Lecture 2

In this lecture I will review some basic notion of differential geometry for surfaces in the space, and more generally for hyper--surfaces in $\mathbb{R}^{n+1}$.

2.1 First fundamental form

For every point p in the surface S, we denote by $Tan(S,p)$ the tangent plane of S at p.
The first fundamental form (of S at p) is the quadratic form on $Tan(S,p)$ associated with the scalar product on $Tan(S,p)$.
In our setting the scalar product is the one induced by the immersion in the Euclidean space and therefore

$$I_p(v) = |v|^2 \quad \forall v \in Tan(S,p)$$

Recall that from this quadratic form you can recover the scalar product:

$$\langle v,w\rangle_p = \frac{1}{2}\left[I_p(v+w) - I_p(v) - I_p(w)\right]$$

Representation of I_p using coordinates

Let $n=2$ and $X : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a param. of the surface S. Since $\partial_1 X, \partial_2 X$ are a basis of $Tan(S,p)$ we can write every

vector $v \in Tan(S,p)$ as

$$v = v_1\, \partial_1 X + v_2\, \partial_2 X$$

Hence

$$I_p(v) = E v_1^2 + 2F v_1 v_2 + G v_2^2$$

where

$$E := \overset{|\partial_1 X|^2}{\overset{\shortparallel}{\partial_1 X \cdot \partial_1 X}}\,, \quad F := \partial_1 X \cdot \partial_2 X\,, \quad G := \overset{|\partial_2 X|^2}{\overset{\shortparallel}{\partial_2 X \cdot \partial_2 X}}$$

in other words

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

is the matrix associated to the quadratic form I_p by chosing $\partial_1 X, \partial_2 X$ as a basis of $Tan(S,p)$.

The use of the letters E, F, G for the coefficients of this matrix is classical and goes back to Gauss. In modern notation one denotes these coefficients by g_{ij}.

Incidentally $|\partial_1 X \wedge \partial_2 X| = \sqrt{EG - F^2}$.

Indeed

$$\begin{aligned} |\partial_1 X \wedge \partial_2 X| &= |\partial_1 X|\,|\partial_2 X| \sin\theta \\ &= \sqrt{|\partial_1 X|^2 |\partial_2 X|^2 (1-\cos^2\theta)} \\ &= \sqrt{|\partial_1 X|^2 |\partial_2 X|^2 - (\partial_1 X \cdot \partial_2 X)^2} \\ &= \sqrt{E \cdot G - F^2}. \end{aligned}$$

Remark

The second fundamental form is <u>intrinsic</u> (does not depend on the ambient space).

2.2 Second fundamental form

Let S be an oriented surface, that is, for every point p it is given a unit normal vector N (depending continuosly on p....)

Fix $p \in S$ and decompose $\mathbb{R}^{n+1}$ as

$$\mathrm{Tan}(S,p) \oplus \mathrm{Nor}(S,p) \simeq \mathrm{Tan}(S,p) \oplus \mathbb{R}$$

and write a point $p \in \mathbb{R}^{n+1}$ as (x,y) accordingly.

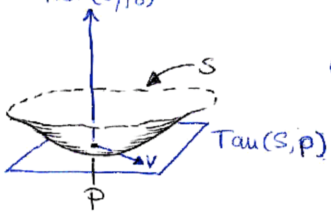
the identification of $\mathrm{Nor}(S,p)$ with $\mathbb{R}$ depends on the choice of the normal vector N....

Close to p, S is described as the graph of some function $f : \mathrm{Tan}(S,p) \longrightarrow \mathbb{R}$ that is by the equation

$$y = f(x) = \underline{\langle Av ; v\rangle} + o(|v|)^2$$

write $x = p+v$

second order Taylor expansion of f at p; $A = A_d$ depends on p.



The second fundamental form of S at p is precisely

$$\mathrm{II}_p(v) := \langle A_p v, v\rangle$$

(The map $p \longmapsto A_p$ is known as Weingarten map).

