac{f'(z)}{g'(z)}$$

which is the thesis. □

REMARK 4.2. The theorem holds true in many other cases:
1) for indeterminate limits of the form $\frac{\infty}{\infty}$;
2) for limits from the right;
3) for limits from the left;
4) when $x \to +\infty$ or $x \to -\infty$.
We will not prove all cases but we will use the other variants of the theorem in all useful situations.

REMARK 4.3. We verify the special limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

This is an indeterminate limit of the form $0/0$, and given that the limit of ratio of derivatives

$$\lim_{x\to 0} \frac{(\sin x)'}{(x)'} = \lim_{x\to 0} \frac{\cos x}{1} = 1$$

exists and is equal to 1, we conclude that also the original limit exists and is equal to 1.

Another verification:

$$\lim_{x\to +\infty} \frac{e^x}{x} = +\infty.$$

This is an indeterminate limit of the form ∞/∞. The limit of ratio of derivatives exists:

$$\lim_{x\to +\infty} \frac{(e^x)'}{(x)'} = \lim_{x\to +\infty} \frac{e^x}{1} = +\infty$$

and this proves the result.

Remark 4.4. Sometimes an application of the theorem produces again an indeterminate limit; but if we continue to apply the theorem, passing to the second derivatives (or thirds etc.) in some cases we arrive at a limit which is no longer indeterminate, so that the method can still be applied. An example: calculate the limit

$$\lim_{x\to 0}\frac{\sin x - x}{x^3}.$$

The ratio of derivatives

$$\frac{\cos x - 1}{3x^2}$$

is again an indeterminate limit $0/0$; we reapply the theorem and once again differentiate the numerator and denominator:

$$\frac{-\sin x}{6x};$$

still an indeterminate limit! (even if we know how to calculate it very well). Differentiating one last time we get

$$\frac{-\cos x}{6} \quad \text{which tends to} \quad -\frac{1}{6}$$

and in conclusion the original limit is equal to $-\frac{1}{6}$.

The Theorem of de l'Hôpital can also be applied to the computation of indeterminate limits of the form $0\cdot\infty$. For example, to calculate the limit

$$\lim_{x\to 0^+} x\log x$$

we can write

$$x\log x = \frac{\log x}{\frac{1}{x}}$$

and in this way we have written the limit in the indeterminate form $\frac{\infty}{\infty}$; hence differentiating numerator and denominator

$$\frac{(\log x)'}{\left(\dfrac{1}{x}\right)'} = \frac{\dfrac{1}{x}}{-\dfrac{1}{x^2}} = -x$$

the limit becomes simply

$$\lim_{x\to 0^+} x\log x = -\lim_{x\to 0^+} x = 0.$$

Problem.

Exercise 4.6. Calculate the following limits (warning! Check first that they are actually indeterminate limits of a suitable form; if they aren't, you should not apply de l'Hôpital's Theorem!).

$$\lim_{x\to 0}\frac{\sin(2x)}{x},\quad \lim_{x\to+\infty}\frac{e^x+x}{x},\quad \lim_{x\to 0}\frac{e^x-1}{\sin x},\quad \lim_{x\to+\infty}\frac{e^x-1}{\sin x}$$

$$\lim_{x\to 2}\frac{x^2-4x+4}{x^4-16}\quad \lim_{x\to 1}\frac{\log x}{x^3-3x^2+3x-1}\quad \lim_{x\to 1}\frac{e^{x-1}-x}{(x-1)^2}$$

$$\lim_{x\to+\infty}\frac{x^3-x^2+x}{-2x^3+x},\quad \lim_{x\to 0}\frac{x^3-x^2+x}{-2x^3+x},\quad \lim_{x\to-\infty}\frac{2x^2+3x-1}{x^2+2x}$$

$$\lim_{x\to 0^+}\frac{\sin(x^2)}{x^3},\quad \lim_{x\to 0^-}\frac{\sin(x^2)}{x^3},\quad \lim_{x\to 0^+}\frac{\log(\sin x)}{x^2},\quad \lim_{x\to 0}\frac{x^2}{e^{x^2}-1}$$

$$\lim_{x\to 0}\frac{\tan x}{x},\quad \lim_{x\to 0}\frac{\sin(x-x^3)}{x},\quad \lim_{x\to 0^+}\frac{\log x}{\sqrt{x}},\quad \lim_{x\to 0^+}\frac{\log x}{x}$$

$$\lim_{x\to 0^+}\frac{e^{1/x}+1}{e^{1/x}-1},\quad \lim_{x\to 0^-}\frac{e^{1/x}+1}{e^{1/x}-1},\quad \lim_{x\to\frac{\pi}{2}^+}\frac{\cos x}{x-\frac{\pi}{2}},\quad \lim_{x\to\frac{\pi}{2}^-}\frac{\cos x}{x-\frac{\pi}{2}}$$

$$\lim_{x\to 2^\pm} \frac{x-2}{x^3-5x+2}, \quad \lim_{x\to 1^\pm} \frac{x-2}{x^3-5x+2}, \quad \lim_{x\to 0^+} \frac{3x+2\sqrt{x}}{x-\sqrt{x}}, \quad \lim_{x\to +\infty} \frac{3x+2\sqrt{x}}{x-\sqrt{x}}$$

$$\lim_{x\to 0} \frac{e^{3x}-1}{2x}, \quad \lim_{x\to 0} \frac{1-e^{2x}}{x}, \quad \lim_{x\to 0} \frac{2x^2}{\sin(x^2)}, \quad \lim_{x\to \pm\infty} \frac{e^x-1}{e^{2x}+1}$$

$$\lim_{x\to 0^+} \frac{e^{\sqrt{x}}}{x^2+1}, \quad \lim_{x\to 0} \frac{1-\cos x}{x^2}, \quad \lim_{x\to 0} \frac{2-2\cos x - x^2}{x^4}, \quad \lim_{x\to 0^\pm} \frac{x^2-\sin x}{x^3}.$$

Exercise 4.7. Calculate the following limits:

$$\lim_{x\to +\infty} x \sin\left(\frac{1}{x}\right), \qquad \lim_{x\to 0^+} \sqrt{x}\cdot \log x, \qquad \lim_{x\to 0^+} \log x \cdot \sin x, \qquad \lim_{x\to +0^+} x\log(x-x^2)$$

5. Study of functions

At this point, we have many tools to study the behavior of a function and draw its approximate graph. The in–depth study of a function can be very complex; but we will consider only quite simple examples, for which the following steps provide sufficient information to understand the behaviour of the graph. Given a function $f(x)$ expressed as a combination of elementary functions,

(1) establish the set of definition of the expression; usually, this set is the union of a finite number of intervals;
(2) check if the function is even, or odd, or periodic;
(3) determine if the function has any zeros, and where it is positive and negative;
(4) calculate the limits of the function at the ends of the intervals where it is defined;
(5) at this stage it is already possible to start drawing a (very approximate) graph which can serve as a guide in the following calculations;
(6) compute the derivative f' e compute (if there are any) its zeros;
(7) determine the intervals on which the derivative is positive or negative and therefore the intervals of increase and decrease of f;
(8) find any maxima and minima of the function;
(9) complete the graph using the elements just calculated.

