# LECTURE 3

In this lecture I will present some non-parametric approaches to minimal surfaces (that is, not based on the idea of parametrizing the surface ~ representing surfaces as maps from a planar domain into the space).
First I will discuss the possibility of a direct strategy, namely minimizing the Hausdorff measure $\mathcal{H}^d$ in suitable classes of sets: we will see that this can be done in a relatively simple way only for $d=1$.
I will then turn to finite perimeter sets.
For lack of time, I will not touch the other main non-parametric approach, namely the one based on the theory of rectifiable currents.

## 3.1 Naive approach

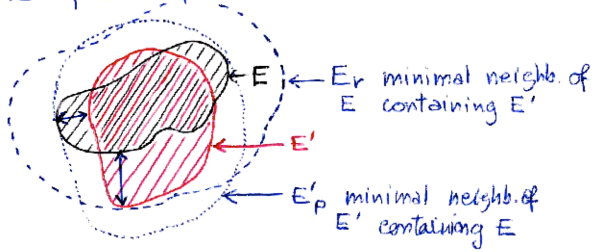
Let $\mathcal{F}$ be the class of all non-empty closed, connected subsets of a fixed compact domain $D$ in $\mathbb{R}^n$.
On $\mathcal{F}$ we consider the Hausdorff distance

$$d_H(E,E') := \inf\left\{ r\in[0,+\infty] \text{ s.t. } E\subset E'_r \;\&\; E'\subset E_r \right\}$$

here $E_r$ is the (open) $r$-neighbourhood of $E$, that is, $E_r := \bigcup_{x\in E} B(x,r)$.



So a set $E'$ is "close" to $E$ in the Hausdorff distance if every point of $E$ is close to some point of $E'$ and every point of $E'$ is close to some point of $E$:




In this example: $d_H(E,E') := r\wedge\rho$

**Compactness:** the class $\mathcal{F}$ with the distance $d_H$ is compact. Moreover for any set $\Gamma\subset D$, the subclass

$$\mathcal{F}_\Gamma := \{E\in\mathcal{F} \text{ s.t. } E\supset\Gamma\}$$

is closed in $\mathcal{F}$, and therefore compact, too.

The key step in the proof is showing that the distance $d_H$ is complete (on $\mathcal{F}$) and totally bounded. Both claims are rather straightforward.

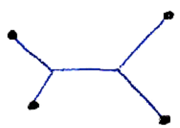
**Semicontinuity:** the 1-dimensional Hausdorff measure $\mathcal{H}^1$ is lower-semicontinuous on $\mathcal{F}$.

This result is known as Gołąb theorem. The proof is rather delicate.

Putting together this semicontinuity and compactness result we immediately obtain

**Existence of sets of minimal length:** for every $\Gamma$ contained in $D$ there exists a connected, closed set $E$ which minimizes the length ($\mathcal{H}^1$) among all closed, connected set that contains $\Gamma$.

Well, of course we must assume that $\Gamma$ is contained in at least one such $E$ with finite length.
The constraint $E$ be connected is required to make the problem non trivial, together with the inclusion $\Gamma \subset E$, it replaces the condition "$\Gamma$ is the boundary of $E$", which does not make sense in this context....
Using this result one can show for instance that for every bounded $\Gamma$ in $\mathbb{R}^n$ there exists a length minimizing compact, connected set $E$ containing $\Gamma$ (one has only to observe that given any closed ball $D$ which contains $\Gamma$, it suffices to consider $E$ contained in $D$, because in any case replacing $E$ with its projection $E'$ on $D$ reduces the length!)



• $\Gamma$

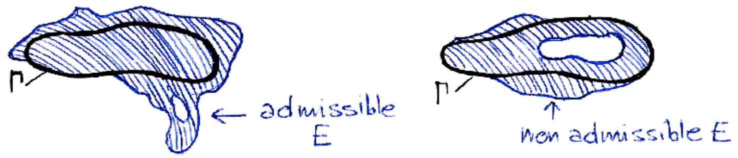
— $E$ : minimal "network" connecting $\Gamma$.



