EXACT AND APPROXIMATE CORRECTORS FOR STOCHASTIC HAMILTONIANS: THE 1–DIMENSIONAL CASE

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Abstract. We perform a qualitative investigation of critical Hamilton–Jacobi equations, with stationary ergodic Hamiltonian, in dimension 1. We show the existence of approximate correctors, give characterizing conditions for the existence of correctors, provide Lax–type representation formulae and establish comparison principles. The results are applied to look into the corresponding effective Hamiltonian and to study a homogenization problem. In the analysis a crucial role is played by tools from stochastic geometry such as, for instance, closed random stationary sets.

1. Introduction

Given a 1–dimensional ergodic dynamical system $(\tau_x)_{x\in\mathbb{R}}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider a stationary Hamiltonian $H(x, p, \omega)$, defined in $\mathbb{R}\times\mathbb{R}\times\Omega$, enjoying suitable continuity, quasiconvexity and coercivity conditions, and the family of stochastic Hamilton–Jacobi equations

$$H(x, v'(x, \omega), \omega) = a \quad \text{in } \mathbb{R},$$

with $a \in \mathbb{R}$. Note that the periodic and the almost–periodic case are particular instances of this setting, see Remark 4.2. The aim of this paper is to perform a qualitative study of the previous equation when the parameter in the right–hand side is set as the minimum for which it has admissible subsolutions, that is to say Lipschitz random functions which are a.e. subsolutions, behave sublinearly at infinity, and possess stationary increments, almost surely with respect to ω. This distinguished value will be denoted from now on by c and termed as (stationary) critical.

The relevance of the critical equation is that it is the unique among those of the family for which admissible (viscosity) solutions or approximate solutions may possibly exist, see Section 4 for precise definitions. As we will discuss later with more detail, these functions enter as exact or approximate correctors in homogenization problems associated with H, and the notions of critical value and effective Hamiltonian, primarily introduced in [15] in the periodic homogenization setting, are strictly related (see Section 7). We will use, in what follows, the term (approximate) corrector in place of admissible (approximate) solution.

The achievements of the paper can be broken down as follows: we prove that approximate correctors do always exist, give characterizing conditions for the existence of correctors, provide Lax–type representation formulae and establish comparison results. We moreover apply this material to study the effective Hamiltonian and a homogenization problem as well.

While it is clear that, due to the lack of compactness, one cannot expect to find, in general, correctors, the problem of the existence of approximate correctors, even in the 1–dimensional case, was not yet clarified. For instance in [16] Lions and Souganidis tackle this issues and exploit a Crandall's result on Eikonal Hamiltonians to construct a 1–dimensional example showing that *bounded* approximate correctors in general do not exist, in contrast to what happens in the almost periodic case, see [2, 13]

The point is that, in the almost periodic setting, the probability space Ω can be endowed with a topology making it compact and the Hamiltonian, because of its stationary character, continuous, which accounts for the existence of bounded approximate correctors. In the general stationary ergodic environment this topological construction cannot be carried out, and consequently bounded approximate correctors are not to be expected. According to our results, (possibly unbounded) approximate correctors in dimension 1 do always exist, while the problem is still open in any space dimension.

Beyond the results obtained, we strongly believe that the method we have employed, based on an interplay between stochastic geometry and viscosity solutions theory, brings some novelties and is capable to be applied in more general contexts. The key idea is to regard the stationary ergodic setting as inducing a stochastic geometry on $\mathbb{R}$, whose fundamental objects are the closed random stationary sets, see Section 3. Such random sets play in a sense the same role as points in the deterministic case. Taking also into account that, as a consequence of the Birkhoff Ergodic Theorem, they are spread with some uniformity at infinity (see Proposition 3.5), we can understand, in the end, that some form of compactness is inscribed in the model.

Bearing the previous considerations in mind, we could adapt to the random environment the analysis performed for the critical equation in the periodic setting or, more generally, when the underlying space is a compact manifold, see [12]. It is based on the definition of a (random) distance intrinsically related to the Hamiltonian, and on the property that any suitably chosen trace on a closed random stationary set can be extended by means of such a distance to the whole $\mathbb{R}$ through a Lax–type formula, yielding a class of fundamental admissible subsolutions.

For special choices of the closed random stationary set appearing in the Lax formula we get, in addition, admissible correctors or approximate correctors. The setup can be described in terms of a generalized Aubry set adapted to our context, see Remark 6.10, and which is, of course, closed random stationary.

Let us recall that such an object has been revealed to be useful, for H deterministic, in various contexts, see [6, 9, 12]. It reduces to the classical Aubry set of dynamical systems when the Hamiltonian is sufficiently regular, but it can be also defined, through the metric approach, under broader assumption on H, without making any reference to the Hamiltonian flow. Indeed, it consists of points such that the intrinsic distance from them is a critical solution.

In our case we find that a corrector does exist if and only if the generalized Aubry set is almost surely nonempty. It may be empty only when the critical value c equals $\sup_{x\in\mathbb{R}} (\min_{p\in\mathbb{R}} H(x,p,\omega))$, almost surely in ω; note that the latter quantity is indeed almost surely constant by the ergodicity condition, and, in addition, is the minimum value of the corresponding effective Hamiltonian, see Section 7. In this situation the Aubry set is made up by the maximizers in $\mathbb{R}$ of the function $x \mapsto \min_{p\in\mathbb{R}} H(x,p,\omega)$, and is accordingly empty if such maximizers do not exist, almost surely in ω (and this can actually happen, even for quasi–periodic Hamiltonians, as shown in [16]). In this case, however, we are able to define approximate Aubry sets that provide approximate correctors via the Lax formula.

When $c > \sup_{x\in\mathbb{R}} (\min_{p\in\mathbb{R}} H(x,p,\omega))$, the Aubry set coincides with the whole $\mathbb{R}$ and then there is a unique admissible subsolution, up to addition of real random variables, which is a corrector of class C^1, almost surely in ω.

Altogether, the quite surprising output is that, in dimension 1, the picture is not so different from that of the periodic case, even if, of course, a tribute must be paid to the lack of compactness in the fact that the generalized Aubry set can be empty.

The only antecedent we know in our line of research, outlined above, is the already cited work by Lions and Souganidis [16], where the existence of correctors is put in relation with the nonemptiness of a random generalization of the Mather set.

As we have already pointed out, the subject of our investigation has a clear connection with the homogenization of Hamilton–Jacobi equations in the stationary ergodic environment. This topic has recently attracted a considerable interest both from a theoretical and an application point of view. The existence of sublinear approximate correctors makes the Evans' perturbed test function method [10, 11] viable for carrying out the homogenization procedure in our setting. This seems new because, so far, all the available homogenization methods for stochastic Hamilton–Jacobi equations are based on representation formulae for solutions of the equations involved, and so require the Hamiltonian to be convex in the second argument, while here we are just assuming quasiconvexity.

More precisely in [18, 19], the authors deal with the homogenization of a time–dependent equation, and the result is obtained by passing to the limit in the Lax–Oleinik formulae representing the solutions of the equations with the oscillating variables, when the oscillation parameter goes to 0, through a Γ–convergence type argument, and by using the Subadditive Ergodic Theorem.

We finally wish to emphasize another peculiarity of our work. Even if some results that we present are not new, in particular in the first two sections which are of introductory nature, often it has been not easy to adapt them to the topic we are dealing with, starting from the form they appear in the literature. So we claim some originality in the presentation of this material and hope that it will be useful for further investigations on the subject. This is the case, for instance, of the stability principle for Lipschitz random functions with stationary increments stated in Theorem 3.8. It has been a valuable tool for our analysis, taking the place, in some sense, of Ascoli Theorem, not valid in the random setting.

The paper is organized as follows: in Section 2 we fix terminology and notations, and introduce some basic material that will be repeatedly used throughout the paper. Section 3 is devoted to illustrate the main properties of some classes of random variables that are of primary importance for our analysis. In Section 4 the requirements on the stochastic Hamiltonian are detailed and discussed, the intrinsic distance is defined and the key notions of stationary and free critical value are given. In Section 5 we provide a version of Lax formulae for representing admissible subsolutions of stochastic Hamilton–Jacobi equations. Section 6 contains the main results on exact and approximate correctors, and, finally, the application to stationary ergodic homogenization is presented in Section 7.

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2. Preliminaries

We start by introducing some notations and concepts that will be used throughout the paper.

We denote by $\mathbb{R}_+$ the set of nonnegative real numbers and by $\mathbb{Q}$ the set of rational numbers. We will call *modulus* any nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ vanishing and continuous at 0. A real function θ defined either on $\mathbb{R}$ or on $\mathbb{R}_+$ will be called *coercive* if $\lim_{|h|\to+\infty} \theta(h) = +\infty$.

Given some Euclidean ground space and a subset A, we denote by $\overline{A}$ its closure, by $\mathrm{Int}(A)$ its interior and by ∂A its boundary; we furthermore indicate by $B_R(x_0)$ and B_R the closed ball of radius R centered in x_0 and 0, respectively. The symbol $|x|$ stands for the Euclidean norm of x.

For any Lebesgue measurable subset E, we denote by $|E|$ its Lebesgue measure, and we call E *negligible* whenever $|E| = 0$. When $|E|$ is a positive real number and f is an integrable function on E, the notation $\fint_E f \,\mathrm{d}x$ stands for the integral mean value of f on E.

We say that a property holds *almost everywhere* (*a.e.* for short) if it is valid up to a negligible subset. We write χ_E for the characteristic function of a set E, i.e. the function taking the value 1 on E and 0 outside. We write $\varphi_n \rightrightarrows \varphi$ to mean that a sequence of functions $(\varphi_n)_n$ locally uniformly converges to φ.

We will repeatedly make use of the notion of (sub, super) solution to some Hamilton–Jacobi equation in the viscosity sense, see [3, 4]. Given a locally Lipschitz function ϕ on $\mathbb{R}$, we denote by $\partial\phi(x)$ its *Clarke generalized gradient* [8] at $x \in \mathbb{R}$, defined as

$$\partial\phi(x) := \mathrm{co}\{\lim_i \phi'(x_i) \,:\, \phi \text{ differentiable at } x_i,\ \lim_i x_i = x\},$$

where $\mathrm{co}(A)$ stands for the closed convex hull of a subset A of $\mathbb{R}$.

The symbol $(\Omega, \mathcal{F}, \mathbb{P})$ indicates a *probability space*, where $\mathbb{P}$ is the probability measure on Ω and $\mathcal{F}$ the σ–algebra of $\mathbb{P}$–measurable subsets. A property is said to hold *almost surely* (*a.s.* for short) in ω if it is valid up to a subset of probability 0. We indicate by $L^p(\Omega)$, $p \ge 1$, the usual Lebesgue space on Ω with respect to $\mathbb{P}$. If $f \in L^1(\Omega)$, we write $\mathbb{E}(f)$ for the mean of f on Ω, i.e. the quantity $\int_\Omega f(\omega)\,\mathrm{d}\mathbb{P}(\omega)$.

We qualify as *measurable* a map from Ω to itself, or to a topological space $\mathcal{M}$ with Borel σ–algebra $\mathcal{B}(\mathcal{M})$, if the inverse image of any set in $\mathcal{F}$ or in $\mathcal{B}(\mathcal{M})$ belongs to $\mathcal{F}$. The latter will be also called *random variable* with values in $\mathcal{M}$.

In what follows we will be interested in random variables taking values in $\mathrm{C}(\mathbb{R})$ and $\mathrm{Lip}_\kappa(\mathbb{R})$, the spaces of continuous and Lipschitz–continuous functions with Lipschitz constant less than or equal to $\kappa > 0$, defined on $\mathbb{R}$, respectively. Such spaces are both endowed with the metrizable topology of the uniform convergence on compact subsets of $\mathbb{R}$. We will use the expressions *continuous random function*, *Lipschitz random function* (with Lipschitz constant less than or equal to κ), respectively, for the previously introduced random variables. The following characterization of random continuous functions will be used in the sequel.

Proposition 2.1. *Let $\omega \mapsto v(\cdot, \omega)$ be a map from Ω to $\mathrm{C}(\mathbb{R})$. The following are equivalent facts:*

(i) v is a random continuous function;

(ii) for every $x \in \mathbb{R}$, the function $\omega \mapsto v(x,\omega)$ is measurable in Ω;

(iii) the function $(x,\omega) \mapsto v(x,\omega)$ is jointly measurable in $\mathbb{R} \times \Omega$, i.e. measurable with respect to the product σ–algebra $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$.

Proof. *(i) $\Rightarrow$ (ii).* The function appearing in item *(ii)* is the composition of v and the evaluation map assigning to any continuous function its value in x. From the fact that the latter is continuous from $\mathrm{C}(\mathbb{R})$ to $\mathbb{R}$, we derive the claimed implication.

