METRIC TECHNIQUES FOR CONVEX STATIONARY ERGODIC HAMILTONIANS

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Abstract. We adapt the metric approach to the study of stationary ergodic Hamilton–Jacobi equations, for which a notion of admissible random (sub)solution is defined. For any level of the Hamiltonian greater than or equal to a distinguished critical value, we define an intrinsic random semidistance and prove that an asymptotic norm does exist. Taking as source region a suitable class of closed random sets, we show that the Lax formula provides admissible subsolutions. This enables us to relate the degeneracies of the critical stable norm to the existence/nonexistence of exact or approximate critical admissible solutions.

1. Introduction

The main purpose of the paper is to adapt the so–called metric method, which has revealed to be a powerful tool for the analysis of critical Hamilton–Jacobi equations posed on compact spaces, see [5, 8, 16], to the stationary ergodic setting. Loosely speaking, the ergodicity can be viewed as a weaker form of compactness, mostly thanks to some powerful asymptotic results, like Birkhoff and Kingman subadditive Theorems, that we repeatedly employ in our research.

We consider a probability space Ω, on which the action of $\mathbb{R}^N$ gives rise to an N–dimensional ergodic dynamical system, and a random continuous Hamiltonian $H(x, p, \omega)$, which is stationary with respect to such dynamics, and, in addition, convex and coercive in the momentum variable.

We look for *admissible subsolutions* of the corresponding stochastic Hamilton–Jacobi equations at different levels of the Hamiltonians. By this we mean Lipschitz random functions which are almost surely subsolutions either in the *viscosity* sense, or, equivalently, *almost everywhere*, while the term *admissible* refers to the fact that they are stationary or, in a weaker form, that they possess stationary increments and gradient with vanishing mean. Exploiting ergodicity and Birkhoff Theorem, this last property turns out to be equivalent to the almost sure sublinearity at infinity.

Actually, we prove that the infima of the values for which the corresponding equations admit a subsolution of the two types coincide. This quantity is called the *stationary critical value* of H and will be denoted by c. The difference is that, due to lack of stability, the existence of a stationary subsolution to $H(x, Dv, \omega) = c$ can fail already in the one–dimensional setting, see [9], while an Ascoli–type theorem, see Theorem 2.12, adjusted to the random environment, guarantees to find subsolutions of the latter class to the critical equation, cf. [19]. For this reason we will use, from now on, the word admissible in the weak sense.

The relevance of the critical value is in the fact that it is the unique level of H for which the corresponding Hamilton–Jacobi equation can have admissible (viscosity) solutions or approximate solutions, see Section 2.3 for definitions. These objects can be used as exact or approximate correctors in related homogenization procedures implementing the perturbed test function method [11, 12].

Existence and nonexistence issues for exact and approximate solutions are relevant open problems in the field. They have been first addressed in [19], and actually this paper has initially attracted our attention on the subject. So far, the setup has been completely clarified only in the one–dimensional case [9], where we proved the existence of approximate or exact correctors, depending on whether 0 belongs or not to the interior of the flat part of the effective Hamiltonian.

What is disappointing, at first sight, about the metric approach in this context, is that it is, in the starting point, purely deterministic, with ω playing just the role of a parameter. We in fact define for every $\omega \in \Omega$ and $a \in \mathbb{R}$ an intrinsic semidistance S_a starting from the support function of the a–sublevel of $H(x, \cdot, \omega)$.

It is well known that such a distance is finite if and only if $H(x, Du, \omega) = a$ admits (deterministic) subsolutions. A new critical value, depending on ω, say $c_f(\omega)$, then comes at the surface, corresponding to the minimum a for which the equation admits subsolutions. Because of the measurability properties of the Hamiltonian the map $\omega \mapsto c_f(\omega)$ is a random variable, which is, in addition, almost surely constant by the stationarity of H and the ergodicity assumption. Such a constant will be denoted by c_f and called *free critical value* to distinguish it from c. For the same reasons, S_a is, for $a \geq c_f$, a stochastic semidistance, namely a random variable taking value in the family of semidistances endowed with the local uniform convergence. It is apparent from its very definition that $c_f \leq c$, and strict inequality is possible.

From what previously outlined, it is clear that the intrinsic distances S_a cannot be useful *per se* to our analysis, in particular the critical level cannot be detected through the appearance of some degeneracies of the corresponding intrinsic random distance, like in the compact setting. As a matter of fact, such kind of phenomena do not take place, in general, even when $a < c$. Some other steps should therefore be accomplished.

We basically follow two ways: first, we perform an asymptotic analysis of intrinsic distances showing that corresponding (deterministic) *stable norms*, say ϕ_a, do exist for any $a \geq c_f$ and enjoy some relevant properties; secondly, we generalize Lax–type formulae to the stochastic environment providing a class of admissible subsolutions. Through the interplay of these lines of investigation, we establish in the end some of our main results.

To show that there is a stable norm, even in the deterministic case, some kind of subadditive principle is needed, see for instance [3]. Here we use Kingman's Subadditive Ergodic Theorem and mimic the proof given in [21, 22] for the existence of an *effective* (homogenized) *Hamiltonian.* The stable norms ϕ_a are of Minkowski type, possibly degenerate; we actually prove that the critical value is the infimum of the a for which ϕ_a is nondegenerate.

Such norms, being convex and positively homogeneous, are the support functions of some compact convex sets which can be interpreted as the associate dual unit balls. We show that such sets coincide with the corresponding sublevels of the effective Hamiltonian, denoted by $\overline{H}$. In this manner we provide a new simple proof of a result already established in [19] through PDE techniques, namely that the effective Hamiltonian coincides with the function associating with any $P \in \mathbb{R}^N$ the stationary critical value of the Hamiltonian $H(x, P+p, \omega)$, see Theorem 4.7. Moreover we show that the free critical value is the minimum of $\overline{H}$. This is the analogous of a result obtained in [15] for Hamiltonians defined in unbounded spaces and enjoying some form of symmetry, but it seems new in our context.

Regarding the Lax formula, we recall that for any fixed ω a class of fundamental subsolutions to $H(x, Du, \omega) = a$ is built up by

$$\inf\{g(y) + S_a(y, x, \omega) \, : \, y \in C\},$$

where C is a closed subset and g a function defined on it which is 1–Lipschitz continuous with respect to S_a. These kinds of functions are, in addition, solutions outside C. To get through this pattern *admissible* subsolutions, appropriate conditions have to be assumed on the source set, which depends on ω, as well as on the trace, linking them to the stationary ergodic structure.

The key idea, already exploited in [9], is to borrow some tools from stochastic geometry (see [20] for a comprehensive treatment of this topic), and to take as source region a stationary closed random set. That is to say a random variable taking values in the family of closed subsets of $\mathbb{R}^N$ endowed with the Fell topology which, in addition, satisfies a compatibility property with respect to the ergodic dynamics, see (2).

With this choice the Lax formula gives an admissible subsolution for any stationary Lipschitz random function g, provided it takes finite values, see Proposition 4.1. This latter condition is always fullfilled when g is itself an admissible subsolution, see Proposition 4.2. In this instance, the more delicate item to prove is that the function so obtained is sublinear at infinity, and for this scope it is essential the asymptotic formula for random closed stationary sets stated in Proposition 2.3 which, in turn, relies upon Birkhoff Theorem.

We use the information gathered to investigate on the existence of exact correctors when $c = c_f$. At this level, some degeneracies of the intrinsic semidistance may appear. The collection of points around which the latter fails to be equivalent to the Euclidean one form the classical *Aubry set* $\mathcal{A}_f(\omega)$, which, in this setting, turns out to be closed random and stationary. Thus, when it is almost surely nonempty, $\mathcal{A}_f(\omega)$ can be used as source region in the Lax formula to construct an exact corrector. If, on the contrary, it is almost surely empty and, in addition, the stable norm ϕ_c is nondegenerate, in other terms if no metric degeneracies take place at finite points or at infinity, then no correctors can exist, see Theorem 4.8. Note that all known counterexamples to the existence of correctors are in this frame.

When $c = c_f$ and the latter agrees with $\sup_x \inf_p H(x, p, \omega)$ almost surely, we also prove that approximate correctors can be constructed, always exploiting Lax formula, by taking as source region the set of δ–approximate equilibria, see Proposition 4.9. It should be interesting to prove or disprove that such a property holds true whenever $c = c_f$.

Concerning the structure of the paper, we have collected all the material of preliminary nature in Section 2, while Sections 3 and 4 are devoted to present the main results of our research. A more detailed description of the content of each section is the following: the main notations are presented in Section 2.1, while in Section 2.2 we recall the notions of random set and of stationarity for random functions and random set–valued variables. Furthermore, we introduce the class of admissible random functions and describe their main properties. Section 2.3 is focused on deterministic and stochastic Hamilton–Jacobi equations. We introduce the metric tools we will use and we recall some basic facts about Aubry–Mather theory and deterministic Lax formulae. In Section 3 we show the existence of the stable norms associated with the intrinsic distances and we study their connection with the effective Hamiltonian. We begin Section 4 by giving a stochastic version of the Lax formulae, then we exploit the information gathered to prove our main results on the effective Hamiltonian and some partial results about the existence/nonexistence of exact and approximate correctors. The section ends with an example showing some major differences with the one–dimensional setting.

A last comment before concluding this introduction: the material exposed in Section 2.2, and more precisely the part devoted to Lipschitz random functions with stationary gradient, is by no means new, see for instance [18]. Yet, we did not succeed to find it in a form adequate to our needs and with neat proofs in the existing literature, and even if many results have been reconstructed following [18], such a derivation has been definitively neither immediate nor easy. Actually, it took us a considerable effort. To keep the preliminary section as short as possible, we have shifted proofs to the Appendix. We hope that our exposition can be useful for people, especially PDE oriented, working on stochastic Hamilton–Jacobi equations.

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2. PRELIMINARIES

2.1. **Notations and basic results.** We write below a list of symbols used throughout this paper.

N	an integer number
$B_R(x_0)$	the closed ball in $\mathbb{R}^N$ centered at x_0 of radius R
B_R	the closed ball in $\mathbb{R}^N$ centered at 0 of radius R
$\langle \cdot , \cdot \rangle$	the scalar product in $\mathbb{R}^N$
$\lvert \cdot \rvert$	the Euclidean norm in $\mathbb{R}^N$
$\mathbb{R}_+$	the set of nonnegative real numbers
$\mathcal{B}(\mathbb{R}^N)$	the σ–algebra of Borel subsets of $\mathbb{R}^N$
χ_E	the characteristic function of the set E

Given a subset U of $\mathbb{R}^N$, we denote by $\overline{U}$ its closure and, if it is Lebesgue measurable, by $|U|$ its N–dimensional Lebesgue measure. We say that a property holds *almost everywhere* (*a.e.* for short) on $\mathbb{R}^N$ if it holds up to a subset of zero Lebesgue measure. We will write $\varphi_n \rightrightarrows \varphi$ on $\mathbb{R}^N$ to mean that the sequence of functions $(\varphi_n)_n$ uniformly converges to φ on compact subsets of $\mathbb{R}^N$.

With the term *curve*, without any further specification, we refer to a Lipschitz–continuous function from some given interval $[a, b]$ to $\mathbb{R}^N$. The space of all such curves is denoted by $\mathrm{Lip}([a, b]; \mathbb{R}^N)$, while $\mathrm{Lip}_{x,y}([a, b]; \mathbb{R}^N)$ stands for the family of curves γ joining x to y, i.e. such that $\gamma(a) = x$ and $\gamma(b) = y$, for any fixed x, y in $\mathbb{R}^N$. We denote by $W^{1,1}([a, b]; \mathbb{R}^N)$ the space of absolutely continuous curves defined in $[a, b]$. Given a curve γ defined on some interval $[a, b]$, a curve γ' defined on $[a', b']$ will be called a *reparametrization of* γ if there exists an order preserving Lipschitz–continuous map $f : [a', b'] \to [a, b]$ surjective and such that $\gamma' = \gamma \circ f$. The Euclidean length of a curve γ is denoted by $\mathcal{H}^1(\gamma)$.

For a measurable function $g : I \to \mathbb{R}^N$, $\|g\|_\infty$ stands for the usual L^∞ norm.

Throughout the paper, $(\Omega, \mathcal{F}, \mathbb{P})$ will denote a *probability space*, where $\mathbb{P}$ is the probability measure and $\mathcal{F}$ the σ–algebra of $\mathbb{P}$–measurable sets. A property will be said to hold *almost surely* (*a.s.* for short) in ω if it holds up to a subset of probability 0. We will indicate by $L^p(\Omega)$, $p \geq 1$, the usual Lebesgue space on Ω with respect to $\mathbb{P}$. If $f \in L^1(\Omega)$, we write $\mathbb{E}(f)$ for the mean of f on Ω, i.e. the quantity $\int_\Omega f(\omega)\, d\mathbb{P}(\omega)$.

We qualify as *measurable* a map from Ω to itself, or to a topological space $\mathcal{M}$ with Borel σ–algebra $\mathcal{B}(\mathcal{M})$, if the inverse image of any set in $\mathcal{F}$ or in $\mathcal{B}(\mathcal{M})$ belongs to $\mathcal{F}$. This object will be also called *random variable* with values in $\mathcal{M}$.

