Generalized periods of Kähler manifolds

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A semicosimplicial (or presimplicial) differential graded Lie algebra is a covariant functor \( \Delta_{\text{mon}} \to \text{DGLA} \).

\( \Delta_{\text{mon}} \): finite ordinal sets with order-preserving injective maps.

\[
\begin{array}{ccc}
g_0 & \rightarrow & g_1 & \rightarrow & g_2 & \rightarrow & \cdots \\
\partial_{k,i} : g_{i-1} & \rightarrow & g_i, & \text{where} & k = 0, \ldots, i,
\end{array}
\]

such that \( \partial_{k+1,i+1} \partial_{l,i} = \partial_{l,i+1} \partial_{k,i} \), for any \( k \geq l \).
The maps

$$\partial_i = \partial_{0,i} - \partial_{1,i} + \cdots + (-1)^i \partial_{i,i}$$

endow the vector space $\bigoplus_i g_i$ with the structure of a differential complex.

Each $g_i$ is in particular a differential complex

$$g_i = \bigoplus_j g^j_i; \quad d_i : g^j_i \rightarrow g^{j+1}_i$$

has a natural bicomplex structure.

$$g^\bullet = \bigoplus_{i,j} g^j_i$$
The associated total complex

\[ \text{Tot}(g^\triangle) = \bigoplus_i g_i[-i], \quad d_{\text{Tot}} = \sum_{i,j} \partial_i + (-1)^j d_j \]

has no canonical DGLA structure.

Yet, it can be endowed with a canonical $L_\infty$-algebra structure via homotopy transfer by the Thom-Whitney DGLA.

\[ C^{i,j}_{\text{TW}}(g^\triangle) = \{(x_n)_{n \in \mathbb{N}} \in \bigoplus_n \Omega^i_n \otimes g^j_n \mid \delta^{k,n} x_n = \partial_{k,n} x_{n-1}\} \]

where $\Omega_n$ are the polynomial differential forms on the $n$-simplex.
\( \chi : L \to M \) DGLA morphism.

\[
\begin{array}{c}
L \xrightarrow{\chi} M \xrightarrow{0} 0 \xrightarrow{\cdots}
\end{array}
\]

The associated total complex is the *mapping cone* of \( \chi \):

\[ C_\chi = L \oplus M[-1]; \quad d(l, m) = (dl, \chi(l) - dm) \]

One finds an explicit expression for the multilinear brackets

\[ \mu_n : \bigwedge^n C_\chi \to C_\chi[2-n], \quad n \geq 2, \]

defining the \( L_\infty \)-algebra structure on \( C_\chi \).
\[ \mu_1(l, m) = (dl, \chi(l) - dm) \]

\[ \mu_2((l_1, m_1) \wedge (l_2, m_2)) = \left( [l_1, l_2], \frac{1}{2} [m_1, \chi(l_2)] + \frac{(-1)^{\deg(l_1)}}{2} [\chi(l_1), m_2] \right) \]
and for \( n \geq 3 \)

\[
\mu_n((l_1, m_1) \wedge \cdots \wedge (l_n, m_n)) = \\
\left(0, \frac{B_{n-1}}{(n-1)!} \sum_{\sigma \in S_n} \pm [m_{\sigma(1)}, \cdots, [m_{\sigma(n-1)}, \chi(l)_{\sigma(n)}] \cdots]\right)
\]

where the \( B_n \) are the Bernoulli numbers.

Note that \( \pi_1 : C_\chi \to L \) is a linear \( L_\infty \)-morphism.
$L_\infty$-algebras $\leadsto$ Deformation functors

$$\text{Def}_g(A) = \frac{\text{MC}(g \otimes m_A)}{\text{homotopy equivalence}}$$

$(A, m_A)$ local Artin algebra.

