AUBRY SETS FOR WEAKLY COUPLED SYSTEMS OF HAMILTON–JACOBI EQUATIONS

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Abstract. We introduce a notion of Aubry set for weakly coupled systems of Hamilton–Jacobi equations on the torus and characterize it as the region where the obstruction to the existence of globally strict critical subsolutions concentrates. As in the case of a single equation, we prove the existence of critical subsolutions which are strict and smooth outside the Aubry set. This allows us to derive in a simple way a comparison result among critical sub and supersolutions with respect to their boundary data on the Aubry set, showing in particular that the latter is a uniqueness set for the critical system. We also highlight some rigidity phenomena taking place on the Aubry set.

Introduction

In this paper we will consider a *weakly coupled system* of Hamilton–Jacobi equations of the form

$$H_i(x, Du_i) + \sum_{j=1}^{m} b_{ij}(x)u_j(x) = a \quad \text{in } \mathbb{T}^N \qquad \text{for every } i \in \{1, \ldots, m\}, \tag{1}$$

where a is a real constant, $H_1, \ldots, H_m$ are continuous Hamiltonians defined on the cotangent bundle of $\mathbb{T}^N$, convex and coercive in the momentum variable, and $B(x) := \big(b_{ij}(x)\big)$ is a continuous $m \times m$ matrix satisfying

$$b_{ij}(x) \leqslant 0 \quad \text{for } j \neq i, \quad \sum_{j=1}^{m} b_{ij}(x) = 0 \qquad \text{for every } x \in \mathbb{T}^N \text{ and } i \in \{1, \ldots, m\}.$$

Such weakly coupled systems arise naturally in optimal control problems associated with randomly switching costs, where the switching is governed by specific Markov chains, see [22, 36]. In the PDE literature, they have been studied as a particular instance of monotone systems, see [14, 25, 26]. More recently, they have been considered in connection with homogenization problems [5, 21, 30] and for the long–time behavior of the associated evolutionary system [6, 28, 29, 32]. These works are a generalization of results established in the case of a single equation, see [1, 3, 11, 17, 24, 27, 31, 33, 34, 35].

Note that we must assume some further condition on the coupling matrix to solve the system (1) with the same constant a as right–hand side: in the extreme case where $B(x) \equiv 0$, each equation $H_i(x, Du_i) = a_i$ can be solved for a unique a_i, and usually the a_i are different. We will therefore assume that B is *irreducible*, meaning, roughly speaking, that the coupling is non–trivial and the system cannot be split into independent subsystems, see Definition 1.1–*(ii)*. Under this assumption, it has

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been established in [6, 28] that there is a unique a for which the system (1) can be solved. Such a value is denoted by c and termed *critical* in the sequel. These results were proved using the so called *ergodic approximation* in the spirit of [27], or the new adjoint method introduced by Evans [15].

Solutions of (1) are usually not unique, even up to addition of a common constant to all the u_i. Except for the work [6], which addresses the case of Hamiltonians of particular form, little attention has been devoted to the issue of non–uniqueness of solutions for systems. In the case of a single equation, the study of these non-uniqueness phenomena is part of weak KAM theory, see for example [18]. Inspired by these results, we have addressed these questions and we have drawn a weak KAM analog for weakly coupled systems of the form (1). The main contribution of this paper is to define the Aubry set for systems and to establish its main properties. Due to a lack of suitable variational formulae for the solutions of the system, we have relied more on the PDE methods for weak KAM theory.

Our study is based on a different definition of the critical value c, given as the minimal $a \in \mathbb{R}$ for which the corresponding weakly coupled system admits viscosity subsolutions. This characterization was already known, see for instance [28], but we provide here a new proof to the existence of solutions at the critical level, based on a fixed point argument in the spirit of [16, 18].

Next we show that the obstruction to the existence of globally strict subsolutions of the corresponding critical system is not spread indistinctly on the torus, but concentrates on a closed set $\mathcal{A}$, that we call *Aubry set* in analogy to the case of a single equation. We prove existence of critical subsolutions smooth and strict outside the Aubry set and we show that they are dense, with respect to the topology of uniform convergence, in the family of critical subsolutions. This allows us to derive in a simple way a comparison result among critical sub and supersolutions satisfying suitable "boundary" conditions on $\mathcal{A}$, see Theorem 5.5. In particular, we infer that the Aubry set is a uniqueness set for the critical system, i.e. two critical solutions that coincide on $\mathcal{A}$ do coincide on the whole torus. We furthermore show that the trace of any critical subsolution on $\mathcal{A}$ can be extended on the whole torus in such a way that the output is a critical solution, see Theorem 5.7.

Our study also highlights some rigidity phenomena taking place on the Aubry set. First, we show that any pair of critical subsolutions differ, at each point y of $\mathcal{A}$, by a vector of the form $k\,(1, 1, \ldots, 1)$, see Proposition 5.1. This accounts for the kind of symmetries already observed in [6] for the particular class of Hamiltonians therein considered, see Section 6.1 for more details. A second rigidity phenomenon that we point out is when the Hamiltonians are additionally assumed strictly convex in the momentum: in this case we prove that, at any point of the Aubry set, the intersection of the reachable gradients of all the critical subsolutions is always nonempty, see Proposition 4.4. This can be regarded as a weak version of a result holding in the scalar case, where it is known that, under suitable regularity assumptions on the Hamiltonian, the critical subsolutions are all differentiable on the Aubry set and have the same gradient, see [18, 19, 20].

This paper is organized as follows. In Section 1 we fix the notations and assumptions, and we give a brief overview of existing results on weakly coupled systems. Section 2 is devoted to the definition of the critical value and to the study of its main properties. In Section 3 we give the definition of Aubry set and explore its properties. The first part of Section 4 is devoted to the regularization of subsolutions outside of the Aubry set, while in the second part we prove a rigidity phenomenon enjoyed by reachable gradients of critical subsolutions on the Aubry set. Another rigidity phenomenon is instead presented at the beginning of Section 5, where we also prove the comparison principle. In Section 6 we illustrate our theory on some examples. Appendix A contains the more technical proofs of Section 2.

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1. Preliminaries

1.1. **Notations.** Throughout the paper, we will denote by $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$ the N–dimensional flat torus, where N is an integer number. The scalar product in $\mathbb{R}^N$ will be denoted by $\langle \cdot , \cdot \rangle$, while the symbol $|\cdot|$ stands for the Euclidean norm. Note that the latter induces a distance on $\mathbb{T}^N$, denoted by $d(\cdot,\cdot)$, defined as

$$d(x,y) := \min_{\kappa\in\mathbb{Z}^N} |x-y+\kappa| \qquad \text{for every } x,y\in\mathbb{T}^N.$$

We will denote by $B_R(x_0)$ and B_R the open balls in $\mathbb{T}^N$ of radius R centered at x_0 and 0, respectively.

With the symbols $\mathbb{N}$ and $\mathbb{R}_+$ we will refer to the sets of positive integer numbers and nonnegative real numbers, respectively. We say that a property holds *almost everywhere* (*a.e.* for short) in a subset E of $\mathbb{T}^N$ if it holds up to a *negligible* subset of E, i.e. a subset of zero N–dimensional Lebesgue measure.

We will denote by $\|g\|_\infty$ the usual L^∞–norm of g, where the latter is a measurable real function defined on $\mathbb{T}^N$. We will write $g_n \rightrightarrows g$ in $\mathbb{T}^N$ to mean that the sequence of functions $(g_n)_n$ uniformly converges to g in $\mathbb{T}^N$, i.e. $\|g_n-g\|_\infty\to 0$. We will denote by $\big(\mathrm{C}(\mathbb{T}^N)\big)^m$ the Banach space of continuous functions $\mathbf{u}=(u_1,\dots,u_m)^T$ from $\mathbb{T}^N$ to $\mathbb{R}^m$ (where the upper–script symbol T stands for the transpose), endowed with the norm

$$\|\mathbf{u}-\mathbf{v}\|_\infty = \max_{1\leqslant i\leqslant m}\|u_i-v_i\|_\infty, \qquad \mathbf{u},\mathbf{v}\in\big(\mathrm{C}(\mathbb{T}^N)\big)^m.$$

We will write $\mathbf{u}^n \rightrightarrows \mathbf{u}$ in $\mathbb{T}^N$ to mean that $\|\mathbf{u}^n-\mathbf{u}\|_\infty\to 0$. A function $\mathbf{u}\in\big(\mathrm{C}(\mathbb{T}^N)\big)^m$ will be termed Lipschitz continuous if each of its components is κ–Lipschitz continuous, for some $\kappa>0$. Such a constant κ will be called a *Lipschitz constant* for $\mathbf{u}$. The space of all such functions will be denoted by $\big(\mathrm{Lip}(\mathbb{T}^N)\big)^m$.

We will denote by $\mathbb{1}=(1,\cdots,1)^T$ the vector of $\mathbb{R}^m$ having all components equal to 1. We consider the following partial relations between elements $\mathbf{a},\mathbf{b}\in\mathbb{R}^m$: $\mathbf{a}\leqslant\mathbf{b}$ (respectively, $\mathbf{a}<\mathbf{b}$) if $a_i\leqslant b_i$ (resp., $<$) for every $i\in\{1,\dots,m\}$. Given two functions $\mathbf{u},\mathbf{v}:\mathbb{T}^N\to\mathbb{R}^m$, we will write $\mathbf{u}\leqslant\mathbf{v}$ in $\mathbb{T}^N$ (respectively, $<$) to mean that $\mathbf{u}(x)\leqslant\mathbf{v}(x)$ $\big($resp., $\mathbf{u}(x)<\mathbf{v}(x)\big)$ for every $x\in\mathbb{T}^N$.

1.2. **Linear algebra.** Here we briefly present some elementary linear algebraic results concerning coupling matrices.

Definition 1.1. Let $B = (b_{ij})_{i,j}$ be a $m \times m$–matrix.

(i) We say that B is a *coupling matrix* if it satisfies the following conditions:
$$b_{ij} \leqslant 0 \text{ for } j \neq i, \qquad \sum_{j=1}^{m} b_{ij} \geqslant 0 \qquad \text{for any } i \in \{1, \ldots, m\}. \tag{C}$$
It is additionally termed *degenerate* if $\sum_{j=1}^{m} b_{ij} = 0$ for any $i = 1, \ldots, m$.

(ii) We say that B is *irreducible* if for every subset $\mathcal{I} \subsetneq \{1, \ldots, m\}$ there exist $i \in \mathcal{I}$ and $j \notin \mathcal{I}$ such that $b_{ij} \neq 0$.

When a coupling matrix is irreducible, we can derive further information on the sign of its diagonal elements:

Proposition 1.2. *Let $B = (b_{ij})_{i,j}$ be an irreducible $m \times m$ coupling matrix. Then $b_{ii} > 0 \quad$ for every $i \in \{1, \ldots, m\}$.*

Proof. Indeed, if $b_{ii} = 0$ for some $i \in \{1, \ldots, m\}$, condition (C) would imply $b_{ij} = 0$ for every $j \in \{1, \ldots, m\}$, in contradiction with the fact that B is irreducible. ☐

The following proposition gives an obstruction to being in the image of a degenerate coupling matrix.

Proposition 1.3. *Let $B = (b_{ij})_{i,j}$ be a degenerate $m \times m$ coupling matrix. Let $\mathbf{a} = B\mathbf{v}$ for some $\mathbf{v} \in \mathbb{R}^m$. Then $\min_i a_i \leqslant 0 \leqslant \max_i a_i$. Moreover, if B is irreducible and $\min_i a_i \geqslant 0$ (resp. $\max_i a_i \leqslant 0$), then $\mathbf{v} = \lambda \mathbb{1}$ for some $\lambda \in \mathbb{R}$ and $\mathbf{a} = \mathbf{0}$.*

Proof. Let $\mathbf{v} = (v_1, \ldots, v_m)^T$ be such that $B\mathbf{v} = \mathbf{a}$ and set $\mathcal{I} := \{k \in \{1, \ldots, m\} : v_k = \min_i v_i\}$. For $k \in \mathcal{I}$ we have
$$a_k = \sum_{j=1}^{m} b_{kj} v_j \leqslant \sum_{j=1}^{m} b_{kj} v_k = 0, \tag{1.1}$$
that is $a_k \leqslant 0$ for every $k \in \mathcal{I}$. In particular, we get $\min_i a_i \leqslant 0$.

Let us additionally assume B irreducible and $\min_i a_i \geqslant 0$. We claim that $\mathcal{I} = \{1, \ldots, m\}$. Indeed, if this where not the case, there would exist $k \in \mathcal{I}$ and $j \notin \mathcal{I}$ such that $b_{kj} \neq 0$. From (1.1) and the hypothesis we get $0 \leqslant \min_i a_i \leqslant a_k \leqslant 0$, i.e. $a_k = 0$ and all the inequalities in (1.1) must be equalities. In particular $b_{kj} v_j = b_{kj} v_k$, yielding $v_j = v_k = \min_i v_i$, i.e. $j \in \mathcal{I}$, a contradiction. Hence $\mathbf{v} = v_1 \mathbb{1}$ and $\mathbf{a} = B\mathbf{v} = \mathbf{0}$ by the degenerate character of B. The analogous results with max in place of min follow at once by replacing $\mathbf{a}$ and $\mathbf{v}$ with $-\mathbf{a}$ and $-\mathbf{v}$. ☐

As a straightforward consequence of Proposition 1.3, we derive the following invertibility criterion:

Proposition 1.4. *Let $B = (b_{ij})_{i,j}$ be an $m \times m$ irreducible coupling matrix. Then*

(i) $\operatorname{Ker}(B) \subseteq \operatorname{span}\{(1, \ldots, 1)^T\} = \mathbb{R}\mathbb{1}$*;*

(ii) $\operatorname{Ker}(B) = \operatorname{span}\{(1, \ldots, 1)^T\} = \mathbb{R}\mathbb{1}$ *if and only if* B *is degenerate.*

In particular, B is invertible if and only if $\sum_{j=1}^{m} b_{ij} > 0$ for some $i \in \{1, \dots, m\}$.

1.3. **Weakly coupled systems.** Throughout the paper, we will call *convex Hamiltonian* a function H satisfying the following set of assumptions:

(H1) $\quad H : \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{R} \qquad$ is continuous;

(H2) $\quad p \mapsto H(x,p) \qquad$ is convex on $\mathbb{R}^N$ for any $x \in \mathbb{T}^N$;

(H3) $\quad \min_{x \in \mathbb{T}^N} H(x,p) \to +\infty \qquad$ as $|p| \to +\infty$.

Property (H3) will be referred as *coercivity* of H in p. The Hamiltonian will be termed *strictly convex* if it additionally satisfies the following stronger assumption:

(H2)$'$ $\quad p \mapsto H(x,p) \qquad$ is strictly convex on $\mathbb{R}^N$ for any $x \in \mathbb{T}^N$.

Moreover, we will denote by $B(x) = \big(b_{ij}(x)\big)_{i,j}$ an $m \times m$–matrix with continuous coefficients $b_{ij}(x)$ on $\mathbb{T}^N$. If not otherwise stated, the following hypotheses will be always assumed:

(B1) $\quad B(x)$ is an irreducible coupling matrix for every $x \in \mathbb{T}^N$;

(B2) $\quad B(x)$ is degenerate for every $x \in \mathbb{T}^N$.

Let $H_1(x,p), \dots, H_m(x,p)$ be convex Hamiltonians, i.e. functions satisfying conditions (H1)–(H3). We are interested in weakly coupled systems of the form

$$H_i(x, Du_i) + \big(B(x)\mathbf{u}(x)\big)_i = a_i \quad \text{in } \mathbb{T}^N \qquad \text{for every } i \in \{1, \dots, m\}, \tag{1.2}$$

for some constant vector $\mathbf{a} = (a_1, \dots, a_m)^T$, where $\mathbf{u}(x) = \big(u_1(x), \dots, u_m(x)\big)^T$ and $\big(B(x)\mathbf{u}(x)\big)_i$ denotes the i–th component of the vector $B(x)\mathbf{u}(x)$, i.e.

$$\big(B(x)\mathbf{u}(x)\big)_i = \sum_{j=1}^{m} b_{ij}(x) u_j(x).$$

Remark 1.5. The weakly coupled system (1.2) is a particular type of *monotone system*, i.e. a system of the form $G_i\big(x, u_1(x), \dots, u_m(x), Du_i\big) = 0$ in $\mathbb{T}^N$ for every $i \in \{1, \dots, m\}$, where suitable monotonicity conditions with respect to the u_j–variables are assumed on the functions G_i, see [5, 14, 23, 25, 26]. In the specific case considered in this paper, the conditions assumed on the coupling matrix imply, in particular, that each function G_i is strictly increasing in u_i and non–increasing in u_j for every $j \neq i$. This kind of monotonicity will be exploited in many points of the paper.