(ii) $\Rightarrow$ (iii). Let $Q_i^n := i/n + [-1,1)/2n$ for every $i \in \mathbb{Z}$, $n \in \mathbb{N}$. Then the functions

$$v_n(x,\omega) := \sum_i \chi_{Q_i^n}(x)\, v(i/n,\omega)$$

are clearly measurable on $\mathbb{R} \times \Omega$ with respect to $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$. Since they converge pointwise to v on $\mathbb{R} \times \Omega$, assertion (iii) follows.

(iii) $\Rightarrow$ (i). We have to show that, for every $r > 0$,

$$\Omega_r := \{\omega \in \Omega \,:\, d(v(\cdot,\omega),0) \le r\} \in \mathcal{F},$$

where d is a distance on $\mathrm{C}(\mathbb{R})$ inducing the topology of the local uniform convergence on $\mathbb{R}$, say

$$d(f,g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f-g\|_{L^\infty(B_n)}}{\|f-g\|_{L^\infty(B_n)}+1} \qquad f,g \in \mathrm{C}(\mathbb{R}).$$

For every $n \in \mathbb{N}$, let $(x_i^n)_{i\in\mathbb{N}}$ be a dense sequence in B_n. By the continuity of $v(\cdot,\omega)$ for every ω, it is easy to check that

$$\Omega_r = \bigcap_{i,\,n=1}^{+\infty} \left\{ \omega \in \Omega \,:\, \frac{|v(x_i^n,\omega)|}{|v(x_i^n,\omega)|+1} \le 2^n\, r \right\}.$$

The claim follows from the fact that, for every fixed $x \in \mathbb{R}$, the map $\omega \mapsto v(x,\omega)$ is measurable by Fubini's Theorem. □

A 1–*dimensional dynamical system* $(\tau_x)_{x\in\mathbb{R}}$ is defined as a family of mappings $\tau_x : \Omega \to \Omega$ which satisfy the following properties:

(1) the *group property:* $\tau_0 = id$, $\quad \tau_{x+y} = \tau_x \circ \tau_y$;

(2) $\tau_x : \Omega \to \Omega$ is measurable and measure preserving for any x, i.e. $\mathbb{P}(\tau_x E) = \mathbb{P}(E)$ for every $E \in \mathcal{F}$;

(3) the map $(x,\omega) \mapsto \tau_x\omega$ from $\mathbb{R}\times\Omega$ to Ω is jointly measurable, i.e. measurable with respect to the product σ–algebra $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$.

We make the crucial assumption that $(\tau_x)_{x\in\mathbb{R}}$ is *ergodic,* i.e. such that one of the following equivalent conditions hold:

(i) every measurable function f defined on Ω with $f(\tau_x\omega) = f(\omega)$ a.s. in Ω, for any $x \in \mathbb{R}$, is almost surely constant;

(ii) every set $A \in \mathcal{F}$ with $\mathbb{P}(\tau_x A \,\Delta\, A) = 0$ for every $x \in \mathbb{R}$ has probability either 0 or 1, where Δ stands for the symmetric difference.

Given a random variable $f : \Omega \to \mathbb{R}$, for any fixed $\omega \in \Omega$ the function $x \mapsto f(\tau_x\omega)$ is said to be a *realization of f*. The following properties follow from Fubini's Theorem, see [14]: if $f \in L^p(\Omega)$, then $\mathbb{P}$–almost all its realizations belong to $L^p_{loc}(\mathbb{R})$; if $f_n \to f$ in $L^p(\Omega)$, then $\mathbb{P}$–almost all realizations of f_n converge to the corresponding realization of f in $L^p_{loc}(\mathbb{R})$. The Lebesgue spaces on $\mathbb{R}$ are with respect to the Lebesgue measure.

The next lemma guarantees that it is possible to modify a random variable on a set of zero probability without affecting its realizations on sets of positive Lebesgue measure on $\mathbb{R}$, almost surely in ω. It will be exploited, for instance, in Theorem 3.7. The proof is based on Fubini's Theorem again, see Lemma 7.1 in [14].

Lemma 2.2. *Let $\widehat{\Omega}$ be a set of full measure in Ω. Then there exists a set of full measure $\Omega' \subseteq \widehat{\Omega}$ such that for any $\omega \in \Omega'$ we have $\tau_x\omega \in \widehat{\Omega}$ for almost every $x \in \mathbb{R}$.*

We finish the section by stating the Birkhoff Ergodic Theorem for ergodic 1–dimensional dynamical systems. It establishes a relation between statistical and spatial means.

Theorem 2.3 (Birkhoff Ergodic Theorem). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\tau_x)_{x\in\mathbb{R}}$ an ergodic group of translations as above. Then, for any $f \in L^1(\Omega)$,*

$$\mathbb{E}(f) = \lim_{t\to+\infty} \fint_{tE} f(\tau_x\omega)\,\mathrm{d}x \qquad a.s. \text{ in } \omega,$$

where E is any Borel subset of $\mathbb{R}$ with $|E| > 0$.

3. Stationary random variables

In this section we give the notion of stationarity for random functions and extend it, via characteristic functions, to random sets. These objects will be of crucial relevance to get Lax-type formulae representing admissible (sub) solutions of Hamilton–Jacobi equations in the stationary ergodic setting. The concept of stationary random set, as well as other material we expose in this section, comes from Stochastic Geometry. We refer the interested reader to [17] for a nice presentation of this topic. In the last part of the section we introduce the class of random Lipschitz–functions with stationary increments, and prove an Ascoli–type stability result for such random functions.

We say that a jointly measurable function v defined in $\mathbb{R} \times \Omega$ is *stationary* if for any $z \in \mathbb{R}$ there exists a set Ω_z of probability 1 such that

$$v(x + z, \omega) = v(x, \tau_z\omega) \quad \text{ for every } x \in \mathbb{R} \text{ and } \omega \in \Omega_z.$$

Proposition 3.1. *Assume a jointly measurable function v defined on $\mathbb{R} \times \Omega$ to be stationary. Then there exist a measurable function ϕ defined on Ω and a set Ω' of probability 1 such that for every $\omega \in \Omega'$*

$$v(x, \omega) = \phi(\tau_x\omega) \quad \textit{for a.e. } x \in \mathbb{R}.$$

Proof. Let us set $\phi(\omega) := v(0, \omega)$, which is measurable by Fubini's Theorem, and

$$E := \{(x, \omega) \,:\, v(x, \omega) \neq \phi(\tau_x\omega)\}.$$

By the fact that $(x, \omega) \mapsto \phi(\tau_x\omega)$ is jointly measurable we get $E \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$. The stationarity assumption implies that the x–section of E at any fixed $x \in \mathbb{R}$ equals $\Omega \setminus \Omega_x$, which is a set of probability 0, so E is negligible in the product measure. This implies that $\mathbb{P}$–almost all the ω–sections of E has 0 Lebesgue measure, which gives the assertion. □

With the term *(graph–measurable) random set* we will denote a set–valued function $X : \Omega \to \mathcal{B}(\mathbb{R})$ such that

$$\Gamma(X) := \{(x, \omega) \in \mathbb{R} \times \Omega \,:\, x \in X(\omega)\}$$

is measurable in $\mathbb{R} \times \Omega$ with respect to the product σ–algebra $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$. In other term, we require the characteristic function $\chi_{\Gamma(X)}$ to be jointly measurable.

A random set X will be qualified as *stationary* if for every $z \in \mathbb{R}$ there exists a set Ω_z of probability 1 such that

$$X(\tau_z \omega) = X(\omega) - z \qquad \text{for every } \omega \in \Omega_z. \tag{1}$$

We use a stronger notion of measurability, which is usually named in the literature after Effros, to define a *closed random set*, say $X(\omega)$. Namely we require $X(\omega)$ to be a closed subset of $\mathbb{R}$ for any ω and

$$\{\omega \,:\, X(\omega) \cap K \neq \emptyset\} \in \mathcal{F}$$

with K varying among the compact (equivalently, open) subsets of $\mathbb{R}$. This condition can be analogously expressed by saying that X is measurable with respect to the Borel σ–algebra related to the Fell topology on the family of closed subsets of $\mathbb{R}$. This, in turn, coincides with the Effros σ–algebra, see [17] for more details. If $X(\omega)$ is measurable in this sense then it is also graph–measurable.

A closed random set X is called stationary if it, in addition, satisfies (1). Note that in this event the set $\{\omega \,:\, X(\omega) \neq \emptyset\}$, which is measurable by the Effros measurability of X, is invariant with respect to the group of translations $(\tau_x)_{x \in \mathbb{R}}$ by stationarity, so it has probability either 0 or 1 by the ergodicity assumption.

A convenient way to produce random closed (stationary) sets in $\mathbb{R}$ is indicated by the next result.

Proposition 3.2. *Let f be a continuous random function and C a closed subset of $\mathbb{R}$. Then*

$$X(\omega) := \{x \,:\, f(x, \omega) \in C\}$$

is a closed random set. If in addition f is stationary, then X is stationary.

Proof. It is clear that $X(\omega)$ is a closed subset of $\mathbb{R}$ for every ω. Let K be a compact subset of $\mathbb{R}$. Pick a dense sequence $(y_k)_{k \in \mathbb{N}}$ in K. By the continuous character of $f(\cdot, \omega)$ for every ω we get

$$\{\omega \,:\, X(\omega) \cap K \neq \emptyset\} = \bigcap_n \bigcup_k \{\omega \,:\, f(y_k, \omega) \in C + B_{1/n}\}.$$

Each random variable $f(y_k, \omega)$ is measurable by Tonelli's Theorem since f is jointly measurable. That implies $\{\omega \,:\, X(\omega) \cap K \neq \emptyset\} \in \mathcal{F}$, as it was to be shown. The fact that X is stationary if f is stationary is immediate. □

Remark 3.3. We record for later use that, for a positive radius r, the set

$$X_r(\omega) := X(\omega) + B_r, \quad \omega \in \Omega,$$

is closed random whenever X is an almost surely nonempty closed random set. In fact, for every compact set K in $\mathbb{R}$, $X_r(\omega) \cap K \neq \emptyset$ if and only if $X(\omega) \cap (K + B_r) \neq \emptyset$, hence

$$\{\omega \,:\, X_r(\omega) \cap K \neq \emptyset\} = \{\omega \,:\, X(\omega) \cap (K + B_r) \neq \emptyset\}$$

and the assertion follows by the Effros measurability of X.

Remark 3.4. In this section, where the tools of Stochastic Geometry we employ are essentially based on Fubini's Theorem, the graph measurability of random sets is all we need to carry out the arguments. But for the metric analysis of stochastic Hamilton–Jacobi equations we will develop in the sequel this notion is too weak since, for instance, the crucial property outlined in the previous remak does not hold in general for random sets that are just graph–measurable. This is the reason we have introduced the more stringent definition of Effros measurability for random sets with closed values.

When the σ–algebra $\mathcal{F}$ is complete, graph and Effros measurability are equivalent, see Theorem 2.3 in [17], but this assumption on $\mathcal{F}$ is definitively too strong in our context since it would exclude the model examples of periodic and quasi–periodic Hamiltonians, see Remark 4.2.

In the sequel, we will use the following notation

$$X^{-1}(x) := \{\,\omega \in \Omega \,:\, x \in X(\omega)\}, \qquad x \in \mathbb{R}.$$

We will call *volume fraction* of a stationary random set X the number $q_X = \mathbb{P}(X^{-1}(0))$. Note that

$$q_X = \mathbb{P}(X^{-1}(x)) \qquad \text{for every } x \in \mathbb{R},$$

for $X^{-1}(x)$ and $\tau_{-x}(X^{-1}(0))$ coincide up to a set of probability 0 by stationarity of X and the probability measure is invariant by translation. We also remark that $q_X > 0$ implies $X(\omega) \neq \emptyset$ a.s. in ω, as a straightforward application of Fubini's Theorem shows.

We next exploit the ergodicity assumption to get, through the Birkhoff Ergodic Theorem, an interesting information on the asymptotic structure of closed random stationary sets. It says that they are spread with some uniformity at infinity. This accounts for the fact that homogenization results can be obtained in this setting.