If $g$ and $\mathfrak{h}$ are quasiisomorphic, then $\text{Def}_g \sim \text{Def}_\mathfrak{h}$

By composing with $DGLA$-morphisms $\leadsto L_\infty$-algebras we get a functor $DGLA$-morphisms $\leadsto$ Deformation functors.
A commutative diagram

\[
\begin{array}{ccc}
L_1 & \xrightarrow{f_L} & L_2 \\
\downarrow{\chi_1} & & \downarrow{\chi_2} \\
M_1 & \xrightarrow{f_M} & M_2 
\end{array}
\]

of morphisms of differential graded Lie algebras induces a natural transformation

\[
\text{Def}_{\chi_1} \to \text{Def}_{\chi_2}.
\]

Moreover, if \( f_L \) and \( f_M \) are quasi-isomorphisms, then \( \text{Def}_{\chi_1} \to \text{Def}_{\chi_2} \) is an isomorphism.
An explicit description of \( \text{Def}_\chi \).

\[
\text{Def}_\chi(A) = \frac{\text{MC}_\chi(A)}{\text{gauge equivalence}},
\]

\[
\text{MC}_\chi(A) = \left\{ (x, e^a) \in (L^1 \otimes m_A) \times \exp(M^0 \otimes m_A) \mid \begin{align*}
    dx + \frac{1}{2}[x, x] &= 0, \\
    e^a \ast \chi(x) &= 0
\end{align*} \right\},
\]

where \( \ast \) is the gauge action of \( \exp(M^0 \otimes m_A) \) on \( M^1 \otimes m_A \).
The gauge action

\[(\exp(L^0 \otimes m_A) \times \exp(dM^{-1} \otimes m_A)) \times MC_\chi(A) \to MC_\chi(A)\]

is

\[(e^l, e^{dm}) \star (x, e^a) = (e^l \star x, e^{dm} e^a e^{-\chi(l)})\].

**Example.** If \(M = 0\), then \(\text{Def}_\chi = \text{Def}_L\)

**Example.** If \(L = 0\) and the differential of \(M\) is trivial, then \(\text{Def}_\chi = \exp(M^0)\), i.e., \(\text{Def}_\chi(A) = \exp(M^0 \otimes m_A)\).
Let $X$ be a compact Kähler manifold, and $A_X$ the differential graded commutative algebra of smooth complex differential forms. We have DGLAs

\[ L = \{ f \in \text{Hom}^*(A_X, A_X) \mid f(\ker \partial) \subseteq \partial A_X \}, \]

and

\[ M = \{ f \in \text{Hom}^*(A_X, A_X) \mid f(\ker \partial) \subseteq \ker \partial \text{ and } f(\partial A_X) \subseteq \partial A_X \} \]

and a commutative diagram

\[
\begin{array}{ccc}
0 & \leftarrow & L \\
\rho \downarrow & & \eta \downarrow \\
\text{Hom}^*(\ker \partial_{\partial A_X}, \ker \partial_{\partial A_X}) & \leftarrow & M \\
\downarrow & & \downarrow \\
& & \text{Hom}^*(A_X, A_X)
\end{array}
\]

\[
\begin{array}{ccc}
L & \rightarrow & L \\
\chi \downarrow & & \\
& & \text{Hom}^*(A_X, A_X)
\end{array}
\]
By the $\partial\overline{\partial}$-lemma, we have quasi-isomorphisms

$$(A_X, d) \leftarrow (\ker \partial, d) \rightarrow \left( \frac{\ker \partial}{\partial A_X}, 0 \right).$$

Hence the horizontal arrows in the commutative diagram

$$\begin{array}{ccc}
0 & \leftarrow & L \\
\rho \downarrow & & \eta \downarrow & \chi \downarrow \\
\text{Hom}^* \left( \frac{\ker \partial}{\partial A_X}, \frac{\ker \partial}{\partial A_X} \right) & \leftarrow & M & \rightarrow & \text{Hom}^* (A_X, A_X)
\end{array}$$

are quasi-isomorphisms.