## 3.2 What about area-minimization?

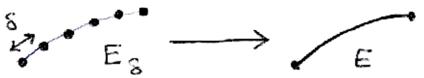
Can we apply a similar strategy to solve Plateau's problem?
The first issue would be to find a correct formulation for Plateau's problem when surfaces are replaced by sets (what would be the boundary of a set?). !
One possibility is:

> Given a closed curve $\Gamma$ in $\mathbb{R}^3$, find a compact set $E$ which minimizes the area $\mathcal{H}^2(E)$ among all set $E$ s.t. $\Gamma$ is homotopic to a constant in $E$ (that is, $\Gamma$ can be shrinked to a point inside $E$).




This program, however, cannot be carried out as before.
! The main obstruction is not extending the compactness argument, but rather the semicontinuity theorem.

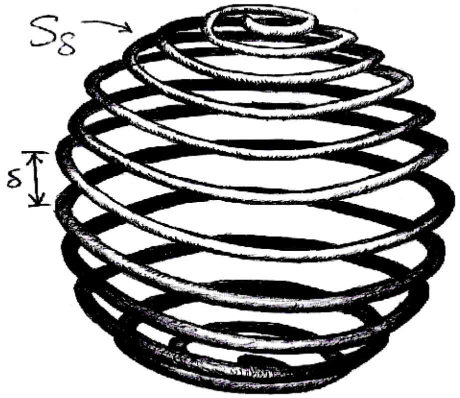
We first remark that the semicontinuity of $\mathcal{H}^1$ on $\mathcal{F}$ stated before depends heavily on the connectedness of the sets in $\mathcal{F}$. Indeed if we drop this assumption semicontinuity fails:




every set E can be approximated in $d_H$ by disrete sets $E_\delta$ and therefore for every $d>0$ we have

$$\lim_{\delta\to 0} \mathcal{H}^d(E_\delta) = 0 < \mathcal{H}^d(E)$$

The real problem is that no topological assumption can ensure the semicontinuity of $\mathcal{H}^2$ (or any $\mathcal{H}^d$ with $d>1$).
Indeed the sphere $S^2$ can be approximated by sphere-like surfaces $S_\delta$ such that $\mathcal{H}^2(S_\delta) \longrightarrow 0$:




$S_\delta$ is a "tube" spiraling down $S^2$. It's long (order $\delta^{-1}$) but if the cross-section is sufficently thin (order $\delta^2$) then the total area tends to 0.



It should be remarked that, despite the failure of lower-semicontinuity, a direct approach to area-minimization along this line has been carried out by E.R. Reifenberg (in the 60's); however the proof of his existence result is rather complicated.

## 3.3 Variants of Plateau's problem

Finite perimeter sets are particularly suited for solving problems related to area minimization which are slightly different from Plateau's one.
A typical problem is

> Find the domain D in $\mathbb{R}^3$ which minimizes
>
> Area$(\partial D)$ + additional integral term (e.g. $\int_D f(x)\,dx$)
>
> \+ additional constraint (e.g. Volume(D) is prescribed.)

### Plan for solving this problem

We want to solve this problem by the usual compactness-and-semicontinuity approach (the so-called direct method in the Calculus of Variations).

To this end we would like to define

a) a class $\mathcal{F}$ of sets $E$ in $\mathbb{R}^d$ endowed with a topology with good compactness properties so that sets with smooth boundaries belong to $\mathcal{F}$ and are dense;

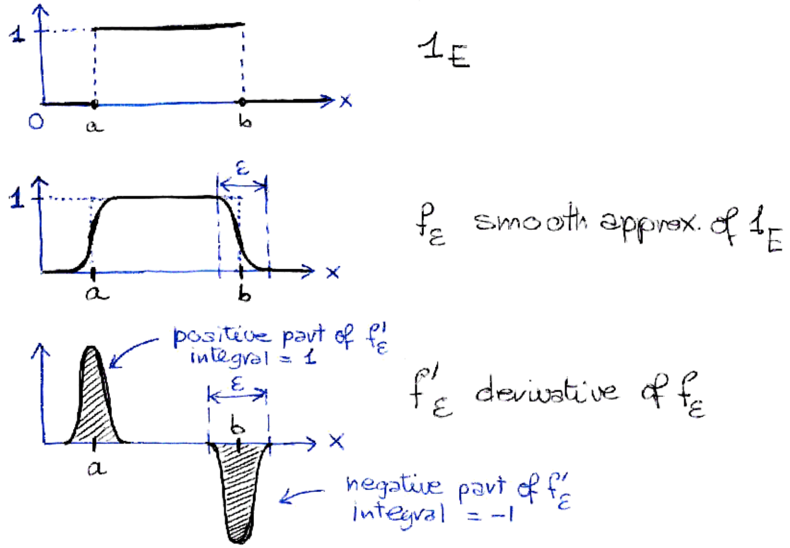
b) a notion of "perimeter" $P(E)$ for every $E$ in $\mathcal{F}$ so that $E \mapsto P(E)$ is lower-semi-continuous on $\mathcal{F}$, and $P$ extends the usual notion of area ($\mathcal{H}^{d-1}$) of the boundary; more precisely we require that $P(E) = \mathcal{H}^{d-1}(\partial E)$ for every $E$ with smooth boundary and for every $E \in \mathcal{F}$ there exists a sequence $E_n \to E$ with smooth boundaries and satisfying $\mathcal{H}^{d-1}(\partial E) \longrightarrow P(E)$.