Proposition 3.5. *Let X be an almost surely nonempty closed random stationary set in $\mathbb{R}$. Then for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that*

$$\lim_{r\to+\infty} \frac{|\,(X(\omega)+B_R)\cap B_r|}{|B_r|} \geq 1-\varepsilon \qquad \textit{a.s. in } \omega,$$

provided $R \geq R_\varepsilon$,

Proof. For each $n \in \mathbb{N}$, set $X_n(\omega) := X(\omega) + B_n$, which are stationary closed random sets by Remark 3.3, and denote by q_n their volume fractions. Exploiting the stationary character of the X_n and arguing as in the proof of Proposition 3.1, we derive that there exists a set Ω' of probability 1 such that for every $\omega \in \Omega'$

$$f_n(x,\omega) := \chi_{X_n(\omega)}(x) = \chi_{X_n(\tau_x\omega)}(0) = f_n(0,\tau_x\omega) \qquad \text{for a.e. } x \in \mathbb{R}.$$

Birkhoff Ergodic Theorem then yields

$$\frac{|\,(X(\omega)+B_n)\cap B_r|}{|B_r|} = \fint_{B_r} f_n(x,\omega)\,\mathrm{d}x = \fint_{B_r} f_n(0,\tau_x\omega)\,\mathrm{d}x \underset{r\to+\infty}{\longrightarrow} q_n \quad \text{a.s. in } \omega.$$

To conclude, it is enough to show that $\sup_n q_n = 1$. But this follows from the Monotone Convergence Theorem for $(\chi_{X_n^{-1}(0)})_n$ is an increasing sequence of functions converging to the function identically equal to 1. $\square$

We say that a random Lipschitz function v has *stationary increments* if for any $z \in \mathbb{R}$ there exists a set Ω_z of probability 1 such that for every $\omega \in \Omega_z$.

$$v(x+z,\omega) - v(y+z,\omega) = v(x,\tau_z\omega) - v(y,\tau_z\omega) \quad \text{for all } x,y \in \mathbb{R}.$$

This is equivalent to requiring

$$v(\cdot + z,\omega) = v(\cdot,\tau_z\omega) + k(\omega) \qquad \text{on } \mathbb{R}$$

a.s. in ω, where k is a random variable depending on z. We define

$$\Delta_v(\omega) := \{\, x \in \mathbb{R} \,:\, v(\cdot,\omega) \text{ is differentiable at } x \,\}\,.$$

We derive from Proposition 3.1.

Proposition 3.6. *Let v be a random Lipschitz function with stationary increments. Then Δ_v is a stationary random set, and there exist $\Phi \in L^\infty(\Omega)$ and a set Ω' of probability 1 such that for every $\omega \in \Omega'$*

$$v'(x,\omega) = \Phi(\tau_x\omega) \quad \textit{for a.e. } x \in \mathbb{R}.$$

Moreover, for every $x \in \mathbb{R}$, $v'(x,\cdot)$ exists a.s. in ω and $\mathbb{E}\big(v'(x,\omega)\big) = \mathbb{E}(\Phi)$.

Proof. Let $\kappa > 0$ be such that $v(\cdot,\omega) \in \mathrm{Lip}_\kappa(\mathbb{R})$ for every ω. Let us set

$$\begin{aligned} \underline{\varphi}(x,\omega) &:= \sup_{n\in\mathbb{N}} \inf_{h\in\mathbb{Q}\cap B_{1/n}} \frac{v(x+h,\omega)-v(x,\omega)}{h}, \\ \overline{\varphi}(x,\omega) &:= \inf_{n\in\mathbb{N}} \sup_{h\in\mathbb{Q}\cap B_{1/n}} \frac{v(x+h,\omega)-v(x,\omega)}{h}. \end{aligned}$$

Clearly, $\underline{\varphi}$, $\overline{\varphi}$ are jointly measurable on $\mathbb{R}\times\Omega$, and stationary since v has stationary increments. Since

$$\Delta_v(\omega) = \{x \in \mathbb{R} \,:\, \underline{\varphi}(x,\omega) = \overline{\varphi}(x,\omega)\,\} \qquad \text{for every } \omega \in \Omega,$$

the asserted property for Δ_v follows. Set $\Phi(\omega) := \underline{\varphi}(0,\omega)$. Clearly $\Phi \in L^\infty(\Omega)$, and, according to Proposition 3.1, there exists a set Ω' of probability 1 such that, for every $\omega \in \Omega'$,

$$\underline{\varphi}(x,\omega) = \Phi(\tau_x\omega) \quad \text{for a.e. } x \in \mathbb{R}.$$

This concludes the proof of the first part of the statement since $v'(\cdot,\omega) = \underline{\varphi}(\cdot,\omega)$ in $\Delta_v(\omega)$, i.e. almost everywhere in $\mathbb{R}$ by the Lipschitz–continuity of $v(\cdot,\omega)$.

To prove the remainder, we note that, by the stationary random character of Δ_v, the set

$$X(\omega) := \{x \in \Delta_v(\omega) \,:\, v'(x,\omega) = \Phi(\tau_x\omega)\} \qquad \omega \in \Omega,$$

is stationary random as well. Since $X(\omega)$ has full measure in $\mathbb{R}$ for every ω, we derive from Fubini's Theorem that

$$\mathbb{P}(X^{-1}(x)) = 1 \qquad \text{for a.e. } x \in \mathbb{R}.$$

The stationarity of X yields that this equality holds for *every* $x \in \mathbb{R}$, in particular $v'(x,\cdot)$ is almost surely well defined and $\mathbb{E}(v'(x,\cdot)) = \mathbb{E}(\Phi)$ for every $x \in \mathbb{R}$, as it was to be shown. □

A converse construction is also possible, namely to any function in $L^\infty(\Omega)$ we can associate an *antiderivative* which is a Lipschitz continuous random function with stationary increments. To this purpose, we introduce a 1–parameter group $(U_x)_{x\in\mathbb{R}}$ of isometries on $L^2(\Omega)$ by setting

$$\begin{array}{rcl} U_x : L^2(\Omega) & \to & L^2(\Omega) \\ \Phi(\omega) & \mapsto & \Phi(\tau_x\omega) \end{array}$$

which is strongly continuous, in the sense that

$$\lim_{x\to 0} \|U_x\Phi - \Phi\|_{L^2(\Omega)} = 0, \qquad \Phi \in \mathrm{L}^2(\Omega).$$

We exploit this property to give a meaning in the Cauchy sense to the following integral

$$\int_0^x \Phi(\tau_s\omega)\,\mathrm{d}s$$

as an element of $L^2(\Omega)$, for every fixed $x \in \mathbb{R}$.

Theorem 3.7. *Let $\Phi \in L^\infty(\Omega)$. There exist a Lipschitz random function v with stationary increments and a set Ω' of probability 1 such that for every $\omega \in \Omega'$*

$$v'(x,\omega) = \Phi(\tau_x\omega) \quad \textit{for a.e. } x \in \mathbb{R}.$$

Proof. Let $\kappa = \|\Phi\|_\infty$. Up to redefining Φ on a measurable set of probability 0 it is not restrictive in view of Lemma 2.2 to assume that $|\Phi(\omega)| \le \kappa$ for every $\omega \in \Omega$. We set

$$v(x,\omega) := \int_0^x \Phi(\tau_s\,\omega)\,\mathrm{d}s,$$

For every ω, the function $x \mapsto \Phi(\tau_x\omega)$ is measurable and bounded on $\mathbb{R}$, therefore $v'(x,\omega) = \Phi(\tau_x\omega)$ for a.e. $x \in \mathbb{R}$, and this in turn shows that $v(\cdot,\omega)$ belongs to $\mathrm{Lip}_\kappa(\mathbb{R})$ for every ω. To prove that v is jointly measurable in $\mathbb{R}\times\Omega$, notice that the function

$$g(x,s,\omega) := \chi_{[0,+\infty)}(s)\,\chi_{[0,+\infty)}(x-s)\,\Phi(\tau_s\omega)$$

is jointly measurable on $\mathbb{R}\times\mathbb{R}\times\Omega$. Since

$$v(x,\omega) = \int_{\mathbb{R}} g(x,s,\omega)\,\mathrm{d}s,$$

the asserted measurability of v follows by Fubini's Theorem. Last, we have that, given $z \in \mathbb{R}$,

$$v(x+z,\omega) - v(y+z,\omega) = \int_{y+z}^{x+z} \Phi(\tau_s\,\omega)\,\mathrm{d}s = \int_y^x \Phi(\tau_{s+z}\,\omega)\,\mathrm{d}s = v(x,\tau_z\omega) - v(y,\tau_z\omega)$$

for every $\omega \in \Omega$ and x, $y \in \mathbb{R}$, which proves that v has stationary increments. □

We call a sequence $(v_n)_n$ of random functions *equiLipschitz* if all the v_n take value in $\mathrm{Lip}_\kappa(\mathbb{R})$, for some fixed $\kappa > 0$

We proceed by stating and proving a crucial stability result for the class of Lipschitz random functions with stationary increments.

Theorem 3.8. *Let $(v_n)_n$ be an equiLipschitz sequence of random functions with stationary increments. Then there exist a random Lipschitz function v with stationary increments, a sequence $w_k = \sum_{n\ge n_k} \lambda_n^k v_n$ of finite convex combinations of the v_n and a sequence g_k of real random variables such that*

$$w_k(\cdot,\omega) + g_k(\omega) \rightrightarrows v(\cdot,\omega) \quad \textit{a.s. in } \omega.$$

In addition the sequence of indices $(n_k)_k$ can be taken diverging.

Proof. Let κ be a Lipschitz constant for all the v_n. We denote by Φ_n the functions of $L^\infty(\Omega)$ associated to v_n through Proposition 3.6. We have $\|\Phi_n\|_\infty \leq \kappa$ for every n, therefore Φ_n weakly converge in $L^2(\Omega)$ to some Φ, and a sequence made up by finite convex combinations of the Φ_n, say $\Psi_k = \sum_{n \geq n_k} \lambda_n^k \Phi_n$, strongly converges to Φ in $L^2(\Omega)$. We can also assume, up to extraction of a subsequence, that Ψ_k a.s. converges to Φ, in particular $\|\Phi\|_\infty \leq \kappa$. Let us consider the set

$$E := \{(x, \omega) \,:\, \lim_k \Psi_k(\tau_x \omega) = \Phi(\tau_x \omega)\},$$

which is clearly measurable with respect to the product σ–algebra $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$. The almost sure convergence of Ψ_k to Φ implies that the x–section of E has probability 1 for every fixed $x \in \mathbb{R}$. By Fubini's Theorem we derive that there exists a set Ω' of probability 1 such that, for every $\omega \in \Omega'$,

$$\Psi_k(\tau_x \omega) \to \Phi(\tau_x \omega) \quad \text{for a.e. } x \in \mathbb{R}. \tag{2}$$

Let w_k be a sequence of random functions defined according to the statement. Note that each w_k is an antiderivative of Ψ_k, for any k. Set $g_k := -w_k(0, \cdot)$, which is a random variable by Proposition 2.1. For every $x \in \mathbb{R}$ we set

$$v(x, \omega) := \int_0^x \Phi(\tau_s \omega) \, \mathrm{d}s \qquad \text{a.s. in } \omega.$$

Then

$$|w_k(x, \omega) + g_k(\omega) - v(x, \omega)| \leq \int_0^x |\Psi_k(\tau_s \omega) - \Phi(\tau_s \omega)| \, \mathrm{d}s,$$

and the integral in the above formula becomes almost surely infinitesimal as k goes to infinity, by (2) and the Dominated Convergence Theorem. This proves the asserted convergence. □

In the sequel, we will be interested in random Lipschitz functions with stationary increments and gradient with mean 0. We deduce from Birkhoff Ergodic Theorem that such a random function has a sublinear behavior for $|x|$ going to infinity.

Theorem 3.9. *Let v be a random Lipschitz function with stationary increments. The following are equivalent facts:*

(i) $\mathbb{E}(v'(x, \cdot)) = 0$ *for every* $x \in \mathbb{R}$*;*

(ii) *for every* $x \in \mathbb{R}$

$$\lim_{t \to \pm\infty} \frac{v(x+t, \omega)}{t} = 0 \qquad \textit{a.s. in } \omega.$$

The following result holds:

Theorem 3.10. *Let v be a random Lipschitz function with stationary increments. The following are equivalent facts:*

(i) $\mathbb{E}(v'(x, \cdot)) = 0$ *for every* $x \in \mathbb{R}$*;*

(ii) $\mathbb{E}(v(x, \cdot) - v(y, \cdot)) = 0$ *for every* x, $y \in \mathbb{R}$*.*

Proof. Set $Q := \mathbb{E}(v'(0, \cdot))$. In view of Proposition 3.6, we have that

$$Q = \mathbb{E}(v'(x, \cdot)) \qquad \text{for every } x \in \mathbb{R}.$$

Let us fix $x, q \in \mathbb{R}$. For every ω we have

$$v(x+q,\omega) - v(x,\omega) = \int_0^1 v'(x+tq,\omega)\, q\, \mathrm{d}t.$$

By integrating on Ω, we get

$$\begin{aligned} \mathbb{E}(v(x+q,\cdot)) - \mathbb{E}(v(x,\cdot)) &= \int_\Omega \left(\int_0^1 v'(x+tq,\omega)\, q\, \mathrm{d}t \right) \mathrm{d}\mathbb{P} \\ &= \int_0^1 q \left(\int_\Omega v'(x+tq,\omega)\, \mathrm{d}\mathbb{P} \right) \mathrm{d}t = q\, Q. \end{aligned} \tag{3}$$

Now, if (i) holds, i.e. if $Q = 0$, then (ii) follows for the choice of q was arbitrary in $\mathbb{R}$. Conversely, if (ii) holds, then it is enough to take $q = Q$ in (3) to get $Q = 0$, i.e. (i). □

Remark 3.11. Theorems 3.9 and 3.10 could be equivalently restated by requiring (i) to hold for some $x \in \mathbb{R}$, in view of Proposition 3.6.

