A REMARK ON UNIFORMLY SYMMETRIZABLE SYSTEMS

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Abstract

We prove that any first order system, in one space variable, with analytic coefficients depending only on time, is smoothly symmetrizable if and only if it is uniformly symmetrizable. Thus any one of this conditions is sufficient for the well posedness in C^∞.

1. Introduction

We consider the Cauchy problem on $[0,T]\times\mathbb{R}_x$

$$u_t = A(t)u_x + B(t,x)u + F(t,x) \qquad \text{on } [0,T]\times\mathbb{R}_x \tag{1.1}$$

$$u(0,x) = u_0(x) \qquad \text{on } \mathbb{R}_x \tag{1.2}$$

in one space variable $x\in\mathbb{R}$. System (1.1) is called *uniformly symmetrizable* if there exists an $N\times N$ matrix $S(t)$, possibly nonsmooth in t, such that

$$\|S(t)\| + \|S(t)^{-1}\| \le M < \infty \quad \text{on} \quad [0,T], \tag{1.3}$$

$$S(t)A(t)S(t)^{-1} \quad \text{is Hermitian.} \tag{1.4}$$

Clearly this is equivalent to assume that $A(t)$ is *uniformly diagonable* with real eigenvalues, since any Hermitian matrix can be diagonalized by a unitary change of basis. In particular this implies $A(t)$ is a (weakly) *hyperbolic* matrix, meaning that its eigenvalues are purely real.

If in addition to (1.3), (1.4) one assumes $S(t)$ is a C^1 function, and in this case the system is called *smoothly* symmetrizable, then it is well known that (1.1), (1.2) is well posed in C^∞. Indeed, well posedness holds for any system

$$u_t = \sum_{j=1}^{n} A_j(t,x)u_{x_j} + B(t,x)u + F(t,x) \tag{1.5}$$

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in any number of variables, provided the $N \times N$ matrix

$$A(t,x,\xi) = \sum_{j=1}^{n} A(t,x)\xi_j$$

has a smooth symmetrizer $S(t,x,\xi)$ belonging to $C^1([0,T];S^0)$ (see e.g. [8]).

On the other hand, when the symmetrizer is nonsmooth with respect to t well posedness may fail to hold, as the following arguments show.

Example 1.1 (The 2×2 case). Tarama [7] (see also [1]) constructed two C^∞ functions $a(t), b(t)$ on $[0,T]$ such that

$$0 < C_1 \leq \frac{a(t)}{b(t)} \leq C_2 \qquad \text{on} \quad [0,T] \tag{1.6}$$

and that the 2×2 system

$$u_t = A(t)u_x, \qquad A(t) = \begin{pmatrix} 0 & a(t) \\ b(t) & 0 \end{pmatrix} \tag{1.7}$$

is not well posed in C^∞. Indeed, an easy computation shows that the 2×2 real valued matrix

$$A(t,\xi) = \begin{pmatrix} d_1(t,\xi) & a(t,\xi) \\ b(t,\xi) & d_2(t,\xi) \end{pmatrix}$$

is uniformly symmetrizable if and only if one of the following equivalent conditions is fulfilled:

$$4ab + (d_1 - d_2)^2 \geq C(a-b)^2 \quad \text{for some} \quad C > 0, \tag{1.8}$$

$$4ab + K(d_1 - d_2)^2 \geq \epsilon(a^2 + b^2) \quad \text{for some} \quad K < 1 \quad \text{and} \quad \epsilon > 0, \tag{1.9}$$

with C, K, ϵ independent of t, ξ. In particular, when $d_1 - d_2 = 0$ as in example (1.7), condition (1.9) is equivalent to (1.6).

In our paper we show that, in the special case of space dimension equal to one, and with the additional condition

$$A(t) \quad \text{is real analytic,} \tag{1.10}$$

the matrix $A(t)$ is in fact uniformly symmetrizable if and only if it is smoothly symmetrizable. As a consequence, we can prove:

Theorem 1.1. *Consider Problem* (1.1), (1.2) *under assumptions* (1.3), (1.4) *and* (1.10), *and assume* $B(t,x) \in C([0,T];W^{s,\infty}(\mathbb{R}))$ *for some* $s \geq 0$. *Then, for any* $u_0 \in H^s(\mathbb{R})$ *and* $F(t,x) \in C([0,T];H^s(\mathbb{R}))$, *Problem* (1.1), (1.2) *has a unique solution* $u \in C^1([0,T];H^s(\mathbb{R}))$.