Given a continuous function u on $\mathbb{T}^N$, we will call *subtangent* (respectively, *supertangent*) of u at x_0 a function ϕ of class C^1 in a neighborhood U of x_0 such that $u - \phi$ has a local minimum (resp., maximum) at x_0. Its gradient $D\phi(x_0)$ will be called a *subdifferential* (resp. *superdifferential*) of u at x_0. The set of sub and superdifferentials of u at x_0 will be denoted $D^-u(x_0)$ and $D^+u(x_0)$, respectively. The function ϕ will be furthermore termed *strict subtangent* (resp., *strict supertangent*) if $u - \phi$ has a *strict* local minimum (resp., maximum) at x_0. Any subtangent (resp., supertangent) ϕ of u can be always assumed strict at x_0 without affecting $D\phi(x_0)$

by possibly replacing it with $\phi - d^2(x_0,\cdot)$ (resp. $\phi + d^2(x_0,\cdot)$). We recall that u is differentiable at x_0 if and only if $D^+u(x_0)$ and $D^-u(x_0)$ are both nonempty. In this instance, $D^+u(x_0) = D^-u(x_0) = \{Du(x_0)\}$. We refer the reader to [7, Chapter 3, Propositions 3.1.5, 3.1.9] for the proofs.

When u is locally Lipschitz in $\mathbb{T}^N$, we will denote by $\partial^* u(x_0)$ the set of *reachable gradients* of u at x_0, that is the set

$$\partial^* u(x_0) = \{\lim_n Du(x_n) \,:\, u \text{ is differentiable at } x_n,\, x_n \to x_0\,\},$$

while the *Clarke's generalized gradient* $\partial^c u(x_0)$ is the closed convex hull of $\partial^* u(x_0)$. The set $\partial^c u(x_0)$ contains both $D^+u(x_0)$ and $D^-u(x_0)$, in particular $Du(x_0) \in \partial^c u(x_0)$ at any differentiability point x_0 of u. We refer the reader to [9] for a detailed treatment of the subject.

Definition 1.6. Let $\mathbf{u} \in \big(\mathrm{C}(\mathbb{T}^N)\big)^m$. We will say that $\mathbf{u}$ is a *viscosity subsolution* of (1.2) if the following inequality holds for every $(x,i) \in \mathbb{T}^N \times \{1,\dots,m\}$:

$$H_i(x,p) + \big(B(x)\mathbf{u}(x)\big)_i \leqslant a_i \quad \text{for every } p \in D^+u_i(x).$$

We will say that $\mathbf{u}$ is a *viscosity supersolution* of (1.2) if the following inequality holds for every $(x,i) \in \mathbb{T}^N \times \{1,\dots,m\}$:

$$H_i(x,p) + \big(B(x)\mathbf{u}(x)\big)_i \geqslant a_i \quad \text{for every } p \in D^-u_i(x).$$

We will say that $\mathbf{u}$ is a *viscosity solution* if it is both a sub and a supersolution.

In the sequel, solutions, subsolutions and supersolutions will be always meant in the viscosity sense, hence the adjective *viscosity* will be omitted.

Due to the convexity of the Hamiltonian H_i, the following equivalences hold:

Proposition 1.7. *Let $a \in \mathbb{R}$, $i \in \{1,\dots,m\}$ and $\mathbf{u} \in \big(\mathrm{Lip}(\mathbb{T}^N)\big)^m$. The following facts are equivalent:*

(i) $\quad H_i(x,p) + \big(B(x)\mathbf{u}(x)\big)_i \leqslant a \quad$ *for every $p \in D^+u_i(x)$ and $x \in \mathbb{T}^N$;*

(ii) $\quad H_i(x,p) + \big(B(x)\mathbf{u}(x)\big)_i \leqslant a \quad$ *for every $p \in D^-u_i(x)$ and $x \in \mathbb{T}^N$;*

(iii) $\quad H_i(x,p) + \big(B(x)\mathbf{u}(x)\big)_i \leqslant a \quad$ *for every $p \in \partial^c u_i(x)$ and $x \in \mathbb{T}^N$;*

(iv) $\quad H_i\big(x,Du_i(x)\big) + \big(B(x)\mathbf{u}(x)\big)_i \leqslant a \quad$ *for a.e. $x \in \mathbb{T}^N$.*

Next, we state a proposition that will be needed in the sequel, see also [14, 26, 23, 25] for similar results.

Proposition 1.8. *Let $\mathcal{F}$ be a subset of $\big(\mathrm{C}(\mathbb{T}^N)\big)^m$ and define the functions $\underline{\mathbf{u}}$, $\overline{\mathbf{u}}$ on $\mathbb{T}^N$ by setting:*

$$\underline{u}_i(x) = \inf_{\mathbf{u}\in\mathcal{F}} u_i(x), \quad \overline{u}_i(x) = \sup_{\mathbf{u}\in\mathcal{F}} u_i(x) \qquad \text{for every } x \in \mathbb{T}^N \text{ and } i \in \{1,\dots,m\}.$$

Assume that $\underline{\mathbf{u}}$ and $\overline{\mathbf{u}}$ belong to $\big(\mathrm{C}(\mathbb{T}^N)\big)^m$ and let $\mathbf{a} \in \mathbb{R}^m$. Then:

(i) if every $\mathbf{u} \in \mathcal{F}$ is a subsolution of (1.2), then $\overline{\mathbf{u}}$ is a subsolution of (1.2);

(ii) if every $\mathbf{u} \in \mathcal{F}$ is a supersolution of (1.2), then $\underline{\mathbf{u}}$ is a supersolution of (1.2).

For a scalar Hamilton–Jacobi equation this proposition is well known, see for instance Section 2.6 in [2]. Using monotonicity, the proof for the scalar case can be easily generalized to our setting.

We will be also interested in the evolutionary counterpart of (1.2), i.e. the system

$$\frac{\partial u_i}{\partial t} + H_i(x, D_x u_i) + \big(B(x)\mathbf{u}(t,x)\big)_i = 0 \quad \text{in } (0,+\infty)\times\mathbb{T}^N \qquad \forall i \in \{1,\dots,m\}, \tag{1.3}$$

where we have denoted by $\mathbf{u}(t,x) = \big(u_1(t,x),\dots,u_m(t,x)\big)^T$.

The following comparison result holds, see for instance [5] for a proof.

Proposition 1.9. *Let $T>0$ and $\mathbf{v}$, $\mathbf{u} \in \big(\mathrm{Lip}([0,T]\times\mathbb{T}^N)\big)^m$ be, respectively, a sub and a supersolution of* (1.3)*. Then, for every $i\in\{1,\dots,m\}$,*

$$v_i(t,x) - u_i(t,x) \leqslant \max_{1\leqslant i\leqslant m} \max_{\mathbb{T}^N} \big(v_i(0,\cdot) - u_i(0,\cdot)\big), \qquad (t,x)\in[0,T]\times\mathbb{T}^N.$$

By making use of this proposition and of Perron's method, it is then easy to prove the following

Proposition 1.10. *Let $\mathbf{u}_0 \in \big(\mathrm{Lip}(\mathbb{T}^N)\big)^m$. Then there exists a unique function $\mathbf{u}(t,x)$ in $\big(\mathrm{Lip}(\mathbb{R}_+\times\mathbb{T}^N)\big)^m$ that solves the system* (1.3) *subject to the initial condition $\mathbf{u}(0,x) = \mathbf{u}_0(x)$ in $\mathbb{T}^N$. Moreover, the Lipschitz constant of $\mathbf{u}(t,x)$ in $\mathbb{R}_+\times\mathbb{T}^N$ only depends on the Hamiltonians $H_1,\dots,H_m$ and on the Lipschitz constant of $\mathbf{u}_0$.*

We will denote by $\mathcal{S}(t)\mathbf{u}_0(x)$ the solution $\mathbf{u}(t,x)$ of (1.3) with initial datum $\mathbf{u}_0$. This defines, for every $t>0$, a map $\mathcal{S}(t) : \big(\mathrm{Lip}(\mathbb{T}^N)\big)^m \to \big(\mathrm{Lip}(\mathbb{T}^N)\big)^m$.

We summarize in the next proposition the properties enjoyed by such maps, which come as an easy application of the above results.

Proposition 1.11. *For every $t,s>0$ and $\mathbf{u},\mathbf{v} \in \big(\mathrm{Lip}(\mathbb{T}^N)\big)^m$ we have:*

(i) **(Semigroup property)** $\mathcal{S}(s)\big(\mathcal{S}(t)\mathbf{u}\big) = \mathcal{S}(t+s)\mathbf{u}$ *in* $\mathbb{T}^N$*;*

(ii) **(Monotonicity)** *if* $\mathbf{v}\leqslant\mathbf{u}$ *in* $\mathbb{T}^N$*, then* $\mathcal{S}(t)\mathbf{v}\leqslant\mathcal{S}(t)\mathbf{u}$ *in* $\mathbb{T}^N$*;*

(iii) **(Non–expansiveness property)** $\|\mathcal{S}(t)\mathbf{v} - \mathcal{S}(t)\mathbf{u}\|_\infty \leqslant \|\mathbf{v}-\mathbf{u}\|_\infty$*;*

(iv) for every $a\in\mathbb{R}$, $\mathcal{S}(t)(\mathbf{u}+a\mathbb{1}) = \mathcal{S}(t)\mathbf{u} + a\mathbb{1}$ *in* $\mathbb{T}^N$*.*

The fact that the coupling matrix $B(x)$ is everywhere degenerate is crucial for assertion *(iv)*.

2. The critical value

In this section we define the critical value and we study the corresponding critical system.

We first establish *a priori* estimates for the subsolutions of the system (1.2).

Proposition 2.1. *Let $\mathbf{a} = (a_1,\dots,a_m)^T \in \mathbb{R}^m$ and $\mathbf{u} \in \big(\mathrm{C}(\mathbb{T}^N)\big)^m$ such that*

$$\big(B(x)\mathbf{u}(x)\big)_i \leqslant a_i \qquad \textit{for every } x\in\mathbb{T}^N \textit{ and } i\in\{1,\dots,m\}. \tag{2.1}$$

Then there exists a constant $M_\mathbf{a}$ only depending on $\mathbf{a}$ and $B(x)$ such that

(i) $\|u_i - u_j\|_\infty \leqslant M_\mathbf{a}$ *for every i, $j\in\{1,\dots,m\}$;*

(ii) $\big|\big(B(x)\mathbf{u}(x)\big)_i\big| \leqslant M_\mathbf{a}$ *for every $x\in\mathbb{T}^N$ and $i\in\{1,\dots,m\}$.*

Proof. It suffices to prove the assertion for $\mathbf{a} = a\,\mathbb{1}$. Let us set

$$\beta_\star = \min_{1\leqslant i\leqslant m}\min_{x\in\mathbb{T}^N} b_{ii}(x), \qquad \beta^\star = \max_{1\leqslant i,j\leqslant m}\max_{x\in\mathbb{T}^N} |b_{ij}(x)|.$$

Such quantities are finite valued. Moreover, $\beta_\star$ is strictly positive in view of Proposition 1.2 and of the fact that $B(x)$ is, for every $x \in \mathbb{T}^N$, an irreducible coupling matrix with continuous coefficients.

Let us now fix $x \in \mathbb{T}^N$ and assume, without any loss of generality that $u_1(x) \leqslant u_2(x) \leqslant \cdots \leqslant u_m(x)$. First notice that, by subtracting $\sum_{j=1}^m b_{mj}(x)u_m(x) = 0$ from both sides of equation (2.1) with $i = m$, one gets $\sum_{j\neq m} -b_{mj}(x)\big(u_m(x) - u_j(x)\big) \leqslant a$, yielding $\big(u_m(x) - \max_{j\neq m} u_j(x)\big)\sum_{j\neq m} -b_{mj}(x) \leqslant a$. Since $B(x)$ is degenerate and $u_1(x) \leqslant u_2(x) \leqslant \cdots \leqslant u_m(x)$ we get

$$0 \leqslant u_m(x) - u_{m-1}(x) \leqslant \frac{a}{b_{mm}(x)} \leqslant \frac{a}{\beta_\star}. \tag{2.2}$$

This proves assertion *(i)* when $m = 2$. To prove it in the general case, we argue by induction: we assume the result true for m and we prove it for $m + 1$. To this aim, we restate equation (2.1) as

$$\sum_{j=1}^{m-1} b_{ij}(x)u_j(x) + \Big(b_{im}(x) + b_{i\,m+1}(x)\Big)u_m(x) + b_{i\,m+1}(x)\Big(u_{m+1}(x) - u_m(x)\Big) \leqslant a,$$

then we exploit (2.2) to get

$$\sum_{j=1}^{m-1} b_{ij}(x)u_j(x) + \Big(b_{im}(x) + b_{i\,m+1}(x)\Big)u_m(x) \leqslant a\left(1 + \frac{\beta^\star}{\beta_\star}\right) \tag{2.3}$$

for every $i \in \{1,\dots,m+1\}$. The irreducible character of $B(x)$ applied to the set $\mathcal{I} = \{m, m+1\}$ implies $b_{im}(x) + b_{i\,m+1}(x) > 0$ for either $i = m$ or $i = m+1$, let us say $i = m$ for definitiveness. Assertion *(i)* now follows by applying the induction hypothesis to the system given by (2.3) with i varying in $\{1,\dots,m\}$, the corresponding coupling matrix being still irreducible and degenerate.

To prove *(ii)* it suffices to note that, for every $i \in \{1,\dots,m\}$,

$$\begin{aligned}
-\big(B(x)\mathbf{u}(x)\big)_i &= -b_{ii}(x)u_i(x) + \sum_{j\neq i}\big(-b_{ij}(x)\big)u_j(x)\\
&\leqslant -b_{ii}(x)u_i(x) + \sum_{j\neq i} -b_{ij}(x)\big(u_i(x) + \|u_i - u_j\|_\infty\big)\\
&\leqslant (m-1)\,\beta^\star\|u_i - u_j\|_\infty,
\end{aligned}$$

and the assertion follows from *(i)* and from hypothesis (2.1). □

As a consequence, we derive the following result:

Proposition 2.2. *Let $\mathbf{u} = (u_1,\dots,u_m)^T \in \big(\mathrm{C}(\mathbb{T}^N)\big)^m$ be a subsolution of* (1.2) *for some $\mathbf{a} \in \mathbb{R}^m$. Then there exist constants $C_{\mathbf{a}}$ and $\kappa_{\mathbf{a}}$, only depending on $\mathbf{a}$, on the Hamiltonians $H_1,\dots,H_m$ and on the coupling matrix $B(x)$, such that*

(i) $\|u_i - u_j\|_\infty \leqslant C_{\mathbf{a}}$ *for every i, $j \in \{1,\dots,m\}$;*

(ii) $\mathbf{u}$ *is* $\kappa_{\mathbf{a}}$*–Lipschitz continuous in* $\mathbb{T}^N$*.*

Proof. For each $i \in \{1, \dots, m\}$, we have

$$H_i(x,p) + \big(B(x)\mathbf{u}(x)\big)_i \leqslant a_i \quad \text{for every } x \in \mathbb{T}^N \text{ and } p \in D^+u_i(x).$$

Since $D^+u_i(x) \neq \varnothing$ for a dense set of points, see [7, Proposition 3.1.9], if we set $\mu := \inf\{H_i(x,p) \,:\, (x,p) \in \mathbb{T}^N \times \mathbb{R}^N,\, i = 1, \dots, m\}$ the continuity of B and $\mathbf{u}$ implies $\big(B(x)\mathbf{u}(x)\big)_i \leqslant a_i - \mu$ for every $x \in \mathbb{T}^N$. In view of Proposition 2.1 we get *(i)* and

$$\left|\big(B(x)\mathbf{u}(x)\big)_i\right| \leqslant C_{\mathbf{a}} \qquad \text{for every } x \in \mathbb{T}^N$$

with $C_{\mathbf{a}} := M_{\mathbf{a}-\mu\mathbb{1}}$. Plugging this inequality in (1.2) we derive that u_i is a viscosity subsolution of

$$H_i(x, Du_i) \leqslant a_i + C_{\mathbf{a}} \qquad \text{in } \mathbb{T}^N$$

and assertion *(ii)* follows as well via a standard argument that exploits the coercivity of $H_i(x,p)$ in p, see for instance [2, Lemma 2.5]. □

Next, we establish a remarkable property of weakly coupled systems.