We will be particulary interested in the case where the range of a random variable is a *Polish* space, namely a complete and separable metric space. By $\mathrm{C}(\mathbb{R}^N)$ and $\mathrm{Lip}_\kappa(\mathbb{R}^n)$ we will denote the Polish space of continuous and Lipschitz–continuous real functions, with Lipschitz constant less than or equal to $\kappa > 0$, defined in $\mathbb{R}^N$, both endowed with the metric of the uniform convergence on compact subsets of $\mathbb{R}^N$. We will use the expressions *continuous random function*, *κ–Lipschitz random function*, respectively, for the previously introduced random variables. Actually, we will usually omit κ and simply write *Lipschitz random function.* The following characterization of random continuous functions holds, see [9]

Proposition 2.1. *Let $\omega \mapsto v(\cdot, \omega)$ be a map from Ω to $\mathrm{C}(\mathbb{R}^N)$. The following are equivalent facts:*

- *(i) v is a random continuous function;*
- *(ii) for every $x \in \mathbb{R}^N$, the map $\omega \mapsto v(x, \omega)$ is measurable in Ω;*
- *(iii) the map $(x, \omega) \mapsto v(x, \omega)$ is jointly measurable in $\mathbb{R}^N \times \Omega$, i.e. measurable with respect to the product σ–algebra $\mathcal{B}(\mathbb{R}^N) \otimes \mathcal{F}$.*

Throughout the paper $(\tau_x)_{x \in \mathbb{R}^N}$ will denote a *N–dimensional dynamical system*, defined as a family of mappings $\tau_x : \Omega \to \Omega$ which satisfy the following properties:

- *(1)* the *group property:* $\tau_0 = id, \quad \tau_{x+y} = \tau_x \circ \tau_y$;
- *(2)* the mappings $\tau_x : \Omega \to \Omega$ are measurable and measure preserving, i.e. $\mathbb{P}(\tau_x E) = \mathbb{P}(E)$ for every $E \in \mathcal{F}$;
- *(3)* the map $(x, \omega) \mapsto \tau_x \omega$ from $\mathbb{R}^N \times \Omega$ to Ω is jointly measurable, i.e. measurable with respect to the product σ–algebra $\mathcal{B}(\mathbb{R}^N) \otimes \mathcal{F}$.

We will moreover assume that $(\tau_x)_{x \in \mathbb{R}^N}$ is *ergodic,* i.e. that one of the following equivalent conditions holds:

- *(i)* every measurable function f defined on Ω such that, for every $x \in \mathbb{R}^N$, $f(\tau_x \omega) = f(\omega)$ a.s. in Ω, is almost surely constant;
- *(ii)* every set $A \in \mathcal{F}$ such that $\mathbb{P}(\tau_x A \,\Delta\, A) = 0$ for every $x \in \mathbb{R}^N$ has probability either 0 or 1, where Δ stands for the symmetric difference.

Notice that for any vector subspace $V \subset \mathbb{R}^N$, $(\tau_x)_{x \in V}$ is still a dynamical system on Ω, but ergodicity does not hold in general.

Given a random variable $f : \Omega \to \mathbb{R}$, for any fixed $\omega \in \Omega$ the function $x \mapsto f(\tau_x \omega)$ is said to be a *realization of f.* The following properties follow from Fubini's Theorem, see [17]: if $f \in L^p(\Omega)$, then $\mathbb{P}$–almost all its realizations belong to $L^p_{loc}(\mathbb{R}^N)$; if $f_n \to f$ in $L^p(\Omega)$, then $\mathbb{P}$–almost all realizations of f_n converge to the corresponding realization of f in $L^p_{loc}(\mathbb{R}^N)$. The Lebesgue spaces on $\mathbb{R}^N$ are understood with respect to the Lebesgue measure.

We will use in the sequel the Birkhoff Ergodic and the Kingman's subadditive Theorems; we refer to [17] and [21], respectively, for it.

2.2. **Stationary Random Variables.** Here we recall the notion of stationarity for random functions and random sets. These objects are of crucial relevance for the extension of Lax–type formulae to the stationary ergodic setting, see Propositions 4.1 and 4.2.

A jointly measurable function v defined in $\mathbb{R}^N \times \Omega$ is said *stationary* if, for every $z \in \mathbb{R}^N$, there exists a set Ω_z with probability 1 such that for every $\omega \in \Omega_z$

$$v(\cdot + z, \omega) = v(\cdot, \tau_z \omega) \quad \text{on } \mathbb{R}^N$$

It is clear that a real random variable ϕ gives rise to a stationary function v by setting $v(x, \omega) = \phi(\tau_x \omega)$. Conversely, according to Proposition 3.1 in [9], a stationary function v is, a.s. in ω, the realization of the measurable function $\omega \mapsto v(0, \omega)$. More precisely, there exists a set Ω' of probability 1 such that for every $\omega \in \Omega'$

$$v(x, \omega) = v(0, \tau_x \omega) \quad \text{for a.e. } x \in \mathbb{R}^N. \tag{1}$$

With the term *(graph–measurable) random set* we indicate a set–valued function $X : \Omega \to \mathcal{B}(\mathbb{R}^N)$ with

$$\Gamma(X) := \big\{ (x, \omega) \in \mathbb{R}^N \times \Omega \, : \, x \in X(\omega) \big\}$$

jointly measurable in $\mathbb{R}^N \times \Omega$. A random set X will be qualified as *stationary* if for every $z \in \mathbb{R}^N$, there exists a set Ω_z of probability 1 such that

$$X(\tau_z \omega) = X(\omega) - z \qquad \text{for every } \omega \in \Omega_z. \tag{2}$$

We use a stronger notion of measurability, which is usually named in the literature after Effros, to define a *closed random set*, say $X(\omega)$. Namely we require $X(\omega)$ to be a closed subset of $\mathbb{R}^N$ for any ω and

$$\{\omega \, : \, X(\omega) \cap K \neq \emptyset\} \in \mathcal{F}$$

with K varying among the compact (equivalently, open) subsets of $\mathbb{R}^N$. This condition can be analogously expressed by saying that X is measurable with respect to the Borel σ–algebra related to the Fell topology on the family of closed subsets of $\mathbb{R}^N$. This, in turn, coincides with the Effros σ–algebra. If $X(\omega)$ is measurable in this sense then it is also graph–measurable, see [20] for more details.

A closed random set X is called stationary if it additionally satisfies (2). Note that in this event the set $\{\omega \, : \, X(\omega) \neq \emptyset\}$, which is measurable by the Effros measurability of X, is invariant with respect to the group of translations $(\tau_x)_{x \in \mathbb{R}^N}$ by stationarity, so it has probability either 0 or 1 by the ergodicity assumption.

A convenient way to produce random closed (stationary) sets in $\mathbb{R}^N$ is indicated by the next result, see [9] for a proof.

Proposition 2.2. *Let f be a continuous random function and C a closed subset of $\mathbb{R}$. Then*

$$X(\omega) := \{x \, : \, f(x, \omega) \in C\}$$

is a closed random set in $\mathbb{R}^N$. If in addition f is stationary, then X is stationary.

For a random stationary set X is immediate, by exploiting that the maps $\{\tau_x\}_{x \in \mathbb{R}^N}$ are measure preserving, that $\mathbb{P}(X^{-1}(x))$ does not depend on x, where

$$X^{-1}(x) = \{\omega \, : \, x \in X(\omega)\}.$$

Such quantity will be called *volume fraction* of X and denoted by q_X.

Exploiting the ergodicity assumption and the Birkhoff Ergodic Theorem, the following information on the asymptotic structure of stationary sets can be obtained, see [9] for a proof:

Proposition 2.3. *Let X be an almost surely nonempty closed stationary set in $\mathbb{R}^N$. Then for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that*

$$\lim_{r \to +\infty} \frac{|\,(X(\omega) + B_R) \cap B_r|}{|B_r|} \geq 1 - \varepsilon \qquad a.s. \text{ in } \Omega,$$

whenever $R \geq R_\varepsilon$.

We proceed by outlining the main properties of Lipschitz random functions with stationary increments. Given a Lipschitz random function v, we set

$$\Delta_v(\omega) := \left\{ x \in \mathbb{R}^N \,:\, v(\cdot,\omega) \text{ is differentiable at } x \right\}.$$

We prove in the Appendix, see Proposition A.2, that Δ_v is a random set.

Definition 2.4. A random Lipschitz function v is said to have *stationary increments* if, for every $z \in \mathbb{R}^N$, there exists a set Ω_z of probability 1 such that

$$v(x+z,\omega) - v(y+z,\omega) = v(x,\tau_z\omega) - v(y,\tau_z\omega) \quad \text{for all } x, y \in \mathbb{R}^N$$

for every $\omega \in \Omega_z$.

If a random Lipschitz function v has stationary increments then Δ_v is a stationary random set with volume fraction 1, see Proposition A.2 in the Appendix.

For equiLipschitz random functions with stationary increments , i.e. taking all values in $\mathrm{Lip}_\kappa(\mathbb{R}^N)$ for some fixed $\kappa > 0$, the following relevant stability result holds true, see the Appendix for a proof.

Theorem 2.5. *Let $(v_n)_n$ be an equiLipschitz sequence of random functions with stationary increments. Then there exist a random Lipschitz function v with stationary increments, a sequence $w_k = \sum_{n \geq n_k} \lambda_n^k v_n$ of finite convex combinations of the v_n and a sequence g_k of real random variables such that*

$$w_k(\cdot,\omega) + g_k(\omega) \rightrightarrows v(\cdot,\omega) \quad a.s. \text{ in } \omega.$$

In addition the sequence of indices $(n_k)_k$ can be taken diverging.

Let v be a Lipschitz random function with stationary gradient. For every fixed $x \in \mathbb{R}^N$, the random variable $Dv(x,\cdot)$ is well defined on $\Delta_v^{-1}(x)$, which has probability 1 since Δ_v is a stationary set with volume fraction 1. Accordingly, we can define the mean $\mathbb{E}(Dv(x,\cdot))$, which is furthermore independent of x, see Proposition A.6–*(i)*. In the sequel, we will be especially interested in the case when this mean is zero.

Definition 2.6. A Lipschitz random function will be called *admissible* if it has stationary increments and gradient with mean 0.

We state three theorems on admissible random functions, see the Appendix for the proofs.

Theorem 2.7. *A Lipschitz random function v with stationary increments has gradient with vanishing mean if and only if it is almost surely sublinear at infinity, namely*

$$\lim_{|x|\to+\infty} \frac{v(x,\omega)}{|x|} = 0 \qquad a.s.\ in\ \omega. \tag{3}$$

Theorem 2.8. *A Lipschitz random function v with stationary increments has gradient with vanishing mean if and only if*

$$x \mapsto \mathbb{E}(v(y,\cdot) - v(x,\cdot)) = 0 \quad \text{for any } x, y \in \mathbb{R}^N. \tag{4}$$

Theorem 2.9. *Any stationary Lipschitz random function v is admissible.*

2.3. **Stochastic Hamilton–Jacobi equations.** We consider a Hamiltonian

$$H : \mathbb{R}^N \times \mathbb{R}^N \times \Omega \to \mathbb{R}$$

satisfying the following conditions:

(H1) the map $\omega \mapsto H(\cdot,\cdot,\omega)$ from Ω to the Polish space $C(\mathbb{R}^N \times \mathbb{R}^N)$ is measurable;
(H2) for every $(x,\omega) \in \mathbb{R}^N \times \Omega$, $H(x,\cdot,\omega)$ is convex on $\mathbb{R}^N$;
(H3) there exist two superlinear functions $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\alpha(|p|) \le H(x,p,\omega) \le \beta(|p|) \qquad \text{for all } (x,p,\omega) \in \mathbb{R}^N \times \mathbb{R}^N \times \Omega;$$

(H4) $H(\cdot + z, \cdot, \omega) = H(\cdot,\cdot,\tau_z\omega)$ for every $(z,\omega) \in \mathbb{R}^N \times \Omega$.

Remark 2.10. Condition (H3) is equivalent to saying that H is superlinear and locally bounded in p, uniformly with respect to (x,ω). We deduce from (H2)

$$|H(x,p,\omega) - H(x,q,\omega)| \le L_R|p-q| \quad \text{for all } x,\ \omega, \text{ and } p,\ q \text{ in } B_R, \tag{5}$$

where $L_R = \sup\{\, |H(x,p,\omega)| \,:\, (x.\omega) \in \mathbb{R}^N \times \Omega,\ |p| \le R+2 \,\}$, which is finite thanks to (H3).

Remark 2.11. Any given periodic, quasi–periodic or almost–periodic Hamiltonian $H_0 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ can be seen as a specific realization of a suitably defined stationary ergodic Hamiltonian, cf. Remark 4.2 in [9].

For every $a \in \mathbb{R}$, we are interested in the stochastic Hamilton–Jacobi equation

$$H(x, Dv(x,\omega), \omega) = a \qquad \text{in } \mathbb{R}^N. \tag{6}$$

The analysis performed on it in [9] stays valid in the present multidimensional setting, with minor adjustments. We basically refer to it, just recalling the main items and pointing out the main differences.

We say that a Lipschitz random function is a *solution* (resp. *subsolution*) of (6) if it is a viscosity solution (resp. a.e. subsolution) a.s. in ω (see [1, 2] for the definition of viscosity (sub)solution in the deterministic case). Notice that any such subsolution takes value in $\mathrm{Lip}_{\kappa_a}(\mathbb{R}^n)$, where

$$\kappa_a := \sup\{\, |p| \,:\, H(x,p,\omega) \le a \text{ for some } (x,\omega) \in \mathbb{R}^N \times \Omega \,\},$$

and this quantity is finite thanks to (H3). We are interested in the class of *admissible subsolutions*, hereafter denoted by $\mathcal{S}_a$, i.e. random functions with stationary increments and zero mean gradient that are subsolutions of (6). An admissible solution will be also named *exact corrector*, remembering its role in homogenization. Further, for any $\delta > 0$, a random function v_δ will be called a *δ–approximate corrector* for the equation (6) if it belongs to $\mathcal{S}_{a+\delta}$ and satisfies the inequalities

$$a - \delta \le H(x, Dv_\delta(x,\omega), \omega) \le a + \delta \tag{7}$$

in the viscosity sense a.s. in ω. We say that (6) has *approximate correctors* if it admits δ–approximate correctors for any $\delta > 0$.