We get an isomorphism of deformation functors

$$\text{Def}_\chi \cong \text{Def}_\eta \cong \text{Def}_\rho = \text{Aut}^0 \left( \frac{\ker \partial}{\partial A_X} \right) \cong \text{Aut}^0 (H^* (X, \mathbb{C})).$$
The isomorphism $\text{Def}_\chi \to \text{Aut}^0(H^*(X, \mathbb{C}))$ has a simple explicit description.

$$(\alpha, e^a) \mapsto \psi_a,$$

where

$$\psi_a([\omega]) = [e^a(\omega_0 + \partial \beta)]$$

for any $\partial$-closed representative $\omega_0$ of the cohomology class $[\omega]$, and any $\beta \in A_X$ such that $de^a(\omega_0 + \partial \beta) = 0$. 
Let $L$ and $M$ DGLAs, $i: L \to M[-1]$ linear map. Let $l = [d, i]: L \to M$, i.e.,

$$l_a = di_a + ida.$$  

The map $i$ is called a *Cartan homotopy* if for every $a, b \in L$ we have:

$$i_{[a,b]} = [i_a, l_b], \quad [i_a, i_b] = 0.$$  

If $i: L \to M[-1]$ is a Cartan homotopy, then $l: L \to M$ is a DGLA morphism.
Example. Let $M$ be a differential manifold, $\mathcal{X}(M)$ be the Lie algebra of vector fields on $M$, and $\mathcal{E}nd^*(\Omega^*(M))$ be the Lie algebra of endomorphisms of the de Rham algebra of $M$. The Lie algebra $\mathcal{X}(M)$ can be seen as a DGLA concentrated in degree zero, and the graded Lie algebra $\mathcal{E}nd^*(\Omega^*(M))$ has a degree one differential given by $[d_{dR}, -]$, where $d_{dR}$ is the de Rham differential. Then the contraction

$$\iota: \mathcal{X}(M) \to \mathcal{E}nd^*(\Omega^*(M))[-1]$$

is a Cartan homotopy and its differential is the Lie derivative

$$[d, \iota] = \mathcal{L}: \mathcal{X}(M) \to \mathcal{E}nd^*(\Omega^*(M)).$$
When the second equation \([i_a, i_b] = 0\) is replaced by the weaker condition \(\sum_{\sigma \in S_3} \pm [i_{x_\sigma(1)}, [i_{x_\sigma(2)}, i_{x_\sigma(3)}]] = 0\) we say that \(i\) is a weak Cartan homotopy.

Now recall \(C_1 = L \oplus M[-1]\) and that \(\pi_1: C_1 \to L\) is a linear \(L_\infty\)-morphism. If \(i: L \to M[-1]\), then \((id, i): L \to C_1\) is a linear map lifting \(id: L \to L\)

\((id, i): L \to C_1\) is a linear \(L_\infty\) morphism if and only if \(i: L \to M[-1]\) is a weak Cartan homotopy.
Let $i: N \to M[-1]$ be a (weak) Cartan homotopy for $l: N \to M$, let $L$ be a subDGLA of $M$ such that $l(N) \subseteq L$, and let $\chi: L \hookrightarrow M$ be the inclusion. Then the linear map

$$\Phi: N \to C_\chi, \quad \Phi(a) = (l_a, i_a)$$

is a linear $L_\infty$-morphism. In particular, the map $a \mapsto (l_a, e^{i a})$ induces a natural transformation of Maurer-Cartan functors $MC_N \to MC_\chi$, and consequently a natural transformation of deformation functors $Def_N \to Def_\chi$.

**Proof.** We have a commutative diagram of differential graded Lie algebras

$$
\begin{array}{ccc}
N & \xrightarrow{l} & L \\
\downarrow{l} & & \downarrow{\chi} \\
M & \cong & M
\end{array}
$$

Domenico Fiorenza

Generalized periods of Kähler manifolds
Let $X$ be a complex manifold. For any integer $(a, b)$ with $a \leq 0$ and $b \geq 0$, let $\mathcal{G}erst_X^{a, b}$ be the sheaf

$$\mathcal{G}erst_X^{a, b} = \mathcal{A}^{0, b}_X(\bigwedge^{-a} T_X).$$

The direct sum $\mathcal{G}erst^*_X = \bigoplus_k \bigoplus_{a+b=k} \mathcal{G}erst_X^{a, b}$ is a sheaf of differential Gerstenhaber algebras, with the wedge product as graded commutative product, the Dolbeault differential $\overline{\partial}$ as differential and the Schouten-Nijenhuis bracket as odd graded Lie bracket.