Theorem 2.3. *Assume that* $\mathbf{v}$, $\mathbf{u} \in \big(\mathrm{C}(\mathbb{T}^N)\big)^m$ *are, respectively a sub and a supersolution of the weakly coupled system* (1.2) *for some* $\mathbf{a} \in \mathbb{R}^m$*. Let* $x_0 \in \mathbb{T}^N$ *be such that*

$$v_i(x_0) - u_i(x_0) = M := \max_{1\leqslant i\leqslant m} \max_{\mathbb{T}^N} (v_i - u_i) \qquad \textit{for some } i \in \{1, \dots, m\}.$$

Then $\mathbf{v}(x_0) = \mathbf{u}(x_0) + M\mathbb{1}$*.*

Proof. In view of Proposition 2.2, we know that $\mathbf{v}$ is Lipschitz continuous. Set

$$\mathcal{I} = \big\{ i \in \{1, \dots, m\} \,:\, \big(v_i(x_0) - u_i(x_0)\big) = M \,\big\}.$$

We want to prove that $\mathcal{I} = \{1, \dots, m\}$. Indeed, if this were not the case, by the irreducible character of the matrix $B(x_0)$ there would exist $i \in \mathcal{I}$ and $k \notin \mathcal{I}$ such that $b_{ik}(x_0) < 0$. We now make use of the method of doubling the variables to reach a contradiction. For every $\varepsilon > 0$, we set

$$\psi^\varepsilon(x,y) = v_i(x) - u_i(y) - \frac{d(x,y)^2}{2\varepsilon^2} - \frac{d(x,x_0)^2}{2}, \qquad x, y \in \mathbb{T}^N.$$

Let $M_\varepsilon = \max_{\mathbb{T}^N \times \mathbb{T}^N} \psi_\varepsilon$ and denote by $(x_\varepsilon, y_\varepsilon)$ a point in $\mathbb{T}^N \times \mathbb{T}^N$ where such a maximum is achieved. By a standard argument in the theory of viscosity solution, see for instance Lemma 2.3 in [2], the following properties hold:

$$x_\varepsilon, y_\varepsilon \to x_0, \quad \frac{d(x_\varepsilon, y_\varepsilon)}{\varepsilon} \to 0 \qquad \text{as } \varepsilon \to 0. \tag{2.4}$$

Furthermore, for $\varepsilon > 0$ small enough,

$$p'_\varepsilon := \frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2} \in D^- u_i(y_\varepsilon), \quad p_\varepsilon := p'_\varepsilon - (x_\varepsilon - x_0) \in D^+ v_i(x_\varepsilon).$$

By the Lipschitz character of v_i we derive that the vectors $\{\, p_\varepsilon \,:\, \varepsilon > 0 \,\}$ are equi–bounded, hence, up to subsequences and in view of the estimates (2.4), we infer p_ε, $p'_\varepsilon \to p_0$ as $\varepsilon \to 0$ for some vector $p_0 \in \mathbb{R}^N$. We now use the fact that $\mathbf{v}$ and $\mathbf{u}$ are a sub and supersolution of (3.1), respectively, to get

$$H_i(x_\varepsilon, p_\varepsilon) + \big(B(x_\varepsilon)\mathbf{v}(x_\varepsilon)\big)_i \leqslant 0 \quad \text{and} \quad H_i(y_\varepsilon, p'_\varepsilon) + \big(B(y_\varepsilon)\mathbf{u}(y_\varepsilon)\big)_i \geqslant 0.$$

By subtracting the above inequalities and by passing to the limit for $\varepsilon \to 0$ we end up with

$$\Big(B(x_0)\big(\mathbf{v}(x_0) - \mathbf{u}(x_0)\big)\Big)_i \leqslant 0, \tag{2.5}$$

that is, since $i \in \mathcal{I}$ and the matrix $B(x_0)$ is degenerate,

$$M\, b_{ii}(x_0) \leqslant \sum_{j\neq i} -b_{ij}(x_0)\big(v_j(x_0) - u_j(x_0)\big) \leqslant M \sum_{j\neq i} -b_{ij}(x_0) = M b_{ii}(x_0).$$

Hence the above inequalities are equalities, in particular $v_k(x_0) - u_k(x_0) = M$ since $b_{ik}(x_0) \neq 0$, in contrast with the fact that $k \notin \mathcal{I}$. □

Remark 2.4. Note that the degeneracy hypothesis on the coupling matrix is only used at the very end of the proof of Theorem 2.3. In particular, the variable–doubling argument still works under broader assumptions, and equation (2.5) is still valid, even for non degenerate coupling matrices.

Definition 2.5. For every $\mathbf{a} \in \mathbb{R}^m$, we denote by $\mathcal{H}(\mathbf{a})$ the set of subsolutions of the weakly coupled system (1.2). We will more simply write $\mathcal{H}(a)$ whenever $\mathbf{a} = a\mathbb{1}$ for some constant $a \in \mathbb{R}$.

Lemma 2.6. *The sets $\mathcal{H}(\mathbf{a})$ are convex and closed in $\big(\mathrm{C}(\mathbb{T}^N)\big)^m$, and increasing with respect to the partial ordering on $\mathbb{R}^m$.*

Proof. Convexity and monotonicity are straightforward. The fact that the $\mathcal{H}(\mathbf{a})$ are closed is a direct consequence of stability of viscosity subsolutions. □

We now focus our attention to the case $\mathbf{a} = a\mathbb{1}$. As a direct consequence of the definition of the semigroup $\mathcal{S}(t)$, we get the following assertion:

Proposition 2.7. *Let $a \in \mathbb{R}$ and $\mathbf{u} \in \big(\mathrm{Lip}(\mathbb{T}^N)\big)^m$. Then $\mathbf{u}$ is a viscosity solution of* (1.2) *with $\mathbf{a} = a\,\mathbb{1}$ if and only if $\mathbf{u} = \mathcal{S}(t)\mathbf{u} + t\,a\mathbb{1}$ in $\mathbb{T}^N$ for every $t > 0$.*

We have the following characterization:

Proposition 2.8. *Let $a \in \mathbb{R}$ and $\mathbf{u} \in \big(\mathrm{Lip}(\mathbb{T}^N)\big)^m$. The following facts are equivalent:*

(i) $\mathbf{u} \in \mathcal{H}(a)$;

(ii) *the map $t \mapsto \mathcal{S}(t)\mathbf{u} + t\,a\mathbb{1}$ is non–decreasing on $[0, +\infty)$.*

In particular, the sets $\mathcal{H}(a)$ are stable under the action of the semigroup $\mathcal{S}(t)$, in the sense that $\mathcal{S}(t)\big(\mathcal{H}(a)\big) \subset \mathcal{H}(a)$.

The proof of this proposition is rather technical and it is postponed to the Appendix A.

Definition 2.9. *The* critical value c *of the weakly coupled system* (1.2) *is defined as*

$$c = \inf\{a \in \mathbb{R} \,:\, \mathcal{H}(a) \neq \varnothing\,\}. \tag{2.6}$$

The following holds:

Proposition 2.10. *The critical value c is finite and $\mathcal{H}(c) \neq \varnothing$.*

Proof. By the compactness of $\mathbb{T}^N$ and the continuity of the Hamiltonians, it is easily seen that the function $\mathbf{u} \equiv (0, \dots, 0)^T$ is a subsolution of (1.2) for $a_0 \mathbb{1}$ with $a_0 \in \mathbb{R}$ big enough.

Let us proceed to show that c is finite valued and that $\mathcal{H}(c) \neq \varnothing$. Let $(a_n)_n$ be a decreasing sequence converging to c and let $\mathbf{u}^n \in \mathcal{H}(a_n)$ for each $n \in \mathbb{N}$. Arguing as in the proof of Proposition 2.2 and taking into account that $\mathbf{u}^n$ is Lipschitz, we obtain that $\mu \leqslant H_i(x, Du_i^n(x)) \leqslant a_n + M_{a_1}$ a.e. in $\mathbb{T}^N$ for every $i \in \{1, \dots, m\}$ and $n \in \mathbb{N}$. This shows that c is finite.

We now exploit Proposition 2.2: by the monotonicity of the sets $\mathcal{H}(a)$ with respect to a, we infer that the functions $\mathbf{u}^n$ are equi–Lipschitz. Up to subtracting a vector of the form $k_n \mathbb{1}$ to each $\mathbf{u}^n$, we can furthermore assume that $u_1^n(0) = 0$ for every $n \in \mathbb{N}$, yielding $\sup_n \|u_1^n\|_\infty \leqslant L$ for some $L \in \mathbb{R}$ by the equi–Lipschitz character of the sequence. Moreover, $\|u_j^n - u_1^n\|_\infty \leqslant C_{a_1}$ for every $j \in \{1, \dots, m\}$ and $n \in \mathbb{N}$, yielding $\|u_j^n\|_\infty \leqslant C_{a_1} + L$ for every $j \in \{1, \dots, m\}$ and $n \in \mathbb{N}$. Up to subsequences, by the Arzela–Ascoli theorem, we infer that $\mathbf{u}^n \rightrightarrows \mathbf{u}$ in $\mathbb{T}^N$ and $\mathbf{u} \in \mathcal{H}(c)$ by stability of the notion of viscosity subsolution. □

We now proceed to show that a weakly coupled system of the kind (1.2) with $\mathbf{a} = a\mathbb{1}$ possesses solutions if and only if a equals the critical value c.

We start with a preliminary result.

Proposition 2.11. *Let $B(x)$ be a continuous irreducible coupling matrix on $\mathbb{T}^N$ and let us assume that $B(x)$ is invertible for every $x \in \mathbb{T}^N$. Let $\mathbf{v}$, $\mathbf{u} \in \big(\mathrm{C}(\mathbb{T}^N)\big)^m$ be, respectively, a sub and a supersolution of the weakly coupled system* (1.2)*, for some $\mathbf{a} \in \mathbb{R}^m$. Then $\mathbf{v}(x) \leqslant \mathbf{u}(x)$ for every $x \in \mathbb{T}^N$.*

Proof. By the continuity of $B(x)\mathbf{v}(x)$ and the coercivity of the Hamiltonians H_i, we easily get that $\mathbf{v}$ is Lipschitz, cf. proof of Proposition 2.2.

Set $M = \max_{1 \leqslant i \leqslant m} \max_{\mathbb{T}^N} (v_i - u_i)$. We want to prove that $M \leqslant 0$. Assume by contradiction that $M > 0$ and pick a point $x_0 \in \mathbb{T}^N$ where such a maximum is attained. Set $\mathcal{I} = \big\{ i \in \{1, \dots, m\} \,:\, v_i(x_0) - u_i(x_0) = M \big\}$. Using a variable–doubling argument as in the proof of Theorem 2.3 (see also remark 2.4) we infer that

$$\Big(B(x_0)\big(\mathbf{v}(x_0) - \mathbf{u}(x_0)\big) \Big)_i \leqslant 0 \qquad \text{for every } i \in \mathcal{I}. \tag{2.7}$$

If $\mathcal{I} = \{1, \dots, m\}$, inequality (2.7) must be an equality since the matrix $B(x_0)$ satisfies condition (C) and this is in contradiction with the fact that it is invertible. If $\mathcal{I} \neq \{1, \dots, m\}$, we choose $i \in \mathcal{I}$ and $k \notin \mathcal{I}$ such that $b_{ik}(x_0) < 0$. From (2.7) and the assumption that $M > 0$ we infer that

$$M\, b_{ii}(x_0) \leqslant \sum_{j \neq i} -b_{ij}(x_0)\big(v_j(x_0) - u_j(x_0)\big) \leqslant M \sum_{j \neq i} -b_{ij}(x_0) \leqslant M b_{ii}(x_0),$$

which implies that $v_k(x_0) - u_k(x_0) = M$, in contrast with the fact that $k \notin \mathcal{I}$. □

The next result implies that solutions to a weakly coupled system of the kind (1.2) with $\mathbf{a} = a\mathbb{1}$ may exist only if a equals the critical value.

Proposition 2.12. *Let a, $b \in \mathbb{R}$ and $\mathbf{v}$, $\mathbf{u} \in \big(\mathrm{C}(\mathbb{T}^N)\big)^m$ such that the following inequalities are satisfied in the viscosity sense:*

$$H_i(x, Dv_i) + \big(B(x)\mathbf{v}(x)\big)_i \leqslant a \quad \text{and} \quad H_i(x, Du_i) + \big(B(x)\mathbf{u}(x)\big)_i \geqslant b \quad \text{in } \mathbb{T}^N$$

for every $i \in \{1, \dots, m\}$. Then $b \leqslant a$.

Proof. Let us assume by contradiction that $b > a$. Up to replacing $\mathbf{v}$ with $\mathbf{v}+k\mathbb{1}$ with $k > 0$ big enough, we can assume $\mathbf{v} > \mathbf{u}$ in $\mathbb{T}^N$. Let $\varepsilon > 0$ such that $b-\varepsilon > a+\varepsilon$. By continuity of the functions $\mathbf{v}$ and $\mathbf{u}$, we can find $\lambda > 0$ such that $\|\lambda\, v_i\|_\infty$, $\|\lambda\, u_i\|_\infty < \varepsilon$ for every $i \in \{1, \ldots, m\}$. Then the following inequalities hold in the viscosity sense in $\mathbb{T}^N$:

$$H_i(x, Du_i) + \big((B(x) + \lambda\,\mathrm{I})\mathbf{u}(x)\big)_i > b-\varepsilon > a+\varepsilon > H_i(x, Dv_i) + \big((B(x) + \lambda\,\mathrm{I})\mathbf{v}(x)\big)_i.$$

For every $x \in \mathbb{T}^N$, the matrix $B(x) + \lambda\,\mathrm{I}$ is irreducible, satisfies (C) and the sum of the elements of each of its rows is strictly positive, hence it is invertible in view of Proposition 1.4. By Proposition 2.11 we conclude that $\mathbf{v} \leqslant \mathbf{u}$ in $\mathbb{T}^N$, achieving a contradiction. □

The next theorem is already known in literature, see [28, 6], and is proved by the ergodic approximation method. We provide here a new proof using an idea introduced in [18].

Theorem 2.13. *There exists a function $\mathbf{u} \in \mathcal{H}(c)$ that solves the weakly coupled system*

$$H_i(x, Du_i) + \big(B(x)\mathbf{u}(x)\big)_i = c \quad \text{in } \mathbb{T}^N \qquad \text{for every } i \in \{1, \ldots, m\} \tag{2.8}$$

in the viscosity sense.

Proof. We have already proved in Proposition 2.10 that $\mathcal{H}(c) \neq \varnothing$. Let us introduce the quotient space $\hat{\mathcal{H}} = \mathcal{H}(c)\backslash\mathbb{R}\mathbb{1}$, where we identify critical subsolutions that differ by a constant vector belonging to $\mathbb{R}\mathbb{1}$. Arguing as in the proof of Proposition 2.10, it is easily seen that $\hat{\mathcal{H}}$ is compact for the topology of uniform convergence. Indeed, it is isomorphic to the subset of $\mathcal{H}(c)$ of subsolutions whose first component vanishes at the point $x = 0$. Moreover, since the viscosity semigroup commutes with the addition of vectors of the form $\lambda\mathbb{1}$ and leaves $\mathcal{H}(c)$ stable, it induces a continuous semigroup, denoted $\hat{S}$, on $\hat{\mathcal{H}}$.