The following stability property of admissible subsolutions is a consequence of Theorem 2.5 along with the remark that, if in the convergence established there the approximating random functions are admissible, the limit too keeps this property. The proof is analogous to that of Theorem 4.3 in [9].

Theorem 2.12. *Let $(a_n)_n$ be a sequence of real numbers and v_n a random function in $\mathcal{S}_{a_n}$ for each n. If a_n converges to some a, there exist $v \in \mathcal{S}_a$ and a sequence $(w_k)_k$ made up by finite convex combinations of the v_n, up to an additive real random variable, such that*

$$w_k(\cdot,\omega) \rightrightarrows v(\cdot,\omega) \quad \textit{a.s. in } \omega.$$

We proceed by defining the *free* and the *stationary critical value*, denoted by $c_f(\omega)$ and c respectively, as follows:

$$c = \inf\{a \in \mathbb{R} \,:\, \mathcal{S}_a \neq \emptyset\}, \tag{8}$$

$$c_f(\omega) = \inf\left\{a \in \mathbb{R} \,:\, (6) \text{ has a subsolution } v \in \mathrm{Lip}(\mathbb{R}^N)\right\}. \tag{9}$$

The set appearing at the right–hand side of (8) is non void, since it contains the value $\sup_{(x,\omega)} H(x,0,\omega)$, which is finite thanks to (H3). We emphasize that in definition (9) we are considering *deterministic* a.e. subsolutions v of the equation (6), where ω is treated as a fixed parameter. Note that $c_f(\tau_z\omega) = c_f(\omega)$ for every $(z,\omega) \in \mathbb{R}^N \times \Omega$, so, by ergodicity, the random variable $c_f(\omega)$ is almost surely equal to a constant, still denoted by c_f. Hereafter we will write Ω_f for the set of probability 1 where $c_f(\omega)$ is equal to c_f. It is apparent that $c \ge c_f$.

In what follows, we mostly focus our attention on the *critical equation*

$$H(x, Dv(x,\omega), \omega) = c \qquad \text{in } \mathbb{R}^N. \tag{10}$$

It follows from Theorem 2.12 that it admits admissible subsolutions, i.e. $\mathcal{S}_c \neq \emptyset$, see also [19]. The relevance of the critical value c is given by the following result, see Theorem 4.5 in [9] for the proof.

Theorem 2.13. *The equation* (6) *has neither exact correctors nor approximate correctors for $a \neq c$.*

We introduce an intrinsic path distance, assuming in next formulae $a \ge c_f$ and $\omega \in \Omega_f$. We start by defining the sublevels

$$Z_a(x,\omega) := \{p \,:\, H(x,p,\omega) \le a\},$$

and the related support functions $\sigma_a(x,q,\omega)$ by

$$\sigma_a(x,q,\omega) := \sup\{\langle q,p\rangle \,:\, p \in Z_a(x,\omega)\}.$$

It comes from (5) (cf. Lemma 4.6 in [9]) that, given $b > a$, we can find $\delta = \delta(b,a) > 0$ with

$$Z_a(x,\omega) + B_\delta \subseteq Z_b(x,\omega) \qquad \text{for every } (x,\omega) \in \mathbb{R}^N \times \Omega_f. \tag{11}$$

This property is used in the proof of Theorems 3.1 and 3.3 (and also in the proof of Theorem 2.13, actually). It is straightforward to check that σ_a is convex in q, upper semicontinuous in x and, in addition, continuous whenever $Z_a(x, \omega)$ has nonempty interior or reduces to a point. We extend the definition of σ_a to $\mathbb{R}^N \times \mathbb{R}^N \times \Omega$ by setting $\sigma_a(\cdot, \cdot, \omega) \equiv 0$ for every $\omega \in \Omega \setminus \Omega_f$. With this choice, the function σ_a is jointly measurable in $\mathbb{R}^N \times \mathbb{R}^N \times \Omega$ and enjoys the stationarity property

$$\sigma_a(\cdot + z, \cdot, \omega) = \sigma_a(\cdot, \cdot, \tau_z \omega) \quad \text{for every } z \in \mathbb{R}^N \text{ and } \omega \in \Omega.$$

Next, for every $a \geq c_f$, we define the semidistance S_a as

$$S_a(x, y, \omega) = \inf \left\{ \int_0^1 \sigma_a(\gamma(s), \dot{\gamma}(s), \omega) \, \mathrm{d}s \; : \; \gamma \in \mathrm{Lip}_{x,y}([0,1]; \mathbb{R}^N) \right\}. \tag{12}$$

The following holds, cf. [16]:

Proposition 2.14. *Let $a \geq c_f$.*

(i) The function S_a is measurable on $\mathbb{R}^N \times \mathbb{R}^N \times \Omega$ with respect to the product σ–algebra $\mathcal{B}(\mathbb{R}^N) \otimes \mathcal{B}(\mathbb{R}^N) \otimes \mathcal{F}$, and satisfies the following properties:

$$\begin{aligned} S_a(x, y, \tau_z \omega) &= S(x + z, y + z, \omega) \\ S_a(x, y, \omega) &\leq S_a(x, z, \omega) + S_a(z, y, \omega) \\ S_a(x, y, \omega) &\leq \kappa_a |x - y| \end{aligned}$$

for all $x, y, z \in \mathbb{R}^N$ and $\omega \in \Omega$.

(ii) Let $\omega \in \Omega_f$. A continuous function ϕ is a subsolution of (6) if and only if

$$\phi(x) - \phi(y) \leq S_a(y, x, \omega) \qquad \text{for all } x, y \in \mathbb{R}^N.$$

Let L be the Lagrangian associated with H via the Fenchel transform, i.e.

$$L(x, q, \omega) := \max_{p \in \mathbb{R}^N} \big(\langle q, p \rangle - H(x, p, \omega) \big), \qquad (x, q, \omega) \in \mathbb{R}^N \times \mathbb{R}^N \times \Omega,$$

and

$$h_t(x, y, \omega) := \inf \left\{ \int_0^t L(\gamma, \dot{\gamma}, \omega) \, \mathrm{d}s \; : \; W^{1,1}([0,t]; \mathbb{R}^N), \, \gamma(0) = x, \, \gamma(t) = y \right\} \tag{13}$$

for every $x, y \in \mathbb{R}^N$, $\omega \in \Omega$ and $t > 0$. Since for $a \geq c_f$

$$L(x, q, \omega) \geq \max_p \{ \langle q, p \rangle - H(x, p, \omega) \; : \; H(x, p, \omega) \leq a \} = \sigma_a(x, q, \omega) - a \quad \text{a.s. in } \omega,$$

by the previous item (ii) and the definition of c_f it is easily seen that

$$h_t(x, x, \omega) + at \geq 0 \qquad \text{for every } x \in \mathbb{R}^N, \, t > 0 \text{ and } \omega \in \Omega_f.$$

We define for every $\omega \in \Omega$ the *classical (projected) Aubry set* (cf. Proposition 3-7.1 in [6] and Theorem 4.8 in [8]) as

$$\mathcal{A}_f(\omega) := \left\{ y \in \mathbb{R}^N \; : \; \liminf_{t \to +\infty} h_t(y, y, \omega) + c_f \, t = 0 \right\}.$$

It plays a crucial role in the study of equation (6) with $a = c_f$. It is easily seen that $\mathcal{A}_f(\omega)$ is closed for every $\omega \in \Omega$.

Given $\omega \in \Omega_f$, $a \geq c_f$, and a closed subset C of $\mathbb{R}^N$, a standard way of producing a deterministic subsolution of (6) is by means of the following Lax formula

$$v(x) := \inf\{ w_0(y) + S_a(y, x, \omega) \; : \; y \in C \} \qquad x \in \mathbb{R}^N, \tag{14}$$

where w_0 is a function defined on C which is 1–Lipschitz continuous with respect to $S_a(\cdot,\cdot,\omega)$, i.e.

$$w_0(x) - w_0(y) \le S_a(y,x,\omega) \qquad \text{for every } x,y \in C.$$

We recall that the function given above is also the maximal subsolution taking the value w_0 on C and hence a solution in $\mathbb{R}^N \setminus C$. Furthermore we have (see [16]):

Theorem 2.15. *For a fixed $\omega \in \Omega_f$, the following holds.*

(i) Let $a = c_f$, $C \subset \mathcal{A}_f(\omega)$ and w_0 1–Lipschitz continuous with respect to S_{c_f}. Then the function v given by (14) *is a viscosity solution on the whole $\mathbb{R}^N$.*

(ii) Let U be a bounded open subset of $\mathbb{R}^N$, $a > c_f$ and w_0 a function defined on ∂U which is 1–Lipschitz continuous with respect to S_{c_f}. Then the function v given by (14) *with $C := \partial U$ is the unique viscosity solution of the Dirichlet Problem:*

$$\begin{cases} H(x, D\phi(x), \omega) = a & \text{in } U \\ \phi(x) = w_0(x) & \text{on } \partial U. \end{cases}$$

(iii) Let U as above, $a = c_f$ and let w_0 be a function defined on $\partial U \cup (U \cap \mathcal{A}_f(\omega))$ which is 1–Lipschitz continuous with respect to S_{c_f}. Then the function v given by (14) *with $C := \partial U \cup (U \cap \mathcal{A}_f(\omega))$ is the unique viscosity solution of the Dirichlet Problem:*

$$\begin{cases} H(x, D\phi(x), \omega) = c_f & \text{in } U \setminus \mathcal{A}_f(\omega) \\ \phi(x) = w_0(x) & \text{on } \partial U \cup (U \cap \mathcal{A}_f(\omega)). \end{cases}$$

We define for every $\omega \in \Omega$ the set of *equilibria* as follows:

$$\mathcal{E}(\omega) := \{ y \in \mathbb{R} \, : \, \min_p H(y,p,\omega) = c_f \, \}.$$

The set $\mathcal{E}(\omega)$ is a (possibly empty) closed subset of $\mathcal{A}_f(\omega)$. Indeed, $h_t(y,y,\omega) = -c_f\, t$ for every $y \in \mathcal{E}(\omega)$, as easily seen by taking as a competitor curve in the definition of $h_t(y,y,\omega)$ the curve constantly equal to y. It is apparent that $c_f \ge \sup_{x\in\mathbb{R}^N} \min_{p\in\mathbb{R}^N} H(x,p,\omega)$ a.s. in ω; we point out that $\mathcal{E}(\omega)$ is nonempty if and only if the previous formula holds with an equality and the sup is a maximum. In this case, $\mathcal{E}(\omega)$ is made up by the points where such a maximum is attained. Note that $\omega \mapsto \sup_{x\in\mathbb{R}^N} \min_{p\in\mathbb{R}^N} H(x,p,\omega)$ is a random variable and consequently, by ergodicity, almost surely constant.

Later on we will give a stochastic version of the Lax–type formula (14) and we will make use of it when the random source set is either $\mathcal{A}_f(\omega)$ or $\mathcal{E}(\omega)$. The following result will be needed.

Proposition 2.16. *$\mathcal{E}$ and $\mathcal{A}_f$ are closed random stationary sets.*

Proof. For every $(x,\omega) \in \mathbb{R}^N \times \Omega$, let us set $c_0(x,\omega) = \min_p H(x,p,\omega)$ and

$$\Phi(x,y,\omega) = \liminf_{t\to+\infty} \, \min\{h_t(x,y,\omega) + c_f\, t, \; 1\}.$$

Then $\mathcal{E}(\omega) = \{\, y \in \mathbb{R}^N \, : \, c_0(y,\omega) = c_f \,\}$ and $\mathcal{A}_f(\omega) = \{\, y \in \mathbb{R}^N \, : \, \Phi(y,y,\omega) = 0 \,\}$. In view of Proposition 2.2, it suffices to show that the random functions $c_0(x,\omega)$ and $\Phi(x,x,\omega)$ are stationary, jointly measurable in (x,ω), and continuous in x for every fixed ω.

The stationary and continuous character of c_0 in x can be directly deduced from its very definition by making use of assumptions (H1), (H3) and (H4); for the measurability issue, simply notice that $c_0(x,\omega) = \inf_{p_k \in \mathbb{Q}} H(x, p_k, \omega)$.

Let us now consider Φ. It is easily seen by definition that

$$h_t(x,y,\tau_z\omega) = h_t(x+z, y+z, \omega) \qquad \text{for every } x,\, y,\, z \in \mathbb{R}^N,\, \omega \in \Omega \text{ and } t > 0,$$

which yields the asserted stationary character of $\Phi(x,x,\omega)$.

Moreover, $\Phi(\cdot,\cdot,\omega)$ is locally Lipschitz in $\mathbb{R}^N \times \mathbb{R}^N$ for every fixed ω, since the family of functions $\{h_t(\cdot,\cdot,\omega) \,:\, t \geq 1\}$ are locally equi–Lipschitz in $\mathbb{R}^N \times \mathbb{R}^N$ (see for instance [7]).