Remarks. We conclude this section with a few remarks.

1. We were not able to extend our method to the case of several space dimensions $n > 1$. We suspect that there exist uniformly but not smoothly symmetrizable matrices $A(t, \xi)$ with analytic coefficients. However, smooth symmetrizability is not necessary for the C^∞ well posedness. Indeed, in the case of 2×2 systems one can prove (Nishitani [5], Nishitani and Colombini [1]) that assumptions (1.3), (1.4) and (1.10) are sufficient for the C^∞ well posedness. The general case $N \geq 3$, $n \geq 2$ is open.
2. We also mention a related result due to Kajitani [3] who proved that any uniformly diagonable hyperbolic system with smooth (not necessarily analytic) coefficients is well posed in the Gevrey classes γ^s for $s < 2$.
3. Another class of hyperbolic systems with analytic coefficients depending only on time, well posed in C^∞, is the class of *pseudosymmetric* systems introduced in [2]. The pseudosymmetricity assumption is in general not comparable with (1.3), (1.4). We recall also the result of Nishitani [4] concerning 2×2 systemswith analytic coefficients depending on $(t, x) \in \mathbb{R}^2$.

2. Proof of Theorem 1.1

Lemma 2.1. *Let $A(z)$ be a $N \times N$ matrix, with coefficients holomorphic on a complex neighbourhood of the real interval $I =]a, b[$, and assume $A(z)$ has real eigenvalues for real $z \in]a, b[$. Then there exist $\lambda_1(z), \dots, \lambda_N(z)$ holomorphic functions on some complex neighbourhood of I, such that the spectrum of $A(z)$ is exactly $\{\lambda_1(z), \dots, \lambda_N(z)\}$ for all z.*

Proof. We shall apply the Schwartz reflection principle in the following form (see e.g. [6]):

Let Ω be an open subset of $\mathbb{C}$ such that $z \in \Omega \iff \bar{z} \in \Omega$, and write

$$\Omega^\pm = \{z \in \Omega : \pm\Im z > 0\}.$$

Let $f(z)$ be holomorphic on Ω^+ and assume

$$\Im f(z_n) \to 0$$

for any sequence $z_n \in \Omega^+$ converging to a point of $\omega \cap \mathbb{R}$.

Then f can be extended to a function F, holomorphic on Ω, such that $F(\bar{z}) = \overline{F(z)}$.

Consider the characteristic polynomial $p(\lambda, z) = \det(\lambda I - A(z))$ of the matrix $A(z)$. As it is well known, apart from isolated exceptional points, each point has a neighbourhood where the roots in λ of $p(\lambda, z) = 0$ can be expressed as N holomorphic functions $\lambda_1(z), \dots, \lambda_N(z)$ (not necessarily distinct). By analytic continuation, any simply connected domain not containing an exceptional point has the same property. Now, denote by D_r the disk $|z - t_0| < r$ for a fixed $t_0 \in I$ and write

$$D_r^{\pm} = \{z \in D_r : \pm\Im z > 0\}.$$

By the above argument, we can express the roots of $p(\lambda, z) = 0$ on D_r^+ as N holomorphic functions $\lambda_1(z), \dots, \lambda_N(z)$, provided r is small enough. Notice that $|\lambda_j(z)| \leq M$ on D_r, with a bound M depending only on the coefficients of $A(z)$.

Now, fix a root $\lambda(z) = \lambda_j(z)$ for some $j = 1, \dots, N$. We shall prove that $\lambda(z)$ extends to a holomorphic function on D_r; this will follow at once from Schwartz' principle, as soon as we prove that $\Im\lambda(z_n) \to 0$ for any sequence $z_n \in D_r^+$ with $z_n \to t^* \in \mathbb{R}$. By a compactness argument, this is equivalent to prove that if $\lambda(z_{n_k}) \to \lambda^*$ for some subsequence z_{n_k}, then $\lambda^* \in \mathbb{R}$; but this follows immediately by continuity, since

$$p(\lambda^*, t^*) = \lim p(\lambda(z_{n_k}), z_{n_k}) = 0$$

and $p(\lambda, t^*)$ has only real roots for $t^* \in \mathbb{R}$ by the hyperbolicity assumption.