By the Schauder–Tychonoff fixed point theorem (see [13, Theorem 2.2, page 414]), $\hat{S}$ possesses a fixed point, that is, there exists an element $\hat{\mathbf{u}} \in \hat{\mathcal{H}}$ such that $\forall t \geqslant 0$, $\hat{S}(t)\hat{\mathbf{u}} = \hat{\mathbf{u}}$. Lifting these relations to $\mathcal{H}(c)$, we get that, for every $t \geqslant 0$, there exists $c_t \in \mathbb{R}$ such that $\mathcal{S}(t)\mathbf{u} = \mathbf{u} + c_t\mathbb{1}$, where $\mathbf{u}$ is any element in the equivalence class of $\hat{\mathbf{u}}$. Since $\mathcal{S}$ is a semigroup, one readily realizes that $c_{t+s} = c_t + c_s$ for every $t, s > 0$. Since $t \mapsto \mathcal{S}(t)\mathbf{u}$ is continuous, we necessarily deduce that $c_t = -t\tilde{c}$ for all $t > 0$ for some constant $\tilde{c} \in \mathbb{R}$. The identity $\mathcal{S}(t)\mathbf{u} = \mathbf{u} - t\tilde{c}\mathbb{1}$, for all $t \geqslant 0$, implies that $\mathbf{u}$ is a viscosity solution of (2.8) with $\tilde{c}$ in place of c, see Proposition 2.7. But then $\tilde{c} = c$ in view of Proposition 2.12 and the statement is proved. □

3. The Aubry set

In this section we start our qualitative analysis on the critical weakly coupled system, i.e. the system (1.2) with $\mathbf{a} = c\mathbb{1}$, where c is defined via (2.6). From now on we will always assume the critical value c to be equal to 0. This renormalization is always possible by replacing each H_i with $H_i - c$. The critical weakly coupled system reads as

$$H_i(x, Du_i) + \big(B(x)\mathbf{u}(x)\big)_i = 0 \quad \text{in } \mathbb{T}^N \qquad \text{for every } i \in \{1, \ldots, m\}. \tag{3.1}$$

Solutions, subsolutions and supersolutions of (3.1) will be termed *critical* in the sequel. The family of critical subsolutions, we recall, is denoted by $\mathcal{H}(0)$.

Our qualitative analysis on the critical weakly coupled system is based on the notion of *Mañé matrix*, defined in analogy with that of the Mañé potential.

Definition 3.1. For all $(x, y, i, j) \in \mathbb{T}^N \times \mathbb{T}^N \times \{1, \ldots, m\} \times \{1, \ldots, m\}$, we define

$$\Phi_{i,j}(y, x) = \sup_{\mathbf{v} \in \mathcal{H}(0)} v_i(x) - v_j(y).$$

The following properties hold:

Proposition 3.2. *The Mañé matrix verifies the following properties:*

(i) it is everywhere finite and Lipschtiz continuous;

(ii) $\Phi_{\cdot,j}(y, \cdot)$ *is a critical solution, for every* $(y, j) \in \mathbb{T}^N \times \{1, \ldots, m\}$*;*

(iii) for every $(y, j) \in \mathbb{T}^N \times \{1, \ldots, m\}$ *and* $\mathbf{v} \in \mathcal{H}(0)$,

$$\mathbf{v} - v_j(y)\mathbb{1} \leqslant \Phi_{\cdot,j}(y, \cdot) \qquad \text{in } \mathbb{T}^N,$$

namely $\Phi_{\cdot,j}(y, \cdot)$ *is the maximal critical subsolution whose* j*–th component vanishes at* y*;*

(iv) the entries of the Mañé matrix are linked by the following triangular inequality:

$$\Phi_{i,k}(x, z) \leqslant \Phi_{j,k}(x, y) + \Phi_{i,j}(y, z)$$

for every $i, j, k \in \{1, \ldots, m\}$ *and* $x, y, z \in \mathbb{T}^N$.

Proof. The fact that the Mañé matrix is well defined directly follows from Proposition 2.2. Lipschitz continuity comes from the equi–Lipschitz character of critical subsolutions.

The second assertion comes from the fact that $\Phi_{\cdot,j}(y, \cdot)$ is, for every fixed (j, y), a supremum of critical subsolutions, hence itself a critical subsolution by Proposition 1.8.

The third point is a direct consequence of the definition.

The last point comes from the fact that $\Phi_{\cdot,j}(y, \cdot)$ is the greatest subsolution whose j–th component vanishes at y. Since $\Phi_{\cdot,k}(x, \cdot) - \Phi_{j,k}(x, y)\mathbb{1}$ is a subsolution whose j–th component vanishes at y we obtain that $\Phi_{\cdot,k}(x, \cdot) - \Phi_{j,k}(x, y)\mathbb{1} \leqslant \Phi_{\cdot,j}(y, \cdot)$, which is the triangular inequality to be proved. □

As in the case of a single critical equation, the Mañé vectors are "almost" critical solutions, in the sense explained below:

Proposition 3.3. *Let* $y_0 \in \mathbb{T}^N$ *and* $i_0 \in \{1, \ldots, m\}$. *Then the function* $\mathbf{u} = \Phi_{\cdot,i_0}(y_0, \cdot)$ *satisfies*

$$H_i(x, Du_i) + \big(B(x)\mathbf{u}(x)\big)_i = 0 \quad \text{in } \{1, \ldots, m\} \times \mathbb{T}^N \setminus \{(i_0, y_0)\}$$

in the viscosity sense.

Proof. We argue by contradiction, following the classical argument of [18] for the classical Mañé potential.

Let (i, y) be such that either $i \neq i_0$ or $y \neq y_0$. Let us assume that the viscosity supersolution condition is violated at (i, y). This means that there exists a C^1 function ψ such that $\psi(x) \leqslant \Phi_{i,i_0}(y_0, x)$ for all x, with equality if and only if $x = y$, and $H_i\big(x, D\psi(y)\big) + \big(B(y)\Phi_{\cdot,i_0}(y_0, y)\big)_i < 0$. Since ψ is C^1, and $B(\cdot)$ and $\Phi_{\cdot,i_0}(x_0, \cdot)$ are continuous, it is clear that this strict inequality continues to hold in a neighborhood of y. We infer that it is possible to find $\varepsilon > 0$ small enough such that the function $w_i := \max\{\Phi_{i,i_0}(y_0,\cdot), \psi+\varepsilon\}$ verifies

$$H_i\big(x, Dw_i(x)\big) + \big(B(x)\mathbf{w}(x)\big)_i \leqslant 0 \qquad \text{for a.e. } x \in \mathbb{T}^N,$$

where $\mathbf{w}$ is the vector whose i–th coordinate is w_i and whose other coordinates are those of $\Phi_{\cdot,i_0}(y_0,\cdot)$. In the case when $i = i_0$ and $y \neq y_0$, we choose $\varepsilon > 0$ small enough in such a way that $w_i(y_0) = \Phi_{i,i_0}(y_0, y_0) = 0$. Moreover, for every $j \neq i$,

$$H_j\big(x, Dw_j(x)\big) + \big(B(x)\mathbf{w}(x)\big)_j \leqslant 0 \qquad \text{for a.e. } x \in \mathbb{T}^N,$$

as it is easily seen from the fact that $b_{ji}(\cdot) \leqslant 0$ in $\mathbb{T}^N$ and $w_i \geqslant \Phi_{i,i_0}(y_0,\cdot)$.

We have thus shown that $\mathbf{w}$ is a critical subsolution with $w_{i_0}(y_0) = 0$, $\mathbf{w} \geqslant \Phi_{i,i_0}(y_0,\cdot)$ and $\mathbf{w} \not\equiv \Phi_{i,i_0}(y_0,\cdot)$, thus contradicting the maximality of $\Phi_{i,i_0}(y_0,\cdot)$ amongst subsolutions whose i_0–th coordinate vanishes at y_0. □

Next, we show a strong invariance property enjoyed by the rows of the Mañé matrix.

Proposition 3.4. *Let $i, j \in \{1,\dots,m\}$ and $y \in \mathbb{T}^N$. If $\Phi_{\cdot,i}(y,\cdot)$ is a critical solution on $\mathbb{T}^N$, then $\Phi_{\cdot,j}(y,\cdot)$ is too.*

Proof. Let us set $\mathbf{v} := \Phi_{\cdot,j}(y,\cdot)$ and $\mathbf{u} := \Phi_{\cdot,i}(y,\cdot) + \Phi_{i,j}(y,y)\,\mathbb{1}$. In view of Proposition 3.3, we only need to show that $H_j(y,p) + \big(B(y)\mathbf{v}(y)\big)_j \geqslant 0$ for every $p \in D^- v_j(y)$.

According to Proposition 3.2, $\mathbf{v} \leqslant \mathbf{u}$ in $\mathbb{T}^N$ and $v_i(y) = u_i(y)$. The functions $\mathbf{v}$ and $\mathbf{u}$ being respectively a critical subsolution and a solution, we can apply Theorem 2.3 to infer that $\mathbf{v}(y) = \mathbf{u}(y)$. This also implies that $D^- v_j(y) \subseteq D^- u_j(y)$. Exploiting again the fact that $\mathbf{u}$ is a critical solution we finally get

$$0 \leqslant H_j(y,p) + \big(B(y)\mathbf{u}(y)\big)_j = H_j(y,p) + \big(B(y)\mathbf{v}(y)\big)_j \quad \text{for every } p \in D^- v_j(y).$$

□

In view of the previous proposition, the following definition is well posed:

Definition 3.5. The Aubry set $\mathcal{A}$ for the weakly coupled system (3.1) is the set defined as

$$\mathcal{A} = \big\{ y \in \mathbb{T}^N \,:\, \Phi_{\cdot,i}(y,\cdot) \text{ is a critical solution} \big\},$$

where i is any fixed index in $\{1,\dots,m\}$.

By the continuity of the Mañé matrix and the stability of the notion of viscosity solution, it is easily seen that $\mathcal{A}$ is closed. The analysis we are about to present will show that the Aubry set is nonempty: as in the corresponding critical scalar case, we will see that $\mathcal{A}$ is the set where the obstruction to the existence of globally strict critical subsolutions concentrates.

Definition 3.6. Let $\mathbf{v} \in \mathcal{H}(0)$. We will say that v_i is *strict at* $y \in \mathbb{T}^N$ if there exist an open neighborhood V of y and $\delta > 0$ such that $H_i\big(x, Dv_i(x)\big) + \big(B(x)\mathbf{v}(x)\big)_i < -\delta$ for a.e. $x \in V$.

We will say that v_i is strict in an open subset U of $\mathbb{T}^N$ if it is strict at y for every $y \in U$.

We start by establishing an auxiliary result that will be needed in the sequel.

By modulus we mean a nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$, vanishing and continuous at 0.

We will say that $(\rho_n)_n$ is a sequence of *standard mollifiers* if $\rho_n(x) := n^N \rho(nx)$ in $\mathbb{R}^N$ for each $n \in \mathbb{N}$, where ρ is a smooth, non–negative function on $\mathbb{R}^N$, supported in B_1 and such that its integral over $\mathbb{R}^N$ is equal to 1.

Lemma 3.7. *Let $\mathbf{w} \in \mathcal{H}(0)$ such that w_i is strict at $y \in \mathbb{T}^N$. Then there exists $\widetilde{\mathbf{w}} \in \mathcal{H}(0)$ such that $\widetilde{w}_i$ is C^∞ and strict in a neighborhood of y.*

Proof. By hypothesis, there exist $r > 0$ and $\delta > 0$ such that

$$H_i\big(x, Dw_i(x)\big) + \big(B(x)\mathbf{w}(x)\big)_i < -\delta \quad \text{for a.e. } x \in B_{2r}(y).$$

Let $\phi : \mathbb{T}^N \to [0,1]$ be a C^∞–function, compactly supported in $B_r(y)$ and such that $\phi \equiv 1$ in $B_{r/2}(y)$. Let us denote by κ a Lipschitz constant for the critical subsolutions and by ω a continuity modulus of H_i in $\mathbb{T}^N \times B_R$ for some fixed $R > \kappa + \|D\phi\|_\infty$. Let $(\rho_n)_n$ be a sequence of standard mollifiers on $\mathbb{R}^N$ and define

$$\psi_n(x) = (\rho_n * w_i)(x) + \|\rho_n * w_i - w_i\|_\infty, \qquad x \in \mathbb{T}^N.$$

Note that $\psi_n \geqslant w_i$ in $\mathbb{T}^N$ for every $n \in \mathbb{N}$ and $d_n := \|\psi_n - w_i\|_\infty \to 0$ as $n \to +\infty$.

Up to neglecting the first terms, we furthermore assume that all the d_n are less than 1. For every $n \in \mathbb{N}$, we define a function $\widetilde{\mathbf{w}}^n \in \big(\mathrm{Lip}(\mathbb{T}^N)\big)^m$ by setting $w_i^n(x) = \phi(x)\psi_n(x) + \big(1 - \phi(x)\big)w_i(x)$ and $\widetilde{w}_j^n(x) = w_j(x)$ if $j \neq i$, for every $x \in \mathbb{T}^N$. It is apparent by the definition that $\widetilde{w}_i^n$ is of class C^∞ in $B_{r/2}(y)$. Moreover the functions $(\widetilde{w}_i^n)_n$, and hence the $(\widetilde{\mathbf{w}}^n)_n$, are equi–Lipschitz. Indeed, for almost every $x \in \mathbb{T}^N$,

$$D\widetilde{w}_i^n(x) = \phi(x)D\psi_n(x) + \big(1 - \phi(x)\big)Dw_i(x) + \big(\psi_n(x) - w_i(x)\big)D\phi(x) \tag{3.2}$$

that is $\|D\widetilde{w}_i^n\|_\infty \leqslant \kappa + \|D\phi\|_\infty$. We want to show that n can be chosen sufficiently large in such a way that $\widetilde{\mathbf{w}}^n \in \mathcal{H}(0)$ and

$$H_i\big(x, D\widetilde{w}_i^n(x)\big) + \big(B(x)\widetilde{\mathbf{w}}^n(x)\big)_i < -\frac{2}{3}\,\delta \qquad \text{for a.e. } x \in B_r(y). \tag{3.3}$$

We first note that, since $\widetilde{w}_i^n \geqslant w_i$ and $b_{ji} \leqslant 0$ in $\mathbb{T}^N$ for every $j \neq i$, we have

$$H_j(x, D\widetilde{w}_j^n(x)) + \big(B(x)\widetilde{\mathbf{w}}^n(x)\big)_j \leqslant 0 \quad \text{in } \mathbb{T}^N \qquad \text{for every } j \neq i. \tag{3.4}$$

Moreover, since $\widetilde{\mathbf{w}}^n$ agrees with $\mathbf{w}$ outside $B_r(y)$, in order to show that $\widetilde{\mathbf{w}}^n$ satisfies (3.4) also for $j = i$, it will be enough, by the convexity of H_i, to prove (3.3).

To this aim, we start by noticing that

$$H_i\big(x, D\widetilde{w}_i^n(x)\big) \leqslant \phi(x)H_i\big(x, D\psi_n(x)\big) + \big(1 - \phi(x)\big)H_i\big(x, Dw_i(x)\big) + \omega\left(d_n\,\|D\phi\|_\infty\right) \tag{3.5}$$

for almost every $x \in \mathbb{T}^N$, in view of (3.2) and of the convexity of H_i. By Jensen's inequality, for every $n > 1/r$ and every $x \in B_r$ we have

$$\begin{aligned}
H_i\big(x, D\psi_n(x)\big) &= H_i\left(x, \int_{B_{1/n}} Dw_i(x-y)\rho_n(y)\,\mathrm{d}y\right) \\
&\leqslant \int_{B_{1/n}} H_i\big(x, Dw_i(x-y)\big)\rho_n(y)\,\mathrm{d}y \\
&\leqslant \omega(1/n) + \int_{B_{1/n}} H_i\big(x-y, Dw_i(x-y)\big)\rho_n(y)\,\mathrm{d}y \\
&\leqslant -\int_{B_{1/n}} \big(B(x-y)\mathbf{w}(x-y)\big)_i\,\rho_n(y)\,\mathrm{d}y - \delta + \omega(1/n) \\
&\leqslant -\big(B(x)\widetilde{\mathbf{w}}^n(x)\big)_i - \delta + \omega(1/n) + \varepsilon_n, \qquad (3.6)
\end{aligned}$$

where

$$\varepsilon_n := \sup_{|z|\leqslant 1/n} \left\| \big(B(\cdot+z)\mathbf{w}(\cdot+z) - B(\cdot)\widetilde{\mathbf{w}}^n(\cdot)\big)_i \right\|_\infty.$$

Since $\widetilde{\mathbf{w}}^n \rightrightarrows \mathbf{w}$ in $\mathbb{T}^N$ and all these functions are equi–Lipschitz, it is easily seen that $\lim_n \varepsilon_n = 0$. Furthermore

$$H_i\big(x, Dw_i(x)\big) \leqslant -\big(B(x)\widetilde{\mathbf{w}}^n(x)\big)_i - \delta + \varepsilon_n \quad \text{for a.e. } x \in B_r(y). \tag{3.7}$$

We now choose $n > 1/r$ sufficiently large such that $\omega\left(d_n \|D\phi\|_\infty\right) + \omega(1/n) + \varepsilon_n < \delta/6$ and plug (3.6) and (3.7) into (3.5) to finally get (3.3). The assertion follows by setting $\widetilde{\mathbf{w}} := \widetilde{\mathbf{w}}^n$ for such an index n. □

The next proposition shows that the i–th component of any critical subsolution fulfills the supersolution test on $\mathcal{A}$.