For the measurability character of Φ, we observe that, since the map $t \mapsto h_t(x,y,\omega)$ is continuous (see [7]), it will be enough to show that h_t is jointly measurable in (x,y,ω) for every fixed $t > 0$. By Proposition 2.1, this amounts to requiring that $\omega \mapsto h_t(x,y,\omega)$ is measurable for every fixed x, $y \in \mathbb{R}^N$, and the latter property is true since

$$h_t(x,y,\omega) = \inf_n \int_0^t L(\gamma_n, \dot{\gamma}_n, \omega)\, \mathrm{d}s$$

where $(\gamma_n)_n$ is any fixed sequence of regular curves dense in

$$\{\gamma \in \mathrm{C}^1\left([0,t];\mathbb{R}^N\right) \,:\, \gamma(0) = x,\, \gamma(t) = y\,\}$$

with respect to the C^1–norm (see Theorem 6.16 in [4]). □

3. Stable norms

In this section, we show the existence of asymptotic norm–type functions associated with S_a, whenever $a \geq c_f$, and explore their link with the effective Hamiltonian $\overline{H}$. Given $\varepsilon > 0$, we define

$$S_a^\varepsilon(x,y,\omega) = \inf\left\{ \int_0^1 \sigma_a(\gamma(t)/\varepsilon, \dot{\gamma}(t), \omega)\, \mathrm{d}t \,:\, \gamma \in \mathrm{Lip}_{x,y}([0,1];\mathbb{R}^N) \right\}$$

for every $x, y \in \mathbb{R}^N$ and $\omega \in \Omega_f$, where we agree that $S_a^\varepsilon(\cdot,\cdot,\omega) \equiv 0$ when $\omega \in \Omega \backslash \Omega_f$. Note that $S_a^\varepsilon(x,y,\omega) = \varepsilon\, S_a(x/\varepsilon, y/\varepsilon, \omega)$.

Theorem 3.1. *Let $a \geq c_f$. There exists a convex and positively 1–homogeneous function $\phi_a : \mathbb{R}^N \to \mathbb{R}$ such that*

$$S_a^\varepsilon(x,y,\omega) \underset{\varepsilon \to 0}{\rightrightarrows} \phi_a(y-x), \qquad x, y \in \mathbb{R}^N. \tag{15}$$

for any ω in a set Ω_a of probability 1. In addition, ϕ_a is nonnegative for $a = c$, and nondegenerate, i.e. satisfying $\phi_a(\cdot) \geq \delta_a |\cdot|$ for some $\delta_a > 0$, when $a > c$.

With some abuse of terminology, we will refer to the function ϕ_a appearing in the statement above as the *stable norm* associated with S_a, in analogy with the case of periodic Riemannian metrics. The above theorem states that ϕ_a is a *Minkowski norm* (i.e. a norm which fails to be symmetric) when $a > c$; it can possibly degenerate when $a = c$, in the sense that ϕ_c may be identically 0 along some directions.

Proof. The proof is basically divided in two parts. In the first half, we essentially follow the arguments of [21], to which we refer for the details (cf. also [22]). The second half, based on a combined use of Egoroff's and Birkhoff Ergodic Theorems, follows an argument provided in [23], which is also needed in [22] to complete the proof of Theorem 1.

Since for every ω the functions $\{S_a^\varepsilon(\cdot,\cdot,\omega)\}_{\varepsilon>0}$ are equiLipschitz–continuous, the local uniformity of the asserted convergence is a consequence of Ascoli–Arzelá Theorem, once we show that there is pointwise convergence.

The first step is to consider the sequence of random variables $S_a(0, n\,q, \omega)$, where q is any vector of $\mathbb{R}^N$. The subadditive decomposition through the double indexed random variables $S_a(m\,q, n\,q, \omega)$, $0 \le m \le n$, allows to apply the Subadditive Ergodic Theorem, see [21], and deduce the existence of

$$\lim_{n\to\infty} \frac{1}{n}\, S_a(0, n\,q, \omega) = \lim_{n\to\infty} S_a^{1/n}(0, q, \omega)$$

for ω belonging to some set Ω_q of probability 1. The estimate $|S_a(0, nq, \omega)| \le \kappa_a\, n|q|$, which holds for every ω, implies that such limit is almost surely finite. Since for every fixed ω the functions $\{S_a^\varepsilon(\cdot,\cdot,\omega)\}_{\varepsilon>0}$ are equiLipschitz–continuous, we derive that the same limit is attained by $S_a^\varepsilon(0, q, \omega)$, as ε goes to 0, for $\omega \in \Omega_q$, and stays unaffected passing from ω to $\tau_z\omega$ for all $z \in \mathbb{R}^N$, which in turn implies that it is almost surely constant by ergodicity. By possibly redefining Ω_q if necessary, we set

$$\phi_a(q) = \lim_{\varepsilon\to 0} S_a^\varepsilon(0, q, \omega), \qquad \omega \in \Omega_q. \tag{16}$$

By taking a sequence $(q_n)_n$ dense in $\mathbb{R}^N$ and exploiting the equiLipschitz–continuity of $\{S_a^\varepsilon(0,\cdot,\omega)\}_{\varepsilon>0}$ and $\phi_a(\cdot)$, we see that the convergence in (16) takes place for any $q \in \mathbb{R}^N$ whenever $\omega \in \widehat{\Omega}_a := \cap_n \Omega_{q_n}$. In addition, by the Subadditive ergodic Theorem

$$S_a^\varepsilon(0, q, \omega) \to \phi_a(q) \quad \text{in } L^1(\Omega) \qquad \text{for any } q \in \mathbb{R}^N.$$

Let us now fix x, y in $\mathbb{R}^N$. Since $S_a^\varepsilon(x, y, \omega) = S_a^\varepsilon(0, y - x, \tau_{x/\varepsilon}\omega)$ a.s. in ω, and $\tau_{x/\varepsilon}$ is measure preserving, we deduce

$$\mathbb{E}(|S_a^\varepsilon(0, y - x, \cdot) - \phi_a(y - x)|) = \mathbb{E}(|S_a^\varepsilon(x, y, \cdot) - \phi_a(y - x)|),$$

and so

$$S_a^\varepsilon(x, y, \omega) \to \phi_a(y - x) \qquad \text{in } L^1(\Omega).$$

We now proceed to show that there exists a fixed set of probability 1 on which this convergence also holds pointwise, for any pair x, y in $\mathbb{R}^N$. For this, we make a combined use of Egoroff's and Birkhoff Ergodic Theorem.

Since the functions $q \mapsto S_a^\varepsilon(0, q, \omega)$ are equiLipschitz–continuous and locally equibounded for every ω we deduce that, for every $r > 0$,

$$\sup_{|q|\le 2r} |S_a^\varepsilon(0, q, \omega) - \phi_a(q)| \underset{\varepsilon\to 0}{\longrightarrow} 0 \qquad \omega \in \widehat{\Omega}_a.$$

We use Egoroff's Theorem to make this convergence uniform in ω on large sets, when $r \in \mathbb{Q}^+$ (the set of positive rational numbers): for every $\delta > 0$, we find a set A_δ with $\mathbb{P}(\Omega \setminus A_\delta) \le \delta$ such that

$$\sup_{\omega\in A_\delta} \left(\sup_{|q|\le 2r} |S_a^\varepsilon(0, q, \omega) - \phi_a(q)| \right) \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

for every $r \in \mathbb{Q}^+$. The Birkhoff Ergodic Theorem, see [17], applied to the function χ_{A_δ} yields the existence of a set Ω^δ of probability 1 with

$$\lim_{R\to+\infty} ⨍_{B_R} \chi_{A_\delta}(\tau_x\omega)\, \mathrm{d}x = \mathbb{P}(A_\delta) \qquad \text{for every } \omega \in \Omega^\delta,$$

in other terms for every $\omega \in \Omega^\delta$

$$\frac{\left|\{x \in \mathbb{R}^N \,:\, \tau_x\omega \in A_\delta\,\} \cap B_R\right|}{|B_R|} \geq \mathbb{P}(A_\delta) - \delta \geq 1 - 2\delta, \tag{17}$$

whenever R is large enough. We set $\Omega_a := \cap_{\delta \in \mathbb{Q}^+} \Omega^\delta$. Given $\omega_0 \in \Omega_a$, for any $\alpha > 0$ we can therefore find, according to (17), a pair of positive numbers $\delta(\alpha)$ and $R(\alpha)$ such that, if $|z_0| \geq R(\alpha)$, any ball centered at z_0 with radius exceeding $\alpha|z_0|$ must intersect $\{x \,:\, \tau_x\omega_0 \in A_{\delta(\alpha)}\,\}$, or equivalently

$$|z_0 - z| \leq \alpha|z_0| \quad \text{for some } z \text{ with } \tau_z\omega_0 \in A_{\delta(\alpha)}. \tag{18}$$

Now fix $\alpha > 0$, and pick a pair of points x, y in $\mathbb{R}^N$. We assume that they both belong to B_r for some $r \in \mathbb{Q}^+$. Let ε_0 be such that

$$\sup_{\omega \in A_{\delta(\alpha)}} \left(\sup_{|q| \leq 2r} |S_a^\varepsilon(0, q, \omega) - \phi_a(q)| \right) \leq \alpha \quad \text{for } \varepsilon \leq \varepsilon_0$$

and

$$\frac{|x|}{\varepsilon_0} > R(\alpha).$$

We denote, for $\varepsilon \leq \varepsilon_0$, by z_ε a point such that (18) holds true with z_ε, $\frac{x}{\varepsilon}$ in place of z, z_0, respectively. Accordingly $|x - \varepsilon z_\varepsilon| \leq \alpha\, r$, and for $\varepsilon \leq \varepsilon_0$ we have

$$\begin{aligned} |S_a^\varepsilon(x, y, \omega_0) \;-\; \phi_a(y - x)| &\leq |S_a^\varepsilon(x, y, \omega_0) - S_a^\varepsilon(\varepsilon z_\varepsilon, y, \omega_0)| \\ &+ \;\; |S_a^\varepsilon(\varepsilon z_\varepsilon, y, \omega_0) - \phi_a(y - \varepsilon z_\varepsilon)| + |\phi_a(y - \varepsilon z_\varepsilon) - \phi_a(y - x)| \\ &\leq \;\; 2\,\kappa_a\,\alpha\, r + |S_a^\varepsilon(0, y - \varepsilon z_\varepsilon, \tau_{z_\varepsilon}\omega_0) - \phi_a(y - \varepsilon z_\varepsilon)| \leq \alpha\,(2\kappa_a\, r + 1). \end{aligned}$$

As α was arbitrarily chosen, we conclude that

$$\lim_{\varepsilon \to 0} S_a^\varepsilon(x, y, \omega_0) = \phi_a(y - x),$$

as desired.

It comes from its very definition that ϕ_a is positively homogeneous. To prove that it is convex, we pick $\omega \in \Omega_a$, $\lambda \in (0,1)$, x, y in $\mathbb{R}^N$, and we pass to the limit, as ε goes to 0, in the inequality

$$S_a^\varepsilon(0, \lambda\, x + (1 - \lambda)\, y, \omega) \leq S_a^\varepsilon(0, \lambda\, x, \omega) + S_a^\varepsilon(\lambda\, x, \lambda\, x + (1 - \lambda)\, y, \omega).$$

For the sign of ϕ_a, we take $v \in \mathcal{S}_c$. From (11) we know that, for every $a \geq c$, there exists $\delta_a \geq 0$ with

$$S_a^\varepsilon(0, q, \omega) = \varepsilon\, S_a(0, q/\varepsilon, \omega) \geq \delta_a|q| + \frac{v(q/\varepsilon, \omega) - v(0, \omega)}{1/\varepsilon},$$

for any $q \in \mathbb{R}^N$. In addition, $\delta_a > 0$ when $a > c$. By the sublinear character of $v(\cdot, \omega)$ a.s. in ω, we obtain in the limit $\phi_a(q) \geq \delta_a|q|$. ☐

We proceed recalling a result proved in [21, 22].

Proposition 3.2. *Let $h_t(x, y, \omega)$ be the randon function defined via (13). Then there exists a convex and superlinear function $\overline{L} : \mathbb{R}^N \to \mathbb{R}$ such that for any ω in a set Ω_0 of probability 1 the following convergence holds*

$$\frac{h_t(0, tq, \omega)}{t} \underset{t \to +\infty}{\rightrightarrows} \overline{L}(q), \qquad q \in \mathbb{R}^N. \tag{19}$$

The function $\overline{L}$ is called the *effective Lagrangian*, and the *effective Hamiltonian* is accordingly defined through the Fenchel transform as follows:

$$\overline{H}(p) = \max_{q \in \mathbb{R}^N} \big(\langle p, q \rangle - \overline{L}(q) \big) \qquad \text{for every } p \in \mathbb{R}^N.$$

Theorem 3.3. *For every $a \geq c_f$, the stable norm ϕ_a is the support function of the a–sublevel of the effective Hamiltonian $\overline{H}$.*

Proof. We denote by $\overline{\sigma}_a(\cdot)$ the support function of the a–sublevel of $\overline{H}$. From the inequality $L(x, q, \omega) + a \geq \sigma_a(x, q, \omega)$, holding for any $a \geq c_f$ and $(x, q, \omega) \in \mathbb{R}^N \times \mathbb{R}^N \times \Omega_f$, we infer

$$\frac{h_t(0, \lambda\, t\, q, \omega) + a\, t}{\lambda\, t} \geq \frac{1}{\lambda\, t} S_a(0, \lambda\, t\, q, \omega)$$

for every $\lambda > 0$ and $t > 0$. Passing to the limit for t going to $+\infty$ we find

$$\lambda^{-1} \, \big(\overline{L}(\lambda\, q) + a\big) \geq \phi_a(q)$$

and, taking into account the identity

$$\overline{\sigma}_a(q) = \inf_{\lambda > 0} \left\{ \lambda^{-1} \, \big(\overline{L}(\lambda\, q) + a\big) \right\},$$

we conclude that $\overline{\sigma}_a(\cdot) \geq \phi_a(\cdot)$. We divide the proof of the converse inequality in two steps.