Warning: for certain functions, it may happen that some of the previous properties are too difficult to study; in such cases, one tries to understand the behavior of the function using only the remaining information. For example, often it is very difficult to calculate the zeroes of a function, but knowing the precise location of zeros is not essential for an approximate study. If the function contains absolute values, it is recommended as a preliminary step, to split the domain into zones where it is possible to resolve the absolute values, and then study the functions thus obtained separately.

Example 5.1. Study of the function

$$f(x) = \frac{1}{x}e^{\frac{1}{x}}.$$

The expression is not defined if $x = 0$ but all other points present no problem; the definition set is

$$\text{D.S.} \ = \ \{x \neq 0\}.$$

The function is not even or odd, moreover it does not vanish at any point. We immediately see that the sign of the function is exactly the sign of x since the exponential is positive:

$$f(x) > 0 \ \text{ for } \ x > 0, \qquad f(x) < 0 \ \text{ for } \ x < 0.$$

We calculate the limits at the extrema of the definition intervals, that is, we study the limits for $x \to \pm\infty$ and $x \to \pm 0$. For $x \to +\infty$ the function $1/x \to 0$, thus we have it right away

$$\lim_{x\to +\infty} \frac{1}{x}e^{\frac{1}{x}} = 0 \cdot e^0 = 0$$

and similarly

$$\lim_{x\to-\infty}\frac{1}{x}e^{\frac{1}{x}}=0\cdot e^0=0.$$

Moreover

$$x\to 0^+ \implies \frac{1}{x}\to+\infty$$

and therefore

$$\lim_{x\to 0^+}\frac{1}{x}e^{\frac{1}{x}}=+\infty.$$

Finally,

$$x\to 0^- \implies \frac{1}{x}\to-\infty, \quad e^{\frac{1}{x}}\to 0$$

and therefore we are faced with the indeterminate form $(-\infty)\cdot 0$; but after the change of variable

$$y=\frac{1}{x}$$

the limit becomes

$$\lim_{x\to 0^-}\frac{1}{x}e^{\frac{1}{x}}=\lim_{y\to-\infty}ye^y=0.$$

Knowing the limits and the sign of the function, we can already draw a very rough graph:

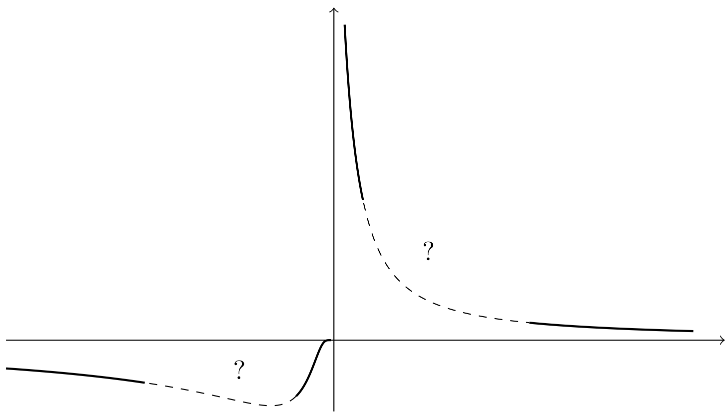


FIGURE 4.6. Behavior at the extrema of the graph of $\frac{1}{x}e^{\frac{1}{x}}$.

Even if it is not precise, the graph suggests that the function could have a relative minimum in a certain point $x<0$. To proceed, we calculate the derivative:

$$\left(\frac{1}{x}e^{\frac{1}{x}}\right)'=-\frac{1}{x^2}e^{\frac{1}{x}}+\frac{1}{x}e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)=-\frac{x+1}{x^3}e^{\frac{1}{x}}$$

that is

$$f'(x)=-\frac{x+1}{x^3}e^{\frac{1}{x}}.$$

The derivative vanishes when $(x+1)$ is null, that is

$$f'(x)=0 \iff x=-1.$$

We then study the sign of f'; the exponential is always positive, so we just study the sign of

$$-\frac{x+1}{x^3}.$$

We see that $(x+1)$ is positive for $x > -1$ and negative for $x < -1$, while x^3 has the same sign as x, therefore

$$f'(x) > 0 \text{ for } -1 < x < 0, \qquad f'(x) < 0 \text{ elsewhere.}$$

Then we conclude that

$$f \text{ is increasing on } -1 < x < 0, \text{ decreasing on } x < -1 \text{ and on } x > 0$$

and of course

$$f \text{ has a minimum for } x = -1.$$

In conclusion, we have enough elements to draw the graph:

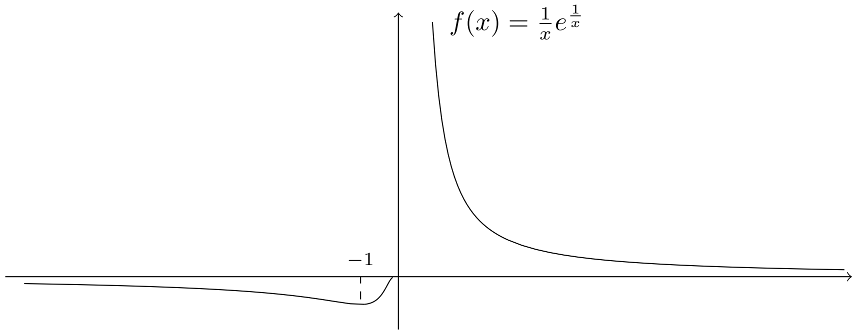


FIGURE 4.7. Graph of the function $\frac{1}{x}e^{\frac{1}{x}}$

EXAMPLE 5.2. Study of the function

$$f(x) = \frac{x^2+2}{x^2-9}.$$

The function is defined as the ratio of two polynomials; the only problem is the denominator that could vanish. Hence the set of definition is given by the condition

$$x^2 - 9 \neq 0 \iff x \neq \pm 3.$$

The function is even: in fact

$$f(-x) = \frac{(-x)^2+2}{(-x)^2-9} = \frac{x^2+2}{x^2-9} = f(x).$$

Hence the graph is symmetric with respect to the ordinate axis. The numerator x^2+2 never vanishes and is always strictly positive, thus the function $f(x)$ can never be equal to zero, and has the same sign as the denominator:

$$x^2 - 9 > 0 \iff x > 3 \text{ or } x < -3$$

and therefore

$$f(x) > 0 \iff x > 3 \text{ or } x < -3$$
$$f(x) < 0 \iff -3 < x < 3.$$

We calculate the limits at the ends of the definition intervals: since the excluded points are $x = \pm 3$, we need to calculate six limits:

$$x \to \pm\infty, \qquad x \to -3^{\pm}, \qquad x \to 3^{\pm}.$$

The first two are obtained by factoring out the maximum degree:

$$\frac{x^2+2}{x^2-9} = \frac{x^2\left(1+\frac{2}{x^2}\right)}{x^2\left(1-\frac{9}{x^2}\right)} = \frac{1+\frac{2}{x^2}}{1-\frac{9}{x^2}}$$

and therefore

$$\lim_{x\to\pm\infty} f(x) = \frac{1}{1} = 1.$$

We calculate the limits as $x \to 3^{\pm}$. When $x \to 3^+$ the denominator tends to 0 and is positive, that is

$$x-3 \to 0^+$$

while the numerator approaches 11, so the function tends to $+\infty$:

$$\lim_{x\to 3^+} f(x) = +\infty.$$

When $x \to 3^-$ the denominator tends to 0 and is negative, that is

$$x-3 \to 0^-$$

while the numerator approaches 11, so the function tends to $-\infty$:

$$\lim_{x\to 3^-} f(x) = -\infty.$$

Now we calculate the limits as $x \to -3^{\pm}$. When $x \to -3^+$ the denominator tends to 0 and is negative, that is

$$x-3 \to 0^-$$

while the numerator again tends to 11, so that the function tends to$-\infty$:

$$\lim_{x\to -3^+} f(x) = -\infty.$$

When $x \to -3^-$ the denominator tends to 0 and is positive, that is

$$x-3 \to 0^+$$

while also this time the numerator tends to 11, hence the function tends to $+\infty$:

$$\lim_{x\to -3^-} f(x) = +\infty.$$

We can draw a very rough first graph based on the limits we calculated:

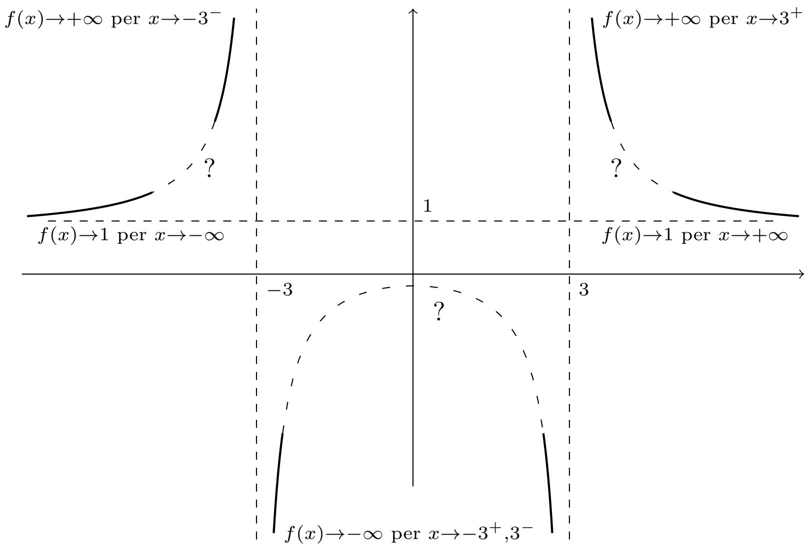


FIGURE 4.8. Limits at the extrema of the function $\frac{x^2+2}{x^2-9}$.

To check our suspicions about the shape of the graph we calculate the derivative: we obtain

$$f'(x) = -\frac{22x}{(x^2-9)^2}$$

and we see that f' vanishes only for $x = 0$, while

$$f' < 0 \iff x > 0, \qquad f' > 0 \iff x < 0,$$

that is

$$f \text{ is increasing for } x < 0 \text{ and decreasing for } x > 0.$$

In particular, we have

$$x = 0 \;\text{ Is a local maximum point.}$$

Now we can plot the graph of f:

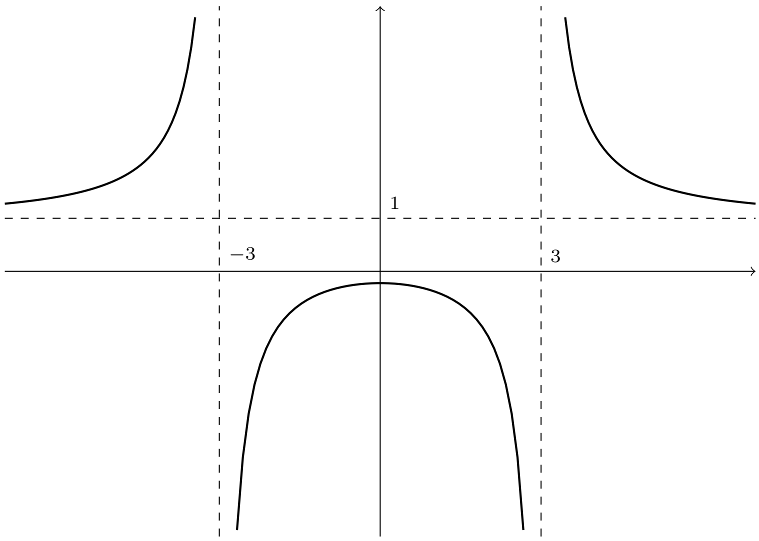


FIGURE 4.9. Graph of the function $\dfrac{x^2+2}{x^2-9}$.

Problems.

EXERCISE 4.8. Study the functions listed in Exercise 3.11 and plot an approximate graph.

EXERCISE 4.9. Study the following functions and plot a rough graph of them.

$$xe^x, \quad xe^{-x}, \quad xe^{1/x}, \quad e^{-1/x}, \quad e^{-x^2}, \quad x^3-x, \quad \frac{e^x-1}{e^x}$$

$$2x+|x|, \quad xe^{-1/x}, \quad x^x, \quad \frac{1}{x^2-1}, \quad \frac{1}{e^{2x}}, \quad \frac{x^2}{e^{2x}}, \quad \frac{1}{x}e^{x-1}$$

$$\sqrt{\frac{x+1}{x-1}}, \quad \log\left(\frac{1}{2-x}\right), \quad e^{\frac{1}{x-2}}, \quad e^{\frac{x}{x+1}}, \quad \log(e^x-1)$$

$$e^{x^2-x}, \quad e^{2x^2-x^3}, \quad \frac{x}{2x-3}, \quad \frac{x^2}{2-x}, \quad \frac{x+1}{x^2-1},$$

$$x^3-3x^2+2x, \quad (x-1)e^{1/x}, \quad \frac{x}{x^2-4}, \quad \frac{e^x}{x-2},$$

$$\frac{x+1}{e^x}, \quad \frac{x}{e^{2/x}}, \quad \frac{x^2}{e^{1/x}}, \quad \log(1-e^x), \quad \log(x^2-2x+3)$$

$$x\log x, \quad \log(2-2x^2), \quad x\log\frac{1}{x}, \quad x\log\left(\frac{1}{x-2}\right)$$

$$\frac{e^x+1}{e^x-1}, \quad \frac{e^{-x}-1}{e^x}, \quad \frac{e^{2x}+1}{e^x+1}, \quad \frac{1}{(x+2)(x-2)},$$

$$\sqrt{1-2x}, \qquad \frac{1}{|x|}, \qquad \frac{1}{\sqrt{x}}, \qquad (x-1)^2 - \log x, \qquad e^{x+1/x},$$

$$\log(x+x^2), \qquad \sqrt{1+x^2}, \qquad \sqrt{1-e^{-x}}, \qquad \log(e^x + e^{-x})$$

$$|x| + |x-1|, \qquad x + |x|, \qquad |x-1| + |2x+3|, \qquad (x + |3-2x|)^2.$$

6. Taylor's polynomial

We have seen that if a function $f\colon I \to \mathbb{R}$ defined on an interval I is differentiable at all points of I, we can define a new function $f'\colon I \to \mathbb{R}$, the derivative of f. If f' is differentiable at a point x, we denote this derivative with $f''(x)$ (instead of writing $(f')'(x)$); this is called the *second derivative* of f in x. Of course if f' is differentiable at all points of I we obtain a new function f'', called *second derivative* of f. We also say that f is *twice differentiable*. The procedure may to continue; the successive derivatives are indicated with f''' (third derivative), $f^{(4)}$ (fourth derivative: note that from 4 on we use numbers!) and so on. These functions are called as a whole, the *higher order derivatives* of the function f.