Definition 3.12. We call *admissible* any Lipschitz random function with stationary increments and zero mean gradient. The class of such random functions will be denoted by $\mathcal{S}$.

The following property is immediate.

Proposition 3.13. *The class of admissible random functions is stable under the addition of real random variables.*

Remark 3.14. Looking at the proof of Theorem 3.8 it is easy to see that if the sequence in the statement is made up by admissible random functions then the limit is admissible too.

If v is a stationary function then clearly $\mathbb{E}(v(x,\cdot))$ does not depend on x, and so we derive:

Corollary 3.15. *Any stationary random Lipschitz function is admissible.*

4. Stochastic Hamilton–Jacobi equations

We consider a Hamiltonian

$$H : \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$$

satisfying the following conditions:

(H1) the map $\omega \mapsto H(\cdot,\cdot,\omega)$ from Ω to $C(\mathbb{R} \times \mathbb{R})$, endowed with the topology induced by the local uniform convergence, is measurable;

(H2) $Z_a(x,\omega) := \{\, p \in \mathbb{R} \,:\, H(x,p,\omega) \le a \,\}$ is a (possibly empty) interval;

(H3) $\partial Z_a(x,\omega) = \{\, p \in \mathbb{R} \,:\, H(x,p,\omega) = a \,\}$;

(H4) for every $R > 0$ there exists a modulus η_R such that

$$|H(x,p,\omega) - H(x,q,\omega)| \le \eta_R(|p-q|) \qquad \text{for all } p, q \in B_R;$$

(H5) there exist two continuous coercive functions $\theta, \Theta : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\theta(|p|) \le H(x,p,\omega) \le \Theta(|p|) \qquad \text{for all } p \in \mathbb{R};$$

(H6) $\quad H(\cdot + z, \cdot, \omega) = H(\cdot, \cdot, \tau_z \omega)$,

for any a, x, z in $\mathbb{R}$, and $\omega \in \Omega$.

Remark 4.1. Condition (H5) amounts to requiring H to be coercive and locally bounded in p, uniformly with respect to (x, ω). We point out that conditions (H2) and (H4) are always fulfilled by Hamiltonians convex in p and satisfying (H5). In this case, we can choose $\eta_R(h) = L_R \, h$ with

$$L_R := \sup\{ \, |H(x,p,\omega)| \, : \, (x.\omega) \in \mathbb{R} \times \Omega, \, |p| \leq R + 2 \, \},$$

which is finite thanks to (H5). For a convex Hamiltonian, condition (H3) is equivalent to the set of minimizers of $H(x, \cdot, \omega)$ being a singleton, for any x, ω.

Remark 4.2. Any given periodic, quasi–periodic or almost–periodic Hamiltonian $H_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ can be seen as a specific realization of a suitably defined stationary ergodic Hamiltonian.
The periodic and quasi–periodic case consist in assuming $H_0(x,p) = \sum_{i=1}^m H_i(x,p)$, $m \geq 1$, where each H_i is λ_i–periodic in x, and $\lambda_1, \ldots, \lambda_m$ are rationally independent. We take as Ω the set $\Pi_{i=1}^m [0, \lambda_i]$, as $\mathbb{P}$ the m–dimensional Lebesgue measure restricted to Ω, renormalized to be a probability measure, and as $\mathcal{F}$ the σ–algebra of Borel subset of Ω. The action of $\mathbb{R}$ on Ω is defined as

$$(\tau_x \omega)_i = \omega_i + x \pmod{\lambda_i} \qquad i = 1, \cdots, m$$

for any $\omega = (\omega_1, \ldots, \omega_m) \in \Omega$, $x \in \mathbb{R}$. A stationary Hamiltonian is obtained by setting

$$H(x,p,\omega) = \sum_{i=1}^m H_i(x + \omega_i, p).$$

In the almost–periodic case, we can choose as Ω the Bohr compactification of $\mathbb{R}$, as $\mathbb{P}$ the Haar measure on Ω, renormalized to be a probability measure, and as $\mathcal{F}$ the σ–algebra of Borel subsets of Ω. As well known (see for instance [1]), $\mathbb{R}$ embeds as a dense subset in Ω and the usual addition on $\mathbb{R}$ uniquely extends to an addition operation on Ω, still denoted by $+$, which gives it the structure of topological group, containing $\mathbb{R}$ as a dense subgroup. The family of translations on Ω, defined by $\tau_x \omega := x + \omega$ for every $x \in \mathbb{R}$, $\omega \in \Omega$, turns out to be ergodic with respect to $\mathbb{P}$. By exploiting the fact that any almost–periodic function on $\mathbb{R}$ extends uniquely to a continuous function on Ω, it is not hard to show that H_0 extends uniquely to a continuous function $\underline{H} : \Omega \times \mathbb{R} \to \mathbb{R}$. A stationary Hamiltonian is readily obtained by setting $H(x,p,\omega) = \underline{H}(x + \omega, p)$.

We point out that in all these three cases, Ω is a compact topological space with $\mathbb{P}(U) > 0$ for every nonempty open subset U of Ω, and that the map $\omega \mapsto H(\cdot,\cdot,\omega)$ from Ω to $C(\mathbb{R} \times \mathbb{R})$ is continuous with respect to the uniform convergence on $\mathbb{R} \times B_R$, for every $R > 0$.

Let us recall the definition of viscosity (super, sub) solution in the deterministic case. For a fixed ω_0, a continuous function $f(x)$ is said to be a *viscosity subsolution* (resp. *supersolution*) of $H(x, v', \omega_0) = a$ if $H(x_0, \varphi'(x_0), \omega_0) \leq a$ (resp. $\geq a$) for any C^1 function $\varphi : \mathbb{R} \to \mathbb{R}$ and any local maximizer (resp. minimizer) x_0 of $f - \varphi$. Due to the quasiconvex and coercive character of the Hamiltonian, a continuous function is a subsolution in the previous sense if and only if it is a Lipschitz–continuous almost everywhere subsolution. Finally a continuous function is called *viscosity solution* if it is sub and supersolution at the same time.

For $a \in \mathbb{R}$, we focus our attention on the stochastic equation

$$H(x, v'(x,\omega), \omega) = a \qquad \text{in } \mathbb{R}. \tag{4}$$

A continuous random function is called *subsolution* to (4) if it is a viscosity subsolution a.s. in ω. In particular, by what previously noticed, any subsolution to (4) belongs to $\mathrm{Lip}_{\kappa_a}(\mathbb{R})$ almost surely, where

$$\kappa_a := \sup\{\, |p| \,:\, p \in Z_a(x,\omega),\ (x,\omega) \in \mathbb{R} \times \Omega \,\}, \tag{5}$$

which is finite thanks to (H4). We are interested in the class of *admissible subsolutions*, hereafter denoted by $\mathcal{S}_a$, i.e. random functions taking values in $\mathrm{Lip}_{\kappa_a}(\mathbb{R})$ with stationary increments and zero mean gradient that are subsolutions of (4).

A solution to (4) is nothing but a Lipschitz random functions which solves the equation in the viscosity sense almost surely. An admissible solution will be preferably called an *exact corrector* taking into account the role it plays in the homogenization procedure.

Given $\delta > 0$, a random function v_δ will be called a *δ–approximate corrector* for the equation (4) if it belongs to $\mathcal{S}_{a+\delta}$ and satisfies

$$a - \delta \le H(x, v'_\delta(x,\omega), \omega) \le a + \delta \qquad \text{in } \mathbb{R}$$

in the viscosity sense, a.s. in ω. We say that (4) has *approximate correctors* if there are δ–approximate correctors for any $\delta > 0$.

The application of Theorem 3.8 and Proposition 3.13 in this context yields:

Theorem 4.3. *Let $(a_n)_n$ be a sequence of real numbers and v_n a random function in $\mathcal{S}_{a_n}$ for each n. If a_n converges to some a, there exist $v \in \mathcal{S}_a$ and a sequence $(w_k)_k$ made up by finite convex combinations of the v_n, up to an additive real random variable, such that*

$$w_k(\cdot,\omega) \rightrightarrows v(\cdot,\omega) \quad \textit{a.s. in } \omega. \tag{6}$$

Proof. Since $(a_n)_n$ is bounded, the random functions v_n all belong to $\mathcal{S}_b$ for some b, and therefore are equiLipschitz. By Theorem 3.8 the convergence in (6) takes place for a sequence $w_k(x,\omega) = \sum_{n \ge n_k} \lambda^k_n v_n(x,\omega) + g_k(\omega)$ of finite convex combinations of the v_n plus a real random variable, where $\lim_{k\to+\infty} n_k = +\infty$; in addition, by Remark 3.14, the limit random function v is admissible.

It is left to prove that $v \in \mathcal{S}_a$. For any $\varepsilon > 0$, the $w_k \in \mathcal{S}_{a+\varepsilon}$ for k large enough thanks to the quasiconvex character of the Hamiltonian. Since the subsolutions of a deterministic Hamilton–Jacobi equation are stable under local uniform convergence, we deduce that $v \in \mathcal{S}_{a+\varepsilon}$. The assertion is then obtained since ε was arbitrarily chosen. □

We define the *(stationary) critical value* c as

$$c := \inf\{a \in \mathbb{R} \,:\, \mathcal{S}_a \neq \emptyset\,\}. \tag{7}$$

We note that the set appearing at the right–hand side of (7) is non void since it contains the value $\Theta(0)$. We also consider the critical equation

$$H(x, v'(x,\omega), \omega) = c \qquad \text{in } \mathbb{R}. \tag{8}$$

We obtain as a corollary of Theorem 4.3 (see [16]):

Corollary 4.4. *$\mathcal{S}_c \neq \emptyset$.*

Therefore the infimum appearing in (7) is indeed a minimum, and the critical value can be equivalently defined via

$$c = \min_{v \in \mathcal{S}} \left(\operatorname*{ess\,sup}_{x \in \mathbb{R}} \; H(x, v'(x,\omega), \omega) \right) \qquad \text{a.s. in } \omega,$$

where, we recall, $\mathcal{S}$ stands for the class of all admissible random functions.

The relevance of the critical value is given by the following:

Theorem 4.5. *The equation (4) has neither exact corrector nor approximate correctors if $a \neq c$.*

A lemma is preliminary.

Lemma 4.6. *Let $b > a$. Then there exists $\delta = \delta(b,a) > 0$ such that*

$$Z_a(x,\omega) + B_\delta \subseteq Z_b(x,\omega)$$

for every $(x,\omega) \in \mathbb{R} \times \Omega$ for which $Z_a(x,\omega) \neq \emptyset$.

Proof. Choose η_R as in assumption (H4) with $R := \sup\{|p| \, : \, \theta(|p|) \leq b\}$, and set $\delta = \inf\{ t \geq 0 \, : \, \eta_R(t) \geq b - a \}$, which is strictly positive as η_R is continuous and vanishing at 0. Fix $(x,\omega) \in \mathbb{R} \times \Omega$ such that $Z_a(x,\omega) \neq \emptyset$. As $Z_a(x,\omega) \subseteq Z_b(x,\omega) \subseteq B_R$, for every $p_b \in \partial Z_b(x,\omega)$ and $p_a \in \partial Z_a(x,\omega)$ we get

$$\eta_R(|p_b - p_a|) \geq H(x, p_b, \omega) - H(x, p_a, \omega) = b - a,$$

yielding $|p_b - p_a| \geq \delta$. The assertion follows by the convexity of $Z_a(x,\omega)$ and $Z_b(x,\omega)$. $\square$

Proof of Theorem 4.5. It suffices to prove the assertion for approximate correctors. Let us assume that equation (4) admits approximate correctors, i.e. for every $\delta > 0$ there exists a δ–approximate corrector v_δ for (4). We aim to show that $a = c$. Passing to the limit of the v_δ, for $\delta \to 0$, we find an admissible subsolution to (4) via Theorem 4.3, which implies $a \geq c$.

We assume for purposes of contradiction that the strict inequality $a > c$ holds. We select a δ with $a - \delta > c$ and denote by v, to notations, the corresponding δ–approximate corrector. Recall that v satisfies

$$H(x, v'(x,\omega), \omega) \geq a - \delta > c \quad \text{in the viscosity sense}$$

a.s. in ω. We finally denote by u an admissible critical subsolution.

