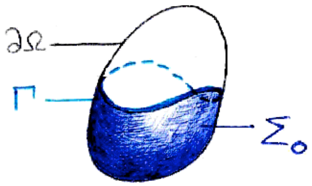
LECTURE 4

In this lecture I will briefly show how to use the theory of finite perimeter sets in a concrete (and simple) situation.

4.1 Finding surfaces with minimal area and prescribed boundary

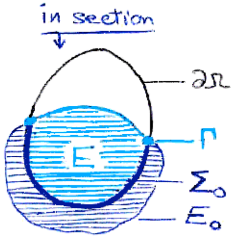
Let Ω be a bounded, open, convex set in $\mathbb{R}^3$ (more generally, in $\mathbb{R}^d$ ....)

Let Γ be a curve on $\partial\Omega$ which agrees with the boundary (relative to $\partial\Omega$) of a set Σ_0.



We first construct a smooth, bounded open set E_0 in $\mathbb{R}^3 \setminus \Omega$ such that $\partial E_0 \cap \partial\Omega = \Sigma_0$.

(We assume that $\partial\Omega$, Γ, Σ_0 are sufficiently regular .....)



Then we minimize $P(E)$ among all sets E with finite perimeter in $\mathbb{R}^3$ such that $E \setminus \Omega = E_0$.

The existence of a minimizer is ensured by the usual compactness theorem for finite perimeter sets and the fact that the constraint $E\setminus\Omega = E_0$ — or, more precisely, $\mathcal{L}^3((E\setminus\Omega)\Delta E_0) = 0$ — is closed.

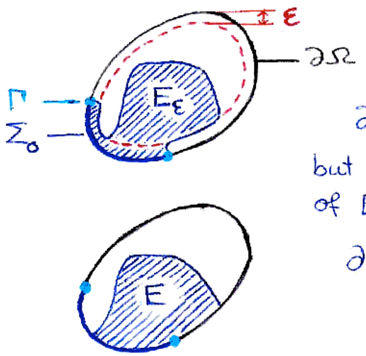
If E is a minimizer, regularity theory shows that ∂E is smooth inside Ω and Lipschitz on $\partial\Omega$ (these results are not at all trivial) and then it follows that the surface $S := \partial E \cap \overline{\Omega}$ minimizes the area among all surfaces with bdry Γ.
To verify this one first notices that a competitor S' must be contained in the convex set Ω, otherwise projecting S' on Ω would reduce the area without modifying the boundary. Secondly, the surface $S' \cup (\partial E_0 \setminus \overline{\Omega})$ turns out to be a compact Lipschitz surface without boundary in $\mathbb{R}^3$, and therefore it is oriented and bounds a set E' with finite perimeter s.t. $E'\setminus\Omega = E_0$.

Remarks

- This approach imposes strong constraints on the geometry of the bounding curve Γ. The point is that finite perimeter sets are not really suited for Plateau's problem.

- This approach can be extended to higher dimension to obtain minimal hypersurfaces with prescribed boundary. However regularity cannot be expected if $d \geq 8$....

- One might wonder why we did not follow a simpler way, namely taking the set E which minimizes the perimeter among all sets E contained in $\overline{\Omega}$ such that $\boxed{\partial_* E \cap \partial\Omega = \Sigma_0}$
The reason is that the constraint $\partial_* E \cap \partial\Omega = \Sigma_0$, or better $\boxed{\mathcal{H}^2((\partial_* E \cap \partial\Omega) \Delta \Sigma_0) = 0}$, is not closed.
This can be reduced to the fact that the trace operator on the space BV is well-defined but, unlike what happens with Sobolev spaces, is only continuous w.r.t the norm topology, and not the dual topology. Here directly, consider the following example:

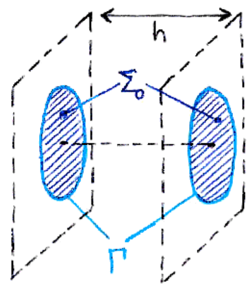


$\partial E_\varepsilon \cap \partial\Omega = \Sigma_0$
but for the limit E of E_ε as $\varepsilon \to 0$
$\partial E \cap \partial\Omega \neq \Sigma_0$