For example, if $f(x) = \sin x$, we have immediately:

$$f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x, \quad f^{(4)}(x) = \sin x.$$

We see that in this case starting from the fourth derivative we get back the same series of derivatives. If we try instead to differentiate a polynomial, for example $f(x) = x^3 - 3x + 2$, we get:

$$f'(x) = 3x^2 - 3, \quad f''(x) = 6x, \quad f'''(x) = 6, \quad f^{(4)}(x) = 0$$

and all derivatives from the fourth onwards are null. Note that this phenomenon happens for all polynomials: if f is a polynomial of degree n, then f' is a polynomial of degree $n-1$, f'' of degree $n-2$ and so on, until you get $f^{(n+1)} = 0$.

Successive derivatives can be used to discover further properties of the function f and its graph. For example, the sign of f'' is linked to the properties of convexity of the graph of f. Here however we focus only on one very important application of the successive derivatives, namely the Taylor polynomial.

Definition 6.1. Suppose $f\colon I \to \mathbb{R}$ is defined on the open interval I and is differentiable n times, and let $x_0 \in I$. The *Taylor polynomial* of f of order n at the point x_0 is the polynomial

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

For example, the Taylor polynomial $P_1(x)$ at the point $x_0 = 0$ is given by

$$P_1(x) = f(0) + f'(0)x$$

while the Taylor polymial at x_0 is

$$P_1(x) = f(x_0) + f'(x_0)(x - x_0).$$

Note that $P_1(x)$ is precisely the tangent line to f at x_0.

Example 6.2. We compute the Taylor polynomial $P_3(x)$ at the point 0 for the function $f(x) = \sin x$. We first observe that

$$f(x) = \sin x, \qquad f'(x) = \cos x, \qquad f''(x) = -\sin x, \qquad f'''(x) = -\cos x,$$

from which we obtain that

$$f(0) = 0, \qquad f'(0) = 1, \qquad f''(0) = 0, \qquad f'''(0) = -1.$$

Substituting the values thus obtained, we obtain

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!} = x - \frac{x^3}{6}.$$

Taylor's polynomial is an indispensable tool to *approximate* the values of a function f. In fact it is much easier to compute a polynomial than an arbitrary function (especially for a computer!), and on the other hand the previous definition ensures that, at least if x remains close to the point x_0, the values of $P_n(x)$ are very close to those of $f(x)$. To measure the distance between $f(x)$ and $P_n(x)$ we calculate the difference:

Definition 6.3. Let x_0, f and $P_n(x)$ be as in the previous definition. The *Taylor remainder* of f of order n in the point x_0 is the difference

$$R_n(x) = f(x) - P_n(x).$$

Our hope is that the remainderR_n is small (in an appropriate sense) and therefore that the Taylor polynomial is a good approximation for $f(x)$; if instead the remainder were big, the approximation would be bad. We state without proof the fundamental result on Taylor polynomials, which responds positively to our expectations.

Theorem 6.4. *Suppose $f\colon I \to \mathbb{R}$ is defined on the open interval I and is differentiable $n+1$ times, and let $x_0 \in I$. Then we have*

$$\lim_{x \to x_0} \frac{R_n(x)}{(x - x_0)^n} = 0.$$

This theorem shows that the remainder is very small for x close to x_0 (even null if $x = x_0$) and increases if we move away from point x_0. More precisely, we see that $R_n(x)$ tends to 0 for $x \to x_0$ *faster than* $(x - x_0)^n$. It can be proved that the Taylor polynomial is the only polynomial of degree n with this property. In other words, if replace $f(x)$ by $P_n(x)$, for n large enough and for x close enough to x_0, we make a small error (which can be computed using $R_n(x)$).

In many practical applications of Calculus, it is necessary to approximate quantities and ensure that the error is not larger than a given threshold. Theorem 6.4 is still not sufficient for this purpose, and we need a more precise estimate of the remainder $R_n(x)$. To this end, we state without proof the second fundamental result on Taylor polynomials.

Theorem 6.5. *Suppose $f\colon I \to \mathbb{R}$ is defined on the open interval I, f is differentiable $n+1$ times, with continuous derivatives, and $x_0, x \in I$. Then there exists a point ξ between x_0 and x such that*

$$R_n(x) = \frac{f^{(n+1)}(\xi)(x - x_0)^{n+1}}{(n+1)!}.$$

Example 6.6. . We show how to approximate the number $\sin\frac{1}{10}$ with an error of less than $1/10000$. It is convenient to use the Taylor polynomial of the function $f(x) = \sin x$ of order n at the point $x_0 = 0$. We observe that the function $f(x) = \sin x$ is differentiable infinite times. Also, for any $n \in \mathbb{N}$, we see easily that $f^{(n+1)}(x) = \pm \sin x$ or $f^{(n+1)}(x) = \pm \cos x$. It follows that

$$\left| f^{(n+1)}(x) \right| \leq 1, \qquad \text{for every } x \in \mathbb{R}.$$

Using Theorem 6.5 with $x_0 = 0$, $x = \frac{1}{10}$, we have therefore

$$\left| R_n\left(\frac{1}{10}\right) \right| \leq \frac{1}{(n+1)!10^{n+1}}.$$

If we choose n such that $(n+1)!10^{n+1} > 1000$, the error R_n will therefore be of the required size. We are immediately convinced that for $n = 3$ we have

$$\left| R_3\left(\frac{1}{10}\right) \right| \leq \frac{1}{240000} \leq \frac{1}{10000},$$

as requested. In conclusion, a good approximation of $\sin\frac{1}{10}$ is given by the polynomial of order 3, which we calculated earlier, that is, $P_3(x) = x - x^3/6$, which in $x = \frac{1}{10}$ returns as a result

$$\sin\frac{1}{10} = \frac{1}{10} - \frac{1}{6000} + \text{ error} = \frac{599}{6000} + \text{ error} = 0,0998\overline{3} + \text{ error},$$

with an error not greater than $1/10000$. In conclusion $\sin\frac{1}{10} \simeq 0,0998...$ with four correct decimal digits.

Problems.

EXERCISE 4.10. Compute the Taylor polynomial $P_3(x)$ at the point 0 for the functions $f(x) = \sin x$, $f(x) = e^x$, $f(x) = x^3 - 3x + 2$, $f(x) = \log(1 + x)$.

EXERCISE 4.11. Approximate the numbers $\cos \frac{1}{100}$, $e^{\frac{1}{10}}$ with error less than 1/10000.

7. Functions of several variables

The notions seen so far, are easily extended to the case of functions of several variables, even if of course the things are a bit more complicated. We work for simplicity with functions of *two* variables; the extension to the general case of n variables is done without particular difficulties.

First of all, $\mathbb{R}^2$ is the well-known vector space studied in Linear Algebra and in Section 7 whose points are pairs of real numbers $x = (x_2, x_2)$. We recall that the *distance* between two points $x = (x_2, x_2)$ and $y = (y_2, y_2)$ is the number, positive or null,

$$d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y^2)^2}.$$

Another common notation is

$$|x - y| \quad \text{instead of} \quad \|x - y\|.$$

In particular, the distance of the point x from the origin $0 = (0, 0)$ is precisely the length of the vector from the origin to point x.