2.3 Differential of the Gauss map

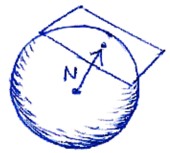
Let S be an oriented surface, and for every $p \in S$ let $N = N(p)$ the orienting normal (unit) vector. The Gauss map of S is precisely

$$N : S \longrightarrow S^n \subset \mathbb{R}^{n+1}$$

Hence the differential of N at $p \in S$ is a linear map

$$dN(p) : \mathrm{Tan}(S,p) \longrightarrow \mathrm{Tan}(S^n, N(p)) \subset \mathbb{R}^{n+1}$$

$$\mathrm{Tan}(S^n, N(p)) \cong \mathrm{Tan}(S,p)$$



Note that the tangent plane of S^{n-1} at N is exactely $N^\perp = \mathrm{Tan}(S,p)$

Recall that the differential $dN(p)$ is related to the first order Taylor expansion of N at p: if we write a point p' "close" to p as $p+v+o(v)$ with $v \in \mathrm{Tan}(S,p)$, then

$$\boxed{N(p') = N(p+v+o(v)) = N(p) + dN(p)\cdot v + o(v)}$$

Moreover, given any curve γ on S starting from p we have

$$\boxed{\frac{d}{dt} N(\gamma)\Big|_{t=0} = dN(p) \cdot \dot\gamma(0)}$$

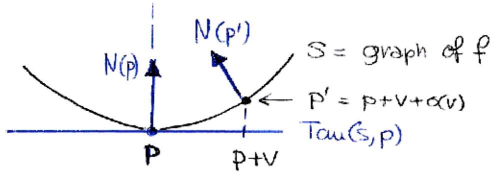
(Of couse similar identities hold for the differential of <u>any</u> map, there is nothing specific of the Gauss map here.)

Fundamental identity

The differential of the Gauss map is related to the second fundamental form by the following identity: for $v \in Tan(S,p)$

$$\boxed{\mathrm{II}_p(v) = \langle -dN(p)\, v, v \rangle}$$

Proof



As before we identify $\mathbb{R}^{n+1}$ with $Tan(S,p) \oplus \mathbb{R}$

$$N(p') = N(p+v+o(v)) = \frac{(-\nabla f(v), 1)}{\sqrt{1+|\nabla f(v)|^2}}$$

$$= \frac{(-A_p v + o(v), 1)}{\sqrt{1+|A_p v + o(v)|^2}} = [(-A_p v, 1) + o(v)][1 + O(v^2)]$$

$$= \underbrace{(0,1)}_{= N(p)} + \underbrace{(-A_p v, 0)}_{= dN(p)\, v} + o(v)$$

Hence $-dN(p) = A_p$ and in particular

$$\mathrm{II}_p(v) := \langle A_p v ; v \rangle = \langle -dN(p)\, v ; v \rangle \quad \square$$

2.4 Curvatures

The second fundamental form II_p is a quadratic form on $Tan(S,p)$ associated to the self adjoint linear map $\underset{\substack{\| \\ -dN(p)}}{A_p} : Tan(S,p) \to Tan(S,p)$

Being self-adjoint, A_p admits n real eigenvalues $\lambda_1, \dots, \lambda_n$ with the corresponding eigenvectors $e_1, \dots, e_n$.
(note that λ_i and e_i do not depend on the choice of a basis on $Tan(S,p)$).

$\lambda_1, \dots, \lambda_n$	principal curvatures (of S at p)
$e_1, \dots, e_n$	principal directions

In fact one is often more interested in the coefficients of the characteristic polynom. of A_p, that is, the elementary symmetric functions of the eigenvalues λ_i:

$$P_A(\lambda) = \det(\lambda Id - A)$$
$$= \prod (\lambda - \lambda_i)$$
$$= \lambda^n - \boxed{(\lambda_1 + \dots + \lambda_n)}\, \lambda^{n-1} + \boxed{(\lambda_1\lambda_2 + \dots + \lambda_{n-1}\lambda_n)}\, \lambda^{n-2}$$
$$\dots\dots + (-1)^n \boxed{(\lambda_1 \lambda_2 \cdots \lambda_n)}$$

In particular

$$\lambda_1 + \cdots + \lambda_n = \text{trace of } A$$

is called mean curvature (of S at p) and usually denoted by H.

Remarks

- the mean curvature is sometimes defined (more properly) as $\frac{1}{n}(\lambda_1 + \cdots + \lambda_n)$. ← !
- It is sometimes convenient to define the mean curvature vector $\vec{H} = H \cdot N$. While H depends on the choice of the orientation, $\vec{H}$ does not.