Let

$$i: \mathcal{G}erst^{a, b}_X \to \mathcal{H}om^{a, b}(\mathcal{A}_X, \mathcal{A}_X), \quad \xi \mapsto i_\xi, \quad i_\xi(\omega) = \xi \lrcorner \omega,$$

be the \textit{contraction map}.
\( i \) is a morphism of sheaves of bigraded associative algebras:

\[ i_{\xi \wedge \eta} = i_{\xi} i_{\eta} \]

Since \((\text{Gerst}^*_X, \wedge)\) is a graded commutative algebra, \([i_\xi, i_\eta] = 0\), so iterated contractions give a symmetric map

\[ i^{(n)}: \bigotimes^n \text{Gerst}^*_X \to \mathcal{H}om^*(A_X, A_X) \]

\[ \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \mapsto i_{\xi_1} i_{\xi_2} \cdots i_{\xi_n} \]
$\mathcal{P}oly^*_X = \mathcal{G}erst[-1]^*_X$

is a sheaf of differential graded Lie algebras. Note that, due to
the shift, the differential $D$ in $\mathcal{P}oly^*_X$ is $-\bar{\partial}$.

The sheaf $\mathcal{K}S^*_X = A^{0,*}(T_X)$ is a sheaf of subDGLAs of $\mathcal{P}oly^*_X$, called the _Kodaira-Spencer_ sheaf of $X$.!
The contraction map \( i: Gerst_X^* \rightarrow \mathcal{H}om^*(A_X, A_X) \) can be seen as a linear map

\[
i: Poly_X^* \rightarrow \mathcal{H}om^*(A_X, A_X)[-1].
\]

- It is a Cartan homotopy
- The induced morphism \( I \) of sheaves of differential graded Lie algebras is the \textit{holomorphic Lie derivative} \( I_\xi = [\partial, i_\xi] \).

\[
I_\xi = d_{\text{Hom}} i_\xi + i D_\xi = [d, i_\xi] - i \bar{\partial} \xi = [d, i_\xi] - [\bar{\partial}, i_\xi].
\]
\[ L = \{ f \in \text{Hom}^*(A_X, A_X) \mid f(\ker \partial) \subseteq \partial A_X \}, \]
\[ \chi : L \leftrightarrow \text{Hom}^*(A_X, A_X) \]

\[ \iota : \text{Poly}^*_X \rightarrow \text{Hom}^*(A_X, A_X)[-1]; \quad \iota(\text{Poly}^*_X) \subseteq L \]

So we have a natural transformation of deformation functors \( \text{Def}_{\text{Poly}^*_X} \rightarrow \text{Def}_\chi \) induced, at the Maurer-Cartan level, by the map \( \xi \mapsto (\iota \xi, e^{i\xi}) \).

\( \text{Def}_{\text{Poly}^*_X} \cong \text{Def}_\chi \) (generalized deformation of \( X \))

\( \text{Def}_\chi \cong \text{Aut}^0(H^*(X; \mathbb{C})) \)
Theorem. The linear map

\[ \text{Poly}_X^* \to \tilde{\mathcal{C}}(\chi), \quad \xi \mapsto (l_\xi, i_\xi) \]

is a linear $L_\infty$-morphism and induces a natural transformation of functors

\[ \Phi : \tilde{\text{Def}}_X \to \text{Aut}^0(H^*(X; \mathbb{C})), \]

given at the level of Maurer-Cartan functors by the map $\xi \mapsto \psi_{i_\xi}$. 
Via the natural identifications $H^1(Poly^*_X) = \bigoplus_{i \geq 0} H^i(\wedge^i T_X)$ and $H^*(X; \mathbb{C}) = \bigoplus_{p,q} H^q(X; \Omega^p_X)$ given by the Dolbeault’s theorem and the $\partial\bar{\partial}$-lemma, the differential of $\Phi$,