Case 1: $a > c_f$. Clearly, it is enough to show that $\overline{\sigma}_a(q) \leq \phi_a(q)$ for every $q \in \mathbb{S}^{N-1}$. Let us fix such a q and pick an ω_0 such that both (19) and (15) hold. For every $n \in \mathbb{N}$, let $\gamma_n : [0, \ell_n] \to \mathbb{R}^N$ be a curve parameterized by the arc–length with $\gamma_n(0) = 0$, $\gamma_n(\ell_n) = nq$ and such that

$$1 + S_a(0, nq, \omega_0) > \int_0^{\ell_n} \sigma_a(\gamma_n, \dot{\gamma}_n, \omega_0) \, \mathrm{d}t.$$

We first claim that there exists a constant C such that $n \leq \ell_n \leq C\, n$ for every $n \in \mathbb{N}$. Indeed, let $v \in \mathrm{Lip}(\mathbb{R}^N)$ such that $H(x, Dv(x), \omega_0) \leq c_f$ a.e. in $\mathbb{R}^N$. As $a > c_f$, there exists by (11) a constant $\delta_a > 0$ such that

$$\int_0^{\ell_n} \sigma_a(\gamma_n, \dot{\gamma}_n, \omega_0) \, \mathrm{d}t \geq v(nq) - v(0) + \delta_a \, \ell_n,$$

so the claim follows with $C = (1 + 2\, \kappa_a)/\delta_a$. According to the results proved in [7] (cf. Lemma 3.4, Proposition 3.7 and Lemma 3.14) there exists a Borel–measurable function $\underline{\lambda}_a : \mathbb{R}^N \times \mathbb{S}^{N-1} \to [0, +\infty)$ such that

$$L(x, \underline{\lambda}_a(x, v)\, v, \omega_0) = \sigma_a(x, \underline{\lambda}_a(x, v)\, v, \omega_0) - a \qquad \text{for every } (x, v) \in \mathbb{R}^N \times \mathbb{S}^{N-1}.$$

Furthermore $\underline{\lambda}_a$ enjoys the following inequality

$$\frac{1}{\lambda_a} \leq \underline{\lambda}_a(x, v) \leq \lambda_a \qquad \text{for every } (x, v) \in \mathbb{R}^N \times \mathbb{S}^{N-1},$$

where λ_a is a positive real constant depending on H and a only. Set

$$f_n(s) := \int_0^s \frac{1}{\underline{\lambda}_a(\gamma_n(\varsigma), \dot{\gamma}_n(\varsigma))} \, \mathrm{d}\varsigma \qquad \text{for any } s \in [0, \ell_n],$$

and $\varphi_n = f_n^{-1}$ on $[0, f_n(\ell_n)]$. It is easily seen that φ_n is a strictly increasing bi–Lipschitz homeomorphism from $[0, f_n(\ell_n)]$ to $[0, \ell_n]$, and that $n/\lambda_a \leq f_n(\ell_n) \leq C\lambda_a\, n$. Arguing as in [7], we get that the curve

$$\xi_n(s) := (\gamma_n \circ \varphi_n)\,(s), \qquad s \in [0, f_n(\ell_n)]$$

is a reparameterization of γ_n such that

$$\int_0^{\ell_n} \sigma_a(\gamma_n, \dot\gamma_n, \omega_0)\,\mathrm{d}t = \int_0^{f_n(\ell_n)} \Big(L(\xi_n, \dot\xi_n, \omega_0) + a \Big)\,\mathrm{d}t.$$

For each $n \in \mathbb{N}$, let $f_n(\ell_n) = \lambda_n\, n$ with $\lambda_n \in [1/\lambda_a, C\lambda_a]$. Up to subsequence, we can assume that $\lambda_n \to \lambda$ as $n \to +\infty$. Then we get

$$\frac{1 + S_a(0, nq, \omega_0)}{n} \geq \frac{h_{\lambda_n n}(0, nq, \omega_0)}{n} + \lambda_n\, a = \lambda_n\, \frac{h_{\lambda_n n}(0, \lambda_n n\, q/\lambda_n, \omega_0)}{\lambda_n n} + \lambda_n\, a$$

and sending $n \to +\infty$ we finally obtain

$$\phi_a(q) \geq \lambda \left(\overline{L}(q/\lambda) + a \right) \geq \overline{\sigma}_a(q).$$

Step 2: $a = c_f$. We want to show that $\phi_{c_f}(\cdot) \equiv \overline{\sigma}_{c_f}(\cdot)$. By the previous step and by definition of $\overline{\sigma}_a$ we have

$$\overline{\sigma}_{c_f}(q) = \inf_{a > c_f} \overline{\sigma}_a(q) = \inf_{a > c_f} \phi_a(q) \qquad \text{for every } q \in \mathbb{R}^N.$$

We therefore get the assertion showing

$$\phi_{c_f}(q) = \inf_{a > c_f} \phi_a(q) \quad \text{ for every } q \in \mathbb{R}^N.$$

The inequality $\phi_{c_f}(q) \leq \inf_{a > c_f} \phi_a(q)$ comes directly from the monotonicity of $a \mapsto S_a(0, nq, \omega)$. For the converse, we fix q and invoke Kingman's Subadditive Ergodic Theorem to get

$$\phi_a(q) = \inf_{n \in \mathbb{N}} \frac{1}{n}\, \mathbb{E}\big(S_a(0, nq, \cdot)\big) \qquad \text{for every } a > c_f.$$

Since $S_{c_f}(0, nq, \omega) = \inf_{a > c_f} S_a(0, nq, \omega)$ for every ω, the Monotone Convergence Theorem then implies

$$\frac{1}{n}\, \mathbb{E}(S_{c_f}(0, nq, \cdot) = \frac{1}{n} \inf_{a > c_f} \mathbb{E}(S_a(0, nq, \cdot)) \geq \inf_{a > c_f} \phi_a(q),$$

and sending $n \to +\infty$ we obtain $\phi_{c_f}(q) \geq \inf_{a > c_f} \phi_a(q)$, as claimed. □

4. Critical equation and effective Hamiltonian

In this section we exploit the previous analysis on stable norms to establish relevant properties of the effective Hamiltonian as well as some existence/nonexistence result for exact and approximate correctors.

We start giving a stochastic version of the Lax formulae recalled in Subsection 2.3. Let $C(\omega)$ be an almost surely nonempty stationary closed random set in $\mathbb{R}^N$. Take a Lipschitz random function g and set, for $a \geq c_f$,

$$u(x, \omega) := \inf\{ g(y, \omega) + S_a(y, x, \omega) \,:\, y \in C(\omega) \} \quad x \in \mathbb{R}^N, \tag{20}$$

where we agree that $u(\cdot, \omega) \equiv 0$ when either $C(\omega) = \emptyset$ or the infimum above is $-\infty$. The following holds:

Proposition 4.1. *Let g be a stationary Lipschitz random function and $C(\omega)$, u as above. Let us assume that, for some $a \geq c_f$, the infimum in* (20) *is finite a.s. in ω. Then u is a stationary random variable belonging to $\mathcal{S}_a$ and satisfies $u(\cdot, \omega) \leq g(\cdot, \omega)$ on $C(\omega)$ a.s. in ω. Moreover, u is a viscosity solution of* (6) *in $\mathbb{R}^N \setminus C(\omega)$ a.s. in ω.*

When g is itself an admissible subsolution of (6), the infimum in (20) is always finite, a.s. in ω, and the following holds.

Proposition 4.2. *Let g be a random function belonging to $\mathcal{S}_a$ and $C(\omega)$, u as above. Then u belongs to $\mathcal{S}_a$. In addition, it is a viscosity solution of* (6) *in $\mathbb{R}^N \setminus C(\omega)$, and takes the value $g(\cdot,\omega)$ on $C(\omega)$ a.s. in ω.*

As already pointed in the Introduction, the property of being C a closed stationary set is of crucial importance to show that formula (20) defines an admissible Lipschitz random function. The proofs of the above results are analogous to those of Proposition 5.2. and 5.3 in [9], respectively, where the case $N = 1$ is considered.

We proceed by proving a result analogous to one proved in [15] for Hamiltonians enjoying some kind of symmetry

Theorem 4.3. $\min_{\mathbb{R}^N} \overline{H} = c_f$.

Proof. The inequality $\min_{\mathbb{R}^N} \overline{H} \leq c_f$ is immediate since the c_f–sublevel of $\overline{H}$ is nonempty, being $\overline{\sigma}_{c_f} = \phi_{c_f}$ finite by Theorems 3.1 and 3.3.

Let us prove $\min_{\mathbb{R}^N} \overline{H} \geq c_f$. Pick an ω in Ω_f. By Proposition 2-1.1 in [6] we know that, for every $a < c_f$,

$$\inf_{t>0} \left(h_t(0,0,\omega) + at\right) = -\infty,$$

in particular there exists $t_0 > 0$ such that $h_{t_0}(0,0,\omega) + at_0 < 0$. Since $h_{t+s}(0,0,\omega) \leq h_t(0,0,\omega) + h_s(0,0,\omega)$ for every $s, t > 0$ by the definition of h_t, we infer

$$\liminf_{t\to+\infty} \frac{h_t(0,0,\omega) + at}{t} \leq \lim_{n\to+\infty} \frac{n\left(h_{t_0}(0,0,\omega) + at_0\right)}{nt_0} < 0.$$

In view of Proposition 3.2, we get $\overline{L}(0) + a < 0$ for every $a < c_f$, that is $-\overline{L}(0) \geq c_f$. The assertion follows since $-\overline{L}(0) = \min_{\mathbb{R}^N} \overline{H}$. □

We exploit Proposition 4.1 to get

Proposition 4.4. *Let $a \geq c_f$ such that the corresponding stable norm is nondegenerate. Then equation* (6) *admits admissible stationary subsolutions.*

Proof. By hypothesis there exists $\delta_a > 0$ such that $\phi_a(q) \geq \delta_a |q|$ for every $q \in \mathbb{R}^N$. By Theorems 3.1 and 3.3 we derive

$$\liminf_{|y|\to+\infty} \frac{S(y,x,\omega)}{|y-x|} \geq \delta_a \qquad \text{for every } x \in \mathbb{R}^N$$

a.s. in ω. In particular $\inf_{y\in\mathbb{R}^N} S_a(y,x,\omega) > -\infty$ a.s. in ω for every fixed $x \in \mathbb{R}^N$. According to Proposition 4.1, an admissible stationary subsolution of (6) is obtained via (20) with $g(\cdot,\omega) \equiv 0$ and $C(\omega) = \mathbb{R}^N$ for every ω. □

We make use of the previous results to give a characterization of the stationary critical value.

Theorem 4.5. $c = \inf\{a \geq c_f \,:\, \phi_a$ *is nondegenerate*$\}$. *If $c > c_f$ then ϕ_c is degenerate but nonnegative.*

Proof. Let us call μ the infimum appearing in the statement. According to Theorem 3.1, $c \geq \mu$. The converse inequality is apparent by Proposition 4.4 since equation (6) admits admissible subsolutions for every $a > \mu$.

Assume by contradiction that $c > c_f$ and ϕ_c is nondegenerate. Then, since ϕ_a coincide with a–sublevel of $\overline{H}$, by continuity the same property holds for ϕ_a with $a < c$ and suitably close to c. That is in contradiction with Proposition 4.4. $\square$

Remark 4.6. In view of Theorem 3.3, the property of being ϕ_a non–degenerate is equivalent to the condition $0 \in \mathrm{Int}(\overline{Z}_a)$. So Theorem 4.5 can be equivalently restated by saying that $c = \inf\{a \geq c_f \,:\, 0 \in \mathrm{Int}(\overline{Z}_a)\}$, with $0 \in \partial \overline{Z}_c$ whenever $c > c_f$. See [9] for the analogy with the 1–dimensional case.

We derive from Proposition 4.4 and Theorem 4.5

$$c := \inf\{a \in \mathbb{R} \,:\, (6) \text{ admits stationary subsolutions}\}.$$

The infimum appearing in the formula is not necessarily a minimum, namely we cannot expect, in general, to find stationary critical subsolutions.

The next theorem relates the effective Hamiltonian to the stationary critical value. The result has been already established by Lions and Souganidis in [19] through PDE techniques. We propose a new, simpler proof based on the properties of the intrinsic metrics and on Theorems 3.1, 3.3.

Theorem 4.7. *$\overline{H}$ coincides with the function associating to any $P \in \mathbb{R}^N$ the stationary critical value of the Hamiltonian $H(x, P + p, \omega)$.*

Proof. We first prove the assertion for $P = 0$, i.e., with the notation used so far, that $\overline{H}(0) = c$. We know by Theorem 3.1 that $\phi_c(q) \geq 0$ for every $q \in \mathbb{R}^N$. This means, in view of Theorem 3.3, that $0 \in \overline{Z}_c$, i.e. $c \geq \overline{H}(0)$. On the other hand the inequality $c > \overline{H}(0)$, i.e. $0 \in \mathrm{Int}(\overline{Z}_c)$, may occur only when $c = c_f$ by Remark 4.6, but this is not possible since $\overline{H}(0) \geq c_f$ by Theorem 4.3.

To prove the assertion for any $P \in \mathbb{R}^N$, we apply the previous argument to the Hamiltonian $H_P(x, p, \omega) := H(x, P + p, \omega)$ and derive that $\overline{H}_P(0) = c_P$, where $\overline{H}_P$ and c_P are the associated effective Hamiltonian and stationary critical value. To get the assertion, it is left to show that $\overline{H}_P(0) = \overline{H}(P)$. To this purpose, denote by L_P the Lagrangian associated with H_P. It is easily seen that $L_P(x, q, \omega)$ coincides with $L(x, q, \omega) - \langle P, q\rangle$, so the claimed equality follows from the definition of effective Hamiltonian by exploiting Proposition 3.2 with L_P in place of L. $\square$

We now address our attention to the issue of the existence/nonexistence of exact or approximate correctors.

Theorem 4.8.

(i) *If $c = c_f$ and $\mathcal{A}_f(\omega) \neq \emptyset$ a.s. in ω, then there exists a corrector for* (10).