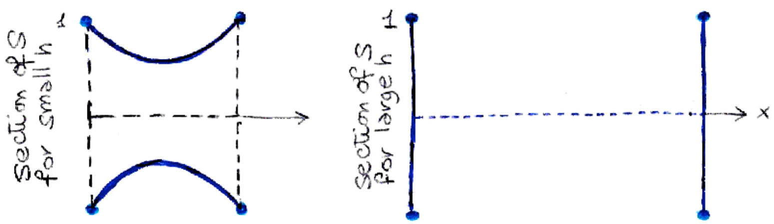
- The lack of closure of this constraint means in particular that if we take a minimizing

sequence E_n, there is no way to ensure that the limit E still satisfy the constraint, that is, that the surface $\partial E \cap \Omega$ has boundary equal to Γ.

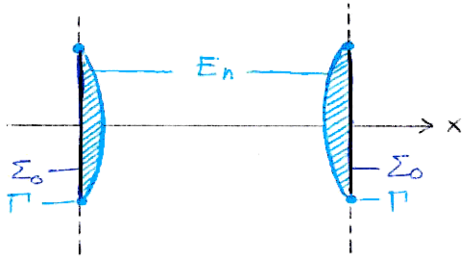
This is not just a technical problem, but corresponds to a "real" phenomenon which has already been introduced in prof. Hildebrandt lectures.
Let Ω be bounded by two parallel planes at distance h, and let Γ be the boundary of two discs of radius 1, one on each plane (Σ_0)



It can be proved that the area-minimizing surface S with bdry Γ is radially symmetric (wrt. the axis passing through the centers of the discs) and if h is sufficiently large it consists just of the two discs, that is, $S = \Sigma_0$

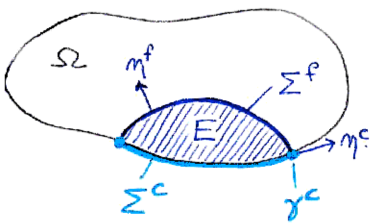


Now, if we minimize $P(E)$ among all sets $E \subset \overline{\Omega}$ s.t. $\partial_* E \cap \partial\Omega = \Sigma_0$, there exists no minimizer, and every minimizing sequence E_n converge (in the L^1-distance) to the empty set, for which $\partial^* E = \emptyset$ and $P(E)=0$.



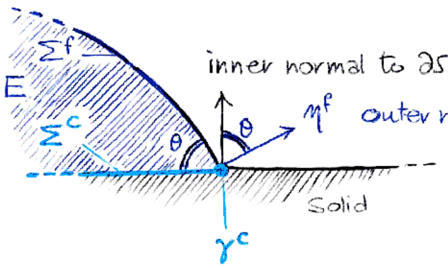
Viceversa everything works if we proceed as suggested at the beginning (in that case the minimizer is the set E_0, and the corresponding minimal surface is indeed $\partial E_0 \cap \partial\Omega = \Sigma_0$).

4.2 Capillarity – Setting of the problem



The "container" is an open, bounded, smooth set Ω in $\mathbb{R}^3$. A "drop" is represented by a set E in Ω (which we assume for the time being as smooth as needed). The free surface, or liquid-vapour interface, is $\Sigma^f := \partial E \cap \Omega$;

the contact surface, or liquid-solid interface, is $\Sigma^c := \partial E \cap \partial\Omega$. Finally the contact line γ^c is the common boundary of Σ^f and Σ^c.
At every point of the contact line γ_c we define the contact angle θ as in the picture:



The capillary energy associated to E is

$$\boxed{\begin{aligned} F = F(E) = {} & \sigma_{LV}\,|\Sigma^f| && \leftarrow \text{area of free surf.}\\ & + \sigma_{LS}\,|\Sigma^c| && \leftarrow \text{area of contact surf.}\\ & + \sigma_{VS}\,|\partial\Omega \setminus \Sigma^c| \\ & + V \end{aligned}}$$

$\uparrow$ area of the solid-vapour interface

$\uparrow$ volume contribution of the energy, e.g. integral over E of the gravitational potential.