We proceed by picking an $\omega_0 \in \Omega$ for which $v(\cdot,\omega_0)$, $u(\cdot,\omega_0)$ are sublinear at infinity, and all the previously listed almost sure properties of the random variables H, v, u hold. We denote by H_0, v_0, u_0 the corresponding deterministic objects obtained by fixing $\omega = \omega_0$. We invoke the following property of viscosity supersolutions, see Proposition 4.3 in [5]: since $H_0(x, v_0') \geq a - \delta$ in the viscosity sense, for any Lipschitz function φ and any x_0 local minimizer of $v_0 - \varphi$, there exist $p_0 \in \partial\varphi(x_0)$ satisfying $H_0(x_0, p_0) \geq a - \delta$. We set $u_0^\varepsilon(x) := u_0(x) - \varepsilon|x|$ for $\varepsilon > 0$, and we find

$$\partial u_0^\varepsilon(x) \subset \partial u_0(x) + B_\varepsilon \quad \text{for any } x \in \mathbb{R},$$

so that, thanks to Lemma 4.6, ε can be chosen in such a way that

$$H_0(x,p) < a - \delta \quad \text{for any } x \in \mathbb{R} \text{ and } p \in \partial u_0^\varepsilon(x). \tag{9}$$

Since both v_0 and u_0 are sublinear at infinity, there are minimizers of $v_0 - u_0^\varepsilon$ in the whole $\mathbb{R}$, and at any of such points a contradiction comes out from (9) and the previous recalled property of viscosity supersolutions. $\square$

Another critical value, that we call free to distinguish it from c, will be relevant in our analysis. To introduce it we start by considering the random variable

$$c_f(\omega) := \sup_{x\in\mathbb{R}} \left(\min_{p\in\mathbb{R}} H(x,p,\omega) \right).$$

Since $c_f(\tau_z\omega) = c_f(\omega)$ for every $(z,\omega) \in \mathbb{R}\times\Omega$, we deduce, by the ergodicity assumption, that $c_f(\omega)$ is almost surely equal to a constant denoted c_f, which is actually the announced *free critical value.* It is apparent by the definition that $c_f \le c$.

To study the properties of subsolutions of (4) and the problem of the existence of admissible correctors for (8), we will make use of the so called *metric approach*, adopted for instance in [12]. Let Ω_f be the set of probability 1 where $c_f(\omega)$ is equal to the free critical value c_f. For every $a \ge c_f$, we define the support function σ_a of the a–sublevel of H by setting

$$\sigma_a(x,q,\omega) := \sup\{ q\,p \,:\, p \in Z_a(x,\omega) \},$$

where we agree that $\sigma_a(\cdot,\cdot,\omega) \equiv 0$ when $\omega \in \Omega\setminus\Omega_f$.

Proposition 4.7. *For every $a \ge c_f$, the function σ_a is jointly measurable in $\mathbb{R}\times\mathbb{R}\times\Omega$, i.e. with respect to the product σ–algebra $\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{F}$. Moreover, it is continuous in x, convex in q, and enjoys*

$$\sigma_a(\cdot+z,\cdot,\omega) = \sigma_a(\cdot,\cdot,\tau_z\omega) \quad \textit{for every } z\in\mathbb{R} \textit{ and } \omega\in\Omega.$$

Proof. The measurable and convex character of σ_a are easily derived from its very definition. To show that σ_a is continuous in x, it suffices to notice that, for every $\omega \in \Omega_f$, the set $Z_a(x,\omega)$ has either nonempty interior or is a singleton in view of assumption (H3). Last, the stationary character of σ_a follows by its very definition in view of assumption (H6) and of the fact that $\tau_z\Omega_f = \Omega_f$ for every $z \in \mathbb{R}$. □

Next, we define for any x, y and ω

$$S_a(x,y,\omega) = \int_0^1 \sigma_a\big((1-t)x+ty, y-x,\omega\big)\,\mathrm{d}t$$

The function S_a is jointly measurable in $\mathbb{R}\times\mathbb{R}\times\Omega$, and satisfies $S_a(x,x,\omega)=0$ for every $x\in\mathbb{R}$ and $\omega\in\Omega$. In addition, it is a semidistance, as it can be deduced from the following result.

Lemma 4.8. *Let $a \ge c_f$. Then*

$$S_a(x,y,\omega) + S_a(y,x,\omega) \ge 0 \quad \textit{for any } x,\, y\in\mathbb{R}$$

a.s. in ω. If, in addition, $a > c_f$ and $x \ne y$, then the previous formula holds with strict inequality.

Proof. Let ω in Ω_f and $x, y \in \mathbb{R}$. We can assume that $x \ne y$, being the assertion trivially satisfied otherwise for $S_a(x,x,\omega) = 0$. We observe that, for any $z\in\mathbb{R}$, $\sigma_a(z,1,\omega)$, $-\sigma_a(z,-1,\omega)$ are given by the right and the left endpoint of $Z_a(z,\omega)$, respectively. From this we get

$$\sigma_a(z,1,\omega) + \sigma_a(z,-1,\omega) \ge 0 \quad \text{for any } z\in\mathbb{R}.$$

If, in addition, $Z_a(z,\omega)$ does not reduce to a singleton, which is the case when $a > c_f$, the inequality in the previous formula is strict. Let $\gamma(t) := (1-t)x + ty$. After a change of variable in the second integral, we deduce from (4) and from the positive 1–homogeneous character of $\sigma_a(x,\cdot,\omega)$ that

$$S_a(x,y,\omega)+S_a(y,x,\omega)=\int_0^1 \sigma_a\big(\gamma(t),y-x,\omega\big)+\sigma_a\big(\gamma(t),-(y-x),\omega\big)\,\mathrm{d}t\geq 0,$$

with strict inequality holding if $a>c_f$. □

We derive from Lemma 4.8

$$S_a(x,y,\omega) \quad\leq\quad S_a(x,z,\omega)+S_a(z,y,\omega)$$

for all $x,y,z\in\mathbb{R}$ and $\omega\in\Omega$, where the equality holds when z lies between x and y. We also have by the very definition of S_a

$$S_a(x,y,\omega)\leq\kappa_a|x-y|\quad\text{for any } x,\ y\in\mathbb{R},\ \omega\in\Omega.$$

As a consequence we see that $S_a(y,\cdot,\omega)$ belongs to $\mathrm{Lip}_{\kappa_a}(\mathbb{R})$ for every $y\in\mathbb{R}$ and $\omega\in\Omega$. In addition, by the previously recalled joint measurability property of S_a and Proposition 2.1, $\omega\mapsto S_a(y,\cdot,\omega)$ is a Lipschitz random function for every fixed y. Same argument applies to $\omega\mapsto S_a(\cdot,y,\omega)$. Taking into account the corresponding properties holding in the deterministic case (see [12]), we also have:

Proposition 4.9. *A Lipschitz random function v is a subsolution of (4) if and only if*

$$v(x,\omega)-v(y,\omega)\leq S_a(y,x,\omega)\quad\textit{for all } x,y\in\mathbb{R},$$

a.s. in ω. In particular, $S_a(y,\cdot,\omega)$ and $-S_a(\cdot,y,\omega)$ are both subsolutions of (4).

Furthermore,

$$S_a(x+z,y+z,\omega)=S_a(x,y,\tau_z\omega)\qquad\text{for every } x,y,z\in\mathbb{R}\text{ and }\omega\in\Omega,$$

which clearly follows from the stationary character of σ_a.

Remark 4.10. When the Hamiltonian is convex in p and a Lagrangian L can be defined through the Fenchel transform, the following identity holds true (cf. [9])

$$S_a(x,y,\omega)=\inf\left\{\int_0^t \big(L(\xi,\dot\xi,\omega)+a\,\big)\,ds\right\},$$

where the infimum is taken letting ξ vary in the family of Lipschitz–continuous parametrizations in $[0,t]$ of the segment joining x to y, for every $t>0$.

5. Lax–type formulae

In this section we adapt Lax formulae to the stationary ergodic setting. We start by recalling the setup for deterministic Hamilton–Jacobi equations, see [12]. For every ω, let

$$\mathcal{E}(\omega):=\{y\in\mathbb{R}\,:\,\min_p H(y,p,\omega)=c_f\,\}\tag{10}$$

the (possibly empty) *set of equilibria.*

Theorem 5.1. *Let us fix $\omega\in\Omega_f$, $a\geq c_f$. We have* *(i) Let K, h be a closed subset of $\mathbb{R}$ and a continuous function defined on it, respectively. We define*

$$\psi(x) = \inf_{y \in K} \big(h(y) + S_a(y, x, \omega)\big), \qquad x \in \mathbb{R}.$$

If $\psi \not\equiv -\infty$ then it is a subsolution to (4) in $\mathbb{R}$ and a viscosity solution in $\mathbb{R} \setminus K$.

(ii) Let U be a bounded open subset of $\mathbb{R}$. We assume that either $a > c_f$ or $a = c_f$ and $\mathcal{E}(\omega) \cap U = \emptyset$, and consider a function w_0 on ∂U with

$$w_0(x) - w_0(y) \le S_a(y, x, \omega) \quad \text{for every } x, y \in \partial U.$$

Then the function

$$w(x) := \inf_{y \in \partial U} \big(w_0(y) + S_a(y, x, \omega)\big), \qquad x \in U$$

is the unique viscosity solution of the Dirichlet Problem:

$$\begin{cases} H(x, \phi'(x), \omega) = a & \text{in } U \\ \phi(x) = w_0(x) & \text{on } \partial U. \end{cases}$$

We proceed giving a stochastic version of the previous properties. Let $C(\omega)$ be an almost surely nonempty stationary closed random set and g a Lipschitz random function. For $a \ge c_f$ set

$$u(x, \omega) := \inf\{g(y, \omega) + S_a(y, x, \omega) \, : \, y \in C(\omega)\}, \quad x \in \mathbb{R}, \tag{11}$$

where we agree that $u(\cdot, \omega) \equiv 0$ when either $C(\omega) = \emptyset$ or the infimum above equals $-\infty$. The following holds:

Proposition 5.2. *Let g be a stationary Lipschitz random function and $C(\omega)$, u as above. Let us assume that, for some $a \ge c_f$, the infimum in* (11) *is finite a.s. in ω. Then u is a stationary random variable belonging to $\mathcal{S}_a$ and satisfies $u(\cdot, \omega) \le g(\cdot, \omega)$ on $C(\omega)$ a.s. in ω. Moreover, u is a viscosity solution of* (4) *in $\mathbb{R} \setminus C(\omega)$ a.s. in ω.*

Proof. We first show that u is jointly measurable in (x, ω). Let us denote by $\widetilde{u}(x, \omega)$ the right–hand side term of (11), where we agree that $\widetilde{u}(x, \omega) = 0$ when $C(\omega) = \emptyset$, and set $\Omega_0 := \{\omega \, : \, C(\omega) \neq \emptyset, \, \widetilde{u}(\cdot, \omega) \not\equiv -\infty\}$. Since $u(x, \omega) = \chi_{\Omega_0}(\omega) \, \widetilde{u}(x, \omega)$ in $\mathbb{R} \times \Omega$, it is enough to show that $\widetilde{u}$ is jointly measurable in (x, ω), as this in turn implies that Ω_0 is measurable too. To this aim, for each $n \in \mathbb{N}$ we set $C_n(\omega) := C(\omega) + B_{1/n}$ and

$$u_n(x, \omega) := \inf_{y_k \in \mathbb{Q}} \big(g(y_k, \omega) + S_a(y_k, x, \omega) + \delta_{\Gamma(C_n)}(y_k, \omega)\big), \qquad (x, \omega) \in \mathbb{R} \times \Omega,$$

where as usual we agree that $u_n(\cdot, \omega) \equiv 0$ when $C(\omega) = \emptyset$. Here $\delta_{\Gamma(C_n)}$ denotes the function identically equal to 0 on $\Gamma(C_n)$ and $+\infty$ outside. Since C_n is a closed random set by Remark 3.3, its graph is measurable in $\mathbb{R} \times \Omega$ and this in turn implies that u_n is jointly measurable in $\mathbb{R} \times \Omega$. By using the fact that the functions $g(\cdot, \omega)$ and $S_a(\cdot, x, \omega)$ are equi–Lipschitz in $\mathbb{R}$, uniformly with respect to x and ω, we derive that u_n converges to $\widetilde{u}$ pointwise in $\mathbb{R} \times \Omega$, which is thus jointly measurable.

In view of Theorem 5.1 we know that $u(\cdot, \omega)$ is a viscosity subsolution of (4), and a solution as well in $\mathbb{R} \setminus C(\omega)$ for every $\omega \in \Omega_0$. It is also apparent by the definition that $u(x, \omega) \le g(x, \omega)$ for every $x \in C(\omega)$.