2.5 Mean curvature and Gauss curvature

Let now S be a 2-dim. surface ($n=2$). Given $p \in S$, let λ_1, λ_2 be the principal curvatures of S at p. Then we set

depends on the orientation
$$\boxed{\begin{aligned} H &= \text{mean curvature} \\ &:= \lambda_1 + \lambda_2 = \text{trace of } A_p = -\text{trace } dN(p) \end{aligned}}$$

does not depend on the orientation
$$\boxed{\begin{aligned} K &= \text{Gauss curvature} \\ &:= \lambda_1 \cdot \lambda_2 = \det. \text{ of } A_p = \det(dN(p)) \end{aligned}}$$

And using coordinates....

If $X : D \to \mathbb{R}^3$ is a parametrization of S we can write every $v \in \mathrm{Tan}(S,p)$ as

$v = v_1 \partial_1 X + v_2 \partial_2 X$. Then the second fund. form is given by

$$\mathrm{II}_p(v) = e v_1^2 + 2f v_1 v_2 + g v_2^2$$

where

$$e := \mathrm{II}_p(\partial_1 X) = \langle -dN\, \partial_1 X, \partial_1 X\rangle$$
$$= \langle N, \partial_1^2 X\rangle \leftarrow \text{derive the identity } \langle N, \partial_1 X\rangle = 0$$

$$f := \langle -dN\, \partial_1 X, \partial_2 X\rangle = \langle N, \partial_1\partial_2 X\rangle$$

$$g := \mathrm{II}_p(\partial_2 X) = \langle -dN\, \partial_2 X, \partial_2 X\rangle$$
$$= \langle N, \partial_2^2 X\rangle \leftarrow \text{derive } \langle N, \partial_2 X\rangle = 0$$

In other words $\begin{pmatrix} e & f \\ f & g \end{pmatrix}$ is the matrix assoc. to the quadratic form II_p by chosing $\partial_1 X, \partial_2 X$ as a basis of $\mathrm{Tan}(S,p)$, that is

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \underline{dX^* \cdot A \cdot dX}$$

↑ note that dX is a linear map from $\mathbb{R}^2$ to $\mathrm{Tan}(S,p)$

Hence

$$eg - f^2 = \det\begin{pmatrix} e & f \\ f & g \end{pmatrix}$$
$$= (\det dX)^2 \det A$$
$$= (EG - F^2) \cdot K$$

that is

$$\boxed{K = \frac{eg - f^2}{EG - F^2}}$$

Note that e, f, g, E, F, G can be computed using X and its first and second order derivatives.

If in addition X is conformal...

that is, $E = G = |\partial_1 X|^2 = |\partial_2 X|^2 = \frac{1}{2}|\nabla X|^2$ and $F = \partial_1 X \cdot \partial_2 X = 0$, then we have a simple formula also for the mean curvature H. Indeed in this case dX is of the form cR with $R: \mathbb{R}^2 \to Tan(S,p)$ an isometry and $c = \frac{1}{2}|\nabla X|$. Hence

$$e+g = \operatorname{tr}\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \operatorname{tr}(dX^* A\, dX)$$
$$= \operatorname{tr}(c^2 R^* A R) = c^2 \operatorname{tr}(A)$$
$$= c^2 H$$

that is

$$\boxed{H = \frac{e+g}{c^2} = \frac{\langle N; \partial_1^2 X + \partial_2^2 X\rangle}{\frac{1}{4}|\nabla X|^2} = \frac{\langle N; \Delta X\rangle}{\frac{1}{4}|\nabla X|^2}}$$

Next we use curvatures to write two useful formulas; one for the volume of the tubular neighbourhood of a surface, and one for the first variation of the area of a surface.

2.6 Volume of the tubular neighbourhood

Let Ω be a bounded open set in $\mathbb{R}^3$ with a regular boundary $S = \partial\Omega$ oriented by the outer normal N. (at least of class C^2)

For every $r > 0$ let Ω_r be the r-neighbourhood of Ω

$$\Omega_r := \{x \text{ s.t. } \operatorname{dist}(x, \Omega) \leq r\}$$

We want to compute the volume of Ω_r.