This is exactly what finite perimeter sets do!



## 3.6 Finite perimeter sets
(R. Caccioppoli, E. DeGiorgi)

The key observation behind this definition is that the boundary of a set is related to the distributional derivative of its charcteristic function:
If $E$ is the interval $[a,b]$ in $\mathbb{R}$ then the distributional derivative of $1_E$ is given by a Dirac mass at $a$ minus a Dirac mass at $b$ (this can be easily checked by approximating $1_E$ by smooth functions:




In the following we assume some familiarity with the notion of distributional derivative!

## DEFINITION

A Borel set $E$ in $\mathbb{R}^d$ (with finite Lebesgue measure) has finite perimeter if the distributional derivative of $1_E$ is a vector-valued measure on $\mathbb{R}^d$

$\boxed{\text{i.e.}}$

there exist $\mu = (\mu_1, \dots, \mu_d)$ with $\mu_i$ a real-valued Radon measure s.t.

$$\int_E \frac{\partial \varphi}{\partial x_i} = \int_{\mathbb{R}^d} \frac{\partial \varphi}{\partial x_i} 1_E = -\int_{\mathbb{R}^d} \varphi \, d\mu_i$$

for every test function $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$\boxed{\text{or, equivalently}}$

here we integrate w.r.t. Lebesgue measure

$$\int_E \operatorname{div} \phi = \int_{\mathbb{R}^d} \operatorname{div} \phi \cdot 1_E = -\int_{\mathbb{R}^d} \sum_i \phi_i \, d\mu_i$$

$$\| \quad \int_{\mathbb{R}^d} \langle \phi, \eta \rangle \, d|\mu|$$

for every test vector-field $\phi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$.

as usual, it's convenient to write the vector-valued measure $\mu$ as the product of the positive bounded measure $|\mu|$ and a unit vectorfield $\eta$

The measure $\mu$ is unique and therefore is denoted by $D1_E$.



We then define

$$\text{Perimeter of } E = P(E)$$

$$:= \| D1_E \| = |\mu|(\mathbb{R}^d)$$

total variation of the vector measure $D1_E$

$$= \sup_{|\phi| \le 1} \int_{\mathbb{R}^d} \langle \phi \cdot \eta \rangle \, d|\mu|$$

here we use the definition of distributional derivative.

$$= \sup_{|\phi| \le 1} \int_E \operatorname{div} \phi \, .$$

Note that $\phi$ is of class $C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$ or, equivalently $C_c^1(\mathbb{R}^d, \mathbb{R}^d)$.

The class $\mathcal{F}$ of all finite perimeter sets in $\mathbb{R}^n$ is endowed for the rest of this lecture with the distance inherited by the immersion in $L^1$

$$d(E, E') := |E \Delta E'| = \| 1_E - 1_{E'} \|_{L^1(\mathbb{R}^d)}$$

symmetric difference: $E \Delta E' := (E \setminus E') \cup (E' \setminus E)$

## 3.5 Remarks, examples, basic properties

- If $E$ is a bounded (open) set in $\mathbb{R}^d$ with smooth boundary, then the divergence theorem yields

$$\int_E \operatorname{div} \phi = -\int_{\partial E} \langle \phi \cdot \nu_E \rangle \, d\mathcal{H}^{d-1}$$

for every vector-field $\phi$ of class $C^1$ on $\mathbb{R}^d$, where $\nu_E$ is the inner (unit) normal of $\partial E$.