Let us show that u is stationary. Let us fix $z \in \mathbb{R}$. Since g is stationary, we have that $g(\cdot, \tau_z\omega) = g(\cdot + z, \omega)$ on $\mathbb{R}$ almost surely in ω. Hence, a.s. in ω, we have

$$\begin{aligned} u(x, \tau_z\omega) &= \inf_{\zeta \in C(\tau_z\omega)} \{g(\zeta, \tau_z\omega) + S_a(\zeta, x, \tau_z\omega)\} \\ &= \inf_{\zeta+z \in C(\omega)} \{g(\zeta + z, \omega) + S_a(\zeta + z, x + z, \omega)\} = u(x + z, \omega) \end{aligned}$$

for every $x \in \mathbb{R}$. In the second equality, we have exploited the fact that $C(\omega)$ is stationary. The conclusion follows in view of Corollary 3.15. $\square$

We go on by providing a variation of the previous result which will allow us to deduce a uniqueness principle for exact correctors in Section 6.

Proposition 5.3. *We use the same notations of the previous proposition, with the difference that now g is assumed to be in $\mathcal{S}_a$. Then u belongs to $\mathcal{S}_a$. In addition, it is a viscosity solution of* (4) *in $\mathbb{R} \setminus C(\omega)$, and takes the value $g(\cdot, \omega)$ on $C(\omega)$ a.s. in ω.*

Proof. The joint measurability of u follows arguing as in the proof of the previous proposition. Set $\Omega_0 := \{\, \omega : C(\omega) \neq \emptyset,\ g(\cdot, \omega) \text{ is a subsolution of } (4)\}$. Let $\omega \in \Omega_0$. Since by Proposition 4.9

$$g(x, \omega) - g(y, \omega) \leq S_a(y, x, \omega) \quad \text{for any } x,\ y \in \mathbb{R},$$

we derive that $u(\cdot, \omega) = g(\cdot, \omega)$ in $C(\omega)$, and consequently $u(\cdot, \omega) \not\equiv -\infty$. We also infer by Theorem 5.1 that $u(\cdot, \omega)$ is a viscosity subsolution of (4), and a solution as well in $\mathbb{R} \setminus C(\omega)$.

We proceed to prove that u has stationary increments. Since g has stationary increments, for every fixed z in $\mathbb{R}$ there exists a random variable k with $g(\cdot, \tau_z\omega) = g(\cdot + z, \omega) + k(\omega)$ on $\mathbb{R}$, a.s. in ω. Then we argue as in the proof of the previous proposition to infer that

$$u(\cdot, \tau_z\omega) = u(\cdot + z, \omega) + k(\omega) \quad \text{on } \mathbb{R}$$

a.s. in ω, which gives the claim.

Let us finally show that u has sublinear behavior at infinity a.s. in ω. For this we will essentially use the asymptotic formula for random stationary sets given in Proposition 3.5. Let us denote by $\widehat{\Omega}_0$ a subset of Ω_0 of probability 1 such that, for every $\omega \in \widehat{\Omega}_0$, $g(\cdot, \omega)$ is sublinear and

$$\lim_{r \to +\infty} \frac{|\,(C(\omega) + B_n) \cap B_r|}{|B_r|} > 1 - \varepsilon_n \qquad \text{for each } n \in \mathbb{N},$$

where $(\varepsilon_n)_n$ is a sequence decreasing to 0, according to Proposition 3.5. Fix $\omega \in \widehat{\Omega}_0$ and $n \in \mathbb{N}$. Then for every $x \in \mathbb{R}$ with $|x|$ large enough, we have

$$\frac{|\,(C(\omega) + B_n) \cap B_{2|x|}|}{|B_{2|x|}|} > 1 - \varepsilon_n.$$

For n sufficiently large $B_{2\varepsilon_n|x|}(x) \subseteq B_{2|x|}$, and from the above inequality we infer that

$$B_{2\varepsilon_n|x|}(x) \cap (C(\omega) + B_n) \neq \emptyset,$$

i.e. there exists $y = y(x,n)$ in $C(\omega)$ such that $|y - x| < 2\,\varepsilon_n |x| + n$. Then, recalling that $u(y,\omega) = g(y,\omega)$, we get

$$\begin{aligned}|u(x,\omega)| &\leq |u(x,\omega) - u(y,\omega)| + |g(y,\omega) - g(x,\omega)| + |g(x,\omega)| \\ &\leq 2\,\kappa_a\,(2\,\varepsilon_n\,|x| + n) + |g(x,\omega)|.\end{aligned}$$

From this we obtain, by the sublinear character of g,

$$\limsup_{|x|\to+\infty} \frac{|u(x,\omega)|}{|x|} \leq 4\,\kappa_a\,\varepsilon_n,$$

and the claim follows letting $n \to +\infty$. □

As a simple consequence of Proposition 5.2 we deduce:

Theorem 5.4. *For any $a > c$ there is a stationary subsolution to (4).*

Proof. We first aim to show that

$$\inf_{x\in\mathbb{R}} S_a(x,x_0,\omega) > -\infty \quad \text{for any } x_0 \in \mathbb{R} \tag{12}$$

a.s. in ω. We start by proving

$$\lim_{|x|\to+\infty} S_a(x,x_0,\omega) = +\infty \quad \text{for any } x_0 \in \mathbb{R}$$

a.s. in ω, from which (12) can be immediately deduced. We denote by w a critical admissible subsolution, and take ω such that $w(\cdot,\omega)$ is sublinear at infinity and satisfies $H(x,w'(x),\omega) \leq c$ for a.e. $x \in \mathbb{R}$. Let us fix x_0. By Lemma 4.6 there is $\delta > 0$ with

$$w'(y,\omega)q + \delta|q| \leq \sigma_a(y,q,\omega) \quad \text{for a.e. } y \in \mathbb{R},$$

and consequently

$$w(x_0,\omega) - w(x,\omega) + \delta|x - x_0| \leq S_a(x,x_0,\omega) \quad \text{for any } x \in \mathbb{R}.$$

Dividing both sides by $|x - x_0|$, and exploiting the sublinear character of $w(\cdot,\omega)$ in the limit for $|x| \to +\infty$, we obtain

$$\liminf_{|x|\to+\infty} \frac{S_a(x,x_0,\omega)}{|x - x_0|} \geq \delta,$$

which, finally implies (5) and, consequently, (12).

Let u be the random function defined trough (11) with $C(\omega) = \mathbb{R}$ and $g(\cdot,\omega) \equiv 0$ for every ω. By (12) the infimum in (11) is almost surely finite, and the assertion follows by Proposition 5.2. □

We derive from the previous theorem a new definition for the critical value c:

$$c := \inf\{a \in \mathbb{R} \,:\, (4) \text{ admits stationary subsolution}\}. \tag{13}$$

In this case the infimum appearing in the formula is not necessarily a minimum, namely we cannot expect, in general, to find stationary critical subsolutions.

6. Existence of exact and approximate correctors

Here we prove that exact correctors do always exist if $c > c_f$. In the case $c = c_f$, we provide characterizing conditions for their existence involving the random set of equilibria. If there are no correctors, we however show the existence of approximate correctors, which are constructed by means of approximate sets of equilibria and a Lax–type formula. Finally, in Remark 6.10, we interpret the previous results in the light of a generalized notion of Aubry set adapted to the random setting.

We write, for every $a \geq c_f$,

$$Z_a(x,\omega) = [\alpha_a(x,\omega),\, \beta_a(x,\omega)], \qquad (x,\omega) \in \mathbb{R} \times \Omega_f.$$

We extend the definition of α_a and β_a on the whole $\mathbb{R} \times \Omega$ by setting $\alpha_a(\cdot,\omega) \equiv 0$, $\beta_a(\cdot,\omega) \equiv 0$ when $\omega \in \Omega \setminus \Omega_f$. Note that $\alpha_a(x,\omega) = -\sigma_a(x,-1,\omega)$ and $\beta_a(x,\omega) = \sigma_a(x,1,\omega)$, so we directly derive from Proposition 4.7

Proposition 6.1. *For any $a \geq c_f$, α_a and β_a are stationary random continuous functions.*

As a consequence of the previous proposition, the means $\mathbb{E}\big(\alpha_a(x,\cdot)\big)$, $\mathbb{E}\big(\beta_a(x,\cdot)\big)$ do not depend on x, so in the sequel we will simply write $\mathbb{E}(\alpha_a)$, $\mathbb{E}(\beta_a)$.

We define the *averaged a–sublevel*

$$\mathbb{E}[Z_a] := [\mathbb{E}(\alpha_a),\, \mathbb{E}(\beta_a)] \quad \text{for any } a \geq c_f.$$

We have:

Proposition 6.2. *The functions $a \mapsto -\mathbb{E}(\alpha_a)$, $a \mapsto \mathbb{E}(\beta_a)$ are strictly increasing, continuous and coercive in $[c_f, +\infty)$.*

Proof. We prove the statement for the function $a \mapsto \mathbb{E}(\beta_a)$. We immediately see that it is strictly increasing and coercive because of its very definition, by (H5) and by Lemma 4.6. We therefore focus our attention on the continuity issue, which is more delicate. We will prove the continuity from the right, being the proof of the remaining case analogous. For this we claim

$$\mathbb{E}(\beta_b) = \inf_{a>b} \mathbb{E}(\beta_a) \qquad \text{for every } b \in [c_f, +\infty). \tag{14}$$

We define, for every $a \geq c_f$ and $\omega \in \Omega$ the function

$$f_a(\omega) = \int_0^1 \beta_a(t,\omega)\,\mathrm{d}t,$$

which is measurable and bounded in Ω. From the continuity properties of H in p, inherited by $a \mapsto \beta_a(t,\omega)$, we derive

$$\lim_{a\to b^+} \beta_a(t,\omega) = \beta_b(t,\omega) \quad \text{for every } (t,\omega) \in \mathbb{R} \times \Omega,$$

and we deduce through the Monotone Convergence Theorem

$$\lim_{a\to b^+} f_a(\omega) = f_b(\omega) \quad \text{for every } \omega \in \Omega.$$

Bearing in mind this limit relation and using again the Monotone Convergence Theorem, we get

$$\int_\Omega f_b(\omega)\,\mathrm{d}\mathbb{P} = \inf_{a>b} \int_\Omega f_a(\omega)\,\mathrm{d}\mathbb{P}.$$

To see that this equality is equivalent to (14), we exploit the joint measurability of β_a in (x,ω) to obtain, for any $a \geq c_f$,

$$\int_\Omega f_a(\omega)\,\mathrm{d}\mathbb{P} = \int_\Omega \left(\int_0^1 \beta_a(t,\omega)\,\mathrm{d}t \right) \mathrm{d}\mathbb{P} = \int_0^1 \left(\int_\Omega \beta_a(t,\omega)\,\mathrm{d}\mathbb{P} \right) \mathrm{d}t = \mathbb{E}(\beta_a).$$

□

In order to apply the results of the previous section, we will also need the following property of the set of equilibria $\mathcal{E}$, see (10) for the definition.

Proposition 6.3. *$\mathcal{E}$ is a closed random stationary set.*

Proof. In view of Proposition 3.2, it is enough to show that the stationary function $f(x,\omega) := \min_p H(x,p,\omega)$ is jointly measurable in (x,ω) and is continuous in x for every fixed ω. For the first assertion, simply notice that $f(x,\omega) = \inf_{p_k \in \mathbb{Q}} H(x,p_k,\omega)$ by the continuity of H in p. The second assertion can be directly deduced from the definition of f by making use of assumptions (H1) and (H5). □

Our subsequent analysis depends on whether 0 is in the interior of $\mathbb{E}[Z_{c_f}]$, or not. If the first instance occurs we further distinguish the subcases when the set of equilibria is almost surely empty or nonempty. The following two results depict this setup.