If r is sufficiently small, then $\Omega_r \setminus \Omega$ is parametrized by

$$\psi : S \times [0,r] \longrightarrow \mathbb{R}^3$$
$$\psi : (p,t) \longmapsto p + t\, N(p)$$

outer normal at p

To compute $Vol(\Omega_r)$ we need to compute the differential of ψ and its determinant.

$$d\psi(p,t) : Tan(S,p) \times \mathbb{R} \to \mathbb{R}^3$$

$$d\psi = \underbrace{dp + t\, dN}_{\text{Tangential component}} + \underbrace{N dt}_{\text{normal component}}$$

Now we identify $\mathbb{R}^3$ with $Tan(S,p) \times Nor(S,p)$, choose e_1, e_2 orthonormal basis of $Tan(S,p)$

and use e_1, e_2, N as an orthonormal basis of $\mathbb{R}^3$. With respect to this basis the matrix associated to $d\psi$ is

$$\begin{array}{cc} \begin{array}{c} 2+1 \end{array} & \\ M = \left(\begin{array}{c|c} I - tA & 0 \\ \hline 0 & 1 \end{array}\right) & \begin{array}{c} 2 \\ + \\ 1 \end{array} \end{array}$$

where A is the matrix associated to $-dN$ and therefore represents the second fundamental form (of S at p).

Hence
$$\det M = \det\left(\begin{array}{c|c} I - tA & 0 \\ \hline 0 & 1 \end{array}\right)$$
$$= \det(I - tA)$$
$$= 1 - \mathrm{tr}(A)\cdot t + \det(A)\cdot t^2$$
$$= 1 - H\cdot t + K t^2$$

and then
$$Vol(\Omega_r) = Vol(\Omega) + Vol(\Omega_r \setminus \Omega)$$
$$= Vol(\Omega) + \int_{p\in S}\int_0^r \det M \; dt\, dp$$
$$= Vol(\Omega) + \int_{p\in S}\int_0^r 1 - H(p)t + K(p)t^2 \, dt\, dp$$
$$= Vol(\Omega) + \int_{p\in S} r - \tfrac{1}{2}H(p)r^2 + \tfrac{1}{3}K(p)r^3 \, dp$$

and finally

$$\boxed{Vol(\Omega_r) = Vol(\Omega) + Area(S)\cdot r - \frac{1}{2}\int_S H\cdot r^2 + \frac{1}{3}\int_S K \; r^3}$$

Thus the volume of Ω_r is a polynomial of degree three in r, at least for r suff. small, that is, within the validity of the tubular neighbourhood theorem (for every $x \in \Omega_r$ there exists a <u>unique</u> point $\pi(x)$ on $S = \partial\Omega$ which minimizes the distance from x).

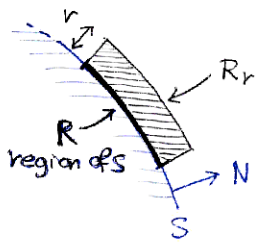
In particular is Ω is convex the previous formula holds for ALL $r > 0$.

Then by approximation one can show that for every convex body Ω, not necessarily with a regular boundary, $Vol(\Omega_r)$ is a polynomial with deg=3 in r.

This way one can give a meaning to $\int_S H$ and $\int_S K$ for every convex surface S.

In fact, there is a local version of the previous formula:

$$Vol(R_r) = Area(R)\cdot r - \frac{1}{2}\int_R H\cdot r^2 + \frac{1}{3}\int_R K\cdot r^3$$



Using this formula one can give a meaning to $\int_R H$ and $\int_R K$ for every region R of every convex surface S.

This remark is the starting point for the definition of curvature measures of convex surfaces (Alexandrov, Federer).

2.7 First variation of the area

Now we want to compute the "first variation of the area". This means that given an hypersurface S in $\mathbb{R}^{n+1}$ we want to find a formula for

$$\frac{d}{dh} \underbrace{Vol_n}(S_h)\Big|_{h=0}$$

n-dimensional volume of S_h. the area for $n=2$

where S_h is a one-parameter family of surfaces such that $S_0 = S$, or, if you like, a curve in the space of surfaces passing through S at time $h=0$ (thus the previous derivative can be seen as the partial derivative of the function Vol_n at S in the direction defined by the curve $h \mapsto S_h$).

To this end, we must first a) define the class of "admissible variations" that is, which maps $h \mapsto S_h$ to consider, and then b) compute explicitely $Vol(S_h)$.

a) We choose a vectorfiel η normal to S (not necessarily with norm $= 1$!). ← η will be as regular as needed in the following comput.