The *triangle inequality* states that one side of a triangle with vertices x, y, z is always shorter than the sum of the other two. In formulas we can write

$$\|x - z\| \leq \|x - y\| + \|y - z\| \quad \text{for all points} \quad x, y, z \in \mathbb{R}^2.$$

The *open ball* with center x and radius $r > 0$ in $\mathbb{R}^2$ is a set of the type

$$B(x, r) = \{y : \|x - y\| < r\}.$$

This is of course the disk with center x and radius r, without the boundary. An *open* set is a subset $A \subseteq \mathbb{R}^2$ with the following property:

if A contains a point x then it contains also a ball $B(x, r)$.

It is easy to see that an open ball is an open set. Other very useful concepts when working on $\mathbb{R}^2$ are the following:

- a set C is *closed* if its complementary (= the set of points *outside* C) is open
- a set is *bounded* if it is contained in a ball
- a set is *compact* if it is closed and bounded.

A *real-valued function of two variables* is simply a function

$$f : A \to \mathbb{R}, \quad \text{where } A \text{ is a subset of } \mathbb{R}^2.$$

The notions of limit and continuity for a function of two variables are virtually identical to the case of a single variable:

Definition 7.1. Let $f : A \setminus \{x_0\} \to \mathbb{R}$ and $x_0 \in A$. We say that f has *limit* L as x *tends to* x_0 if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|f(x) - L\| < \epsilon \quad \text{for every } x \in A \setminus \{x_0\} \text{ such that } \|x - x_0\| < \delta.$$

We write then

$$\lim_{x \to x_0} f(x) = L,$$

or also $f \to L$ for $x \to x_0$.

If f is defined also in x_0 and $\lim_{x \to x_0} f(x) = f(x_0)$, we say that f is *continuous in* x_0; if f is continuous in every point of A we say simply that f is *continuous*.

Also the properties of limits and continuous functions remain the same as in the case of a single variable, and can be proved in same way:

- if $f \to L$ and $g \to M$ as $x \to x_0$, then we have also

$$f + g \to L + M, \quad f - g \to L - M, \quad fg \to LM \quad \text{and (if } M \neq 0) \quad f/g \to L/M$$

Se $M \neq 0$ si ha anche $f/g \to L/M$, e se $M > 0$ si ha $g^f \to M^L$

- as a consequence, if f, g are continuous functions then also $f+g$, $f-g$, fg, f/g (if $g \neq 0$) and g^f (if $g > 0$) are continuous
- Weierstrass' Theorem: if $f : C \to \mathbb{R}$ is continuous and C is a compact subset of $\mathbb{R}^2$, then f has maximum and minimum on C.

When we define derivatives, we encounter the first important difference from the case of a single variables. For example, consider the function of two variables

$$f(x, y) = x^2 y - \cos(x + y^2).$$

A very natural idea is to keep the variable y fixed, as if it were a constant, and differentiate with respect to the variable x: applying the known rules, we get the function

$$2xy + \sin(x + y^2).$$

But we can also keep$x$ fixed instead, and differentiate with respect to y, obtaining

$$x^2 + \sin(x + y^2) \cdot 2y.$$

The result is different; these two derivatives are indicated with

$$\frac{\partial f}{\partial x} = 2xy + \sin(x + y^2) \qquad \text{and} \qquad \frac{\partial f}{\partial y} = x^2 + \sin(x + y^2) \cdot 2y.$$

We give a precise definition:

Definition 7.2. Let A be an open set of $\mathbb{R}^2$, $f(x, y) : A \to \mathbb{R}$ a function and (x_0, y_0) a point of A. The *partial derivative* of f with respect to x at the point (x_0, y_0) is the limit

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

while the *partial derivative* of f with respect to y at the point (x_0, y_0) is the limit

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}.$$

When both limits exist we say that the function *has partial derivatives* at (x_0, y_0), and if this happens at all points of A we say that f *has partial derivatives.*

Here the difficulties begin: in the case of several variables, knowing that a function has partial derivatives is not as strong as in the case of a single variable: for example, it may happen that the function is not continuous. To get results comparable to those obtained previously we need the stronger notion of a *differentiable function*, which we now define.

Assume that $f(x, y)$ has partial derivarives in a point (x_0, y_0) and let L be the function defined on $\mathbb{R}^2$ as follows:

$$L(h, k) = c_1 h + c_2 k \quad \text{where} \quad c_1 = \frac{\partial f}{\partial x}(x_0, y_0) \quad \text{and} \quad c_2 = \frac{\partial f}{\partial y}(x_0, y_0).$$

In other words, L is the scalar product of the vector (h, k) with the vector of partial derivatives at the point (x_0, y_0).

Definition 7.3. We say that f is *differentiable in* (x_0, y_0) if the following limit exists and is equal to 0:

$$\lim_{(h,k) \to (0,0)} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0) - L(h, k)}{\|(h, k)\|} = 0.$$

If this property holds at every point of A we say that f is *differentiable.* The map L (which depends on the point (x_0, y_0)) is called the *differential* of f in (x_0, y_0).

The differential is the "true" generalization of the derivative for functions of several variables. We list without proof some basic properties for a function $f : A \to \mathbb{R}$ defined on an open set A.

- If f is differentiable at a point then it is also continuous at that point

- If f has partial continuous derivatives then f is differentiable
- Assume A is a ball. If f is differentiable and the partial derivatives arezero everywhre, then f is a constant functions.

Problems.

EXERCISE 4.12. Compute the partial derivatives with respect to x and y for the functions

- $f(x,y) = \cos(xy) + \exp(x-y)$ $\qquad f(x,y) = \exp(\frac{xy}{1+x^2})$
- $f(x,y) = \log(1 + x^2 + 2y^2)$ $\qquad f(x,y) = e^{x-y}\cos(x+2y)$
- $f(x,y) = \arctan\frac{y}{x}$ $\qquad f(x,y) = \arctan\frac{x}{y}$
- $f(x,y) = \frac{x-y}{x+y}$ $\qquad f(x,y) = \frac{x^2+y^2}{x^3+y^3}$
- $f(x,y) = \frac{x}{y^2} - \frac{y}{x^2}$ $\qquad f(x,y) = \arctan(x^2y) + \arctan(xy^2)$
- $f(x,y) = e^{x^2+xy-y^2}$ $\qquad f(x,y) = \frac{xy}{x^2+y^2}$.

8. Complex numbers

8.1. The complex plane. The set of complex numbers $\mathbb{C}$ is simply $\mathbb{R}^2$, the set of *ordered* pairs of real numbers (x, y) i.e. the "Cartesian plane", with a new notation and an additional operation.

The new notation is the following: we denote the vector $(0, 1)$ by i and call it *the imaginary unit.* Then a generic vector (x, y) is written $z = x + iy = (x, y)$. The components x and y are called *real part* and *imaginary part* of z:

$$z = x + iy, \quad x = \Re z, \quad y = \Im z.$$

The sum of two vectors $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ in the new notations becomes

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

from which we see that $\Re(z_1 + z_2) = \Re(z_1) + \Re(z_2)$ and $\Im(z_1 + z_2) = \Im(z_1) + \Im(z_2)$. Furthermore, the*conjugate* of $z = x + iy$ is the vector $\overline{z} = \overline{x + iy} = x - iy$. We stop calling $z = (x, y) = x + iy$ a *vector* and begin to call it a *complex number*.