Thus we have proved that, in a complex neighbourhood of each point $t_0 \in I$, we can represent the roots of $p(\lambda, z) = 0$ as holomorphic functions $\lambda_1(z), \dots, \lambda_N(z)$; by analytic continuation we conclude the proof. $\square$

For any $N \times N$ matrix A with distinct eigenvalues $\lambda_1, \dots, \lambda_\nu$, $\nu \leq N$, we can define the projections P_j on the corresponding eigenspaces using the Dunford integrals

$$P_j(A) = \frac{1}{2\pi i} \int_{\Gamma_j} (\zeta I - A)^{-1} d\zeta, \qquad j = 1, \dots, \nu \tag{2.1}$$

where Γ_j is the boundary of a disk containing λ_j but not λ_i with $i \neq j$. We have the following well known properties:

$$P_i P_j = \delta_{ij} P_j,$$
$$P_j A = \lambda_j P_j,$$
$$\sum_{j=1}^{\nu} P_j = I.$$

Define now the operator

$$Q \equiv Q(A) = \sum_{j=1}^{\nu} P_j^* P_j, \tag{2.2}$$

which enjoys the property

$$QA = \sum_{j=1}^{\nu} P_j^* P_j A = \sum_{j=1}^{\nu} \lambda_j P_j^* P_j. \tag{2.3}$$

We have then:

Lemma 2.2. *Let A be an $N \times N$ matrix, and assume S symmetrizes A, i.e.,*

$$SAS^{-1} \quad \textit{is Hermitian.}$$

Then

$$C_0^{-1} I \le Q \le C_0 I$$

with

$$C_0 = \|S\|^2 \cdot \|S^{-1}\|^2.$$

Proof. Since the spectrum of A and SAS^{-1} is the same, we have

$$\begin{aligned} P_j(SAS^{-1}) =& \frac{1}{2\pi i} \int_{\Gamma_j} (\zeta I - SAS^{-1})^{-1} d\zeta \\ =& SP_j(A)S^{-1} \end{aligned}$$

as it is readily seen. Since SAS^{-1} is Hermitian, the operator $\pi_j = P_j(SAS^{-1})$ is an orthogonal projection. Thus

$$\begin{aligned} (Qv, v) &= \sum |P_j(A)v|^2 = \sum |S\pi_j(A)S^{-1}v|^2 \\ &\le \|S\|^2 \sum |\pi_j S^{-1} v|^2 = \|S\|^2 |S^{-1}v|^2 \le C_0 |v|^2 \end{aligned}$$

and conversely

$$\begin{aligned} |v|^2 &\le \|S^{-1}\|^2 |Sv|^2 = \|S^{-1}\|^2 \sum |\pi_j S v|^2 \\ &= \|S^{-1}\|^2 \sum |SP_j v|^2 \le C_0 \sum |P_j v|^2 = C_0 (Qv, v). \end{aligned}$$

□

We are now ready to prove Theorem 1.1. By Lemma 2.1 we know that the eigenvalues $\lambda_1(z), \dots, \lambda_N(z)$ of $A(z)$ are holomorphic functions on a neighbourhood of the real interval $[0, T]$. In particular, for any $i \ne j$ two cases are possible: either $\lambda_i \equiv \lambda_j$ everywhere, or $\lambda_i = \lambda_j$ only at isolated points. Thus we may define ν holomorphic functions $\lambda_1(z), \dots, \lambda_\nu(z)$, $\nu \le N$, such that $\lambda_i = \lambda_j$ only at isolated points and

$$\operatorname{spec}(A(z)) = \{\lambda_1(z), \dots, \lambda_\nu(z)\}.$$

By possibly restricting the complex neighbourhood Ω of the real interval $[0, T]$, we may assume that the $\lambda_j(z)$ are holomorphic on Ω and may coincide only at a finite number of real points $t_1, \dots, t_k \in [0, T]$, while they are distinct for $z \in \Omega \setminus \{t_1, \dots, t_k\}$.