(ii) *If ϕ_c is nondegenerate, then $c = c_f$ and a corrector for* (10) *exists if and only if $\mathcal{A}_f(\omega) \neq \emptyset$ a.s. in ω.*

(iii) *If ϕ_c is nondegenerate and $\mathcal{A}_f(\omega) \neq \emptyset$ a.s. in ω, then the classical Aubry set is an uniqueness set for the critical equation, in the sense that two correctors agreeing on $\mathcal{A}_f(\omega)$ a.s. in ω, coincide on the whole $\mathbb{R}^N$ a.s. in ω. More precisely, any corrector u can be written as*

$$u(x, \omega) = \inf\{u(y, \omega) + S_c(y, x, \omega) \,:\, y \in \mathcal{A}_f(\omega)\} \quad \text{a.s. in } \omega. \tag{21}$$

Proof. *(i)* By Proposition 2.16 $\mathcal{A}_f(\omega)$ is a stationary closed random set. Hence, for any $g \in \mathcal{S}_c$, the function u given by the Lax formula with $a = c$, $\mathcal{A}_f(\omega)$ as source set and trace g on it, is a corrector by Theorem 2.15 (i).

(ii) The equality $c = c_f$ follows from Theorem 4.5. If $\mathcal{A}_f(\omega) \neq \emptyset$ a.s. in ω, a corrector for (10) exists by assertion (i). To prove the converse implication, let us assume by contradiction that a corrector u does exist and that $\mathcal{A}_f(\omega) = \emptyset$ a.s. in ω. Pick ω such that $\mathcal{A}_f(\omega) = \emptyset$ and $u(\cdot, \omega)$ is a viscosity solution of (10). Take $n \in \mathbb{N}$. Since the classical Aubry set is empty, we derive by Theorem 2.15 that $u(\cdot, \omega)$ is the unique viscosity solution of the Dirichlet Problem

$$\begin{cases} H(x, D\phi(x), \omega) = c_f & \text{in } B_n \\ \phi(x) = u(x, \omega) & \text{on } \partial B_n, \end{cases}$$

and

$$u(x, \omega) = \min_{y \in \partial B_n} \{u(y, \omega) + S_{c_f}(y, x, \omega)\} \qquad x \in B_n.$$

We deduce that there exists a diverging sequence $(y_n)_n$ such that

$$S_{c_f}(y_n, 0, \omega) = u(0, \omega) - u(y_n, \omega) \qquad \text{for every } n \in \mathbb{N}.$$

By exploiting the fact that $u(\cdot, \omega)$ is sublinear a.s. in ω, we derive

$$\min_{q \in \mathbb{S}^{N-1}} \phi_{c_f}(q) = \liminf_{|y| \to +\infty} \frac{S_{c_f}(y, 0, \omega)}{|y|} \leq \liminf_{n \to +\infty} \frac{u(0, \omega) - u(y_n, \omega)}{|y_n|} = 0 \qquad \text{a.s. in } \omega,$$

in contrast with the hypothesis that ϕ_{c_f} is nondegenerate.

(iii) We take a corrector u and fix an $\omega \in \Omega_f$ such that $u(\cdot, \omega)$ is a solution to (10) sublinear at infinity and $\mathcal{A}_f(\omega) \neq \emptyset$. Arguing as for item (ii), we see that for any given $x_0 \in \mathbb{R}^N$ there is $n = n(\omega)$ such that

$$u(x_0, \omega) < \inf\{u(z, \omega) + S_c(z, x_0, \omega) \,:\, z - x_0 \in \partial B_n\}.$$

According to Theorem 2.15 (iii), we deduce the existence of $y_0 \in \mathcal{A}_f(\omega)$ with

$$u(x_0, \omega) = u(y_0, \omega) + S_c(y_0, x_0, \omega),$$

and consequently

$$u(x_0, \omega) \geq \inf\{u(y, \omega) + S_c(y, x_0, \omega) \,:\, y \in \mathcal{A}_f(\omega)\}.$$

On the other side by Theorem 2.15 (i) the right hand–side of the previous formula is the maximal subsolution to (10) taking the value $u(\cdot, \omega)$ on $\mathcal{A}_f(\omega)$, which implies (21). □

In the case where $c = c_f = \sup_{x \in \mathbb{R}^N} \min_{p \in \mathbb{R}^N} H(x, p, \omega)$ a.s. in ω, item (i) of the previous theorem can be complemented as follows:

Proposition 4.9. *Assume* $c = c_f = \sup_{x \in \mathbb{R}^N} \min_{p \in \mathbb{R}^N} H(x, p, \omega)$ *and* $\mathcal{A}_f(\omega) = \emptyset$ *a.s. in* ω. *Then equation* (10) *admits approximate correctors.*

Proof. We fix $g \in \mathcal{S}_c$ and define, for any $\delta > 0$

$$\mathcal{E}_\delta(\omega) = \{x \in \mathbb{R}^N \,:\, \min_{p \in \mathbb{R}^N} H(x, p, \omega) \geq c - \delta\}.$$

Arguing as we did for $\mathcal{E}$ in Proposition 2.16, we see that $\mathcal{E}_\delta(\omega)$ is a closed random stationary set and is, in addition, a.s. nonempty. We claim that Lax formula with $\mathcal{E}_\delta(\omega)$ as source set and g as trace on it, provides a δ–approximate corrector.

We denote by v_δ the random function constructed as above indicated. By Proposition 4.1 we already know that v_δ is an admissible subsolution to (10), and a solution as well on $\mathbb{R}^N \setminus \mathcal{E}_\delta(\omega)$ a.s. in ω. Further, if ϕ is a C^1 test function touching $v_\delta(\cdot,\omega)$ at $y \in \mathcal{E}_\delta(\omega)$ from below, then $H(y, D\phi(y,\omega),\omega) \geq c - \delta$ by the very definition of $\mathcal{E}_\delta(\omega)$, as it was to be shown. $\square$

A class of critical stochastic equations satisfying the assumptions of the previous theorem are those of Eikonal type

$$|Du(x,\omega)|^2 = V(x,\omega)^2 \qquad \text{in } \mathbb{R}^N, \tag{22}$$

where $V : \mathbb{R}^N \times \Omega \to \mathbb{R}$ is a jointly measurable function satisfying:

(1) $V(x+z,\omega) = V(x,\tau_z\omega)$ for every $x, z \in \mathbb{R}^N$ and $\omega \in \Omega$;

(2) $V(\cdot,\omega)$ is continuous on $\mathbb{R}^N$ for every ω;

(3) $0 = \inf_{\mathbb{R}^N} V(\cdot,\omega) < \sup_{\mathbb{R}^N} V(\cdot,\omega) < +\infty$ a.s. in ω;

(4) $V(x,\omega) > 0$ for every $x \in \mathbb{R}^N$ a.s. in ω.

We can show, by exploiting an example in [19], that a random function of this type does always exist in any space dimension, cf. Example 4.10 below.

If we add to the assumptions of Proposition 4.9 the nondegeneracy of the critical stable norm, then we can also assert, according to Theorems 4.8 (ii), the nonexistence of exact correctors. In dimension 1 a sufficient conditions for such a nondegeneracy is that there is a strict critical admissible subsolution; this role is for instance played by the null function for the above described Eikonal class.

To better explain this point, let us introduce, for any $a \geq c_f$, the averaged a–sublevel $\mathbb{E}[Z_a]$ of H, given by

$$\mathbb{E}[Z_a] = \{\mathbb{E}(\Phi) \,:\, \Phi \text{ measurable selection of } \omega \mapsto Z_a(0,\omega)\}.$$

The map $\omega \mapsto Z_a(0,\omega)$ is a random closed set taking compact convex values and (see [20])

$$\sigma_{\mathbb{E}[Z_a]}(q) = \mathbb{E}\big(\sigma_a(0,q,\omega)\big) \qquad \text{for every } q \in \mathbb{R}^N,$$

where σ indicates the support function.

If a strict critical subsolution v exists, then, exploiting the stationarity, it is not hard to see that $Dv(0,\omega)$ is defined and belongs to the interior part of $Z_c(0,\omega)$ a.s. in ω, yielding

$$\sigma_c(0,q,\omega) > \langle Dv(0,\omega),\, q\rangle \qquad \text{a.s. in } \omega,$$

for any fixed $q \in \mathbb{R}^N$. Recalling that $Dv(0,\omega)$ has mean zero, we get by integration

$$\sigma_{\mathbb{E}[Z_c]}(q) > 0 \qquad \text{for every } q \in \mathbb{R}^N,$$

i.e. $0 \in \operatorname{Int}(\mathbb{E}[Z_c])$. When the averaged critical sublevel $\mathbb{E}[Z_c]$ coincides with the critical sublevel $\overline{Z}_c$ of the effective Hamiltonian, this amounts to saying that the critical stable distance is nondegenerate.

This is always the case when $N = 1$, as proved in [9]. The situation is quite different in the multidimensional setting, where we can just assert that the sublevels of $\overline{H}$ are contained in the corresponding averaged sublevels of H. The next example shows that this inclusion can be strict and that the critical stable norm can be (even completely) degenerate in presence of a strict admissible critical subsolution.

Example 4.10. We provide below an example in dimension $N = 2$ of a function V satisfying assumptions (1)–(4) such that, for every ω in a set of probability 1,

$$\lim_{t \to +\infty} \frac{S_0(0, tq, \omega)}{t} = 0 \qquad \text{for every } q \in \mathbb{S}^{N-1},$$

where S_0 is the distance associated with $H(x, p, \omega) := |p|^2 - V(x, \omega)^2$ via (12) with $a = 0$. According to the results obtained in the previous section, we derive that the corresponding stable norm is null, i.e. completely degenerate. Note that the null function is a strict admissible critical subsolution.

To this purpose, we start by defining a function $V_0 : \mathbb{T}^2 \to \mathbb{R}$ as follows:

$$V_0(z_1, z_2) = 2 - \cos(2\pi z_1) - \cos(2\pi z_2) \qquad (z_1, z_2) \in \mathbb{T}^2.$$

Let us choose a $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ and set $v_1 = (1, 0),\ v_2 = (\lambda, 0),\ v_3 = (0, 1),\ v_4 = (0, \lambda)$. Next we choose as Ω the torus $\mathbb{T}^4$, as $\mathbb{P}$ the Lebesgue measure restricted to $\mathbb{T}^4$, and as $\mathcal{F}$ the σ–algebra of Borel subsets of Ω. We define a group $(\tau_x)_{x \in \mathbb{R}^2}$ of translations on Ω as follows:

$$\big(\tau_x(\omega)\big)_i \equiv \omega_i + \langle v_i, x\rangle \pmod 1 \qquad \text{for every } \omega \in \Omega \text{ and } i \in \{1, 2, 3, 4\}.$$

The group of translations $(\tau_x)_{x \in \mathbb{R}^2}$ is ergodic, see for instance Appendix A in [10]. We define a function V on $\mathbb{R}^2 \times \Omega$ as

$$V(x, \omega) = V_0(\omega_1 + x_1, \omega_2 + \lambda x_1)\, V_0(\omega_3 + x_2, \omega_4 + \lambda x_2),$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $\omega = (\omega_1, \omega_2, \omega_3, \omega_4) \in \Omega$. Clearly, V is a jointly measurable function satisfying the above assumptions (1)–(3). Furthermore, V verifies assumption (4). Indeed, the function $V(\cdot, \omega)$ attains its infimum on $\mathbb{R}^2$ (that is, the value 0) if and only if $\omega \in \big(\Sigma \times \mathbb{T}^2\big) \cup \big(\mathbb{T}^2 \times \Sigma\big)$, where

$$\Sigma := \{(z_1, z_2) \in \mathbb{T}^2 \,:\, V_0(z_1 + t, z_2 + \lambda t) = 0 \text{ for some } t \in \mathbb{R}\,\}.$$

We claim that $|\Sigma| = 0$. More precisely,

$$\Sigma = \{(z_1, z_2) \in \mathbb{T}^2 \,:\, \lambda z_1 - z_2 = \lambda n - m \text{ for some } n, m \in \mathbb{Z}\,\}.$$

To see this, note that $(z_1, z_2) \in \Sigma$ if and only if there exists $t \in \mathbb{R}$ such that

$$z_1 + t = n,\ z_2 + \lambda t = m \quad \text{for some } n, m \in \mathbb{Z},$$

and this occurs if and only if $\lambda z_1 - z_2 = \lambda n - m$ for some $n, m \in \mathbb{Z}$, as it was claimed.

According to the results of Section 3, we know that, for every ω in a set Ω_0 of probability 1, we have

$$\overline{\sigma}_0(q) = \lim_{t \to +\infty} \frac{S_0(x, x + tq, \omega)}{t} \qquad \text{for every } x \in \mathbb{R}^N \text{ and } q \in \mathbb{S}^{N-1}. \tag{23}$$

To prove that $\overline{\sigma}_0(\cdot) \equiv 0$, it is enough, by the properties enjoyed by $\overline{\sigma}_0$, to show that $\overline{\sigma}_0(\mathbf{e_1}) \leq 0$ and $\overline{\sigma}_0(\mathbf{e_2}) \leq 0$, where $\{\mathbf{e_1}, \mathbf{e_2}\}$ is the canonical basis of $\mathbb{R}^2$. We only prove the assertion for $\mathbf{e_1}$, being the other case analogous. Pick an $\omega = (\omega_1, \omega_2, \omega_3, \omega_4) \in \Omega_0$, and fix an $\varepsilon > 0$. The orbit $\{(\omega_3 + x_2, \omega_4 + \lambda\, x_2) \,:\, x_2 \in \mathbb{R}\}$ is dense in $\mathbb{T}^2$ for λ is irrational, so there exists a point $x_\varepsilon = (0, y_\varepsilon) \in \mathbb{R}^2$ such that $V_0((\omega_3, \omega_4), (0, y_\varepsilon)) < \varepsilon$. By moving along the segment joining x_ε to $x_\varepsilon + t\mathbf{e_1}$, we obtain

$$S_0(x_\varepsilon, x_\varepsilon + t\mathbf{e_1}, \omega) \leq \int_0^1 V(x_\varepsilon + s\, t\mathbf{e_1}, \omega)|t\, \mathbf{e_1}|\, \mathrm{d}s \leq 4\varepsilon\, t,$$

from which we derive $\overline{\sigma}_0(\mathbf{e_1}) \leq 4\varepsilon$ by (23). This concludes the proof as ε is an arbitrarily chosen positive number.