We will always imagine that $\mathbb{R}$ is *immersed* in $\mathbb{C}$ in the natural way: that is, we shall identify the real number x with the complex number $(x, 0) = x + i0 = x$. Then we say that a complex number is *real* if its imaginary part is zero, and that it is *purely imaginary* if its real part is zero.

The new operation is the *product* of complex numbers, defined as follows:

$$z_1 z_2 = (x + iy_1)(x + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

In particular we have $i^2 = i\, i = -1$, thus in this new system of numbers there exist numbers whose square is -1! The *module* of z is just the length of the vector $z = (x, y)$:

$$|z| = |x + iy| = |(x, y)| = \sqrt{x^2 + y^2}.$$

It is easy to check that

$$|z|^2 = z\overline{z}.$$

We know that the length of vectors allows to define a distance on $\mathbb{C} = \mathbb{R}^2$ given by

$$d(z_1, z_2) = |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

This distance satisfies the triangle inequality

$$d(z_1, z_3) \le d(z_1, z_2) + d(z_2, z_3) \quad \text{for all} \quad z_1, z_2, z_3 \in \mathbb{C}$$

which implies (writing $d(z_1 + z_2, 0) \le d(z_1 + z_2, z_2) + d(z_2, 0)$)

$$|z_1 + z_2| \le |z_1| + |z_2|.$$

We can also introduce the scalar product of two complex numbers, defined as

$$(z, w) = z\overline{w}.$$

This is a bilinear form, antisymmetric (i.e. $(z, w) = \overline{(w, z)}$), satisfying

$$(z, z) = |z|^2 \ge 0$$

and the *Cauchy-Schwartz inequality*

$$|(z, w)| \le \sqrt{(z, z)}\sqrt{(w, w)} = |z|\ |w|$$

(actually we have $|(z, w)| = |z| \cdot |\overline{w}| = |z| \cdot |w|$). The distance d is exactly the distance corresponding to this scalar product.

For each $z \ne 0$, we can define the *inverse* of z as the unique complex number w such that $zw = 1$. Explicitly we can write

$$z^{-1} = \frac{1}{z} = \frac{\overline{z}}{|z|^2}.$$

More generally, if z, w are two complex numbers and w is not zero, we can divide z by w, setting $z/w = zw^{-1}$. In this way we have defined the *field of complex numbers* $(\mathbb{C}, +, \cdot)$, endowed with the operations of sum and product, the corresponding inverse operations (subtraction and division) and moreover endowed with a scalar product and a distance.

An alternative, very useful way to write a complex number z is via *polar coordinates* (ρ, θ). The definition of ρ is very simple:

$$\rho = |z|$$

thus ρ is precisely the modulus of z. The definition of θ is more delicate; θ it is the angle formed by the axis of abscissas and the vector $z = (x, y)$ (the angle must be measured in radians starting from the axis of abscissas). A precise definition is the following: given $z = x + iy$, it is always possible to determine a real number θ such that

$$\begin{cases} x = \rho \ \cos(\theta) \\ y = \rho \ \sin(\theta) \end{cases}$$

that is, $z = \rho(\cos(\theta) + i\sin(\theta))$. When $z = 0$ we have $\rho = 0$ so that any real number θ satisfies the relations and we say that then θ is not defined. When $z \neq 0$ we have an infinite number of values of θ satisfying these conditions, since sin and cos are periodic functions; we denote the set of such values by $\operatorname{Arg}(z)$ (the argument of z). If θ is one of such values, all the others are given by the formula $\theta + 2k\pi$ for any $k \in \mathbb{Z}$; however, there is only one of these values in the interval $(-\pi, \pi]$, and we shall call it the *principal value* of $\operatorname{Arg}(z)$ and denote it by $\arg(z)$.

To sum up: given $z \in \mathbb{C}$, $z \neq 0$, we can write

$$z = \rho(\cos\theta + i\sin\theta)$$

for a unique couple (ρ, θ) where $\rho = |z| > 0$ and $\theta = \arg(z) \in (-\pi, \pi]$. This is called the *polar representation* of z.

One introduces the notation

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad \theta \in \mathbb{R}.$$

Then we can write

$$z = \rho e^{i\theta} = \rho(\cos\theta + i\sin\theta), \qquad \rho = |z|, \quad \theta \in \operatorname{Arg}(z).$$

It is easy to check that $|e^{i\theta}| = 1$, and for all θ_1, θ_2 one has

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}.$$

These relationships have nothing magical about them, in fact the definition just given of the exponential of $i\theta$ can be justified in various other equivalent ways. Some examples:

$$1 = 1 \cdot e^{i0} = e^{2\pi i}, \qquad i = e^{\frac{\pi}{2}i} = e^{\frac{5}{2}\pi i}, \qquad -1 = e^{\pi i} = e^{3\pi i}, \qquad -i = e^{-\frac{\pi}{2}i} = e^{\frac{3}{2}\pi i},$$

Polar coordinates allow us to give a clear geometric meaning to the operations introduced at the beginning. If $z_1 = \rho_1 e^{i\theta_1}$ and $z_2 = \rho_2 e^{i\theta_2}$ we have

$$z_1 z_2 = \rho_1 \rho_2 e^{i\theta_1} e^{i\theta_2} = \rho_1 \rho_2 e^{i(\theta_1+\theta_2)}$$

which means

$$|z_1 z_2| = |z_1|\ |z_2|, \qquad \operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2).$$

(Note that the last formula fails for $\arg(z)$ since we need to compute the angles modulo 2π). Thus multiplying two complex numbers means to multiply their moduli and add their angles. In a similar way, for $z = \rho e^{i\theta}$ we have

$$\overline{z} = \rho e^{-i\theta} \qquad \Longrightarrow \qquad |\overline{z}| = |z|, \quad \operatorname{Arg}(\overline{z}) = -\operatorname{Arg}(z),$$

therefore for the inverse of z we have

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{\overline{\rho e^{i\theta}}}{\rho^2} = \frac{1}{\rho} e^{-i\theta} \qquad \Longrightarrow \qquad \left|\frac{1}{z}\right| = \frac{1}{|z|}, \quad \operatorname{Arg}\left(\frac{1}{z}\right) = -\operatorname{Arg}(z)$$

and for the ratio

$$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2} e^{i(\theta_1-\theta_2)} \qquad \Longrightarrow \qquad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad \operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \operatorname{Arg}(z_1) - \operatorname{Arg}(z_2).$$

Complex numbers were introduced in order to extract roots of any real number; but the result exceeded expectations, because there is *always* more than one root of any nonzero number. If $n \geq 1$ is an integer, finding the n -th roots of z means finding all solutions w of the equation $w^n = z$. In polar coordinates $w = re^{i\phi}$ and $z = \rho e^{i\theta}$ this equation becomes

$$r^n e^{in\phi} = \rho e^{i\theta}.$$

In this equation, the number $z = \rho e^{i\theta}$ is given and the unknown are r and ϕ. Equating $r^n = \rho$ and $n\phi = \theta$ we obtain immediately that $r = \sqrt[n]{\rho}$ is the positive n-th rooth of ρ in the sense of real numbers. To compute the argument ϕ, as usual, we have the problem that $\theta \in \operatorname{Arg}(z)$ has an infinite number of possible values $\theta + 2k\pi$, so that

$$\phi = \frac{\theta + 2k\pi}{n}, \qquad k \in \mathbb{Z}.$$

However, these values give only n distinct solutions. Choosing for instance $k = 0, \dots, n-1$ we obtain exactly n different roots of z:

$$z^{1/2} = \left\{ \sqrt[n]{\rho}\ e^{i\left(\frac{\theta+2k\pi}{n}\right)} : 0 \leq k \leq n-1 \right\}.$$

This is called *De Moivre's formula* (1722).