Theorem 6.4. *If $0 \in \operatorname{Int} \mathbb{E}[Z_{c_f}]$ and $\mathcal{E}(\omega) \neq \emptyset$ a.s. in ω, then $c = c_f$ and* (8) *admits an exact corrector, which is, in addition, stationary.*

Proof. The relevant consequence of the assumption on 0 that we are going to exploit is

$$\min\{-\mathbb{E}(\alpha_{c_f}),\, \mathbb{E}(\beta_{c_f})\} > 0.$$

We set

$$u(x,\omega) := \inf\{S_{c_f}(y,x,\omega) \,:\, y \in \mathcal{E}(\omega)\,\}, \qquad (x,\,\omega) \in \mathbb{R} \times \Omega, \tag{15}$$

where we agree that $u(\cdot,\omega) \equiv 0$ when either $\mathcal{E}(\omega) = \emptyset$ or the infimum above is equal to $-\infty$. We know by Proposition 5.2 that u is a stationary admissible subsolution, provided $u(\cdot,\omega) \not\equiv -\infty$ a.s. in ω. To see this, first note that, from the very definition of S_{c_f},

$$\begin{aligned} \frac{S_{c_f}(y,0,\omega)}{|y|} &= \frac{1}{|y|}\int_y^0 \beta_{c_f}(s,\omega)\,\mathrm{d}s \qquad &\text{for } y < 0,\\ \frac{S_{c_f}(y,0,\omega)}{|y|} &= -\frac{1}{|y|}\int_y^0 \alpha_{c_f}(s,\omega)\,\mathrm{d}s \qquad &\text{for } y > 0. \end{aligned}$$

By applying the Birkhoff Ergodic Theorem we find

$$\liminf_{|y|\to+\infty} \frac{S_{c_f}(y,0,\omega)}{|y|} \geq \min\{-\mathbb{E}(\alpha_{c_f}),\, \mathbb{E}(\beta_{c_f})\} > 0$$

for every ω in a set Ω' of probability 1, which, in turn, implies

$$\lim_{|y|\to+\infty} S_{c_f}(y,0,\omega) = +\infty \quad \text{for every } \omega \in \Omega'. \tag{16}$$

Exploiting for instance the fact that $S_{c_f}(\cdot,\cdot,\omega)$ enjoys the triangular inequality, we readily derive that (16) keeps holding when 0 is replaced by any fixed point $x \in \mathbb{R}$. In particular, we conclude that the infimum in (15) is attained for every $\omega \in \Omega'$ such that $\mathcal{E}(\omega) \neq \emptyset$, hence $u(x,\omega) \in \mathbb{R}$ a.s. in ω. For such an ω, we furthermore know that $u(\cdot,\omega)$ is a viscosity solution on $\mathbb{R}\setminus\mathcal{E}(\omega)$ and a subsolution in $\mathbb{R}$ of equation (4) with $a = c_f$. It is left to show that $u(\cdot,\omega)$ is a viscosity supersolution on $\mathcal{E}(\omega)$. We then pick up $y\in\mathcal{E}(\omega)$ and let ϕ be a C^1 test function touching $u(\cdot,\omega)$ in y from below. Since y is an equilibrium point, we clearly have that $H(y,\phi'(y),\omega)\geq \min_p H(y,p,\omega) = c_f$.

This proves that u is an exact corrector of (4) with $a = c_f$, therefore $c = c_f$ in view of Theorem 4.5, as it was claimed. □

If the random set of equilibria is almost surely empty, we have:

Theorem 6.5. *If* $0\in\operatorname{Int}\mathbb{E}[Z_{c_f}]$ *and* $\mathcal{E}(\omega)=\emptyset$ *a.s. in* ω*, then* $c = c_f$ *and* (8) *does not have admissible solutions. However there exist* δ*–approximate correctors for every* $\delta>0$ *which are, in addition, stationary.*

Proof. Fix a $\delta>0$ and set

$$\begin{aligned}\mathcal{E}_\delta(\omega) &:= \{y\in\mathbb{R}\,:\,\min_p H(y,p,\omega)\geq c_f-\delta\,\}\\ u_\delta(x,\omega) &:= \inf\{S_{c_f}(y,x,\omega)\,:\,y\in\mathcal{E}_\delta(\omega)\,\},\qquad (x,\omega)\in\mathbb{R}\times\Omega,\end{aligned}$$

where we agree that $u_\delta(\cdot,\omega)\equiv 0$ when either $\mathcal{E}_\delta(\omega)=\emptyset$ or the infimum above is equal to $-\infty$. By arguing as in Proposition 6.3, we see that $\mathcal{E}_\delta(\omega)$ is an almost surely nonempty stationary closed random set. Arguing as in the proof of Theorem 6.4, we see that $u_\delta(\cdot,\omega)\not\equiv-\infty$ a.s., hence it is an admissible subsolution of (4) with $a = c_f$, and a solution as well on $\mathbb{R}\setminus\mathcal{E}_\delta(\omega)$ a.s.. In particular, we get that $c_f\geq c$, and so $c_f = c$ for the converse inequality trivially holds.

To prove that u_δ is a δ–approximate corrector, we need to show that the inequality $H(y,Du_\delta(y,\omega),\omega)\geq c_f-\delta$ in the viscosity sense for every $y\in\mathcal{E}_\delta(\omega)$ a.s in ω. To this aim, fix ω such that $\mathcal{E}_\delta(\omega)\neq\emptyset$ and pick up a $y\in\mathcal{E}_\delta(\omega)$. Let ϕ be a C^1 test function touching $u_\delta(\cdot,\omega)$ in y from below. Then $H(y,\phi'(y),\omega)\geq\min_p H(y,p,\omega)\geq c_f-\delta$, as was to be shown.

Finally, to prove that equation (8) does not admit exact correctors, we argue by contradiction. Let u be an admissible corrector, we pick up an ω such that $u(\cdot,\omega)$ is a viscosity solution of (8). We apply Theorem 5.1 with $w_0 = u$, $a = c$ and $U = (-n,n)$ for each $n\in\mathbb{N}$ to derive that there exists a diverging sequence $(y_n)_n$ such that

$$S_c(y_n,0,\omega) = u(0,\omega)-u(y_n,\omega)\qquad\text{for every } n\in\mathbb{N},$$

which implies, thanks to the sublinearity of $u(\cdot,\omega)$, $\liminf_{|y|\to+\infty} S_c(y,0,\omega)/|y| = 0$, in contrast with the limit relation

$$\liminf_{|y|\to+\infty}\frac{S_c(y,0,\omega)}{|y|}\geq\min\{-\mathbb{E}(\alpha_c),\,\mathbb{E}(\beta_c)\}>0\quad\text{a.s. in }\omega,$$

derived from Birkhoff Ergodic Theorem. □

Remark 6.6. We stress that the functions u_δ defined in the previous proof are actually admissible critical subsolutions.

If 0 is not in the interior of the averaged c_f–sublevel we have:

Theorem 6.7. *If* $0\notin\operatorname{Int}\mathbb{E}[Z_{c_f}]$*, then*

$$c = \inf\{a\geq c_f\,:\,0\in\mathbb{E}[Z_a]\,\}$$

and (8) *admits an exact corrector (not necessarily stationary), almost surely of class* C^1.

Proof. For c defined as in the statement, we have, by Proposition 6.2, that either $\mathbb{E}(\beta_c)$ or $\mathbb{E}(\alpha_c)$ vanish. Let us assume, to fix ideas, that the first case occurs. We set

$$u(x,\omega) = \int_0^x \beta_c(s,\omega)\,\mathrm{d}s \qquad \text{a.s. in } \omega, \tag{17}$$

and we see that $u'(x,\omega) = \beta_c(x,\omega)$ a.s. in ω, and consequently u has stationary gradient with mean 0. Moreover $u(\cdot,\omega)$ is almost surely of class C^1 for $\beta_c(\cdot,\omega)$ is a.s. continuous, according to Proposition 6.1. In particular, it is a classical solution of (8) a.s. in ω, which finally implies that c is the critical value by Theorem 4.5 and u is the sought exact corrector. The proof is complete. □

We finish the section by showing some uniqueness results for exact correctors.

Theorem 6.8. *Let us assume that either $0 \notin \mathbb{E}[Z_{c_f}]$, or $0 \in \partial\,\mathbb{E}[Z_{c_f}]$ and $\mathcal{E}(\omega) = \emptyset$ a.s. in ω. If u is an admissible solution of* (8)*, then $u(\cdot,\omega)$ is almost surely of class C^1, and*

$$\mathcal{S}_c = \{u + k \;:\; k : \Omega \to \mathbb{R} \textit{ measurable}\},$$

i.e. u is the unique admissible subsolution, up to an additive real random variable.

Proof. We know by what seen above that $0 \in \partial\mathbb{E}[Z_c]$. Let us assume, to fix ideas, that $\mathbb{E}(\beta_c) = 0$, and let u be the random Lipschitz function defined via (17). According to Theorem 6.7, u is an exact corrector almost surely of class C^1. Given a critical subsolution v, we set

$$Y(\omega) := \{x \in \Delta_u(\omega) \cap \Delta_v(\omega) \;:\; u'(x,\omega) \neq v'(x,\omega)\},$$

and notice that it is a random stationary set by Proposition 3.6. By the fact that v is admissible, we have

$$\mathbb{E}(v'(x,\cdot)) = 0 = \mathbb{E}(\beta_c(x,\cdot)) \quad \text{for any } x,$$

and this together with the inequality $v'(x,\cdot) \le \beta_c(x,\cdot)$, holding a.s. in Ω because v is a critical subsolution, yields $v'(x,\cdot) = \beta_c(x,\cdot)$ a.s. in Ω. In other terms $\mathbb{P}(Y^{-1}(x)) = 0$, and consequently $|Y(\omega)| = 0$ a.s. in ω by Fubini's Theorem. Summing up:

$$u'(x,\omega) = v'(x,\omega) \quad \text{for a.e. } x \in \mathbb{R}$$

a.s. in ω, and so u and v coincide up to a real random variable. □

Theorem 6.9. *Assume that $0 \in \mathbb{E}[Z_{c_f}]$ and $\mathcal{E}(\omega) \neq \emptyset$ a.s. in ω. Then $\mathcal{E}$ is a uniqueness set for the critical equation in the sense that if two exact correctors u and v agree on $\mathcal{E}(\omega)$ a.s. in ω, then they coincide on the whole $\mathbb{R}$ a.s. in ω.*

Proof. Given an exact corrector u, we will show that

$$u(x,\omega) = \inf\{u(y,\omega) + S_{c_f}(y,x,\omega) \;:\; y \in \mathcal{E}(\omega)\}, \quad x \in \mathbb{R}$$

almost surely in ω. We take into account Proposition 5.3 and the argument used in the proof of Theorem 6.4 to see that the right hand–side of the previous formula gives an exact corrector coinciding with u on $\mathcal{E}(\omega)$ a.s. in ω. We will denote it by v.

We pick an ω for which $\mathcal{E}(\omega) \neq \emptyset$ and all the previous relations hold. Let $x_0 \notin \mathcal{E}(\omega)$. Since $\mathcal{E}(\omega)$ is closed there are $y_1 < y_2$ in $\mathcal{E}(\omega)$ with

$$x_0 \in [y_1, y_2] \quad , \quad (y_1, y_2) \cap \mathcal{E}(\omega) = \emptyset.$$

By Theorem 5.1–(ii) we have

$$u(x_0,\omega) = \min_{i=1,2}\{u(y_i,\omega) + S_{c_f}(y_i,x_0,\omega)\},$$

which implies $u(x_0,\omega) \geq v(x_0,\omega)$. To prove the opposite inequality, we note that being $u(\cdot,\omega)$ a subsolution yields, by Proposition 4.9,

$$u(x_0,\omega) \leq u(y,\omega) + S_{c_f}(y,x_0,\omega) \qquad \text{for every } y \in \mathcal{E}(\omega),$$

and we conclude by taking the infimum over $y \in \mathcal{E}(\omega)$ of the right–hand side term. □

Remark 6.10. We propose a notion of Aubry set adapted to the present random environment. We recall, as a starting point, that in the deterministic case the Aubry set is made up by points such that the intrinsic distance from them is a critical solution. The idea here is to replace, as in the spirit of the whole paper, points by closed random stationary set. This leads to the following definition: a closed random stationary set $\mathcal{A}$ is called Aubry set if

(i) the extension of an admissible critical subsolution from any closed random stationary subset of $\mathcal{A}$ via the Lax formula, see (11), yields an exact corrector;
(ii) no closed random stationary set properly containing $\mathcal{A}$ enjoys the previous property.

According to the results of the section, we easily derive that $\mathcal{A} = \mathcal{E}$ if $0 \in \operatorname{int} \mathbb{E}[Z_{c_f}]$, and $\mathcal{A} = \mathbb{R}$ otherwise. Note that this is exactly the setup in the periodic case, apart from the fact that $\mathcal{E}(\omega)$ can be almost surely empty, while the set of equilibria is non void in the periodic setting. We have therefore proved that an exact corrector does exist if and only if $\mathcal{A}(\omega) \neq \emptyset$ a.s. in ω.

7. An application to Stochastic Homogenization

In this section we will exploit the results previously obtained to extend the homogenization result proved by Souganidis [19] to the case of quasiconvex Hamiltonians.