This formula (and the uniqueness of the distributional derivative) implies that $E$ has finite perimeter and

$$D1_E = \nu_E \cdot 1_{\partial E} \cdot \mathcal{H}^{d-1},$$

where $\mu = 1_{\partial E} \cdot \mathcal{H}^{d-1}$ is the restriction of the measure $\mathcal{H}^{d-1}$ to the set $1_{\partial E}$.

Hence

$$P(E) = \mu(\mathbb{R}^d) = \mathcal{H}^{d-1}(\partial E) .$$

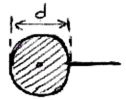
Thus smooth sets have finite perimeter, and the perimeter agrees with the area of the boundary. As required!



- The previous remark can be extended to sets $E$ with Lipschitz boundary (thus including sets with piecewise $C^1$ boundary).

- If $E$ and $E'$ are Lebesgue equivalent (that is $E \Delta E'$ is negligible) then they have the same distrib. deriv. and perimeter. (Indeed the definition of distrib. deriv. involves $E$ only on the level of integration w.r.t. Lebesgue meas.)

- If $E$ is a null set ($\mathcal{L}^d(E)=0$) then $P(E)=0$. (Indeed $E$ is equivalent to the empty set, and therefore has the same perimeter, which is clearly $0$.
This shows in particular that for nonsmooth sets the perimeter is not the area ($\mathcal{H}^{d-1}$) of the boundary (in general there holds $P(E) \le \mathcal{H}^{d-1}(\partial E)$).

- If $E =$  in $\mathbb{R}^2$ then $E$ is equivalent to the disc $E' =$  and therefore $P(E) = P(E') = \pi d$ (the segment does not count).
The same argument applies to the disc minus a diameter.

- COMPACTNESS AND SEMICONTINUITY.

compactness

Let $E_n$ be a sequence of finite perimeter sets contained in the same ball (i.e., uniformly bounded) and with uniformly bounded perimeters. Then there exists a subsequence $E_{n_k}$ such that

$$E_{n_k} \longrightarrow E \quad \text{(in the } L^1 \text{ distance)}$$

where $E$ has finite perimeter.

semicontinuity

Moreover if $E_n \to E$ then

$$\liminf_{n\to\infty} P(E_n) \geq P(E)$$

- APPROXIMATION BY SMOOTH SETS.

For every finite perimeter set $E$ there exists a sequence of smooth sets $E_n$ such that $E_n \to E$ and $P(E_n) \to P(E)$

$$\overset{\shortparallel}{\mathcal{H}^{d-1}(E_n)}$$

- Using these compactness and semicontinuity results one can easily prove the existence of solutions for many variants of Plateau's problem in the setting of finite perimeter sets, e.g.: minimize $F(E) := P(E) + \int_E f(x)\,dx$ among all finite perimeter sets $E$ contained in a given bounded domain $\Omega$ and with



prescribed volume $v$.

Indeed, the volume constraint is closed w.r.t. the ($L^1$) distance in $\mathcal{F}$, while the additional volume integral $\int_E f$ is continuous (we assume that $f \in L^1(\Omega)$), and therefore $F$ is lower-semicontinuous on a class of admissible sets which is compact.

Incidentally, this example shows that the semicontinuity-and-compactness approach is rather robust (adding certain additional terms and constraints requires no additional effort).

- CONCERNING PROOFS: the semicontinuity of perimeter is an immediate consequence of its formulation as supremum of continuous functionals. The compactness result is a direct consequence of the compact embedding of the Banach space BV in $L^1$; the space BV of functions with bounded variations (in $\mathbb{R}^d$) is akin to Sobolev spaces, in particular $W^{1,1}$, and is a dual of a separable space; therefore it enjoys the compactness properties ensured by the Banach–Alaoglu theorem. The proof of the approximation theorem is slightly more complicated: the idea is

these results are "easy" !!