Example 8.1. Thus if $n = 2$ we always obtain two square roots of $z = |z|e^{i\theta}$:

$$z^{1/2} = \sqrt{|z|}e^{i\frac{\theta}{2}} \quad \text{and} \quad \sqrt{|z|}e^{i\frac{\theta}{2}+i\pi}$$

(provided $z \neq 0$). Since $e^{i\pi} = -1$, we can write simply

$$z^{1/2} = \pm\sqrt{|z|}e^{i\frac{\theta}{2}}.$$

In particular, if Δ is a positive real number, we have $\theta = \arg(\Delta) = 0$ and hence

$$\Delta > 0 \qquad \Longrightarrow \qquad \Delta^{1/2} = \pm\sqrt{\Delta}.$$

On the other hand, if Δ is a negative real number, we have $\theta = \arg(\Delta) = \pi$; since $e^{i\frac{\pi}{2}} = i$, we obtain

$$\Delta < 0 \qquad \Longrightarrow \qquad \Delta^{1/2} = \pm\sqrt{|\Delta|}i.$$

We can apply this result to give a full representation of the solutions of the second order equation with real coefficients $a, b, c \in \mathbb{R}$

$$az^2 + bz + c = 0, \qquad a \neq 0.$$

Then the usual method of "completing the square" can be used also in the complex case and we obtain the formula

$$z = \frac{-b + \Delta^{1/2}}{2a}.$$

When $\Delta \geq 0$ this gives the usual formula

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

When $\Delta < 0$ we have instead

$$z = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}.$$

We see that if $\Delta \neq 0$ we have always two solutions, which are two complex conjucate numbers in the "impossible" case $\Delta < 0$.

8.2. Elementary functions. Previously we defined the meaning of the symbol $e^{i\theta}$, using sines and cosines, and for an incredible coincidence, the new function checked the usual calculation rules valid for the traditional exponential. Not only we have $e^{i(\theta_1+\theta_2)} = e^{i\theta_1}e^{i\theta_2}$, but one can immediately verify that if we differentiate $e^{i\theta}$ with respect to θ we get $ie^{i\theta}$. Whatever test you do it seems that our definition makes sense: $e^{i\theta}$ seems just the "true" exponential of an imaginary number.

Of course there is something underneath. We don't have the tools to address the problem in detail, but at least we can say what follows: if we have a function $f(x)$ of a real variable x and we would like to extend it to $\mathbb{C}$,

1) this is not always possible i.e. not all $f(x)$ extend to $f(z)$ in a natural way;
2) whenever possible, the extension is unique.

We have already seen how the exponential $e^{i\theta}$ is defined for a pure imaginary exponent $i\theta$, in a way which preserves the multiplicative properties of the exponential. The most natural way to extend this definition to the exponential of any complex number is to set

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y). \tag{8.1}$$

From the previous verification it follows that we have also in this case $e^{z+w} = e^z e^w$. One can even define the derivative in complex sense and verify that the formula $(e^z)' = e^z$ is still true.

We now try to extend $\sin x$ and $\cos x$. Since we have $e^{i\theta} = \cos\theta + i\sin\theta$ we have also $e^{-i\theta} = \cos\theta - i\sin\theta$ (inserting $-\theta$ in the previous formula); summing and subtracting these two formulas we see that

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}.$$

This suggests the following definition (if it works, this is the only possible definition, in view of the uniqueness property mentioned above):

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz}-e^{-iz}}{2i}.$$

Using the relations $(e^z)' = e^z$ and $e^{z+w} = e^z e^w$ one obtains easily

$$(\sin z)' = \cos z, \quad (\cos z)' = -\sin z, \quad (\sin z)^2 + (\cos z)^2 = 1.$$

Note that the functions $\sin z$ and $\cos z$ *are not bounded!* Indeed we can write

$$\sin z = \frac{e^y+e^{-y}}{2}\sin x + i\frac{e^y-e^{-y}}{2}\cos x,$$
$$\cos z = \frac{e^y+e^{-y}}{2}\cos x - i\frac{e^y-e^{-y}}{2}\sin x.$$

Closely connected functions are the *hyperbolic functions*

$$\operatorname{ch} z = \frac{e^z+e^{-z}}{2}, \qquad \operatorname{sh} z = \frac{e^z-e^{-z}}{2}, \tag{8.2}$$

which are called *hyperbolic cosine and sine*, respectively. We have the properties

$$\sin z = -i\operatorname{sh}(iz), \qquad \cos z = \operatorname{ch}(iz)$$

Exercise 4.13. For which values of z the complex tangent $\tan z = \sin z/\cos z$ is defined? How would you define the *hyperbolic tangent* th z? Which is the relation between z and $\tan z$?

Problems.

Exercise 4.14. Compute real and imaginary part of the following numbers:

$$\frac{i-1}{i+1}, \qquad i^n \ (n \in \mathbb{Z}), \qquad \frac{(i+1)^4}{(i-1)^3}, \qquad \frac{1}{(3i-4)^4}$$

Exercise 4.15. Compute all solutions $z \in \mathbb{C}$ of the equations:

$$z^7 + i = 0, \qquad z^{11} + iz^4 = 0, \qquad z^6 + \overline{z}^6 = 0, \qquad z^2 + \frac{1}{z^2} = 0.$$

Exercise 4.16. Write all the possible values of the complex powers

$$(2i)^{-i} \qquad 3^i, \qquad i^1, \qquad 1^i, \qquad 1^1.$$

CHAPTER 5

Solutions to selected problems

Numbers in **boldface** refer to the number of the corresponding problem.

1.1. We can simplify the intersection by writing

$$[1,10] \cap]4,12] =]4,10].$$

The intervals $]-2,-1[$ e $[0,5]$ are separate, and it is not possible to simplify their union. in a similar way, we can not simplify $[1,2] \cup]3,4[$.

1.8.

- $12° \; 27' \; 30'' = 12,458° = 0,217$ rad;
- $17° \; 2' \; 50'' = 17,047° = 0,297$ rad;
- $30° \; 12' \; 30'' = 30,208° = 0,527$ rad.