Proposition 3.8. *Let $y \in \mathcal{A}$. Then, for every $i \in \{1, \dots, m\}$ and $\mathbf{w} \in \mathcal{H}(0)$,*

$$H_i(y,p) + \big(B(y)\mathbf{w}(y)\big)_i = 0 \qquad \text{for every } p \in D^- w_i(y). \tag{3.8}$$

Proof. Pick $\mathbf{w} \in \mathcal{H}(0)$ and set $\mathbf{u} = \Phi_{\cdot,i}(y,\cdot) + w_i(y)\mathbb{1}$. According to Proposition 3.2, $\mathbf{w} \leqslant \mathbf{u}$ and, by definition of $\mathbf{u}$, $w_i(y) = u_i(y)$, in particular $D^- w_i(y) \subseteq D^- u_i(y)$. Now we exploit the fact that $\mathbf{u}$ and $\mathbf{w}$ are a critical solution and subsolution, respectively: from Theorem 2.3 we infer that $\mathbf{w}(y) = \mathbf{u}(y)$, while Proposition 1.7 implies

$$0 \geqslant H_i(y,p) + \big(B(y)\mathbf{w}(y)\big)_i = H_i(y,p) + \big(B(y)\mathbf{u}(y)\big)_i \geqslant 0 \quad \forall p \in D^- w_i(y).$$

Hence all the inequalities must be equalities and the statement follows. □

A converse of this result is given by the following

Proposition 3.9. *Let $i \in \{1, \dots, m\}$. The following facts are equivalent:*

(i) $y \notin \mathcal{A}$;

(ii) *there exists $\mathbf{w} \in \mathcal{H}(0)$ such that w_i is strict at y.*

Moreover, w_i can be taken of class C^1 in a neighborhood of y.

Proof. Let us assume *(i)*. Since $y \notin \mathcal{A}$, the supersolution test for $\Phi_{\cdot,i}(y,\cdot)$ is violated at (i,y). This means that there exists a C^1 function ψ such that $\psi(x) \leqslant \Phi_{i,i}(y,x)$ for all x, with equality if and only if $x = y$, and $H_i\big(x, D\psi(y)\big) + \big(B(y)\Phi_{\cdot,i}(y,y)\big)_i < 0$.

We define a function $\mathbf{w} \in \big(\mathrm{Lip}(\mathbb{T}^N)\big)^m$ by setting

$$w_i(\cdot) = \max\{\Phi_{i,i}(y,\cdot), \psi + \varepsilon\}, \qquad w_j(\cdot) = \Phi_{j,i}(y,\cdot) \quad \text{for } j \neq i.$$

Arguing as in the proof of Proposition 3.3 we see that it is possible to choose $\varepsilon > 0$ in a such a way that $\mathbf{w}$ is a critical subsolution. Moreover, since w_i agrees with $\psi + \varepsilon$ in a neighborhood of y, there exist $\delta > 0$ and an open neighborhood W of y such that w_i is of class C^1 in W and

$$H_i\big(x, Dw_i(x)\big) + \big(B(x)\mathbf{w}(x)\big)_i < -\delta \quad \text{for every } x \in W.$$

Conversely, let assume *(ii)*. According to Lemma 3.7, there exists $\widetilde{\mathbf{w}} \in \mathcal{H}(0)$ such that $\widetilde{w}_i$ is smooth and strict in a neighborhood of y, in particular $H_i\big(y, D\widetilde{w}_i(y)\big) + \big(B(y)\widetilde{\mathbf{w}}(y)\big)_i < 0$.

In view of Proposition 3.8 we conclude that $y \notin \mathcal{A}$. □

Remark 3.10. Proposition 3.8 expresses the fact, roughly speaking, that the i–th component of a critical subsolution cannot be strict at y. However, since the supersolution test (3.8) is void when $D^-w_i(y)$ is empty, this fact cannot be directly used to prove the equivalence stated in Proposition 3.9. This is the reason why we needed the regularization Lemma 3.7.

We proceed by proving a global version of the previous proposition. We give a definition first.

Definition 3.11. Let $\mathbf{v} \in \mathcal{H}(0)$. We will say that $\mathbf{v}$ is *strict at y* if v_i is strict at y for every $i \in \{1, \ldots, m\}$. We will say that $\mathbf{v}$ is strict in an open subset U of $\mathbb{T}^N$ if it is strict at y for every $y \in U$.

Theorem 3.12. *There exists $\mathbf{v} \in \mathcal{H}(0)$ which is strict in $\mathbb{T}^N \setminus \mathcal{A}$. In particular, the Aubry set $\mathcal{A}$ is closed and nonempty.*

Proof. Fix $i \in \{1, \ldots, m\}$. We first construct a critical subsolution $\mathbf{v}^i$ whose i–th component is strict in $\mathbb{T}^N \setminus \mathcal{A}$. According to Proposition 3.9, for every $y \in \mathbb{T}^N \setminus \mathcal{A}$ there exist an open neighborhood W_y of y, a critical subsolution $\mathbf{w}^y$ and $\delta_y > 0$ such that

$$H_i\big(x, Dw_i^y(x)\big) + \big(B(x)\mathbf{w}^y(x)\big)_i < -\delta_y \quad \text{for a.e. } x \in W_y \tag{3.9}$$

The family $\{W_y \,:\, y \in \mathbb{T}^N \setminus \mathcal{A}\}$ is an open covering of $\mathbb{T}^N \setminus \mathcal{A}$, from which we can extract a countable covering $(W_n)_n$ of $\mathbb{T}^N \setminus \mathcal{A}$. For each $n \in \mathbb{N}$, let us denote by $(\mathbf{w}^n, \delta_n)$ the corresponding pair in $\mathcal{H}(0) \times (0, +\infty)$ that satisfies (3.9) in W_n. Up to subtracting to each critical subsolution $\mathbf{w}^n$ a vector of the form $k_n\mathbb{1}$, we can moreover assume that $w_1^n(0) = 0$. Hence the functions $\mathbf{w}^n$ are equi–Lipschitz and equi–bounded in view of Proposition 2.2, in particular the function

$$\mathbf{v}^i(x) = \sum_{n=1}^{\infty} \frac{1}{2^n}\, \mathbf{w}^n(x), \qquad x \in \mathbb{T}^N$$

is well defined and belongs to $\big(\mathrm{Lip}(\mathbb{T}^N)\big)^m$. By convexity of the Hamiltonians, for almost every $x \in \mathbb{T}^N$ we get

$$H_i\big(x, Dv_i^i(x)\big) + \big(B(x)\mathbf{v}^i(x)\big)_i \leqslant \sum_{n=1}^{\infty} \frac{1}{2^n}\Big(H_i\big(x, Dw_i^n(x)\big) + \big(B(x)\mathbf{w}^n(x)\big)_i\Big) \leqslant 0.$$

Moreover, the above inequalities hold with $-\delta_k/2^k$ in place of 0 almost everywhere in W_k, for every $k \in \mathbb{N}$. This shows that $\mathbf{v}^i$ is a critical subsolution, strict in $\mathbb{T}^N \setminus \mathcal{A}$. Now set $\mathbf{v}(x) = \sum\limits_{i=1}^{m} \frac{1}{m}\mathbf{v}^i(x)$, for $x \in \mathbb{T}^N$. A similar argument shows that $\mathbf{v}$ is a critical subsolution that satisfies the assertion.

If $\mathcal{A} = \varnothing$, by compactness we would have $\mathcal{H}(-\delta) \neq \varnothing$ for some $\delta > 0$, contradicting the definition of the critical value $c = 0$. □

In view of Proposition 3.9, we have the following characterization:

Theorem 3.13. *Let $y \in \mathbb{T}^N$. The following are equivalent facts:*

(i) $y \notin \mathcal{A}$;

(ii) *there exists* $\mathbf{w} \in \mathcal{H}(0)$ *which is strict at* y;

(iii) *there exists* $\mathbf{w} \in \mathcal{H}(0)$ *and* $i \in \{1, \ldots, m\}$ *such that* w_i *is strict at* y.

We end this section by extending to weakly coupled systems a result which is well known in the case of a single critical equation.

Proposition 3.14. *The following equality holds:*

$$\mathcal{A} = \bigcap_{\mathbf{w}\in\mathcal{H}(0)} \left\{ y \in \mathbb{T}^N \,:\, \big(\mathcal{S}(t)\mathbf{w}\big)(y) = \mathbf{w}(y) \quad \textit{for every } t > 0 \right\}.$$

Proof. Let us denote by $\mathcal{A}'$ the set appearing at the right–hand side of the above equality. Fix a point $y \in \mathcal{A}$ and let $\mathbf{w}$ be any critical subsolution. For every fixed index $i \in \{1,\ldots,m\}$, the function $\mathbf{u}^i = \Phi_{\cdot,i}(y,\cdot) + w_i(y)\mathbb{1}$ satisfies $\mathbf{w} \leqslant \mathbf{u}^i$ in $\mathbb{T}^N$ and $w_i(y) = u_i(y)$. Moreover, $\mathbf{u}^i$ is a critical solution, hence it is a fixed point for the semigroup $\mathcal{S}(t)$ by Proposition 2.7. By monotonicity of the semigroup, we have

$$w_i(y) \leqslant \big(\mathcal{S}(t)\mathbf{w}\big)_i(y) \leqslant \big(\mathcal{S}(t)\mathbf{u}^i\big)_i(y) = u^i_i(y) \quad \text{for every } t > 0,$$

hence all the inequalities must be equalities, in particular $\big(\mathcal{S}(t)\mathbf{w}\big)_i(y) = w_i(y)$ for every $t > 0$. This being true for every $i \in \{1,\ldots,m\}$ and $\mathbf{w} \in \mathcal{H}(0)$, we conclude that $y \in \mathcal{A}'$.

To prove the converse inclusion, pick $y \in \mathcal{A}'$ and assume by contradiction that $y \notin \mathcal{A}$. Fix $i \in \{1,\ldots,m\}$ and take a critical subsolution $\mathbf{v}$ such that v_i is of class C^1 and strict in a neighborhood of y, according to Proposition 3.9. By Proposition 2.8, the map $(t,x) \mapsto v_i(x)$ is a subtangent to $\big(\mathcal{S}(t)v\big)_i(x)$ at (t_0,y) for every $t_0 > 0$ and since the latter is a solution of the evolutionary system (1.3) we get $H_i(y, Dv_i(y)) + \big(B(y)\mathcal{S}(t_0)\mathbf{v}(y)\big)_i \geqslant 0$.

By sending $t_0 \to 0^+$ we get a contradiction with the fact that v_i is strict at y. □

4. Regularization

In this section we obtain critical subsolutions which are smooth and strict outside the Aubry set. Note that in the scalar case, under some regularity assumptions on the Hamiltonian, there exist critical subsolutions that are of class C^1, or even of class $C^{1,1}$, on the whole torus and strict outside the Aubry set, see [4, 19, 20]. For systems, we are not able to do it because we do not know how to prove the differentiability of critical subsolutions on the Aubry set. However, we end the section with some more precise behavior of their Clarke derivative on the Aubry set, when all Hamiltonians $H_i(x,p)$ are strictly convex in p.

The existence of critical subsolutions, smooth and strict outside the Aubry set, is obtained by regularization like in [19, 20]. We provide a proof mostly for the reader's convenience.

We start with a local regularization argument.

Lemma 4.1. *Let $\mathbf{u} \in \mathcal{H}(0)$ and assume that, for some $r > 0$, $\delta > 0$ and $y \in \mathbb{T}^N \setminus \mathcal{A}$ and for every $i \in \{1,\ldots,m\}$,*

$$H_i\big(x, Du_i(x)\big) + \big(B(x)\mathbf{u}(x)\big)_i < -\delta \qquad \textit{for a.e. } x \in B_{2r}(y).$$

Then, for every $\varepsilon > 0$, there exists $\mathbf{u}^\varepsilon \in \mathcal{H}(0)$ such that

(i) $\|\mathbf{u}^\varepsilon - \mathbf{u}\|_\infty < \varepsilon$;

(ii) $\mathbf{u}^\varepsilon = \mathbf{u}$ *in* $\mathbb{T}^N \setminus B_r(y)$;

(iii) $\mathbf{u}^\varepsilon$ *is of class* C^∞ *in* $B_{r/2}(y)$ *and satisfies*

$$H_i\big(x, Du_i^\varepsilon(x)\big) + \big(B(x)\mathbf{u}^\varepsilon(x)\big)_i < -\frac{2}{3}\,\delta \qquad \textit{for every } x \in B_{r/2}(y). \tag{4.1}$$

Proof. Let $\phi : \mathbb{T}^N \to [0,1]$ be a C^∞ function, compactly supported in $B_r(y)$ and such that $\phi \equiv 1$ in $B_{r/2}(y)$. Let $(\rho_n)_n$ be a sequence of standard mollifiers on $\mathbb{R}^N$. For every $n \in \mathbb{N}$, we define a function $\mathbf{w}^n \in \big(\mathrm{Lip}(\mathbb{T}^N)\big)^m$ by setting

$$w_i^n(x) = \phi(x)(\rho_n * u_i)(x) + \big(1-\phi(x)\big)u_i(x) \qquad \text{for every } x \in \mathbb{T}^N \text{ and } i \in \{1,\dots,m\}.$$

It is apparent by the definition that $\mathbf{w}^n$ is of class C^∞ in $B_{r/2}(y)$ and agrees with $\mathbf{u}$ outside $B_r(y)$. Arguing as in the proof of Lemma 3.7, we see that it is possible to choose n large enough in such a way that $\mathbf{w}^n$ is a critical subsolution and satisfies (4.1). Since $\mathbf{w}^n \rightrightarrows \mathbf{u}$ in $\mathbb{T}^N$, the assertion follows by setting $\mathbf{u}^\varepsilon := \mathbf{w}^n$ for a sufficiently large n. $\square$

We now prove the announced regularization result.

Theorem 4.2. *There exists a critical subsolution which is strict and* C^∞ *in* $\mathbb{T}^n \setminus \mathcal{A}$. *More precisely, for every critical subsolution* $\mathbf{v}$ *which is strict in* $\mathbb{T}^N \setminus \mathcal{A}$ *and for every* $\varepsilon > 0$, *there exists* $\mathbf{v}^\varepsilon \in \mathcal{H}(0)$ *such that*

(i) $\|\mathbf{v}^\varepsilon - \mathbf{v}\|_\infty < \varepsilon$;

(ii) $\mathbf{v}^\varepsilon = \mathbf{v}$ *on* $\mathcal{A}$;

(iii) $\mathbf{v}^\varepsilon$ *is* C^∞ *and strict in* $\mathbb{T}^n \setminus \mathcal{A}$.

Moreover, the set of such smooth and strict subsolutions is dense in $\mathcal{H}(0)$.