to regularize by convolution the characteristic function $1_E$ and choose a suitable superlevel $E_\varepsilon^t$ of the regularized function $f_\varepsilon := 1_E * \rho_\varepsilon$: indeed by Sard theorem almost all superlevel sets $E_\varepsilon^t := \{x : f_\varepsilon(x) \geq t\}$ have a smooth boundary; moreover one has

$$\| 1_E - f_\varepsilon \|_1 = \int_0^1 |E \Delta E_\varepsilon^t| \, dt$$

and

$$\|\nabla f_\varepsilon\|_1 = \int_0^1 \mathcal{H}^{d-1}(f_\varepsilon = t)\, dt = \int_0^1 P(E_\varepsilon^t)\, dt$$

(the second identity is a particular case of the "coarea formula" for smooth functions), and since as $\varepsilon \to 0$

$$\| 1_E - f_\varepsilon \|_1 \to 0, \quad \|\nabla f_\varepsilon\|_1 \to P(E)$$

it is clearly possible to choose $t$ for every $\varepsilon$ so that $|E \Delta E_\varepsilon^t| \to 0$ and $P(E_\varepsilon^t) \to P(E)$.

- I would like to underline the relevance of the approximation result: it ensures that, at some level, the minimizer we find in the class of finite perimeter sets is not just an abstract object with no relation with the original problem!



- The original definition of finite perimeter set in $\mathbb{R}^d$ (due to R. Caccioppoli) is slightly different from ours but turns out to be equivalent: a set $E$ in $\mathbb{R}^d$ has finite perimeter if the quantity

$$P(E) := \sup_{|\phi| \leq 1} \int_E \operatorname{div} \phi$$

is finite.

- We have only defined finite perimeter sets in $\mathbb{R}^d$. However one can define finite per. sets in any open set $\Omega \subset \mathbb{R}^d$. In this case the perimeter of $E$ in $\Omega$, $P(E,\Omega)$, is essentially the area of the boundary of $E$ relative to $\Omega$, that is, the part of $\partial E$ contained in $\partial\Omega$ is not counted (we are speaking of sets with regular bdry).

- REGULARITY RESULTS. To close the circle one would like to show that the minimizer found in the class of finite perimeter sets is actually smooth, and therefore solves the minimization problem in the original class of sets with smooth bdry.
However regularity results are notoriously quite hard to prove (in some sense, this theory "hides" here some of the deep

techinical difficulties inherent to Plateau's problem). Moreover regularity results are necessarily partial. To explain the situation, consider the simplest issue: the regularity of a set $E$ in $\mathbb{R}^d$ which minimizes the perimeter w.r.t. all compact-support perturbations, that is, for <u>every</u> ball $B$ in $\mathbb{R}^d$ $E$ has finite per in $B$, and for <u>every</u> set $E'$ s.t. $E \Delta E'$ is relatively compact in $B$ there holds

$$P(E,B) \leq P(E',B) \quad \longleftarrow \text{perimeter of } E' \text{ in } B$$

Then, modulo replacing $E$ with a suitably chosen set which is Lebegue-equivalent, the closure of the essential bdry agrees with the topolog. bdry and $\mathcal{H}^{d-1}(\partial E \setminus \partial_* E) = 0$. Moreover if $d \leq 7$ then $\partial E$ is smooth (in fact, analytical). However, if $d \geq 8$ then it can only be proved that $\partial E \setminus S$ is smooth (in $\mathbb{R}^d \setminus S$) where $S$ is a closed set with $\dim(S) \leq 8$. An example showing that this result is somewhat optimal is the cone

$$E := \{(x,y) \in \mathbb{R}^4 \times \mathbb{R}^4 \text{ s.t. } |x| = |y|\}.$$

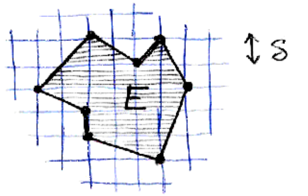
This example shows that, no matter which framework one chooses, minimal surfaces in high dimension cannot be smooth: indeed if $B$ is the unit ball in $\mathbb{R}^8 = \mathbb{R}^4 \times \mathbb{R}^4$ with center $0$ and radius $1$, and we set $\Gamma := \partial E \cap \partial B$, then every minimizing sequence of 7-dim. surfaces $S_n$ with $\partial S_n = \Gamma$ converge to the singular cone $S := \partial E \cap \bar{B}$.



This regularity result is due to many authors: DeGiorgi, Fleming, Federer, Simons. The cone example was conjectured by Simons and its minimality was finally proved by Bombieri, DeGiorgi and Giusti.