Appendix A

The aim of the Appendix is to provide complete proofs of some results used throughout the paper, in particular Theorems 2.5, 2.7 and 2.8.

We will use in what follows the following classical result, which can be obtained as a direct application of Fubini's theorem, see [20].

Theorem A.1 (Robbins' Theorem). *Let X be a random set in $\mathbb{R}^N$. If μ is a locally finite measure on Borel sets, then $\mu(X)$ is a random variable and*

$$\int_\Omega \mu(X(\omega))\,\mathrm{d}\mathbb{P} = \int_{\mathbb{R}^N} \mathbb{P}(X^{-1}(x))\,\mathrm{d}\mu,$$

in the sense that if one side is finite, then so is the other and they are equal.

The following holds:

Proposition A.2. *Let v be a Lipschitz random function, then Δ_v is a random set. In addition, it is stationary with volume fraction 1 whenever v has stationary increments.*

Proof. The property of Δ_v of being a random set can be proved via standard measure theoretic arguments, see for instance Lemma 2.5 in [14] for a short proof. If v has stationary increments then, for any fixed $z \in \mathbb{R}^N$,

$$v(\cdot + z, \omega) - v(\cdot, \tau_z \omega) \quad \text{is constant on } \mathbb{R}^N$$

whenever ω belongs to some set Ω_z with probability 1. This implies that $x + z$ is a differentiability point for $v(\cdot, \omega)$ if and only if x is a differentiability point for $v(\cdot, \tau_z\omega)$, which, in turn, means that Δ_v is a stationary. Since $\Delta_v(\omega)$ has full measure in $\mathbb{R}^N$ for every ω by Rademacher's Theorem, Robbin's Theorem with μ equal the Lebesgue measure restricted to some ball of $\mathbb{R}^N$ readily implies that it volume fraction is equal to 1. $\square$

We define an N–parameter group $(U_x)_{x\in\mathbb{R}^N}$ of isometries on $\left(L^2(\Omega)\right)^N$ via

$$\begin{array}{rcl} U_x : \left(L^2(\Omega)\right)^N & \to & \left(L^2(\Omega)\right)^N \\ \Phi(\omega) & \mapsto & \Phi(\tau_x\omega) \end{array}$$

for every $x \in \mathbb{R}^N$. The group $(U_x)_{x\in\mathbb{R}^N}$ is strongly continuous, in the sense that

$$\lim_{x\to 0} \|U_x\Phi - \Phi\|_{(L^2(\Omega))^N} = 0, \qquad \Phi \in \left(L^2(\Omega)\right)^N,$$

see [17]. Using this property, it is easy to prove:

Lemma A.3. *Let $\gamma : [0,T] \to \mathbb{R}^N$ be of class C^1. Then for every $\Phi \in \left(L^2(\Omega)\right)^N$ the map $t \mapsto \langle U_{\gamma(t)}\Phi, \dot\gamma(t)\rangle$, from $[0,T]$ to $L^2(\Omega)$, is continuous.*

Hence, for every curve $\gamma : [0,T] \to \mathbb{R}^N$ of class C^1 we can give a meaning, in the Cauchy sense, to the integral

$$\int_0^T \langle U_{\gamma(t)}\Phi, \dot\gamma(t)\rangle\,\mathrm{d}t = \int_0^T \langle \Phi(\tau_{\gamma(t)}\omega), \dot\gamma(t)\rangle\,\mathrm{d}t \tag{24}$$

as an element of $L^2(\Omega)$. We note that (24) is invariant under changes of parameterization; moreover, it makes sense even when γ is piecewise C^1, i.e. it is continuous and of class C^1 on $[0,T]$ up to a finite set of points. For any such γ we write

$$\int_\gamma \Phi(\omega) := \int_0^T \langle U_{\gamma(t)}\Phi(\omega), \dot\gamma(t)\rangle \,\mathrm{d}t \qquad \text{a.s. in } \omega.$$

The following result holds:

Lemma A.4. *Let $\gamma : [0,T] \to \mathbb{R}^N$ be a piecewise C^1 curve. Then the map*

$$\Phi \mapsto \int_\gamma \Phi$$

is linear and continuous from $\big(L^2(\Omega)\big)^N$ to $L^2(\Omega)$.

It is a direct consequence of the previous lemma that, if $\Phi_n \to \Phi$ in $(L^2(\Omega))^N$, then, for any given piecewise C^1 curve γ, one can extract a subsequence Φ_{n_k} with

$$\int_\gamma \Phi_{n_k}(\omega) \to \int_\gamma \Phi(\omega) \quad \text{a.s. in } \omega.$$

Note that here the subsequence depends on the curve γ. The step forward in the next result is to show that, under suitable additional assumptions, the sequence $(n_k)_k$ can be chosen in such a way that the above convergence takes place for any curve.

Lemma A.5. *Let $(\Phi_n)_n$ be a sequence in $(L^\infty(\Omega))^N$ with $\sup_n \|\Phi_n\|_\infty \leq \kappa$ for some $\kappa > 0$. If Φ_n converges in $(L^2(\Omega))^N$ to some function Φ, then, up to extraction of a subsequence,*

$$\int_\gamma \Phi_n(\omega) \to \int_\gamma \Phi(\omega) \quad \textit{a.s. in } \omega$$

for all piecewise C^1 curve γ.

Proof. Up to extraction of a subsequence, Φ_n a.s. converges to Φ. The set

$$\{(x,\omega) \,:\, \lim_n \Phi_n(\tau_x\omega) = \Phi(\tau_x\omega)\,,\, |\Phi_n(\tau_x\omega)| \leq \kappa \text{ for any } n\},$$

is clearly measurable with respect to the product σ–algebra $\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{F}$, and so the almost sure convergence of Φ_n to Φ and the boundedness assumption on Φ_n imply that its x–sections have probability 1 for any fixed $x \in \mathbb{R}^N$. We derive from Fubini's Theorem that there exists a set Ω' of probability 1 such that, if $\omega \in \Omega'$

$$\Phi_n(\tau_x\omega) \to \Phi(\tau_x\omega), \quad \sup_n |\Phi_n(\tau_x\omega)| \leq \kappa \qquad \text{for a.e. } x \in \mathbb{R}^N. \tag{25}$$

The set

$$X(\omega) = \{x \,:\, \Phi_n(\tau_x\omega) \to \Phi(\tau_x\omega), \quad \sup_n |\Phi_n(\tau_x\omega)| \leq \kappa\}$$

is accordingly a stationary random set with volume fraction 1. Therefore, given a piecewise C^1 curve γ, by applying the Robbin's Theorem with μ equal to $\mathcal{H}^1$ restricted to γ, we see that $\mathcal{H}^1(\gamma \setminus X(\omega)) = 0$ for ω belonging to some subset of Ω of probability 1. For such an ω the claimed convergence on γ holds true thanks to the Dominated Convergence Theorem. □

We proceed by giving a closer look to the differentiability properties of Lipschitz random function with stationary increments.

Proposition A.6. *Let v be a Lipschitz random function with stationary increments. Then there exists $\Phi \in (L^\infty(\Omega))^N$ such that:*

(i) for every ω in a set of probability 1

$$Dv(x,\omega) = \Phi(\tau_x \omega) \quad \textit{for a.e. } x \in \mathbb{R}^N; \tag{26}$$

(ii) for every closed piecewise C^1 curve γ

$$\int_\gamma \Phi(\omega) = 0 \qquad \textit{a.s. in } \omega.$$

In addition, the equality (26) holds, for any *fixed x, a.s. in ω.*

Proof. Let κ be a positive constant such that $v(\cdot,\omega)$ is κ–Lipschitz for every ω. For each $i \in \{1, \cdots, N\}$ we define

$$w_i(x,\omega) = \sup_{n\in\mathbb{N}} \inf_{h \in \mathbb{Q}\cap B_{1/n}} \frac{v(x + h\, e_i, \omega) - v(x,\omega)}{h}.$$

Such function is jointly measurable in $\mathbb{R}^N \times \Omega$, satisfies $|w_i(x,\omega)| \leq \kappa$ for every $x \in \mathbb{R}^N$ and $\omega \in \Omega$, and, in addition, it is stationary, being v with stationary increments. By (1), there exists a set Ω' of probability 1 such that

$$w_i(x,\omega) = w_i(0, \tau_x\omega) \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and any } \omega \in \Omega'.$$

Moreover, for every $\omega \in \Omega$,

$$w_i(x,\omega) = \partial_{x_i} v(x,\omega) \quad \text{for every } x \in \Delta_v(\omega).$$

Since $\Delta_v(\omega)$ has full measure in $\mathbb{R}^N$ by the Lipschitz character of $v(\cdot,\omega)$, the equality (26) is obtained by setting $\Phi_i(\cdot) = w_i(0,\cdot)$.

For any $\omega \in \Omega$ we set

$$X(\omega) = \{x \in \Delta_v(\omega) \,:\, Dv(x,\omega) = \Phi(\tau_x\omega)\}.$$

By taking into account that Δ_v is a stationary random set (cf. Proposition A.2) and that v has stationary increments, we see that X is a random stationary set. Furthermore, from (26) we deduce that its volume fraction is 1. By Fubini's Theorem, we deduce that for any fixed $x \in \mathbb{R}^N$ the equality (26) holds a.s. in ω.

Given a piecewise C^1 closed curve $\gamma : [0,T] \to \mathbb{R}^N$, we invoke Robbins' Theorem with μ equal to $\mathcal{H}^1$ restricted to γ to deduce that $\mathcal{H}^1(\gamma \setminus X(\omega)) = 0$ for ω belonging to some subset of Ω with probability 1. For such an ω we get

$$\int_\gamma \Phi(\omega) = \int_0^T \langle Dv(\gamma(t),\omega), \dot\gamma(t)\rangle \, \mathrm{d}t = v(\gamma(T),\omega) - v(\gamma(0),\omega)$$

and the assertion follows as $\gamma(T) = \gamma(0)$. □

Conversely, we have (cf. [18])

Proposition A.7. *Let $\Phi \in (L^\infty(\Omega))^N$ with*

$$\int_\gamma \Phi(\omega) = 0 \qquad \textit{a.s. in } \omega$$

for every closed piecewise C^1 curve. Then there exist a Lipschitz random function v with stationary increments and a set Ω_0 of probability 1 such that, for any $\omega \in \Omega_0$,

$$Dv(x,\omega) = \Phi(\tau_x\omega) \qquad \textit{for a.e. } x \in \mathbb{R}^N.$$

Proof. Let us set

$$v(x,\omega) = \int_0^1 \langle \Phi(\tau_{tx}\,\omega), x\rangle \,\mathrm{d}t, \qquad (x,\omega) \in \mathbb{R}^N \times \Omega.$$

Note that v is jointly measurable in (x,ω). By assumption we also have that, for every fixed $x \in \mathbb{R}^N$,

$$v(x,\omega) = \int_0^T \langle \Phi(\tau_{\gamma(t)}\,\omega), \dot\gamma(t)\rangle \,\mathrm{d}t, \qquad \text{a.s. in } \omega \tag{27}$$

whenever γ is a piecewise C^1 curve, defined in some interval $[0,T]$, joining 0 to x. For every $i = 1, \cdots, N$ we derive from (27)

$$\frac{v(x + h\,e_i,\omega) - v(x,\omega)}{h} = \int_0^1 \Phi_i(\tau_{x+t\,h\,e_i}\,\omega)\,\mathrm{d}t \qquad \text{for any } x\text{, a.s. in } \omega,$$

where h is a discrete positive parameter. Hence

$$\int_\Omega \left| \frac{v(x + h\,e_i,\omega) - v(x,\omega)}{h} - \Phi_i(\tau_x\omega)\right|^2 \mathrm{d}\mathbb{P} = \int_\Omega \left| \int_0^1 \Phi_i(\tau_{x+t\,h\,e_i}\,\omega) - \Phi_i(\tau_x\omega)\,\mathrm{d}t\right|^2 \mathrm{d}\mathbb{P}$$

$$\le \int_\Omega\int_0^1 |\Phi(\tau_{x+t\,h\,e_i}\,\omega) - \Phi(\tau_x\omega)|^2 \,\mathrm{d}t\,\mathrm{d}\mathbb{P} = \int_0^1 \|U_{t\,h\,e_i}\Phi - \Phi\|^2_{(L^2(\Omega))^N}\,\mathrm{d}t,$$

where in the last equality we have used Fubini's Theorem and the fact that the probability measure $\mathbb{P}$ is invariant under τ_x. The previous relation implies, by the strong continuity of the group $(U_y)_{y\in\mathbb{R}^N}$

$$\frac{v(x + h\,e_i,\omega) - v(x,\omega)}{h} \underset{h\to 0}{\longrightarrow} \Phi_i(\tau_x\omega) \qquad \text{in } L^2(\Omega),$$

for every $x \in \mathbb{R}^N$; accordingly, thanks to Fubini's Theorem

$$\frac{v(x + h\,e_i,\omega) - v(x,\omega)}{h} \underset{h\to 0}{\longrightarrow} \Phi_i(\tau_x\omega) \qquad \text{in } L^2_{loc}(\mathbb{R}^N),$$

a.s. in ω. This is, in turn, equivalent to the equality $\partial_{x_i} v(x,\omega) = \Phi_i(\tau_x\omega)$, for $i = 1, \cdots, N$, in the sense of distributions, a.s. in ω. Therefore, being Φ essentially bounded, $v(\cdot,\omega)$ is Lipschitz–continuous with $Dv(x,\omega) = \Phi(\tau_x\omega)$ for a.e. $x \in \mathbb{R}^N$ and every ω in a set Ω_0 of probability 1. By suitably assigning the value of $v(\cdot,\omega)$ on $\Omega\setminus\Omega_0$, we extend the Lipschitz character of such function to all ω. Therefore v, being also jointly measurable in $\mathbb{R}^N\times\Omega$, is a Lipschitz random function, as asserted.