Proof. We first show how to regularize a subsolution which is strict outside the Aubry set. Let $\mathbf{v}$ be such a subsolution (given by Theorem 3.12) and fix $\varepsilon > 0$. Since $\mathbf{v}$ is strict in $\mathbb{T}^N \setminus \mathcal{A}$, there exists a continuous and non–negative function $\delta : \mathbb{T}^N \to \mathbb{R}$ with $\delta^{-1}(\{0\}) = \mathcal{A}$ such that $H_i(x, Dv_i) + \big(B(x)\mathbf{v}(x)\big)_i \leqslant -\delta(x)$ in $\mathbb{T}^N$ for every $i \in \{1,\dots,m\}$. Clearly, it is not restrictive to assume that the inequality $\delta(x) \leqslant \min\{\varepsilon/2, d(x,\mathcal{A})^2\}$ holds for every $x \in \mathbb{T}^N$, where $d(x,\mathcal{A}) := \inf_{y\in\mathcal{A}} d(x,y)$. In view of Lemma 4.1, we can find a locally finite covering $(U_n)_n$ of $\mathbb{T}^N \setminus \mathcal{A}$ by open sets compactly contained in $\mathbb{T}^N \setminus \mathcal{A}$ and a sequence $(\mathbf{u}^n)_n$ of critical subsolutions such that each $\mathbf{u}^n$ is C^∞ in U_n and satisfies

$$\begin{aligned} H_i(x, Du_i^n) + \big(B(x)\mathbf{u}^n(x)\big)_i \leqslant -\frac{2}{3}\,\delta(x) \qquad &\text{for every } x \in U_n,\\ |\mathbf{u}^n(x) - \mathbf{v}(x)| \leqslant \delta(x) \qquad &\text{for every } x \in \mathbb{T}^N. \end{aligned} \tag{4.2}$$

Set $\delta_n := \inf_{x\in U_n} \delta(x)$ for every $n \in \mathbb{N}$ and choose a sequence $(\eta_n)_n$ in $(0,1)$ such that, for every $x \in \mathbb{T}^N$ and $n \in \mathbb{N}$, the following holds:

$$|H(x,p) - H(x,p')| < \frac{\delta_n}{6} \qquad \text{for all } p, p' \in B_{\kappa+1} \text{ with } |p-p'| < \eta_n, \tag{4.3}$$

where κ denotes a common Lipschitz constant for the critical subsolutions, in particular for all the $\mathbf{u}^n$. Last, take a smooth partition of unity $(\varphi_n)_n$ subordinate to $(U_n)_n$ and choose the functions $\mathbf{u}^n$ in such a way that the quantities $\|\mathbf{u}^n - \mathbf{v}\|_\infty$, which can be be made as small as desired, satisfy

$$\sum_{\substack{k\in\mathbb{N}\\ U_k\cap U_n\neq\varnothing}} \|\mathbf{u}^k - \mathbf{v}\|_\infty \, \|D\varphi_k\|_\infty < \eta_n \qquad \text{for every } n \in \mathbb{N}. \tag{4.4}$$

That is always possible since the covering $(U_n)_n$ is locally finite.

We now define $\mathbf{v}^\varepsilon : \mathbb{T}^N \to \mathbb{R}^m$ by setting $\mathbf{v}^\varepsilon(x) = \sum_{n=1}^{\infty} \varphi_n(x)\mathbf{u}^n(x)$ in $\mathbb{T}^N \setminus \mathcal{A}$ and $\mathbf{v}^\varepsilon(x) = \mathbf{v}(x)$ on $\mathcal{A}$. By definition, $\mathbf{v}^\varepsilon$ satisfies assertion *(ii)* and is C^∞ in $\mathbb{T}^N \setminus \mathcal{A}$. From (4.2) we infer that $|\mathbf{v}^\varepsilon(x) - \mathbf{v}(x)| \leqslant \delta(x)$ in $\mathbb{T}^N \setminus \mathcal{A}$, which shows at once that $\mathbf{v}^\varepsilon$ is continuous in $\mathbb{T}^N$ and that it satisfies assertion *(i)*. Moreover, by taking into account (4.4) and the fact that $\sum D\varphi_k \equiv 0$, one obtains, for every $x \in U_n$ and $i \in \{1, \ldots, m\}$, that

$$\Big| Dv_i^\varepsilon(x) - \sum_{\substack{k\in\mathbb{N}\\ U_k\cap U_n\neq\varnothing}} \varphi_k(x) Du_i^k(x) \Big| = \Big| \sum_{\substack{k\in\mathbb{N}\\ U_k\cap U_n\neq\varnothing}} \big(u_i^k(x) - v(x)\big) D\varphi_k(x) \Big| < \eta_n, \quad (4.5)$$

in particular

$$|Dv_i^\varepsilon(x)| \leqslant \eta_n + \sum_{\substack{k\in\mathbb{N}\\ U_k\cap U_n\neq\varnothing}} \varphi_k(x) |Du_i^k(x)| \leqslant 1 + \kappa.$$

We infer that $\mathbf{v}^\varepsilon$ is Lipschitz–continuous in $\mathbb{T}^N$. In order to prove that $\mathbf{v}^\varepsilon$ is a critical subsolution and is strict in $\mathbb{T}^N \setminus \mathcal{A}$, it will be enough to show that

$$H_i(x, Dv_i^\varepsilon(x)) + \big(B(x)\mathbf{v}^\varepsilon(x)\big)_i \leqslant -\frac{\delta(x)}{2} \qquad \text{for a.e. } x \in \mathbb{T}^N,$$

for all $i \in \{1, \ldots, m\}$.

Recall the Lipschitz functions $\mathbf{v}^\varepsilon$ and $\mathbf{v}$ coincide on the Aubry set. Setting $\mathbf{w}^\varepsilon = \mathbf{v}^\varepsilon - \mathbf{v}$, we infer that if $x_0 \in \mathcal{A}$, then $|\mathbf{w}^\varepsilon(x) - \mathbf{w}^\varepsilon(x_0)| \leqslant d(x, \mathcal{A})^2 \leqslant d(x, x_0)^2$. Hence $\mathbf{w}^\varepsilon$ is differentiable on $\mathcal{A}$ with vanishing differential and $D\mathbf{v}^\varepsilon(x) = D\mathbf{v}(x)$ for almost every $x \in \mathcal{A}$. Hence, it suffices to establish the claim in the complementary of $\mathcal{A}$. To this aim, by recalling the definition of η_n and by making use of (4.5) and of Jensen inequality, we get that, for every $x \in U_n$ and $i \in \{1, \ldots, m\}$,

$$\begin{aligned}
&H_i\big(x, Dv_i^\varepsilon(x)\big) + \big(B(x)\mathbf{v}^\varepsilon(x)\big)_i \leqslant \\
&\qquad \leqslant H_i\Big(x, \sum_{\substack{k\in\mathbb{N}\\ U_k\cap U_n\neq\varnothing}} \varphi_k(x) Du_i^k(x)\Big) + \frac{\delta_n}{6} + \sum_{\substack{k\in\mathbb{N}\\ U_k\cap U_n\neq\varnothing}} \varphi_k(x) \big(B(x)\mathbf{u}^k(x)\big)_i \\
&\qquad \leqslant \sum_{\substack{k\in\mathbb{N}\\ U_k\cap U_n\neq\varnothing}} \varphi_k(x) \Big(H_i\big(x, Du_i^k(x)\big) + \big(B(x)\mathbf{u}^k(x)\big)_i\Big) + \frac{\delta_n}{6} \\
&\qquad < -\frac{2}{3}\,\delta(x) + \frac{\delta_n}{6} \leqslant -\frac{\delta(x)}{2}.
\end{aligned}$$

This concludes the proof of the first part of the statement.

For the density, let $\mathbf{u}$ be any critical subsolution. Let $\mathbf{v}$ be a critical subsolution which is strict outside the Aubry set (whose existence is assured by Theorem 3.12). Then, for any $\lambda \in (0, 1)$, the function $(1 - \lambda)\mathbf{u} + \lambda\mathbf{v}$ is a subsolution which is strict outside the Aubry set. This subsolution can be therefore regularized using the above procedure, giving a subsolution $\mathbf{w}$ which is strict and smooth outside the Aubry set. Moreover, both these steps can be done in such a way that $\|\mathbf{u} - \mathbf{w}\|_\infty$ is as small as wanted. This establishes the density. □

We now additionally assume the Hamiltonians H_i to be strictly convex in p and derive some further information on the behavior of Clarke's generalized gradients of the critical subsolutions on the Aubry set.

We start with a preliminary lemma.

Lemma 4.3. *Let $y \in \mathcal{A}$ and let $\mathbf{u}^1, \cdots, \mathbf{u}^\ell$ be critical subsolutions. Then, for all $i \in \{1, \ldots, m\}$, the set $\bigcap_{k=1}^{\ell} \partial^c u_i^k(x)$ is nonempty. Moreover, it contains a vector p_i which is extremal for all the sets $\partial^c u_i^k(x)$ and which satisfies*

$$H_i(y, p_i) + \big(B(y)\mathbf{u}^k(y)\big)_i = 0 \qquad \textit{for every } k \in \{1, \ldots, \ell\}.$$

Proof. Let $\mathbf{w} = \frac{1}{\ell}\sum_{k=1}^{\ell} \mathbf{u}^k \in \mathcal{H}(0)$ and let $p_i \in \partial^c w_i(y)$ be such that $H_i(y,p) + \big(B(y)\mathbf{w}(y)\big)_i = 0$. Such a p_i must exist because otherwise w_i would be strict at y. Note that, by strict convexity of H_i, the vector p_i must be an extremal point of $\partial^c w_i(x)$, hence it is a reachable gradient of w_i. Let $y_n \to y$ be such that u_i^k is differentiable at y_n for every $k \in \{1, \ldots, \ell\}$ and $n \in \mathbb{N}$, and

$$Dw_i(y_n) = \frac{1}{\ell}\sum_{k=1}^{\ell} Du_i^k(y_n) \to p_i.$$

Up to extraction of a subsequence, we can assume that $Du_i^k(y_n) \to q_k$ for all $k \in \{1, \ldots, \ell\}$. Then one readily obtains, by Jensen's inequality, that

$$0 = H_i(y, p_i) + \big(B(y)\mathbf{w}(y)\big)_i \leqslant \frac{1}{\ell}\sum_{k=1}^{\ell}\Big(H_i(y, q_k) + \big(B(y)\mathbf{u}^k(y)\big)_i\Big) \leqslant 0.$$

Therefore, all the inequalities $H_i(y, q_k) + (B(y)\mathbf{u}^k(y))_i \leqslant 0$ summing to an equality, we deduce, by strict convexity of H_i, that $q_1 = \cdots = q_l = p_i$. Moreover, since

$$H_i(y, q_k) + \big(B(y)\mathbf{u}^k(y)\big)_i = 0 \qquad \text{for every } k \in \{1, \ldots, \ell\},$$

and because of the strict convexity of H_i, one sees that p_i is extremal, and thus reachable, for all the u_i^k. ☐

We now extend the previous result as follows:

Proposition 4.4. *Let $y \in \mathcal{A}$. Then, for each $i \in \{1, \ldots, m\}$, there exists a vector $p_i \in \mathbb{R}^N$ which is a reachable gradient of u_i at y for every $\mathbf{u} \in \mathcal{H}(0)$ and which satisfies $H_i(y, p_i) + \big(B(y)\mathbf{u}(y)\big)_i = 0$.*

Proof. For each critical subsolution $\mathbf{u}$, let us denote by $P_i^{\mathbf{u}}$ the set of reachable gradients p of u_i at y that satisfy $H_i(y,p) + \big(B(y)\mathbf{u}(y)\big)_i = 0$. This set is not empty and compact. The proposition amounts to proving that $\bigcap_{\mathbf{u} \in \mathcal{H}(0)} P_i^{\mathbf{u}} \neq \varnothing$. If this were not the case, by compactness we could extract a finite empty intersection. But this would violate the previous lemma. ☐

5. Rigidity of the Aubry set and Comparison Principle

We start with the following consequence of Theorem 2.3.

Proposition 5.1. *Let $y \in \mathcal{A}$ and $i \in \{1, \ldots, m\}$. Then $\mathbf{v}(y) = \Phi_{\cdot,i}(y,y) + v_i(y)\mathbb{1}$ for every $\mathbf{v} \in \mathcal{H}(0)$.*

In particular, $\quad \mathbf{v}(y) - \mathbf{w}(y) \in \mathbb{R}\mathbb{1} \quad$ *for any* $\mathbf{v}$, $\mathbf{w} \in \mathcal{H}(0)$.

Proof. Take $\mathbf{v} \in \mathcal{H}(0)$ and set $\mathbf{u} := \Phi_{\cdot,i}(y, \cdot) + v_i(y)\mathbb{1}$. According to Proposition 3.2, $\mathbf{u}$ is a critical solution satisfying $\mathbf{v} \leqslant \mathbf{u}$ in $\mathbb{T}^N$ and $v_i(y) = u_i(y)$. By applying Theorem 2.3 with $x_0 := y$ we get the assertion. ☐

Remark 5.2. On the other hand, the above property does not hold when $y \notin \mathcal{A}$. Indeed, the proof of Lemma 3.7 shows that any critical subsolution $\mathbf{v}$ which is strict at y can be modified in such a way that the output is a critical subsolution all of whose components except one coincide at y with those of $\mathbf{v}$.

We derive two corollaries:

Corollary 5.3. *Let $y \in \mathcal{A}$. Then the matrix $\Phi(y,y)$ is antisymetric.*

Proof. Apply the previous theorem to the weak KAM solution $\Phi_{\cdot,j}(y,\cdot)$ and get $\Phi_{\cdot,j}(y,y) = \Phi_{\cdot,i}(y,y) + \Phi_{i,j}(y,y)\mathbb{1}$. In particular, $0 = \Phi_{j,j}(y,y) = \Phi_{j,i}(y,y) + \Phi_{i,j}(y,y)$. □

Corollary 5.4. *Let $y \in \mathcal{A}$. Then the critical solutions $\Phi_{\cdot,j}(y,\cdot)$ differ by a constant function. More precisely: $\Phi_{\cdot,i}(y,\cdot) = \Phi_{\cdot,j}(y,\cdot) + \Phi_{j,i}(y,y)\mathbb{1}$.*

Proof. Let us apply the last point of Proposition 3.2 twice:

$$\Phi_{k,i}(y,z) \leqslant \Phi_{j,i}(y,y) + \Phi_{k,j}(y,z) \quad \text{and} \quad \Phi_{k,j}(y,z) \leqslant \Phi_{i,j}(y,y) + \Phi_{k,i}(y,z).$$

In particular, we obtain

$$\Phi_{k,i}(y,z) \leqslant \Phi_{j,i}(y,y) + \Phi_{k,j}(y,z) \leqslant \Phi_{j,i}(y,y) + \Phi_{i,j}(y,y) + \Phi_{k,i}(y,z) = \Phi_{k,i}(y,z),$$

thanks to the previous corollary. Therefore all inequalities are equalities and that gives the result. □

Next, we derive a comparison principle for sub and supersolutions of the critical weakly coupled system (3.1) which generalizes to our setting an analogous result established in [6] for Hamiltonians of a special Eikonal form, see Subsection 6.1 for more details. In particular, we obtain that $\mathcal{A}$ is a uniqueness set for the critical system.

Theorem 5.5. *Let $\mathbf{v}$, $\mathbf{u} \in \big(\mathrm{C}(\mathbb{T}^N)\big)^m$ be a sub and a supersolution of the critical weakly coupled system* (3.1)*, respectively. Assume that*

$$\text{for every } x \in \mathcal{A} \quad \text{there exists } i \in \{1,\dots,m\} \text{ such that} \quad v_i(x) \leqslant u_i(x). \tag{5.1}$$

Then $\mathbf{v}(x) \leqslant \mathbf{u}(x)$ for every $x \in \mathbb{T}^N$. In particular, two critical solutions that coincide on $\mathcal{A}$ coincide on the whole $\mathbb{T}^N$.

Remark 5.6. The above theorem also implies that two critical solutions $\mathbf{u}$ and $\mathbf{v}$ are actually the same if (5.1) holds with an equality. This is consistent with Proposition 5.1, which assures that this "boundary" condition amounts to requiring that $\mathbf{u} = \mathbf{v}$ on $\mathcal{A}$.