σ_{LV} is a surface energy density and is "responsible" for surface tension.
σ_{LS} and σ_{VS} account for the interaction of air and liquid with the wall of the container.

the coefficients $\sigma_{LV}, \sigma_{LS}, \sigma_{VS}$ are assumed to satisfy the "wetting conditions"

$$|\sigma_{LS} - \sigma_{VS}| \leq \sigma_{LV}$$

Once this is assumed, we can define the Young angle $\theta_Y \in [0,\pi]$ by

$$\cos\theta_Y := \frac{\sigma_{VS} - \sigma_{LS}}{\sigma_{LV}}$$

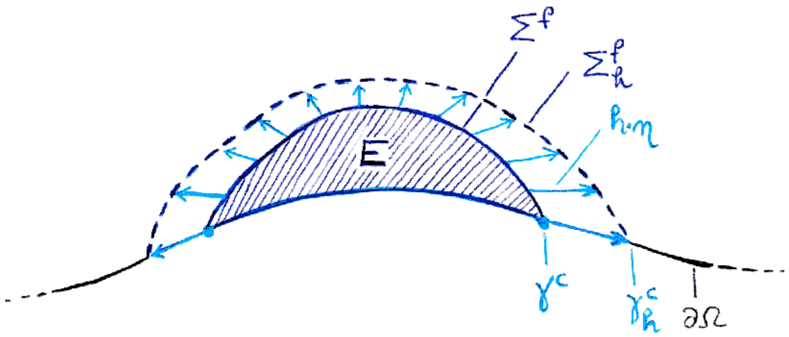
Hence, up to a constant the capillary energy F is given by

$$\boxed{F(E) = \sigma_{LV}\left(|\Sigma^f| - \cos\theta_Y\,|\Sigma^c|\right) + V}$$

4.3 First variation of capillary energy

A drop is at equilibrium if it is a (local) minimizer of the capillary energy at prescribed volume. To derive the equilibrium conditions we compute the first variation of the capill. energy (with respect to volume-preserving variations of E).

Given a drop E, consider a vectorfield η on Σ^f which is tangent to $\partial\Omega$ at every point of $\gamma^c = \Sigma^c \cap \partial\Omega$, and for every h (...) let E_h be the variation of E obtained by moving each point $x \in \Sigma^f$ by $h\,\eta$.



For every h, we denote by Σ^f_h, Σ^c_h and so on the free surface, the contact surface etc. of E_h.

Let us compute the corresponding variations of the various geometric quantities involved in the definition of the capillary energy F. To this end we write

$$\varphi^f := \langle \eta ; \eta^f \rangle \quad \text{normal component of } \eta \text{ on } \Sigma^f$$

$$\varphi^c := \langle \eta ; \eta^c \rangle \quad \text{component of } \eta \text{ tangent to } \partial\Omega \text{ on } \gamma^c$$

Then the usual computations yield

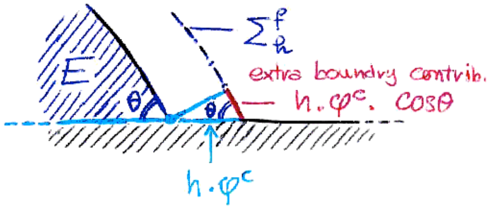
$$\underbrace{\frac{d}{dh}|\Sigma^c_h|}_{\text{area of } \Sigma^c_h}\Big|_{h=0} = \int_{\gamma^c} \varphi^c$$

$$\underbrace{\frac{d}{dh}|E_h|}_{\text{volume of } E_h}\Big|_{h=0} = \int_{\Sigma^f} \varphi^f$$

$$\underbrace{\frac{d}{dh}\left|\Sigma^f_h\right|\Big|_{h=0}}_{\text{area of } \Sigma^f} = -\int_{\Sigma^f} H\cdot\varphi^f + \int_{\gamma^c} \cos\theta\cdot\varphi^c$$

contact angle

This term does not appear in the usual formula for the variation of the area, and is due to the fact that η is not normal to Σ^f at the boundary of Σ^f:



Finally, if the volume term in the energy is

$$V = V(E) := \int_E v(x)\,dx$$

then

$$\frac{d}{dh} V(E_h)\Big|_{h=0} = \int_{\Sigma^f} v\cdot\varphi^f .$$

Putting together all these pieces we get

$$\boxed{\frac{d}{dh} F(E_h)\Big|_{h=0} = \int_{\Sigma^f}\left(-\sigma_{LV} H + v\right)\varphi^f + \int_{\gamma^c}\sigma_{LV}\left(\cos\theta - \cos\theta_Y\right)\varphi^c}$$