It is left to show that it has stationary increments. Let us fix $z \in \mathbb{R}^N$. Then, by (27), for every $x, y \in \mathbb{R}^N$

$$v(x+z,\omega) \;-\; v(y+z,\omega) = \int_0^1 \langle \Phi(\tau_{tx+(1-t)y+z}\,\omega), x-y\rangle\,\mathrm{d}t \tag{28}$$

$$= \int_0^1 \langle \Phi(\tau_{tx+(1-t)y}(\tau_z\,\omega)), x-y\rangle\,\mathrm{d}t = v(x,\tau_z\omega) - v(y,\tau_z\omega).$$

for every ω in a set $\Omega_{x,y}$ of probability 1. Such a set does not depend only on z, as required to prove the claim, but also on x and y. To overcome this difficulty, we set

$$\Omega_z = \Omega_0 \cap \left(\bigcap_{x,y\in\mathbb{Q}^N} \Omega_{x,y}\right).$$

Clearly $\mathbb{P}(\Omega_z) = 1$, and by the continuity of $v(\cdot, \omega)$ we derive that, for any $\omega \in \Omega_z$, the equality (28) now holds for all $x, y \in \mathbb{R}^N$, as required. This ends the proof. $\square$

Proof of Theorem 2.5. The scheme of the proof is similar to that of Theorem 3.8 in [9]. We denote by Φ_n the functions of $(L^\infty(\Omega))^N$ associated to v_n through Proposition A.6. Since the Φ_n are bounded in $(L^2(\Omega))^N$, a sequence made up by finite convex combinations of them, say $\Psi_k = \sum_{n \geq n_k} \lambda_n^k \Phi_n$, converges to some Φ in $(L^2(\Omega))^N$. By Lemma A.4, up to extraction of a subsequence

$$\int_\gamma \Psi_k(\omega) \to \int_\gamma \Phi(\omega) \quad \text{a.s. in } \omega, \tag{29}$$

for any piecewise C^1 curve γ. This implies, in particular, $\int_\gamma \Phi(\omega) = 0$ a.s. in ω for any closed curve γ. We can thus associate to Φ a Lipschitz random function v with stationary increments using Proposition A.7, and we can further assume $v(0, \omega) = 0$ a.s. in ω. Let w_k be a sequence of Lipschitz random functions with stationary increments defined as in the statement and set $g_k(\omega) = -w_k(0, \omega)$. Given a point x and a piecewise C^1 curve γ connecting 0 to x, we have, a.s. in ω,

$$\begin{aligned} w_k(x, \omega) + g_k(\omega) &= \int_\gamma \Psi_k(\omega) \quad \text{for any } k \in \mathbb{N} \\ v(x, \omega) &= \int_\gamma \Phi(\omega). \end{aligned}$$

Therefore $w_k(\cdot, \omega) + g_k(\omega)$ converges pointwise to $v(\cdot, \omega)$ a.s. in ω, by (29). By construction, the sequence $w_k(\cdot, \omega) + g_k(\omega)$ is almost surely equiLipschitz–continuous and locally equibounded. By Ascoli Theorem, it must indeed locally uniformly converge to $v(\cdot, \omega)$ a.s. in ω, as it was to be proved. $\square$

Proof of Theorem 2.7. Let κ be a Lipschitz constant for $v(\cdot, \omega)$ for every ω and let us denote by $\Phi \in (L^\infty(\Omega))^N$ the function associated with v through Proposition A.6. We start by proving the sublinearity property assuming the gradient to have vanishing mean; that is, we want to show the existence of a set Ω' of probability 1 such that

$$\lim_{t \to +\infty} \frac{v(tx, \omega)}{t} = 0 \qquad \text{for any } x \in \mathbb{R}^N \text{ and } \omega \in \Omega'. \tag{30}$$

First, we fix $x \in \mathbb{R}^N$. Then

$$\frac{v(tx, \omega) - v(0, \omega)}{t} = ⨍_0^t \langle \Phi(\tau_{sx}\, \omega), x \rangle \, \mathrm{d}s \qquad \text{for every } t > 0,$$

a.s. in ω. By applying Birkhoff Ergodic Theorem to the function $\omega \mapsto \langle \Phi(\omega), x \rangle$ and the dynamical system $(\tau_{sx})_{s \in \mathbb{R}}$, we get that

$$\lim_{t \to +\infty} \frac{v(tx, \omega) - v(0, \omega)}{t}$$

does exist a.s. in ω and is almost surely equal to some measurable function $k(\omega)$. It is easy to see that, for every $z \in \mathbb{R}^N$, $k(\tau_z \omega) = k(\omega)$ a.s. in ω. Indeed,

$$\frac{v(tx, \tau_z \omega) - v(0, \tau_z \omega)}{t} = \frac{v(tx + z, \omega) - v(z, \omega)}{t} \qquad \text{a.s. in } \omega$$

for v has stationary increments, and

$$\left|\frac{v(tx+z,\omega)-v(z,\omega)}{t}-\frac{v(tx,\omega)-v(0,\omega)}{t}\right|\leq\frac{2\,\kappa|z|}{t}.$$

By ergodicity, we derive that $k(\cdot)$ is a.s. constant, say equal to some $k\in\mathbb{R}$. Using the Dominated Convergence Theorem we infer

$$k=\lim_{t\to+\infty}\int_\Omega\frac{v(tx,\omega)-v(0,\omega)}{t}\,\mathrm{d}\mathbb{P}(\omega)=\lim_{t\to+\infty}\int_\Omega\fint_0^t\langle\Phi(\tau_{sx}\,\omega),x\rangle\,\mathrm{d}s\,\mathrm{d}\mathbb{P}(\omega).$$

By exploiting the fact that $\mathbb{P}$ is invariant with respect to the translations τ_y, we get

$$\int_\Omega\fint_0^t\langle\Phi(\tau_{sx}\,\omega),x\rangle\,\mathrm{d}s\,\mathrm{d}\mathbb{P}(\omega)=\fint_0^t\int_\Omega\langle\Phi(\tau_{sx}\,\omega),x\rangle\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}s=\langle\mathbb{E}(\Phi),x\rangle,$$

and the limit relation in (30) follows for $\mathbb{E}(\Phi)=0$ by hypothesis, at least for some set Ω_x of probability 1 depending on x. We then exploit the Lipschitz character of v to see that (30) holds with $\Omega'=\cap_{x\in\mathbb{Q}^N}\Omega_x$. We pick $\omega\in\Omega'$; the family of functions $y\mapsto\frac{v(ty,\omega)}{t}$, $t\in\mathbb{R}_+$, are equibounded and equiLipschitz continuous, for y varying in ∂B_1, and so it uniformly converges to 0, as $t\to+\infty$, by Ascoli Theorem and (30). Accordingly, given $\varepsilon>0$, we find

$$\frac{|v(x,\omega)|}{|x|}=\frac{\left|v\left(|x|\frac{x}{|x|},\omega\right)\right|}{|x|}<\varepsilon$$

for $|x|$ large enough, as claimed.

We proceed to prove the converse implication. Let us fix an ω for which the convergence (3) takes place and, by Birkhoff Theorem,

$$\mathbb{E}(\Phi_1)=\lim_{R\to+\infty}\fint_{[-R,R]^N}\partial_{x_1}v(x,\omega)\,\mathrm{d}x.$$

Let us denote a point x in $\mathbb{R}^N$ by $(x_1,y)\in\mathbb{R}\times\mathbb{R}^{N-1}$, we have

$$\begin{aligned}\left|\fint_{[-R,R]^N}\partial_{x_1}v(x,\omega)\,\mathrm{d}x\right|&=\left|\fint_{[-R,R]^{N-1}}\left(\fint_{-R}^R\partial_{x_1}v(x_1,y,\omega)\,\mathrm{d}x_1\right)\mathrm{d}y\right|\\&=\left|\fint_{[-R,R]^{N-1}}\left(\frac{v(R,y,\omega)-v(-R,y,\omega)}{R}\right)\mathrm{d}y\right|\leq2\max_{[-R,R]^N}\frac{|v(x,\omega)|}{R},\end{aligned}$$

which implies $\mathbb{E}(\Phi_1)=0$ in force of the assumption. Similarly we show that $\mathbb{E}(\Phi_i)$ vanishes for any $i=2,\cdots N$. □

Proof of Theorem 2.8. Let us fix x, y in $\mathbb{R}^N$ and denote by γ the segment $ty+(1-t)x$, $t\in[0,1]$, and by Q the vector $\mathbb{E}(\Phi)$, where Φ is the function of $(L^\infty(\Omega))^N$ associated with v through Proposition A.6. Using Robin's Theorem, as in Proposition A.6, we get

$$v(y,\omega)-v(x,\omega)=\int_0^1\langle\Phi(\tau_{\gamma(t)}\omega),y-x\rangle\,\mathrm{d}t\quad\text{a.s. in }\omega,$$

and by integrating on Ω

$$\mathbb{E}(v(y,\cdot) - v(x,\cdot)) = \int_\Omega \left(\int_0^1 \langle \Phi(\tau_{\gamma(t)}\omega), y - x \rangle \, \mathrm{d}t \right) \mathrm{d}\mathbb{P} \tag{31}$$
$$= \int_0^1 \langle \int_\Omega \Phi(\tau_{\gamma(t)}\omega) \, \mathrm{d}\mathbb{P}, y - x \rangle \, \mathrm{d}t = \langle Q, y - x \rangle.$$

Now, if v has gradient with vanishing mean , i.e. if $Q = 0$, then (4) follows, conversely, if (4) holds, then it is enough to take $y - x = Q$ in (31) to get $Q = 0$. □

We conclude by providing a proof for Theorem 2.9.

Proof of Theorem 2.9. First note that a stationary Lipschitz random function has stationary increments. Moreover, $\mathbb{E}(v(x,\cdot))$ is independent of x, so when such quantity is finite the assertion is just a consequence of Theorem 2.8. To prove it in the general case, it is enough to show that $v(\cdot,\omega)$ is almost surely sublinear at infinity, in view of Theorem 2.7. Let κ be a Lipschitz constant for $v(\cdot,\omega)$ for every ω. The stationary character of v means, cf. (1), that for any ω in a set of probability 1

$$v(x,\omega) = v(0, \tau_x\omega) \qquad \text{for a.e. } x \in \mathbb{R}^N.$$

We claim that there exists a constant M such that $E := \{\omega \in \Omega \,:\, |v(0,\omega)| \leq M\,\}$ has positive probability. Indeed, if this were not the case, we would have that $E_\infty := \{\omega \,:\, |v(0,\omega)| = +\infty\,\}$ has probability 1. An application of Robbins' Theorem with $\mu = \mathcal{L}^N \llcorner B_r$ for every fixed $r > 0$ would imply that the stationary random set

$$F_\infty(\omega) = \{x \in \mathbb{R}^N \,:\, \tau_x\omega \in E_\infty,\, v(x,\omega) = v(0,\tau_x\omega)\,\}$$

is of full measure in $\mathbb{R}^N$ a.s. in ω, yielding $v(\cdot,\omega) \equiv +\infty$ a.s. in ω, a contradiction.

Let us fix M such that E has positive probability. Then the closed stationary random set

$$C(\omega) = \{x \in \mathbb{R}^N \,:\, |v(x,\omega)| \leq M\,\}$$

is almost surely nonempty. Accordingly, by Proposition 2.3 there exists a set Ω_0 probability 1 such that, for every $\omega \in \Omega_0$,

$$\lim_{r \to +\infty} \frac{|\,(C(\omega) + B_n) \cap B_r|}{|B_r|} > 1 - \varepsilon_n,$$

where $(\varepsilon_n)_n$ is a sequence decreasing to 0. Fix $\omega \in \Omega_0$. Then for every $x \in \mathbb{R}^N$ with $|x|$ large enough, we have

$$|\,(C(\omega) + B_n) \cap B_{2|x|}| > (1 - \varepsilon_n)\,|B_{2|x|}|.$$

For n sufficiently large $B_{2|x|(\varepsilon_n)^{1/N}}(x) \subseteq B_{2|x|}$, and from the above inequality we infer

$$B_{2|x|(\varepsilon_n)^{1/N}}(x) \cap (C(\omega) + B_n) \neq \emptyset,$$

i.e. there exists $y = y(x,n)$ in $C(\omega)$ such that $|y - x| < 2|x|(\varepsilon_n)^{1/N} + n$. Since $|v(y,\omega)| \leq M$, we get

$$|v(x,\omega)| \leq |v(x,\omega) - v(y,\omega)| + |v(y,\omega)| \leq \kappa\left(2|x|(\varepsilon_n)^{1/N} + n\right) + M.$$

From this we obtain

$$\limsup_{|x| \to +\infty} \frac{|v(x,\omega)|}{|x|} \leq 2\kappa\,(\varepsilon_n)^{1/N},$$

and the claim follows letting $n \to +\infty$. □

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