Proof. In view of the density result stated in Theorem 4.2, the critical subsolution $\mathbf{v}$ can be approximated from below by a sequence of critical subsolutions that are, in addition, smooth and strict outside $\mathcal{A}$. Indeed, just pick a sequence $(\mathbf{w}^n)_{n\in\mathbb{N}}$ such that $\|\mathbf{w}^n - \mathbf{v}\|_\infty < n^{-1}$ and then define $\mathbf{v}^n = \mathbf{w}_n - n^{-1}\mathbb{1}$ which then verifies $\mathbf{v}^n \leqslant \mathbf{v}$ and $\|\mathbf{v}^n - \mathbf{v}\|_\infty < 2n^{-1}$. Clearly, each element of the sequence still satisfies the boundary condition (5.1), hence it is enough to prove the statement by additionally assuming $\mathbf{v}$ smooth and strict in $\mathbb{T}^N \setminus \mathcal{A}$.

Let us set $M := \max_{1\leqslant i\leqslant m} \max_{\mathbb{T}^N} (v_i - u_i)$, and pick a point $x_0 \in \mathbb{T}^N$ where such a maximum is attained. By Theorem 2.3 we know that $\mathbf{v}(x_0) = \mathbf{u}(x_0) + M\mathbb{1}$. If

$x_0 \notin \mathcal{A}$, then v_1 would be a smooth subtangent to u_1 at x_0. The function $\mathbf{u}$ being a supersolution, we would have

$$0 \leqslant H_1\big(x_0, Dv_1(x_0)\big) + \big(B(x_0)\mathbf{u}(x)\big)_1 = H_1\big(x_0, Dv_1(x_0)\big) + \big(B(x_0)\mathbf{v}(x)\big)_1,$$

in contrast with the fact that $\mathbf{v}$ is strict in $\mathbb{T}^N \setminus \mathcal{A}$. Hence $x_0 \in \mathcal{A}$ and by the hypothesis (5.1) we get $M \leqslant 0$, as it was to be proved. □

Last, we show that the trace of any critical subsolution on the Aubry set can be extended to the whole torus in such a way that the output is a critical solution.

Theorem 5.7. *For any* $\mathbf{v} \in \mathcal{H}(0)$*, there exists a unique critical solution* $\mathbf{u}$ *such that* $\mathbf{u} = \mathbf{v}$ *on* $\mathcal{A}$*.*

Proof. The assertion is derived by setting $u_i(x) = \sup_{t>0} \big(\mathcal{S}(t)\mathbf{v}\big)_i(x)$ for every $x \in \mathbb{T}^N$ and $i \in \{1, \dots, m\}$. Indeed, the functions $\{\mathcal{S}(t)\mathbf{v} \,:\, t > 0\}$ are equi–Lipschitz and non–decreasing with respect to t and satisfy $\mathcal{S}(t)\mathbf{v} = \mathbf{v}$ on $\mathcal{A}$ for every $t > 0$ by Proposition 3.14. We infer that $\mathbf{u}$ is a vector valued, Lipschitz continuous function and $\mathcal{S}(t)\mathbf{v} \rightrightarrows \mathbf{u}$ in $\mathbb{T}^N$ as $t \to +\infty$. Last, $\mathbf{u}$ is a critical solution for it is a fixed point of the semigroup $\mathcal{S}(t)$. □

6. Examples

The critical value and the Aubry set for a weakly coupled system of the kind studied in this paper have, in general, no connections with those of each Hamiltonian, considered individually. This happens also in simple situations, see Remark 6.1 below. In this section, we present some examples where more explicit results may be obtained for the critical value and for the Aubry set. In what follows, the coupling matrix $B(x)$ will be always assumed irreducible and degenerate.

6.1. **The setting of** [6]**.** The first example we propose corresponds to the setting considered in [6]. Assume that all the Hamiltonians are of the form $H_i(x, p) = F_i(x, p) - V_i(x)$, where:

(a) F_i and V_i take non–negative values;
(b) F_i is convex and coercive in p;
(c) $F_i(x, 0) = 0$ for all $x \in \mathbb{T}^N$ and $i \in \{1, \dots, m\}$;
(d) $\bigcap_{i=1}^m V_i^{-1}(\{0\}) \neq \varnothing$.

Under these hypotheses, we claim that the critical value is 0 and that the Aubry set is nothing but

$$\mathcal{A} = \bigcap_{i=1}^{m} V_i^{-1}(\{0\}).$$

Indeed, it is easily seen that the null function $\mathbf{u}^0$ always belongs to $\mathcal{H}(0)$. Therefore, $\mathcal{H}(0) \neq \varnothing$ and the critical value verifies $c \leqslant 0$. To see that there is actually equality, consider a point $x_0 \in \cap V_i^{-1}(\{0\})$ and any (C^1) function $\mathbf{u}$. By Proposition 1.3 we know that $B(x_0)\mathbf{u}(x_0)$ must have a non–negative entry, say i, hence

$$H_i\big(x_0, Du_i(x_0)\big) + \big(B(x_0)\mathbf{u}(x_0)\big)_i = F_i\big(x_0, Du_i(x_0)\big) + \big(B(x_0)\mathbf{u}(x_0)\big)_i \geqslant 0.$$

Therefore, $\mathbf{u}$ cannot belong to a $\mathcal{H}(-\varepsilon)$ for a positive ε. The same argument can be adapted in the viscosity sense for any (non necessarily C^1) function. Therefore 0 is the critical value.

To prove that $\cap V_i^{-1}(\{0\})$ is the Aubry set, first notice that, for every $y \notin \cap V_i^{-1}(\{0\})$, there exists an index j such that $V_j(y) > 0$. Then the j–th component of the null function $\mathbf{u}^0$ is strict at y. In view of Theorem 3.13 we get the inclusion $\mathcal{A} \subseteq \bigcap_{i=1}^m V_i^{-1}(\{0\})$.

The opposite inclusion is obtained as previously. Take any $\mathbf{u} \in \mathcal{H}(0)$ and $x_0 \in \cap V_i^{-1}(\{0\})$. We will do as if $\mathbf{u}$ is differentiable at x_0, but the argument carries on in the general case using test functions and the viscosity subsolution property. At x_0 we must have

$$F_i\big(x_0, Du_i(x_0)\big) + \big(B(x_0)\mathbf{u}(x_0)\big)_i \leqslant 0 \qquad \text{for every } i \in \{1, \dots, m\}.$$

But this is only possible if $F_i(x_0, Du_i(x_0)) = 0$ for all $i \in \{1, \dots, m\}$ and $B(x_0)\mathbf{u}(x_0) = 0$. Indeed, otherwise, $B(x_0)\mathbf{u}(x_0)$ will have a positive entry in view of Proposition 1.3, which is impossible. In particular, the above inequality holds with an equality. Since this happens for any critical subsolution $\mathbf{u}$, we get $x_0 \in \mathcal{A}$ in view of Theorem 3.13. As a byproduct, this also establishes that at any point of $\mathcal{A}$, any critical subsolution must take as value a vector belonging to $\mathbb{R}\,\mathbb{1}$. This is a particular case of Proposition 5.1 and accounts for the type of symmetries already remarked in [6] for the critical solutions obtained via the asymptotic procedure therein considered.

Remark 6.1. It would be interesting to understand what the Aubry set is for the weakly coupled system considered in the previous example when condition (d) is dropped. Unfortunately, we are not able to give an answer to this question. Note that, in this case, $c < 0$. Indeed, if c were greater or equal than 0, then the null function would satisfy condition *(iii)* in Theorem 3.13 at any point $y \in \mathbb{T}^N$, contradicting the fact that the Aubry set is nonempty. We point out that similar examples appear in [28, Remark 3.5] and [32, Example 1.2].

6.2. **The setting of** [6] **revisited.** This second example is taken from [32]: the Hamiltonians are still of the form $H_i(x, p) = F_i(x, p) - V_i(x)$ with F_i as above, but the quantities $\lambda_i := \min_{\mathbb{T}^N} V_i$ are not required to be zero. The analogous condition $\bigcap_{i=1}^m V_i^{-1}(\{\lambda_i\}) \neq \varnothing$ is in force. Moreover, the coupling matrix is taken independent of x. We claim that

$$c = -\pi(\boldsymbol{\lambda}) \qquad \text{and} \qquad \mathcal{A} = \bigcap_{i=1}^m V_i^{-1}(\{\lambda_i\}),$$

where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$ and $\pi(\boldsymbol{\lambda})$ denotes the unique real number such that $\boldsymbol{\lambda} - \pi(\boldsymbol{\lambda})\mathbb{1} \in \mathrm{Im}(B)$. Indeed, by replacing each H_i with $H_i + \lambda_i$ we reduce to the case of Example 6.1 and we conclude by using the following result:

Proposition 6.2. *Let H_i be convex and coercive Hamiltonians for every $i \in \{1, \dots, m\}$ and assume that the coupling matrix is independent of x. For every $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$, denote by $c_{\boldsymbol{\lambda}}$ and $\mathcal{A}_{\boldsymbol{\lambda}}$ the critical value and Aubry set of the weakly coupled system with $H_i + \lambda_i$ in place of H_i for every $i \in \{1, \dots, m\}$. Then*

$$c_{\boldsymbol{\lambda}} = c_{\mathbf{0}} + \pi(\boldsymbol{\lambda}) \qquad \text{and} \qquad \mathcal{A}_{\boldsymbol{\lambda}} = \mathcal{A}_{\mathbf{0}} \qquad \text{for every } \boldsymbol{\lambda} \in \mathbb{R}^m,$$

where $\pi(\boldsymbol{\lambda})$ denotes the unique real number such that $\boldsymbol{\lambda} - \pi(\boldsymbol{\lambda})\mathbb{1} \in \mathrm{Im}(B)$.

Proof. Fix $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$. Then $\boldsymbol{\lambda} = \pi(\boldsymbol{\lambda})\mathbb{1} + B\boldsymbol{\mu}$ for some $\boldsymbol{\mu} \in \mathbb{R}^m$ and for a unique scalar $\pi(\boldsymbol{\lambda})$, for $\mathbb{R}^m \cong \mathrm{Ker}(B) \oplus \mathrm{Im}(B)$ in view of the results of Section 1.2.

If $\mathbf{u}$ is a solution of the critical weakly coupled system associated with $H_1, \dots, H_m$, then $\mathbf{w} := \mathbf{u} - \boldsymbol{\mu}$ is a solution of

$$H_i(x, Dw_i) + \lambda_i + \big(B\mathbf{w}(x)\big)_i = c_{\mathbf{0}} + \pi(\boldsymbol{\lambda}) \quad \text{in } \mathbb{T}^N \qquad \text{for every } i \in \{1, \dots, m\}, \tag{6.1}$$

thus showing that $c_{\boldsymbol{\lambda}} = c_{\mathbf{0}} + \pi(\boldsymbol{\lambda})$. If we now take as $\mathbf{u}$ a subsolution of the critical weakly coupled system associated with $H_1, \dots, H_m$ which is strict outside $\mathcal{A}_{\mathbf{0}}$, we easily see that $\mathbf{w} := \mathbf{u} - \boldsymbol{\mu}$ is a subsolution of (6.1) which is strict outside $\mathcal{A}_{\mathbf{0}}$, thus showing $\mathcal{A}_{\boldsymbol{\lambda}} \subseteq \mathcal{A}_{\mathbf{0}}$. The reverse inclusion can be proved analogously. This concludes the proof. □

6.3. **Commuting Hamiltonians.** In this last example we consider the case when the Hamiltonians are strictly convex and pairwise commute. If the Hamiltonians are of class C^1, that means

$$\{H_i, H_j\}(x,p) := \Big(\frac{\partial H_i}{\partial p}\frac{\partial H_j}{\partial x} - \frac{\partial H_j}{\partial p}\frac{\partial H_i}{\partial x}\Big)(x,p) = 0 \quad \text{in } \mathbb{T}^N \times \mathbb{R}^N$$

for every $i, j \in \{1, \dots, m\}$. If the Hamiltonians are only continuous, the commutation hypothesis must be expressed in terms of commutation of their Lax–Oleinik semigroup, see [12] for more details. We also make the additional assumption that, individually, all the Hamiltonians have 0 as critical value. Then, we claim that 0 is the critical value of the system as well (whatever the coupling is).

Indeed, it is proved in [12, 37] that the Hamiltonians have the same critical solutions. In particular, there exists a function $u \in \operatorname{Lip}(\mathbb{T}^N)$ satisfying

$$H_i(x, Du) = 0 \quad \text{in } \mathbb{T}^N \qquad \text{for every } i \in \{1, \dots, m\}$$

in the viscosity sense. Since the coupling is degenerate, we infer that the function $\mathbf{u}^0 = u\mathbb{1}$ is a solution of

$$H_i(x, Du_i^0) + \big(B(x)\mathbf{u}^0(x)\big)_i = 0 \quad \text{in } \mathbb{T}^N \qquad \text{for every } i \in \{1, \dots, m\}.$$

Therefore, the claim is a direct consequence of Proposition 2.12. Moreover, in this setting, we may localize the Aubry set of the system using those of the individual Hamiltonians. In order to do so, let us recall another result from [12].

Theorem 6.3. *Let $H_1, \cdots, H_m$ be pairwise commuting and strictly convex Hamiltonians, with common critical value equal to* 0*. Then they have the same Aubry set $\mathcal{A}^*$. Moreover, there exists a common critical subsolution v which is smooth outside $\mathcal{A}^*$ and strict for each Hamiltonian, i.e.*

$$H_i\big(x, Dv(x)\big) < 0 \quad \textit{for every } x \in \mathbb{T}^N \setminus \mathcal{A}^* \textit{ and } i \in \{1, \dots, m\}.$$

Using this theorem, we easily see that the inclusion $\mathcal{A} \subseteq \mathcal{A}^*$ holds. Indeed, the function $\mathbf{v}(x) := v(x)\mathbb{1}$ is a critical subsolution for the system which is strict outside $\mathcal{A}^*$.

We also note that, as in the previous example, $\mathbf{u}(y) \in \mathbb{R}\,\mathbb{1}$ for every $y \in \mathcal{A}$ and every $\mathbf{u} \in \mathcal{H}(0)$ in view of Proposition 5.1.

A particular case of this example is when all the H_i are equal. In this case we get the more precise statement:

Proposition 6.4. *Let H be a convex Hamiltonian and assume $H_1 = \cdots = H_m = H$. Then $\mathcal{A} = \mathcal{A}^*$. Moreover, all critical solutions of the system are of the form $\mathbf{u} = u\mathbb{1}$ where u is a critical solution of H.*

Proof. The inclusion $\mathcal{A} \subseteq \mathcal{A}^*$ can be proved arguing as above (note that we do not need the strict convexity assumption here). Let us prove the converse statement. Pick $\mathbf{v} \in \mathcal{H}(0)$ and set $v(x) := \max_i v_i(x) \quad$ for every $x \in \mathbb{T}^N$. We claim that v is a critical subsolution for H. Indeed, let $x \in \mathbb{T}^N$ and $p \in D^+v(x)$. Then $v(x) = v_i(x)$ for some $i \in \{1, \dots, m\}$. Since $v \geqslant v_i$ with equality at x, we get $p \in D^+v_i(x)$. We now use the fact that $\mathbf{v}$ is a subsolution of the system to get

$$H(x,p) \leqslant H(x,p) + \big(B(x)\mathbf{v}(x)\big)_i \leqslant 0, \tag{6.2}$$

where the first inequality comes from the fact that

$$\big(B(x)\mathbf{v}(x)\big)_i = \sum_{j=1}^m b_{ij}(x)v_j(x) \geqslant \sum_{j=1}^m b_{ij}(x)v(x) = 0,$$

which holds true since $b_{ij}(x) \leqslant 0$ and $v_j(x) \leqslant v(x)$ for every $j \neq i$. Let us now assume that $\mathbf{v}$ is strict outside $\mathcal{A}$. Then the right inequality in (6.2) is strict as soon as $x \notin \mathcal{A}$, yielding that v is a subsolution for H which is strict in the complementary of $\mathcal{A}$. This proves that $\mathcal{A}^* \subseteq \mathcal{A}$, hence $\mathcal{A} = \mathcal{A}^*$.