Then if N is the orienting (unitary) normal field, we can write η as

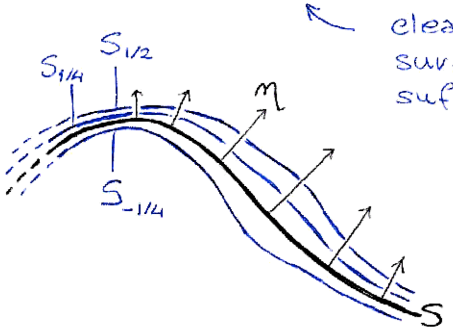
$$\eta = \varphi \cdot N$$

where φ is a given real-valued function on S.

For every $h \in \mathbb{R}$ we set

$$S_h := \{ p + h\,\eta(p) \mid p \in S \}$$

↖ clearly S_h is a regular surface only for h sufficiently small....



Thus S_h can be parametrized by

$$\begin{aligned}\psi_h : S &\longrightarrow S_h\\ p &\longmapsto p + h\,\eta(p)\end{aligned}$$

Let's compute the determinant of $d\psi_h$:

$$d\psi_h(p) = dp + h\, d\eta$$

recall that $\eta = \varphi N \rightarrow$

$$= \underbrace{dp + h\varphi\, dN}_{\text{tangential component}} + \underbrace{h\, N\, d\varphi}_{\text{normal component}}$$

Thus $d\psi_h(p)$ is a linear map from $\mathrm{Tan}(S,p)$ to $\mathbb{R}^{n+1}$, choosing any orthonormal basis $e_1,\dots,e_n$ for $\mathrm{Tan}(S,p)$ and using $e_1,\dots,e_n, N$ as an orthonormal basis of $\mathbb{R}^{n+1}$, we represent $d\psi_h(p)$ by the $(n+1)\times n$ matrix

$$M = \underbrace{\left(\begin{array}{c} \mathrm{Id} - h\varphi A \\ \hline h\,\nabla\varphi^t \end{array}\right)}_{n} \begin{array}{l} \}\, n \\ \}\, 1 \end{array}$$

where $A = -dN$ represent the second fundamental form (of S at p).

Hence

$$M^t M = \left(I - h\varphi A^t \,\middle|\, h\nabla\varphi \right) \left(\begin{array}{c} I - h\varphi A \\ \hline h\,\nabla\varphi^t \end{array}\right)$$

$$= I - h\varphi\,(A^t + A) + O(h^2)\,;$$

then

$$\sqrt{\det(M^t M)} = \sqrt{1 - 2\varphi\, \mathrm{tr}(A)\, h + O(h^2)}$$

we use that $\det(I + hB) = 1 + h\,\mathrm{tr}(B) + O(h^2)$

$$= 1 - \varphi\, \mathrm{tr}(A)\, h + O(h^2)$$

$\sqrt{1+bh} = 1 + \frac{b}{2}h + O(h^2)$

$$= 1 - \varphi H \cdot h + O(h^2)$$

mean curvature of S at p

then

$$\mathrm{Vol}_n(S_h) = \int_S 1 - \varphi H\cdot h + O(h^2)$$

$$= \mathrm{Vol}_n(S) - h\cdot \int_S \varphi H + O(h^2)$$

and finally

$$\boxed{\frac{d}{dh} \mathrm{Vol}_n(S_h)\Big|_{h=0} = -\int_S \varphi H\,.}$$

Conclusions

If S minimize the area (the n-dim. volume) among all surfaces with prescribed boundary Γ then at every point of S there holds

$$H \equiv 0$$

Given indeed any $\varphi : S \to \mathbb{R}$ such that $\varphi = 0$ on ∂S, we have that $\eta = 0$ on ∂S and therefore $\partial S_h = \partial S$ (for h sufficiently small). That is, E_h is an "admissible variation" for the problem at hand.

Hence the minimality of S implies

$$0 = \frac{d}{dh} \mathrm{Vol}_n(S_h)\Big|_{h=0} = \int_S H\varphi$$

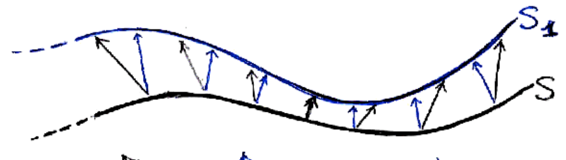
and since this identity holds for every choice of φ with $\varphi = 0$ on ∂S we deduce

$$H \equiv 0 .$$

Remark

Why did we consider only vectorfields η orthogonal to S? because considering non-orthogonal ones would not really give a larger class of variations S_h (at least if we keep the boundary fixed).