$$d\Phi : H^1(Poly^*_X) \rightarrow \text{Hom}^0(H^*(X; \mathbb{C}), H^*(X; \mathbb{C}))$$

is identified with the contraction

$$(\bigoplus_{i \geq 0} H^i(\wedge^i T_X)) \otimes (\bigoplus_{p,q} H^q(X; \Omega^p_X)) \rightarrow \bigoplus_{i,p,q} H^{q+i}(X; \Omega^{p-1}_X)$$
The linear map $\xi \mapsto (l_\xi, i_\xi)$ induces a morphism of obstruction theories $H^2(\text{Poly}^*) \to \text{Hom}^1(H^*(X; \mathbb{C}), H^*(X; \mathbb{C}))$ which is naturally identified with the contraction

$$(\bigoplus_{i \geq 0} H^{i+1}(\wedge^i T_X)) \otimes (\bigoplus_{p, q} H^q(X; \Omega^p_X)) \to \bigoplus_{i, p, q} H^{q+i+1}(X; \Omega^{p-1}_X).$$
Since the deformation functor $\text{Aut}^0(H^*(X; \mathbb{C}))$ is smooth we get:

- The obstructions to extended deformations of a compact Kähler manifold $X$ are contained in the subspace

$$\bigoplus_{i \geq 0} \bigcap_{p,q} \ker \left( H^{i+1}(\wedge^i T_X) \rightarrow \text{Hom} \left( H^q(X; \Omega^p_X), H^{q+i+1}(X; \Omega^{p-1}_X) \right) \right)$$

of $H^2(\text{Poly}^*_X)$. (Kodaira principle: ambient cohomology annihilates obstruction)
Corollary. Extended deformations of compact Calabi-Yau manifolds are unobstructed (Bogomolov-Tian-Todorov’s Lemma)

Proof. If $X$ is an $n$-dimensional compact Calabi-Yau manifold, then for any $i \geq 0$ the contraction pairing

$$H^{i+1}(\wedge^i T_X) \otimes H^{n-i-1}(X; \Omega^i) \to H^n(X; \mathcal{O}_X) \simeq H^n(X; \Omega^n_X)$$

is nondegenerate.
Barannikov-Kontsevich’s generalized periods.

Let $X$ be a Calabi-Yau manifold with volume element $\Omega$. Represent a generalized deformation of $X$ by a Maurer-Cartan element $\xi$ in $\text{Poly}_X^*$ chosen in the Tian’s gauge: $\partial(\xi \lrcorner \Omega) = 0$.

Then

$$\Phi_\xi([\Omega]) = [e^{i\xi}(\Omega)],$$

the generalized period of $(X, \omega)$.
Let $X$ be a compact Kähler manifold, $A_X = F^0 \supseteq F^1 \supseteq \cdots$ be the Hodge filtration of differential forms on $X$.
The inclusion of DGLAs $KS_X \hookrightarrow \text{Poly}^*_X$ induces
$\text{Def}_X \hookrightarrow \widetilde{\text{Def}}_X$.
The Grassmannian $\text{Grass}(H^*(F^p), H^*(X; \mathbb{C}))$ is a homogeneous space for $\text{Aut}^0(H^*(X; \mathbb{C}))$ so we have a natural projection

$$\text{Aut}^0(H^*(X; \mathbb{C})) \to \text{Grass}(H^*(F^p), H^*(X; \mathbb{C}))$$
\[ \overset{\text{Def}_X}{\longrightarrow} \text{Aut}^0(H^*(X; \mathbb{C})) \]

\[ \overset{\text{Def}_X}{\longrightarrow} \text{Grass}(H^*(F^p), H^*(X; \mathbb{C})) \]

\[ \mathcal{P}^p \text{ is the } p\text{-th period map.} \]
Mapping cones and deformation functors
Cartan homotopies
Polyvector fields and generalized periods
Restriction to classical deformations

Generalized periods of Kähler manifolds