1.21.

$$(1+\sqrt{3})^2 - 2\sqrt{3} = 1 + 2\sqrt{3} + \left(\sqrt{3}\right)^2 - 2\sqrt{3} = 4 \quad \text{(integer)},$$

$$(1+\sqrt{2})^2 - \sqrt{2} = 1 + 2\sqrt{2} + \left(\sqrt{2}\right)^2 - \sqrt{2} = 3 + \sqrt{2} \quad \text{(non integer)},$$

$$\frac{\sqrt{200}}{\sqrt{8}} = \frac{\sqrt{8 \cdot 5^2}}{\sqrt{8}} = \sqrt{5^2} = 5 \quad \text{(integer)},$$

$$(1+\sqrt{2})(1+\sqrt[4]{2})(1-\sqrt[4]{2}) = (1+\sqrt{2})(1-(\sqrt[4]{2})^2) = (1+\sqrt{2})(1-\sqrt{2}) = 1-2 = -1 \quad \text{(integer)}.$$

1.36. Since

$$\frac{a}{x} + \frac{b}{x+1} = \frac{a(x+1)+bx}{x(x+1)} = \frac{(a+b)x+a}{x^2+x} \;,$$

a necessary and sufficient condition to have

$$\frac{1}{x^2+x} = \frac{a}{x} + \frac{b}{x+1}$$

is that $a+b=0$, $a=1$. We conclude $a=1$, $b=-1$.

2.1. The second degree equation ax^2+bx+c, $(a \neq 0)$ has only one solution if and only if the discriminant $\Delta = b^2-4ac$ is equal to 0. The discriminant $x^2+kx+(k^2-1)$ is $\Delta = k^2-4(k^2-1) = 4-3k^2$ and the solutions of the equation $0 = \Delta = 4-3k^2$ are $k = \pm 2/\sqrt{3}$.

2.8.

$$a)\ \text{no solution}, \qquad b)\ \{x \leq 0\}, \qquad c)\ \{x=-1\}.$$

2.10.

$$a)\ x = 5, -3, \qquad b)\ x = 2, \qquad c)\ x = 7, \frac{13}{3}.$$

2.14.

$$a)\ x = -2 \qquad b)\ \text{no solution} \qquad c)\ x = 2.$$

2.19. The logarithm is defined only for positive numbers, hence the set of definition it is given by $x-1>0$ and $6-x>0$, that is $1<x<6$. When $1<x<6$ we can write

$$\log(x-1)+\log(6-x)=\log((x-1)(6-x))$$

, thus we have $\log(x-1)+\log(6-x)\geq 0$ if and only if

$$(x-1)(6-x)\geq 1 \iff -x^2+7x-6\geq 1 \iff x^2-7x+7\leq 0\,.$$

The inequality $x^2-7x+7\leq 0$ is solved as usual and we obtain $\frac{1}{2}(7-\sqrt{21})\leq x\leq \frac{1}{2}(7+\sqrt{21})$. To sum up, the expression $\log(x-1)+\log(6-x)$ is defined for $1<x<6$, vanishes for $x=\frac{1}{2}(7\pm\sqrt{21})$ and is positive for $\frac{1}{2}(7-\sqrt{21})<x<\frac{1}{2}(7+\sqrt{21})$.

3.3.

(1) $f(g(x))=\sqrt{x^2-3x}$, $\quad g(f(x))=|x|-3\sqrt{x}$.

(2) $f(g(x))=\dfrac{1}{2x-2}$, $\quad g(f(x))=\dfrac{2x-3}{2x-4}$.

(3) $f(g(x))=\dfrac{2x-2}{2x-4}$, $\quad g(f(x))=\dfrac{5-x}{x-1}$.

3.6. The function $\sin(x)$ has no limit because it oscillates around the abscissa axis and never flattens out on it, that is, it is bounded but has no horizontal asymptote. The set of definition of the functions $\sqrt{1-x}, \log(1+\cos(x)), \dfrac{1}{\cos(\log(x))}$ does not contain any interval of the form $]J,+\infty[$ and for these functions it doesn't make sense to talk about a limit as $x\to+\infty$.

Finally, we have $\lim_{x\to+\infty}\dfrac{\sin(x)}{\log(x)}=0$. Indeed, for every fixed $\epsilon>0$ we consider $K=e^{\frac{1}{\epsilon}}$ and note that for any $x>K$ we have

$$-1\leq\sin(x)\leq 1,\qquad -\epsilon<\frac{\sin(x)}{\log(x)}<\epsilon\,.$$

3.7.

$$\lim_{x\to+\infty}\frac{x^4-3x^2}{8x^3+x^2}=+\infty,\qquad \lim_{x\to+\infty}\frac{2x^4+x^5}{3x^5-x^2}=\frac{1}{3},\qquad \lim_{x\to+\infty}\frac{x^4-x^5}{x^4-x^6}=0,$$

$$\lim_{x\to-\infty}\frac{x^4-3x^2}{8x^3+x^2}=-\infty,\qquad \lim_{x\to-\infty}\frac{2x^4+x^5}{3x^5-x^2}=\frac{1}{3},\qquad \lim_{x\to-\infty}\frac{x^4-x^5}{x^4-x^6}=0.$$

3.12.

$$\lim_{x\to+\infty}\frac{x+\sin x}{e^x}=0,\qquad \lim_{x\to-\infty}\frac{e^{\sin x}}{x^2}=0,\qquad \lim_{x\to0}x\cdot\sin\left(\frac{1}{x}\right)=0.$$

3.13. In the first limit, with the substitution $y=\pi x$ we find

$$\lim_{x\to0}\frac{e^{\pi x}-1}{x}=\lim_{y\to0}\frac{e^y-1}{y/\pi}=\pi\lim_{y\to0}\frac{e^y-1}{y}=\pi\,.$$

Note that we have just proved a particular case of the general formula

$$\lim_{x\to0}\frac{e^{ax}-1}{x}=a,\qquad a\in\mathbb{R}\,.$$

In the second limit, since $2=e^{\log(2)}$, we have $2^x=(e^{\log(2)})^x=e^{x\log(2)}$ e quindi

$$\lim_{x\to0}\frac{2^x-1}{x}=\lim_{x\to0}\frac{e^{x\log(2)}-1}{x}=\log(2)\,.$$

In the third limit, if $a=0$ then $\sin(ax)=0$ for every x and therefore the limit is also equal to 0. If $a\neq0$, with the change of variable $y=ax$ we obtain

$$\lim_{x\to0}\frac{\sin(ax)}{x}=\lim_{y\to0}\frac{\sin(y)}{y/a}=a\lim_{y\to0}\frac{\sin(y)}{y}=a\,.$$

4.1.

$$(6x^3+2x^2-2x+1)'=18x^2+4x-2, \qquad (8x^7-4x^2)'=56x^6-8x,$$
$$\left(\frac{1}{x}-\frac{1}{x+1}\right)'=\frac{-1}{x^2}+\frac{1}{(x+1)^2}, \qquad \left(\frac{x+1}{x-1}\right)'=\frac{(x-1)-(x+1)}{(x-1)^2}=\frac{-2}{(x-1)^2},$$
$$(e^{2x})'=2e^{2x}, \qquad \left(\frac{1}{x}\right)'=\frac{-1}{x^2}, \qquad (\sqrt{x})'=\frac{1}{2\sqrt{x}},$$
$$(\sqrt{2-3x})'=\frac{-3}{2\sqrt{2-3x}}, \qquad (3^x)'=\log 3\cdot 3^x, \qquad (x^x)'=(\log x+1)x^x,$$
$$(\log\cos x)'=-\tan x, \qquad \arctan{(5-3x^2)}'=\frac{-6x}{1+(5-3x^2)^2}.$$

4.4. The functions which **DO NOT** satisfy the assumptions of Rolle's Theorem on the interval $[-1,1]$ are: x^3+x^2+4, $x^2-|x|$, $x^2+\frac{1}{x^2}$ and $\tan(\pi x)$.