Indeed if S_h is the family generated by a vector field η, then "essentially the same" family can be obtained by replacing η by its normal component η_0



$\nwarrow \eta$ $\uparrow \eta_0$ = normal component of η

"Essentially the same" means that for the family $\tilde{S}_h$ generated by η_0 one has

$$\mathrm{Vol}_n(\tilde{S}_h) = \mathrm{Vol}_n(S_h) + O(h^2)$$

and then

$$\frac{d}{dh} \mathrm{Vol}_n(\tilde{S}_h)\Big|_{h=0} = \frac{d}{dh} \mathrm{Vol}_n(S_h)\Big|_{h=0} .$$

2.8 Variation of the area with prescribed volume

Consider now the following variation of the Plateau's problem:
among all set with prescribed volume find the one which minimizes the area of the boundary (isoperimetric problem).
If A is a solution of this problem and

S is the boundary of A, then at every point of S there holds

$$H = \text{constant}$$

Consider as before the family of boundaries S_ε associated to a normal vector field $\eta = \varphi . N$, and denote by A_ε the corresponding interiors.

Since $\frac{d}{d\varepsilon} \mathrm{Vol}(A_\varepsilon)\Big|_{\varepsilon=0} = \int_S \varphi$

(we omit this computation, which is quite similar to some of the previous ones), then the admissible variations must satisfy $\mathrm{Vol}(A_\varepsilon) = $ constant, that is

$$(*) \qquad \int_S \varphi = 0$$

And conversely for any φ s.t. (*) holds, even if $\mathrm{Vol}(A_\varepsilon)$ is not constant, we can modify A_ε "slightly" so that the volume is constant.

Hence the minimality of A implies that

$$\frac{d}{d\varepsilon} \mathrm{Area}(S_\varepsilon)\Big|_{\varepsilon=0} = \int_S H\varphi = 0$$

for all φ s.t. (*) holds, that is, H is constant.

We can obtain the equation H = constant also using Lagrange multipliers: indeed minimizing Area(S) under the constraint Vol(A) = const., is "equivalent" to minimizing Area(S) − λ Vol(A) (well, it is a matter of critical points, not minimizers,...) and by the previous computations

$$\frac{d}{d\varepsilon}\left[\mathrm{Area}(S_\varepsilon) - \lambda \mathrm{Vol}(A_\varepsilon)\right] = \int_S (H-\lambda)\varphi$$

and imposing this to be null for all admissible variations implies $H - \lambda = 0$ for some λ, that is, H = constant.

2.9 Normal and geodesic curvature of a curve

Let γ be a curve in $\mathbb{R}^3$ parametrized by arc-length.
The the orienting unitary! tangent vector is

$$\tau := \dot\gamma$$

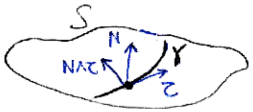
and the total curvature

$$\dot\tau = \ddot\gamma$$

is orthogonal to τ. ← derive the identity $\langle\tau,\tau\rangle = 1$

Now, if γ lies on the surface S, N, τ, $N \wedge \tau$ form an orthonormal system (at every point p in γ).

In particular we can write $\ddot{\tau}$ as combin. of N and $N \wedge \tau$:



$$\ddot{\tau} = c_1 N + c_2 (N \wedge \tau)$$

c_1: called Normal curvature of γ and denoted by k_n

c_2: called Geodesic curvature of γ and denoted by k_g

$$= k_n \cdot N + k_g \cdot (N \wedge \tau)$$

with respect to the parameter of γ

Deriving the identity $\langle \tau, N \rangle = 0$ we get $\langle \dot{\tau}, N \rangle + \langle \tau, \dot{N} \rangle = 0$, that is

$$\begin{aligned} k_n = \langle \dot{\tau}, N \rangle &= -\langle \tau, \dot{N} \rangle \\ &= -\langle \tau, dN \cdot \dot{\gamma} \rangle \\ &= \langle \tau, -dN\, \tau \rangle \\ &= \mathrm{II}_p(\tau) \qquad p = \gamma \end{aligned}$$

Thus

$$\ddot{\tau} = \underbrace{\mathrm{II}_p(\tau) \cdot N}_{} + \underbrace{k_g \, N \wedge \tau}_{}$$

$\mathrm{II}_p(\tau) \cdot N$: depends on S but not on γ, it's not "intrinsic"

$k_g \, N \wedge \tau$: depends on γ and S and is "intrinsic"

2.10 Geodesic curvature and first variation of the length

Let γ be a curve in $\mathbb{R}^3$ (parametrized by arc-length (as always!)) and let η be a normal vector field ($\dot{\gamma} \cdot \eta = 0$).