From these computation it follows that

the equilibrium condition for a drop with prescribed volume is:

$$0 = \frac{d}{dh} F(E_h)\Big|_{h=0} - \underbrace{\lambda\frac{d}{dh}\left|E_h\right|\Big|_{h=0}}$$

Lagrange multiplier due to the volume constraint

$$= \int_{\Sigma^f}\left(-\sigma_{LV} H + v - \lambda\right)\varphi^f + \int_{\gamma^c}\sigma_{LV}\left(\cos\theta - \cos\theta_Y\right)\varphi^c$$

for every admissible choice of η, that is,

(1)	$-\sigma_{LV} H + V = \lambda = \text{constant}$ on Σ^f
(2)	$\theta = \theta_Y$ (i.e. $\cos\theta = \cos\theta_Y$) on γ^c

Equation (1) is known a Laplace's law: it can be interpreted in term of forces as

surface tension $(\simeq \sigma_{LV}\cdot H)$
+
volume forces $(\simeq V)$
=
difference of pressure at the two sides of Σ^f $(\simeq \lambda)$

Equation (2) is known as Young's law: at equilibrium the contact angle is equal to Young's angle at every point of γ^c.

P.S. Well, of course all these computations are quite sketchy, and would need a massive clean-up.

4.4 Existence results in the framework of finite perimeter sets

Let now E be a finite perimeter set in $\mathbb{R}^3$ contained in Ω. Then we can still define the capillary energy F by the formula

$$F(E) := \sigma_{LV}\left(|\Sigma^f| - \cos\theta_Y |\Sigma^c| \right) + \underbrace{V}_{\int_E v\,dx}$$

provided we modify the definition of Σ^c and Σ^c replacing the topological bdry ∂E with the essential bdry $\partial_* E$: $\Sigma^f := \partial_* E \cap \Omega$; $\Sigma^c := \partial_* E \cap \partial\Omega$.

The existence of a minimizer of $F(E)$ with prescribed volume can be proved by the usual semicontinuity-and-compactness method provided one shows that the "surface part" of F is lower semicontinuous w.r.t. the usual (L^1) distance in the class of finite perimeter sets contained in Ω.
(We don't have to worry about the "volume part" V because it is clearly continuous, at least if v is an L^1 function. The volume constr. is not a problem either, because it is obviously closed.)

Proof of semicontinuity

Given

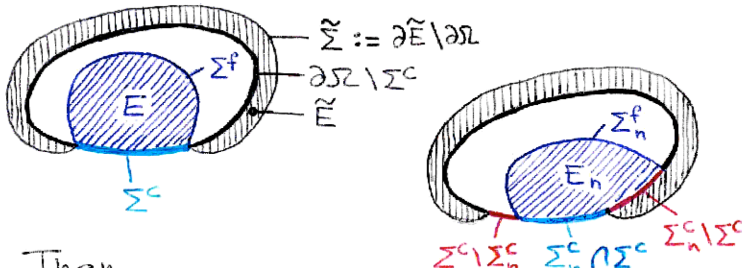
$$F'(E) := |\Sigma^f| - \cos\theta_Y |\Sigma^c|$$

we must show that given a sequence of finite perimeter sets E_n contained in Ω and converging to E, there holds

$$F'(E) \le \liminf_{n\to\infty} F'(E_n) .$$

We first construct a finite perimeter set $\tilde{E}$ contained in $\mathbb{R}^3 \setminus \overline{\Omega}$ such that $\partial_* \tilde{E} \cap \partial\Omega = \partial\Omega \setminus \Sigma^c$ ~ the contact surface of E.

(Since E is a generic finite per. set, this is no trivial task: to construct $\tilde{E}$ we must use a result due to E. Gagliardo, which implies that every Borel set contained in $\partial\Omega$ can be the contact surface of some fin. per. set $\tilde{E}$ contained in $\mathbb{R}^3 \setminus \overline{\Omega}$).