Let now $\mathbf{u}$ be a critical solution for the system. Then $v(x) := \max_i v_i(x)$ is a critical subsolution for H. Moreover, as $\mathbf{u}(x) = u_1(x)\mathbb{1}$ for every $x \in \mathcal{A}$, we deduce that $v = u_1$ on $\mathcal{A}$. Since $\mathcal{A} = \mathcal{A}^*$, there exists a critical solution $\tilde{u}$ for H such that $\tilde{u} = v$ on $\mathcal{A}$. Now the function $\tilde{\mathbf{u}} = \tilde{u}\mathbb{1}$ is a critical solution of the weakly coupled system satisfying $\tilde{\mathbf{u}} = \mathbf{u}$ on $\mathcal{A}$. By the comparison principle, i.e. Theorem 5.5, we conclude that $\mathbf{u} = \tilde{\mathbf{u}}$. □

Appendix A

In this appendix we give a proof of Proposition 2.8.

A function u defined in an open subset U of $\mathbb{R}^k$ will be said to be *semiconcave* if, for every $x \in U$, there exists a vector $p_x \in \mathbb{R}^k$ such that

$$u(y) - u(x) \leqslant \langle p_x, y - x\rangle + d(y,x)\,\omega\big(d(y,x)\big) \qquad \text{for every } y \in U,$$

where ω is a modulus. It can be shown this is equivalent to requiring that for every $x, y \in U$ and $\lambda \in [0,1]$,

$$\lambda u(x) + (1-\lambda)u(y) \leqslant u\big(\lambda x + (1-\lambda)y\big) + \lambda(1-\lambda)\omega\big(d(x,y)\big).$$

The vectors p_x satisfying the above inequality are precisely the elements of $D^+u(x)$, which is thus always nonempty in U. Moreover, $\partial^c u(x) = D^+u(x)$ for every $x \in U$. By the upper semicontinuity of the map $x \mapsto \partial^c u(x)$ with respect to set inclusion, we get in particular that Du is continuous in its domain of definition, see [7]. In what follows, we will be in the case where U is an open subset of either $\mathbb{T}^N$ or $\mathbb{R}_+ \times \mathbb{T}^N$.

We start with the following

Proposition A.1. *Let $T > 0$ and $G : [0,T] \times \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{R}$ be a locally Lipschitz Hamiltonian such that $G(s,\cdot,\cdot)$ is a strictly convex Hamiltonian, for every fixed $s \in [0,T]$. Let $u(t,x)$ be a Lipschitz function in $[0,T] \times \mathbb{T}^N$ that solves the evolutive Hamilton-Jacobi equation*

$$\frac{\partial u}{\partial t} + G(t,x,D_x u) = 0 \qquad \text{in } (0,T) \times \mathbb{T}^N, \tag{A.1}$$

in the viscosity sense. Then

(i) for every $0 < \tau < T$, the function u is semiconcave in $[\tau, T) \times \mathbb{T}^N$;

(ii) if $u(0,\cdot)$ is semi–concave in $\mathbb{T}^N$, then the functions $\{\, u(t,\cdot) \,:\, t\in[0,T)\,\}$ are equi–semiconcave.

Proof. Since u is Lipschitz, up to modifying G outside $[0,T]\times\mathbb{T}^N\times B_R$ for a sufficiently large $R>0$, we can assume that G is superlinear in p, uniformly with respect to (t,x). We are then in the setting considered by Cannarsa and Soner in [8] and item *(i)* follows from their results.

Let us prove *(ii)*. Let us denote by $L(t,x,q)$ the the Lagrangian associated with G through the Fenchel transform and by u_0 the initial datum $u(0,\cdot)$. It is well known, see for instance [7], that the following representation formula holds:

$$u(t,x)=\inf_{\xi(t)=x}\Big(u_0\big(\xi(0)\big)+\int_0^t L\big(s,\xi(s),\dot\xi(s)\big)\mathrm{d}s\Big),\qquad (t,x)\in(0,T)\times\mathbb{T}^N,\quad \text{(A.2)}$$

where the infimum is taken by letting ξ vary in the family of absolutely continuous curves from $[0,t]$ to $\mathbb{T}^N$. Moreover, the minimum is attained by some curve γ, which is, in addition, Lipschitz continuous (actually, of class C^1), see [10].

We claim that there exists a constant κ, only depending on G and on the Lipschitz constant of u in $[0,T]\times\mathbb{T}^N$, such that $\|\dot\gamma\|_\infty\leqslant\kappa$. To this aim, we apply Proposition 2.4 in [24] to the function $u(t,x)$ and the curve $s\mapsto(s,\gamma(s))$ to get

$$\frac{\mathrm{d}}{\mathrm{d}s}\,u\big(s,\gamma(s)\big)=p_t(s)+\langle p_x(s),\dot\gamma(s)\rangle\qquad\text{for a.e. } s\in[0,t],\quad \text{(A.3)}$$

where $s\mapsto\big(p_t(s),p_x(s)\big)$ is a measurable and essentially bounded function on $[0,t]$ such that

$$\big(p_t(s),p_x(s)\big)\in\partial^c u\big(s,\gamma(s)\big)\qquad\text{for a.e. } s\in[0,t].$$

By integrating (A.3) and using the Fenchel inequality we get

$$\begin{aligned}
u(t,x) &= u_0\big(\gamma(0)\big)+\int_0^t p_t(s)+\langle p_x(s),\dot\gamma(s)\rangle\,\mathrm{d}s\\
&\leqslant u_0\big(\gamma(0)\big)+\int_0^t p_t(s)+G\big(s,\gamma(s),p_x(s)\big)+L\big(s,\gamma(s),\dot\gamma(s)\big)\,\mathrm{d}s\\
&\leqslant u_0\big(\gamma(0)\big)+\int_0^t L\big(s,\gamma(s),\dot\gamma(s)\big)\,\mathrm{d}s,
\end{aligned}$$

where in the last inequality we used the fact that u is a (sub)–solution of the time dependent equation, i.e.

$$p_t+G(t,x,p_x)\leqslant 0\qquad\text{for every } (p_t,p_x)\in\partial^c u(t,x)\text{ and }(t,x)\in(0,T)\times\mathbb{T}^N.$$

Since γ is minimizing, all the inequalities must be equalities, in particular we obtain

$$\dot\gamma(s)\in\partial_p G\big(s,\gamma(s),p_x(s)\big)\qquad\text{for a.e. } s\in[0,t].\quad \text{(A.4)}$$

This proves the claim by choosing

$$\kappa:=\sup\big\{|q|\,:\,q\in\partial_p G(s,x,p),\,(s,x)\in[0,T]\times\mathbb{T}^N,\,|p|\leqslant\mathrm{Lip}\,\big(u;[0,T]\times\mathbb{T}^N\big)\ \big\},$$

which is finite since G is bounded on compact subsets of $[0,T]\times\mathbb{T}^N\times\mathbb{R}^N$ and convex in p.

Let us now fix $t\in(0,T)$, $x_1,x_2\in\mathbb{T}^N$, $\lambda\in[0,1]$ and set $x=\lambda x_1+(1-\lambda)x_2$. Note that $x_1=x+(1-\lambda)h$ and $x_2=x-\lambda h$ for $h=x_1-x_2$. Let us denote by γ a curve realizing the infimum in (A.2) for such a pair of (t,x), by K a Lipschitz constant for L restricted to $[0,T]\times\mathbb{T}^N\times B(0,2\kappa)$ and by ω a semi–concavity modulus for u_0. We get

$$\begin{aligned}
&\lambda u(t,x_1)+(1-\lambda)u(t,x_2)-u(t,x)\\
&\qquad\leqslant \lambda\Big(u_0\big(\gamma(0)+(1-\lambda)h\big)+\int_0^t L\big(s,\gamma(s)+(1-\lambda)h,\dot\gamma(s)\big)\mathrm{d}s\Big)\\
&\qquad\quad+(1-\lambda)\Big(u_0\big(\gamma(0)-\lambda h\big)+\int_0^t L\big(s,\gamma(s)-\lambda h,\dot\gamma(s)\big)\mathrm{d}s\Big)\\
&\qquad\quad-\Big(u_0\big(\gamma(0)\big)+\int_0^t L\big(s,\gamma(s),\dot\gamma(s)\big)\mathrm{d}s\Big)\\
&\qquad=\lambda u_0\big(\gamma(0)+(1-\lambda)h\big)+(1-\lambda)u_0\big(\gamma(0)-\lambda h\big)-u_0\big(\gamma(0)\big)\\
&\qquad\quad+\lambda\Big(\int_0^t\Big(L\big(s,\gamma(s)+(1-\lambda)h,\dot\gamma(s)\big)-L\big(s,\gamma(s),\dot\gamma(s)\big)\Big)\mathrm{d}s\\
&\qquad\quad+(1-\lambda)\int_0^t\Big(L\big(s,\gamma(s)-\lambda h,\dot\gamma(s)\big)-L\big(s,\gamma(s),\dot\gamma(s)\big)\Big)\mathrm{d}s\\
&\qquad\leqslant\lambda(1-\lambda)\Big(\omega(d(x_1,x_2))+t\,Kd(x_1,x_2)\Big),
\end{aligned}$$

which proves the assertion. □

The result just proved will be applied to weakly coupled systems as follows:

Proposition A.2. *Let $T>0$ and $\mathbf{u}=(u_1,\dots,u_m)\in\big(\mathrm{Lip}([0,T]\times\mathbb{T}^N)\big)^m$ be a solution of the evolutionary weakly coupled system (1.3). Let $B(x)$ be Lipschitz and H_i be locally Lipschitz and strictly convex, for some fixed index $i\in\{1,\dots,m\}$. Then, for all $0<\tau<T$, the function u_i restricted to $[\tau,T)\times\mathbb{T}^N$ is semiconcave. Moreover, if, the initial condition $u_i(0,\cdot)$ is semiconcave, then the functions $\{\,u_i(t,\cdot)\,:\,t\in[0,T]\,\}$ are equi–semiconcave.*

Proof. The function u_i solves, for the given index $i\in\{1,\dots,m\}$, a Hamilton–Jacobi equation of the kind (A.1) with $G(t,x,p)=H_i(x,p)+\big(B(x)\mathbf{u}(t,x)\big)_i$, with $(t,x,p)\in[0,T]\times\mathbb{T}^N\times\mathbb{R}^N$.

The conclusion follows by applying Proposition A.1. □

We are now ready to prove Proposition 2.8.

Proof of Proposition 2.8. We recall that, by convexity of the Hamiltonians, subsolutions to the critical system coincide with almost everywhere subsolutions. This fact will be repeatedly exploited along the proof.

Assume first that $t\mapsto\mathcal{S}(t)\mathbf{u}+t\,a\mathbb{1}$ is non–decreasing. Pick $t_0>0$ such that the map $(t,x)\mapsto\mathcal{S}(t)\mathbf{u}(x)$ is differentiable at (t_0,x) for almost every $x\in\mathbb{T}^N$ and

$$\partial_t\mathcal{S}(t_0)\mathbf{u}(x)\geqslant -a\mathbb{1}\qquad\text{for a.e. } x\in\mathbb{T}^N.$$

By the Lipschitz character of the map $(t,x)\mapsto\mathcal{S}(t)\mathbf{u}(x)$ and Fubini's theorem, this holds true for almost every $t_0>0$. Using the evolutionary equation, which is verified at every differentiability point of $\mathcal{S}(t)\mathbf{u}(x)$, we deduce that, for every $i\in\{1,\dots,m\}$,

$$H_i\Big(x,D\big(\mathcal{S}(t_0)\mathbf{u}\big)_i(x)\Big)+\big(B(x)\mathcal{S}(t_0)\mathbf{u}(x)\big)_i\leqslant a\quad\text{for a.e. } x\in\mathbb{T}^N,$$

that is, $\mathcal{S}(t_0)\mathbf{u} \in \mathcal{H}(a)$. This being true for almost every $t_0 > 0$, the conclusion follows by stability of viscosity subsolutions.

Let us now assume reciprocally that $\mathbf{u} \in \mathcal{H}(a)$. We first approximate each Hamiltonian H_i with a sequence $(H_i^k)_k$ of convex Hamiltonians that are, in addition, locally Lipschitz in (x,p) and strictly convex in p. This can be done by taking a sequence $(\rho_k)_k$ of standard mollifiers on $\mathbb{R}^N$ and by setting

$$H_i^k(x,p) = \int_{B_1} \rho_k(y) H_i(x-y,p)\,\mathrm{d}y + \frac{|p|^2}{k}, \qquad (x,p) \in \mathbb{T}^N \times \mathbb{R}^N.$$

Analogously, we approximate the matrix $B(x)$ by a sequence of coupling matrixes $\big(B_k(x)\big)_k$ that are Lipschitz in x. Note that, for each index $i \in \{1,\dots,m\}$, $\quad H_i^k \rightrightarrows H_i \quad$ in $\mathbb{T}^N \times \mathbb{R}^N$ and $B_k \rightrightarrows B$ in $\mathbb{T}^N$ as $k \to +\infty$. Let us denote by $\mathcal{H}_k(a)$ the set of $\mathbf{a}$–subsolution of the weakly coupled system (1.2) with $\mathbf{a} = a\mathbb{1}$ and with $H_1^k,\dots,H_m^k$ and B_k in place of $H_1,\dots,H_m$ and B, respectively, and by $\mathcal{S}_k$ the semigroup associated with the corresponding time–dependent equation (1.3).

Next, we approximate $\mathbf{u}$ with a sequence of $(\mathbf{u}^n)_n$ of functions that are component–wise semi–concave by setting

$$u_i^n(x) = \inf_{y\in\mathbb{T}^N} u_i(y) + nd(y,x)^2 \qquad \text{for every } x \in \mathbb{T}^N \text{ and } i = 1,\cdots,m.$$

Fix $\varepsilon > 0$. A standard argument shows that, for n large enough, $\mathbf{u}^n \in \mathcal{H}(a+\varepsilon)$. Moreover, by the Lipschitz character of $\mathbf{u}^n$ and by the local uniform convergence of $(H_1^k,\dots,H_m^k)$ to $(H_1,\dots,H_m)$ and of B_k to B, we also have that $\mathbf{u}^n \in \mathcal{H}_k(a+2\varepsilon)$ for k sufficiently large. We now apply Proposition A.2 to infer that the map $(t,x) \mapsto \mathcal{S}_k(t)\mathbf{u}^n(x)$ is semiconcave in $[0,\tau] \times \mathbb{T}^N$ for every $\tau > 0$. By using the fact that the gradient of a semiconcave function is continuous in its domain of definition and by choosing $\tau > 0$ small enough, we get $\mathcal{S}_k(t)\mathbf{u}^n \in \mathcal{H}_k(a+3\varepsilon)$ for every $0 \leqslant t \leqslant \tau$. By exploiting this information in the evolutive weakly coupled system, we get

$$\frac{\partial}{\partial t}\mathcal{S}_k(t)\mathbf{u}^n(x) \geqslant -(a+3\varepsilon)\mathbb{1} \qquad \text{for a.e. } (t,x) \in (0,\tau) \times \mathbb{T}^N,$$

i.e. $\mathcal{S}_k(t+h)\mathbf{u}^n \geqslant \mathcal{S}_k(t)\mathbf{u}^n - h(a+3\varepsilon)\mathbb{1}$ for every $0 < t < t+h \leqslant \tau$.

Now, by the comparison principle for the evolution equation and by using the fact that the semigroup commutes with the addition of scalar multiples of the vector $\mathbb{1}$, we obtain that $t \mapsto \mathcal{S}_k(t)\mathbf{u}^n - t(a+3\varepsilon)\mathbb{1}$ is non decreasing. We now exploit the fact that

$$\mathcal{S}_k(t)\mathbf{u}^n \underset{k\to+\infty}{\rightrightarrows} \mathcal{S}(t)\mathbf{u}^n \quad \text{and} \quad \mathcal{S}(t)\mathbf{u}^n \underset{n\to+\infty}{\rightrightarrows} \mathcal{S}(t)\mathbf{u} \qquad \text{in } \mathbb{R}_+ \times \mathbb{T}^N$$

to infer that $t \mapsto \mathcal{S}(t)\mathbf{u}^n - t(a+3\varepsilon)\mathbb{1}$ is non–decreasing on $[0,+\infty)$. Being this true for every $\varepsilon > 0$, we finally have that $t \mapsto \mathcal{S}(t)\mathbf{u}^n - ta\mathbb{1}$ is non–decreasing on $[0,+\infty)$.

The last assertion follows from the equivalence just proved, together with the fact that the semigroup $\mathcal{S}(t)$ is non–decreasing and commutes with addition of vectors of the form $a\,\mathbb{1}$ with $a \in \mathbb{R}$. ☐

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