## – APPROXIMATION BY DISCRETE PROBLEMS

Besides proving regularity results, another way to ensure that finding a solution of a minimum problem in the class of finite perimeter sets is not a purely abstract result with little relation to the original problem is to show that the same results would be obtained by any (reasonble) discrete approximation of the original problem. One possible result of this kind (perhaps not the most natural, but one easy to state) is the following: in $\mathbb{R}^2$, fix a grid of parameter $\delta > 0$ and denote by $\mathcal{F}_\delta$ the class of all polygons $E$ with vertices on the grid, $\longrightarrow$ and minimize your preferred functional $F$ on the class $\mathcal{F}_\delta$




(e.g., $F(E) = \mathrm{Per}(E) + \int_E f(x)\,dx$).
Then the minimizers $E_\delta$ converge as $\delta \to 0$

to minimizers of $F$ on $\mathcal{Y}$, while the minimum of $F$ on $\mathcal{Y}_s$ converges to the minimum on $\mathcal{Y}$. (Making this statement precise requires some additional care ....)

## 3.6 Appendix: structure of finite perimeter sets

We conclude this brief review of the theory of finite perimeter sets by describing the structure theorem: the question addressed by this theorem is to what extent a finite perimeter set resembles a regular one; the answer is the first step in the theory of regularity for minimizing sets.

### An example

We begin by showing how "bad" a finite perimeter set can be.

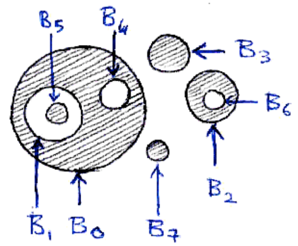
In the plane, choose a dense sequence $x_n$, and for every $n$ chose $r_n > 0$ such that: $0 < r_n < 2^{-n}$, $r_n \neq |x_n - x_m|$ $\forall m$, $r_n < \text{dist}(x_n, x_m)$ $\forall m < n$.

Let $B_n$ be the (open) disc with center $x_n$ and radius $r_n$.

Let $E_0 := B_0$, and $E_{n+1} := E_n \Delta B_{n+1}$ for



every $n$. Then take $E$ the limit (in any suitable sense) of the sets $E_n$.




$\longleftarrow E_7 = (\cdots((B_0 \Delta B_1) \Delta B_2) \Delta \cdots \Delta B_7)$

note that by construction every $B_{n+1}$ is either contained in $E_n$ or disjoint from $E_n$.

In particular each $E_n$ has smooth boundary and $\partial E_n = \partial B_0 \cup \dots \cup \partial B_n$. Hence

$$P(E_n) = 2\pi(r_0 + \cdots + r_n) \leq 2\pi\left(1 + \tfrac{1}{2} + \cdots + \tfrac{1}{2^n}\right) \leq 4\pi.$$

Therefore $P(E) \leq 4\pi$.

Moreover

$$D1_E = \lim_{n\to\infty} D1_{E_n} = \lim_{n\to\infty}\left(\sum_{m=1}^{n} \pm \nu_m \cdot 1_{\partial B_m} \cdot \mathcal{H}^1\right) = \sum_{m=1}^{\infty} \pm \nu_m \cdot 1_{\partial B_m} \cdot \mathcal{H}^1$$

the limit is intended in the sense of measures....

$\nu_m$: inner normal to $B_m$; the sign ($\pm$) depends on whether $B_m \subset E_{m-1}$ or $B_m \cap E_{m-1} = \emptyset$.

In particular, the support of $D1_E$ is the entire plane!

This is different from previous examples, where the distributional derivative of the characteristic function was actually supported on a smooth or piecewise smooth curve....

Keep this example in mind and compare it with the following result:

## Structure theorem for finite perimeter sets (De Giorgi, Federer)

Let $E$ be a finite perimeter set in $\mathbb{R}^d$. Then there exist a set $F$ and a unit vectorfield $\nu$ defined on $F$ such that

0) $F$ and $\nu$ are Borel regular (of course!)