For every $P \in \mathbb{R}$ we define the *effective Hamiltonian* $\overline{H}(P)$ as the stationary critical value of the Hamiltonian $H(\cdot, P + \cdot, \cdot)$, namely

$$\overline{H}(P) := \min_{v \in \mathcal{S}} \left(\operatorname*{ess\,sup}_{x \in \mathbb{R}} \; H(x, P + v'(x,\omega), \omega) \right) \qquad \text{a.s. in } \omega.$$

Proposition 7.1. *The function $\overline{H} : \mathbb{R} \to \mathbb{R}$ is continuous and coercive, with minimum equal to c_f. It is, in addition, quasiconvex in the sense that $\overline{Z}_a := \{p \in \mathbb{R} : \overline{H}(p) \leq a\}$ is a compact interval for $a \geq c_f$, and $\partial \overline{Z}_a = \{p \in \mathbb{R} : \overline{H}(p) = a\}$ for $a > c_f$.*

Proof. We recall from Proposition 6.2 that the functions $a \mapsto \mathbb{E}(\alpha_a)$, $a \mapsto \mathbb{E}(\beta_a)$ admit inverse denoted by f and g, respectively, in $[c_f, +\infty)$ with

$$f : (-\infty, \mathbb{E}(\alpha_{c_f})] \to [c_f, +\infty) \quad , \quad g : [\mathbb{E}(\beta_{c_f}), +\infty) \to [c_f, +\infty),$$

both strictly monotonic and continuous. According to what proved in the previous section, we have

$$\overline{H}(P) = \begin{cases} f(P) & \text{if } P \in (-\infty, \mathbb{E}(\alpha_{c_f})] \\ c_f & \text{if } P \in (\mathbb{E}(\alpha_{c_f}), \mathbb{E}(\beta_{c_f})) \\ g(P) & \text{if } P \in [\mathbb{E}(\beta_{c_f}), +\infty). \end{cases}$$

This formula proves all the points of the assertion. □

The main result of the section is the following:

Theorem 7.2. *Let $H : \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ satisfy (H1)–(H6). Then for any $u_0 \in \mathrm{UC}(\mathbb{R}^N)$, the unique viscosity solutions $u_\varepsilon(\cdot,\cdot,\omega) \in \mathrm{UC}\big((0,+\infty) \times \mathbb{R}^N\big)$ of*

$$\begin{cases} \partial_t u_\varepsilon + H(x/\varepsilon, \partial_x u_\varepsilon, \omega) = 0 & \text{in } (0,+\infty) \times \mathbb{R} \\ u_\varepsilon(0,x,\omega) = u_0(x) & \text{on } \mathbb{R} \end{cases} \tag{18}$$

converge, locally uniformly in $(0,+\infty)\times\mathbb{R}$ as $\varepsilon \to 0^+$, to the unique viscosity solution $u \in \mathrm{UC}\big((0,+\infty) \times \mathbb{R}\big)$ of

$$\begin{cases} \partial_t u + \overline{H}(\partial_x u) = 0 & \text{in } (0,+\infty) \times \mathbb{R} \\ u(0,x) = u_0(x) & \text{on } \mathbb{R} \end{cases} \tag{19}$$

a.s. in ω.

A result is preliminary.

Proposition 7.3. *There exists a set Ω_0 of probability 1 such that equation*

$$H(x, P + v'(x,\omega), \omega) = \overline{H}(P) \tag{20}$$

has approximate correctors, sublinear at infinity, for any $\omega \in \Omega_0$ and $P \in \mathbb{R}$.

Proof. Let $(P_k)_k$ be a dense sequence in $\mathbb{R}$. By the results obtained in Section 6, there exists, for each $k \in \mathbb{N}$, a sequence of random functions $(v_n^k)_n \subseteq \mathcal{S}$ and a set Ω_k of probability 1 such that, given $\omega \in \Omega_k$, the function $v_n^k(\cdot,\omega)$ is sublinear at infinity and solves the following inequalities in the viscosity sense:

$$\overline{H}(P_k) - \frac{1}{n} \le H(x, P_k + (v_n^k)'(x,\omega), \omega) \le \overline{H}(P_k) + \frac{1}{n} \qquad \text{in } \mathbb{R}.$$

We claim that the statement holds true by choosing $\Omega_0 := \cap_k \Omega_k$. Indeed, let us fix $\omega \in \Omega_0$ and $P_0 \in \mathbb{R}$, then a subsequence P_{h_k} converge to P_0 and the functions $v_n^{h_k}(\cdot,\omega)$ are equiLipschitz–continuous for h_k large enough. Therefore we can employ assumption (H4) to control

$$|H(x, P_0 + p, \omega) - H(x, P_{h_k} + p, \omega)|$$

for any x, $p \in \partial v_n^{h_k}(x,\omega)$, h_k large enough. Using this and the fact that $\overline{H}$ is continuous, we can finally select, for any given $\delta > 0$, two indices h_k and n such that the function $\phi(x) := v_n^{h_k}(x,\omega)$ satisfies

$$\overline{H}(P_0) - \delta \le H(x, P_0 + \phi'(x), \omega) \le \overline{H}(P_0) + \delta \qquad \text{in } \mathbb{R},$$

in the viscosity sense, which proves the claim. □

Proof of Theorem 7.2. Let us fix $\omega \in \Omega_0$, where Ω_0 is a set of probability 1 chosen according to Proposition 7.3. The family $\{u_\varepsilon(\cdot,\cdot,\omega) : \varepsilon > 0\}$ of solutions to (18) is made up by equi–uniformly continuous functions in $\mathbb{R}_+ \times \mathbb{R}^N$ that agree at $t = 0$, hence it is precompact in $\mathrm{UC}(\mathbb{R}_+ \times \mathbb{R}^N)$ with respect to the local uniform convergence. In order to get the assertion, it is enough to show that any convergent subsequence has the (unique) solution u of (19) as limit.

The proof of this is based on the *perturbed test function method* (see [10, 11]) and basically exploits the existence of approximate correctors sublinear at infinity, which has been proved in Proposition 7.3. We just sketch it for the reader's convenience.

We show, for instance, that any function v, obtained as limit of a sequence of solutions to (18), say $u_{\varepsilon_k}(\cdot,\cdot,\omega)$, is a viscosity subsolution of (19). Let a point $(\bar{t}, \overline{x})$

be a local strict maximizer of $v - \phi$, with ϕ of class C^1, and let us assume, by contradiction

$$\delta := \partial_t \phi(\bar{t}, \bar{x}) + \overline{H}(\partial_x \phi(\bar{t}, \bar{x})) > 0.$$

We denote by χ a $\delta/2$–approximate corrector of (20) sublinear at infinity, with $P = \partial_x \phi(\bar{t}, \bar{x})$, and set

$$\phi_k(t, x) := \phi(t, x) + \varepsilon_k \chi(x/\varepsilon_k) \qquad \text{for } (t, x) \in \mathbb{R}_+ \times \mathbb{R}.$$

We can find $r > 0$ such that

$$\partial \phi_k(t, x) + H(x/\varepsilon_k, \partial_x \phi_k(t, x), \omega) \geq 0$$

in the viscosity sense in $V_r := (\bar{t} - r, \bar{t} + r) \times (\bar{x} - r, \bar{x} + r)$, for every $k \in \mathbb{N}$. Hence, by the comparison principle,

$$\max_{V_r} \, (u_{\varepsilon_k}(\cdot, \cdot, \omega) - \phi_k) \leq \max_{\partial V_r} \, (u_{\varepsilon_k}(\cdot, \cdot, \omega) - \phi_k).$$

By the sublinear character of χ, $u_{\varepsilon_k} - \phi_k \rightrightarrows v - \phi$ in $\mathbb{R}_+ \times \mathbb{R}$, so that, by passing to the limit in the above inequality, we obtain

$$\max_{V_r} \, (v - \phi) \leq \max_{\partial V_r} \, (v - \phi),$$

in contrast with $(\bar{t}, \bar{x})$ being a local strict maximum point for $v - \phi$. □

When the probability space has also a topological structure and the Hamiltonian is continuous with respect to ω, we can improve Theorem 7.2 by showing that the homogenization results holds for *every* ω. This allows us to recover what is known in the quasi–periodic and almost–periodic setting, see Remark 4.2.

Theorem 7.4. *Let $H : \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ satisfy (H1)–(H6). Let us additionally assume that Ω is a topological space such that*

(i) $\mathbb{P}(U) > 0$ for every nonempty open set $U \subset \Omega$;

(ii) the map $\omega \mapsto H(\cdot, \cdot, \omega)$ from Ω to $C(\mathbb{R} \times \mathbb{R})$ is continuous with respect to the uniform convergence on $\mathbb{R} \times B_R$ for every $R > 0$.

Then the conclusion of Theorem 7.2 holds for every $\omega \in \Omega$.

Proof. Let Ω_0 be a set of probability 1 chosen according to Proposition 7.3. Given $\omega \in \Omega \setminus \Omega_0$, we need to show that equation (20), with such an ω, admits approximate correctors, sublinear ay infinity, for any $P \in \mathbb{R}$.

Let us fix $P \in \mathbb{R}$ and $\delta > 0$. Since $\mathbb{P}$ charges every open neighborhood of ω, there is a sequence ω_k contained in Ω_0 such that $H(\cdot, \cdot, \omega_k)$ converge to $H(\cdot, \cdot, \omega)$ uniformly on $\mathbb{R} \times \mathcal{B}_R$ for every $R > 0$. We denote by u_k a $\delta/2$–approximate corrector, sublinear at infinity, of (20) with ω_k in place of ω and note that such functions are equiLipschitz–continuous, therefore for k large enough we have

$$|H(x, P + p, \omega) - H(x, P + p, \omega_k)| < \frac{\delta}{2} \qquad \text{for any } x \in \mathbb{R},\, p \in \partial u_k(x).$$

This implies that u_k is a δ–approximate correctors of (20) for k large enough. The proof is complete. □

References

[1] L. Ambrosio, H. Frid, Multiscale Young measures in almost periodic homogenization and applications, *Preprint* (available at `http://cvgmt.sns.it/papers/ambfri/`).

[2] M. Arisawa, Multiscale homogenization for first-order Hamilton-Jacobi equations. Proceedings of the workshop on Nonlinear P.D.E., Saitama University, 1998.

[3] M. Bardi, I. Capuzzo Dolcetta, Optimal control and viscosity solutions of Hamilton–Jacobi–Bellman equations. With appendices by Maurizio Falcone and Pierpaolo Soravia. Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 1997.

[4] G. Barles, Solutions de viscositè des équations de Hamilton–Jacobi. Mathmatiques & Applications, 17. Springer–Verlag, Paris, 1994.

[5] F. Camilli, A. Siconolfi, Maximal subsolutions for a class of degenerate Hamilton-Jacobi problems. *Indiana Univ. Math. J.* **48** (1999), no. 3, 1111–1131.

[6] F. Camilli, A. Siconolfi, Effective Hamiltonian and homogenization of measurable Eikonal equations. *Arch. Ration. Mech. Anal.*, **183** (2007), no. 1, 1–20.

[7] G. Castaing, M. Valadier, Convex Analysis and Measurable Multifunctions. Springer–Verlag, Berlin, 1977.

[8] F.H. Clarke, Optimization and nonsmooth analysis. Wiley, New York, 1983.

[9] A. Davini, A. Siconolfi, A generalized dynamical approach to the large time behavior of solutions of Hamilton–Jacobi equations. *SIAM J. Math. Anal.*, Vol. 38, , no. 2 (2006), 478–502.

[10] L.C. Evans, The perturbed test function method for viscosity solutions of nonlinear PDE. *Proc. Roy. Soc. Edinburgh Sect. A* **111** (1989), no. 3-4, 359–375.

[11] L.C. Evans, Periodic homogenisation of certain fully nonlinear partial differential equations. *Proc. Roy. Soc. Edinburgh Sect. A* **120** (1992), no. 3-4, 245–265.

[12] A. Fathi, A. Siconolfi, PDE aspects of Aubry–Mather theory for quasiconvex Hamiltonians. *Calc. Var. Partial Differential Equations* **22**, no. 2 (2005) 185–228.

[13] H. Ishii, Almost periodic homogenization of Hamilton-Jacobi equations. International Conference on Differential Equations, Vol. 1, 2 (Berlin, 1999), 600–605, World Sci. Publ., River Edge, NJ, 2000.

[14] V.V. Jikov, S.M. Kozlov, O.A. Oleinik, Homogenization of differential operators and integral functionals. Translated from the Russian by G.A. Yosifian. Springer-Verlag, Berlin, 1994.

[15] P.L. Lions, G. Papanicolau, S.R.S. Varadhan, Homogenization of Hamilton–Jacobi equations, unpublished preprint (1987).

[16] P.L. Lions, P.E. Souganidis, Correctors for the homogenization of Hamilton-Jacobi equations in the stationary ergodic setting. *Comm. Pure Appl. Math.* **56** (2003), no. 10, 1501–1524.

[17] I. Molchanov, Theory of random sets. Probability and its Applications (New York). Springer-Verlag London, Ltd., London, 2005.

[18] F. Rezakhanlou, J. E. Tarver, Homogenization for stochastic Hamilton-Jacobi equations. *Arch. Ration. Mech. Anal.* **151** (2000), no. 4, 277–309.

[19] P. E. Souganidis, Stochastic homogenization of Hamilton-Jacobi equations and some applications. *Asymptot. Anal.* **20** (1999), no. 1, 1–11.

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