For every h set

$$\gamma_h(t) := \gamma(t) + h\,\eta(t).$$

Then

$$\begin{aligned} \text{Length}(\gamma_h) &= \int |\dot{\gamma}_h| \quad \text{integrated w.r.t. } dt \text{ or length meas. on } \gamma \\ &= \int \sqrt{|\dot{\gamma}|^2 + h^2 |\dot{\eta}|^2 + 2h\, \dot{\gamma} \cdot \dot{\eta}} \\ &= \int \sqrt{1 + 2\, \tau \cdot \dot{\eta}\, h + O(h^2)} \\ &= \int 1 + (\tau \cdot \dot{\eta})\, h + O(h^2) \end{aligned}$$

and then

$$\boxed{\frac{d}{dh} \text{Length}(\gamma_h) = \int \tau \cdot \dot{\eta} = -\int \dot{\tau} \cdot \eta}$$

Assume now that γ lies on the surface S and minimizes length among all curves on S with same endpoints. Then the admissible variations for γ are those for which η is tangent to S.

γ is a geodesic curve on S

Well, this statement requires some justification.....

In other words $\eta = \varphi \cdot (N \wedge \dot{\tau})$ ← recall that $\eta \perp N$ and $\eta \perp \dot{\tau}$.
with φ an arbitrary real function.

Hence the minimality of γ implies

$$0 = \frac{d}{d\varepsilon} \mathrm{Length}(\gamma_\varepsilon)\Big|_{\varepsilon=0}$$

$$= -\int \dot{\tau} \cdot \eta = -\int k_g \cdot \varphi$$

and then

$$\boxed{k_g = 0.}$$

2.11 Gauss-Bonnet theorem

First version: global, no boundary

For every surface S in $\mathbb{R}^3$ compact and without boundary there holds

$$\text{(GB1)} \qquad \int_S \underset{\substack{\uparrow \\ \text{Gauss curvature}}}{\underline{K}} = 2\pi\, \chi(S)$$

where $\chi(S)$ is the Euler characteristic of S.

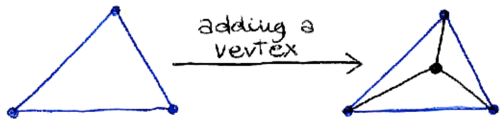
The Euler characteristic of a surface (with or without boundary) is computed — as for polyhedral surfaces — by taking a "reasonable" triangulation of the surface

and then setting

$$\chi(S) = \text{number of faces (triangles)} - \text{number of edges} + \text{number of vertices}.$$

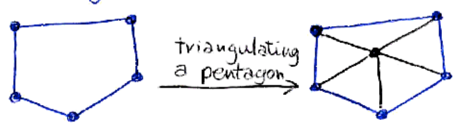
Remarks

- $\chi(S)$ does not depend on the triangulation: one shows that refining a triangulation does not change $\chi(S)$ and then uses that given two triangulations there exists another one finer than both



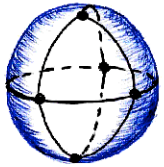
(basic step in refinement: added 2 faces, 3 edges, 1 vertex; χ is the same)

- One can use also other polygons than just triangles.