1) $F$ is $\mathcal{H}^{d-1}$ finite and rectifiable i.e., $F = F_0 \cup F_1 \cup F_2 \cup \dots$ where $\mathcal{H}^{d-1}(F_0) = 0$, and $F_n$ is contained in a hypersurface of class $C^1$ for every $n \geq 1$;

2) the density of $E$ at any $x_0 \notin F$ is either $0$ or $1$, that is

$$\exists \lim_{r \to 0} \frac{\mathcal{L}^d(E \cap B(x_0,r))}{\mathcal{L}^d(\underbrace{B(x_0,r)}_{\text{ball with center } x_0 \text{ and radius } r})} = 1 \text{ or } 0;$$

3) the density of $E$ at $\mathcal{H}^{d-1}$ a.e. $x_0 \in F$ is equal to $1/2$;



4) Setting $E_{x_0,r} := \frac{1}{r}(E - x_0) = \left\{\frac{1}{r}(x - x_0) \text{ s.t. } x \in E\right\}$, then for $\mathcal{H}^{d-1}$ a.e. $x_0 \in F$ there holds

$$1_{E_{x_0,r}} \longrightarrow 1_{H_{\nu(x_0)}} \quad \text{in } L^1_{loc}(\mathbb{R}^d)$$

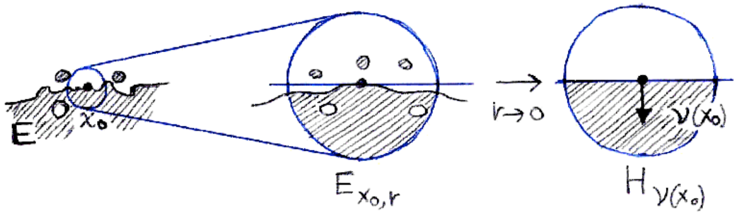
where $H_{\nu(x_0)} = \left\{x \in \mathbb{R}^d \text{ s.t. } \langle x, \nu(x_0)\rangle \geq 0\right\}$.

5) $D1_E = \nu \cdot 1_F \cdot \mathcal{H}^{d-1}$

## Remarks

- Statement 2) is summarized by saying that $F$ is the "measure theoretic boundary" or "essential boundary" of the set $E$.
  The essential boundary is often denoted by $\partial_* E$. The fact that $\mathcal{H}^{d-1}(\partial_* E)$ is finite is specific of finite perimeter sets (in fact, it is a characterization of f. p. sets); for a generic Borel set one would only have $\mathcal{L}^d(\partial_* E) = 0$ (by Lebesgue theorem on points of approximate continuity for $L^1$ functions).

- Statements 3) and 4) make the structure of the essential boundary more precise: by enlarging (or "blowing-up") $E$ close to a generic point $x_0$ of $F = \partial_* E$, it resembles more and more (as $r \to 0$) the half-plane defined by the direction $\nu(x_0)$. Thus $\nu(x_0)$ is rightly called the "measure theoretic" or "approximate" inner normal to $E$ at $x_0$




- Statement 5) shows that the usual formula for the distribut. derivative of $1_E$ when $\partial E$ is smooth can be extended to any finite per. set $E$ provided that we replace the topological bdry $\partial E$ by the essential bdry $\partial_* E$, and the inner normal by the approximate inner normal.

- Strictly related to the essential bdry of $E$ is the "reduced bdry" $\partial^* E$, which is defined as the set of points of approximate continuity of the approximate inner normal $\nu$. It can be proved that $\partial_* E$ and $\partial^* E$ differ only by an $\mathcal{H}^{d-1}$ negligible set.



- The proof of the structure theorem relies on deep results from measure theory (essentially the existence of points of approximate continuity with respect to arbitrary (finite) measures), and is somewhat more complicated than the proof of the semicontinuity and compactness results, which are based only on tools from functional analysis.

- Using the structure theorem one can prove the following "Lusin type" approx. result: given a finite per. set $E$ in $\mathbb{R}^d$ and $\varepsilon > 0$, there exists a set $E'$ with boundary of class $C^1$ such that

$$\mathcal{L}^d(E \Delta E') \le \varepsilon,$$
$$\mathcal{H}^{d-1}(\partial_* E \Delta \partial E') \le \varepsilon,$$
$$\nu_E = \nu_{E'} \text{ at } \mathcal{H}^{d-1} \text{ a.e. } x \in \partial_* E \cap \partial E'.$$

(This result is essentially due to Federer).