Let us now define, for $z \notin \{t_1, \dots, t_k\}$,

$$P_j(z) = P_j(A(z)) = \frac{1}{2\pi i} \int_{\Gamma_j} (\zeta I - A(z))^{-1} d\zeta, \qquad j = 1, \dots, \nu$$

where Γ_j is the boundary of a small disk centered in $\lambda_j(z)$ and not containing $\lambda_i(z)$ for $i \neq j$. Clearly $P_j(z)$ is a matrix valued holomorphic function on

$$\tilde{\Omega} = \Omega \setminus \{t_1, \dots, t_k\}$$

since $\lambda_1(z), \dots, \lambda_\nu(z)$ are continuous and distinct on $\tilde{\Omega}$. Moreover, remarking that

$$(\zeta I - A(z))^{-1} = \frac{{}^{\mathrm{co}}(\zeta I - A(z))}{\det(\zeta I - A(z))},$$

it is easy to prove that the functions $P_j(z)$ may have at most poles at $z = t_1, \dots, t_k$ but no essential singularity. Indeed, we have the estimate

$$\|(\zeta I - A(z))^{-1}\| \leq \frac{C}{|\zeta - \lambda_1(z)| \dots |\zeta - \lambda_\nu(z)|};$$

now we can choose Γ_j such that

$$|\zeta - \lambda_j(z)| = \frac{1}{2} \min_{i \neq \ell} |\lambda_i(z) - \lambda_\ell(z)| \equiv \delta(z).$$

When z approaches one of the possibly singular points t_i, where two of the holomorphic functions $\lambda_j(z)$ coincide, we have nevertheless an estimate like

$$\delta(z) \geq C|z - t_i|^p$$

for some integer $p \geq 1$; in conclusion we obtain

$$\|P_j(z)\| \leq \frac{C}{\delta(z)^{\nu-1}} \leq \frac{C}{|z - t_i|^{p(\nu-1)}}$$

which implies that $P_j(z)$ has a pole at $z = t_i$, i.e.,

$$P_j(z) = \frac{B(z)}{(z - t_i)^M}$$

for some function $B(z)$ holomorphic near t_i, $B(t_i) \neq 0$, and some integer $M \geq 0$, as claimed. We can now apply Lemma 2.2 which gives for real $z = t$ the estimate

$$\|P_j(t)\| \leq C_0^{1/2}$$

and this implies $M = 0$, i.e., $P_j(z) = B(z)$ can be extended to a holomorphic function also at t_i and hence on the whole open set Ω.

Thus we have proved that $Q(z) = Q(A(z))$, defined as

$$Q(z) = \sum_{j=1}^{\nu} P_j^*(z) P_j(z)$$

for $z \in \tilde{\Omega}$ (see (2.2)), can be extended to a C^∞ function on the whole of Ω; actually, $Q(z)$ is a holomorphic function of $(z, \bar{z})$.

Using again Lemma 2.2, we see that we have constructed a function $Q(t) \in C^\infty([0,T])$ such that for $t \in [0,T]$

$$\begin{aligned} &C_0^{-1} I \leq Q(t) \leq C_0 I, \\ &\|Q'(t)\| \leq C, \\ &Q(t)A(t) \text{ is Hermitian.} \end{aligned}$$

In other words, we have proved that Problem (1.1), (1.2) is smoothly symmetrizable, and the conlcusion of the proof follows by well known and standard arguments (see e.g. [8]).

Remark. Notice that in the above proof we are not able to give an estimate of $\|Q'(t)\|$, but we only know it is bounded by the smoothness of $Q(t)$. Hence in the case of several space dimensions we cannot give an estimate uniform in ξ of the analogous matrix $Q'(t,\xi)$, which is essential for the energy estimate.

References

[1] Ferruccio Colombini and Tatsuo Nishitani. Two by two strongly hyperbolic systems and Gevrey classes (to appear).

[2] Piero D'Ancona and Sergio Spagnolo. On pseudosymmetric hyperbolic systems. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 25(1-2):397–418, 1997. Dedicated to Ennio De Giorgi.

[3] Kunihiko Kajitani. The Cauchy problem for uniformly diagonalizable hyperbolic systems in Gevrey classes. In *Hyperbolic equations and related topics (Katata/Kyoto, 1984)*, pages 101–123. Academic Press, Boston, MA, 1986.

[4] Tatsuo Nishitani. Hyperbolicity of two by two systems with two independent variables. *Comm. Partial Differential Equations*, 23(5-6):1061–1110, 1998.

[5] Tatsuo Nishitani. Conference in Tsukuba, February 1999.

[6] Walter Rudin. *Real and complex analysis.* McGraw-Hill Book Co., New York, third edition, 1987.

[7] Shigeo Tarama. Une note sur les systèmes hyperboliques uniformément diagonalisables. *Mem. Fac. Engrg. Kyoto Univ.*, 56(2):9–18, 1994.

[8] Michael E. Taylor. *Pseudodifferential operators.* Princeton University Press, Princeton, N.J., 1981.

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