(triangulation of a pentagon: added 4 faces, 5 edges, 1 vertex: χ is the same)

- Euler characteristic of the sphere



$\chi(S^2) = 6 - 9 + 5 = 2$

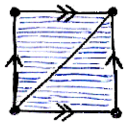
- Euler characteristic of a disc

$\simeq$

$\chi(D^2) = 1 - 3 + 3 = 1$

- Euler charateristic of the torus

$\simeq$



$\chi(T^2) = 2 - 3 + 1 = 0$

T^2 is homeomorphic to the square with opposite edegs identified

- Recall that the Gauss curvature K is the Jacobian determinant (with sign!) of the Gauss map $N : S \to S^2$. Hence the oriented are formula yields

$$\int_S K = \int_{S^2} \deg(N,p)\, dp .$$

If $\partial S = \emptyset$ then the degree $\deg(N,p)$ does not depend on the point p

and therefore

$$\int_S K = \deg(N) \cdot \mathrm{Area}(S^2) = \deg(N) \cdot 4\pi .$$

This argument is almost sufficient to show that $\int_S K$ is a topological invariant: if S and S' are isotopic (that is, there exists a one-parameter family of embedded surfaces S_t such that $S = S_0$ and $S' = S_1$) then $\deg(N) = \deg(N')$ and therefore $\int_S K = \int_{S'} K$.

Second version of Gauss-Bohnet: local with bdry

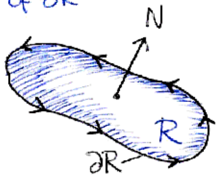
Let R be an embedded disc. Then

$$\text{(GB2)} \qquad \int_R K + \int_{\partial R} k_g = 2\pi$$

Gauss curvature of R

boundary of R, with the canonical orientation induced by R (see figure)

geodesic curvature of ∂R



Remarks

- Using the last formula one can compute the Gauss curvature of a surface S at a point p by taking a small disc-like neighbourhood of p:

$$K(p) \simeq \frac{1}{\mathrm{Area}(R)}\int_R K = \frac{2\pi - \int_{\partial R} k_g}{\mathrm{Area}(R)}$$

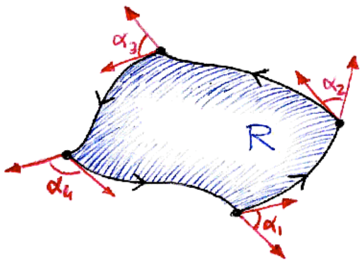
error tending to 0 as diam(R) tends to 0

Now, the geodesic curvature is "intrinsic" (can be computed using only the notion of distance on S, that is, the metric); this formula shows that the Gauss curvature is intrinsic, too. This is Gauss's Theorema Egregium.

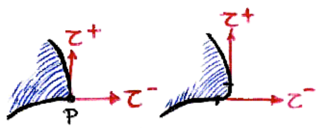
- Formula (GB2) can be extended to the case ∂R is piecewise smooth (we admits corners):

$$(\mathrm{GB2}') \qquad \int_R K + \int_{\partial R} k_g + \sum \alpha_i = 2\pi$$

The term $\sum \alpha_i$ accounts for the geodesic curvature of ∂R "concentrated" at the corners.



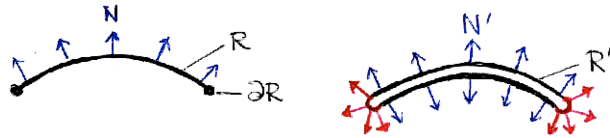
Formula (GB2') can be obtained from formula (GB2) by "smoothing out" corners.



Smoothing a corner p

Alternatively, note that the angle α_i correspond to the "jump" of the tangent vector τ to ∂R at a corner point p_i — that is, the distance on the sphere S^2 between the tangent vectors τ^+ and τ^- at the two sides of p_i. Formally this "jump" corresponds to the derivative of τ (and therefore the curvature of ∂R) concentrated at p_i.....

- To prove (GB2), consider the sphere-like surface R' obtained by taking two copies of R and glueing them at the boundary as in the figure



Then a standard computation yields

$$2\left[\int_R K + \int_{\partial R} k_g\right] = \int_{R'} K$$

$$= \int_{S^2} \deg(N', p)\, dp$$

proceed as before →

$$= \deg(N') \cdot 4\pi = 4\pi$$

if R is "sufficiently flat" then $\deg(N')$ is obviously 1

Third version of G.-B.: global with bdry

Let S be any compact surface with boundary. Then

$$(GB3) \qquad \int_S K + \int_{\partial S} k_g = 2\pi\, \chi(S)$$

∂S: canonically oriented

$\chi(S)$: Euler charct. of S

- It's obvious how to modify this formula to include piecewise smooth boundaries.

- Formula (GB3) can be proved by taking a triangulation of S and applying (GB2') to each triangle.

- Formula (GB1) is a particular case of (GB3).