Then

$$P(E \cup \tilde{E}) = |\Sigma^f| + |\tilde\Sigma| + |\partial\Omega|$$

$$P(E_n \cup \tilde{E}) = |\Sigma^f_n| + |\tilde\Sigma| + |\partial\Omega| - |\Sigma^c \Delta \Sigma^c_n|$$

Hence

$$F'(E_n) - F'(E) = |\Sigma^f_n| - \cos\theta_Y\,|\Sigma^c_n| - |\Sigma^f| + \cos\theta_Y\,|\Sigma^c|$$

Use the identities
$|\Sigma^c_n| = |\Sigma^c_n \cap \Sigma^c| + |\Sigma^c_n \setminus \Sigma^c|$
$|\Sigma^c| = |\Sigma^c_n \cap \Sigma^c| + |\Sigma^c \setminus \Sigma^c_n|$

$$\to \quad = |\Sigma^f_n| - |\Sigma^f| + \cos\theta_Y\,|\Sigma^c \setminus \Sigma^c_n| - \cos\theta_Y\,|\Sigma^c_n \setminus \Sigma^c|$$

use that $-1 \leq \cos\theta_Y \leq 1$

$$\to \quad \geq |\Sigma^f_n| - |\Sigma^f| - |\Sigma^c_n \,\Delta\, \Sigma^c|$$

$$= P(E_n \cup \tilde{E}) - P(E \cup \tilde{E})$$

and by the semicontinuity of perimeter

$$\liminf_{n\to\infty} F'(E_n) - F'(E) \geq$$
$$\geq \liminf_{n\to\infty} P(E_n \cup \tilde{E}) - P(E \cup \tilde{E}) \geq 0.$$

$\square$

About wetting conditions

We have assumed so far that the coeffic. σ_{LV}, σ_{LS}, σ_{VS} in the first formulation of the capillary energy satisfy the wetting inequality $|\sigma_{LS} - \sigma_{VS}| \leq \sigma_{LV}$.

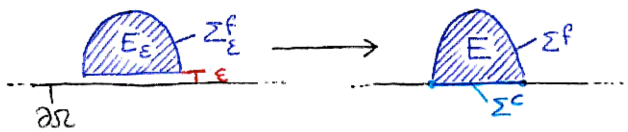
This inequality turns out to be necessary to make the capillary energy F lower-semicontinuous (w.r.t. the convergence of finite perimeter sets).

Indeed F can always be written (up to a constant, and neglecting the volume part)

as $F' = |\Sigma^f| - c\,|\Sigma^c|$ where $c := \dfrac{\sigma_{VS} - \sigma_{LS}}{\sigma_{LV}}$.

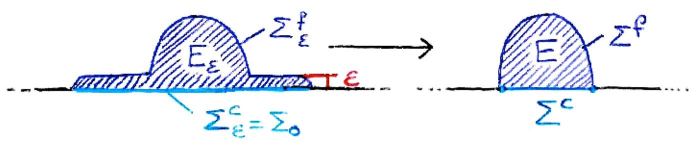
However, the proof of the semicontinuity only works if $-1 \leq c \leq 1$, that is, only if the wetting inequality holds. And indeed F' is not lower semicontinuous otherwise:

if $c < -1$, that is, $\sigma_{VS} - \sigma_{LS} < -\sigma_{LV}$ the semicontin. fails for the following sequence E_ε:



$$F'(E_\varepsilon) = |\Sigma^f_\varepsilon| \to |\Sigma^f| + |\Sigma^c| < |\Sigma^f| - c|\Sigma^c| = F'(E)$$

Viceversa, if $c > 1$, that is, $\sigma_{VS} - \sigma_{LS} > \sigma_{LV}$, then semicontinuity fails for this sequence:



$$F'(E_\varepsilon) = |\Sigma^f_\varepsilon| - c|\Sigma^c_\varepsilon| \to |\Sigma^f| + |\Sigma_0 \setminus \Sigma^c| - c|\Sigma_0|$$
$$< |\Sigma^f| - c|\Sigma^c| = F'(E)$$

Put this way, the wetting inequality may seem a purely technical condition, but it actually admits a "physical" interpretation: if $c < -1$, i.e. $\sigma_{VS} + \sigma_{LV} < \sigma_{LS}$ a drop never touches the wall of the container, because

it is always more convenient (energy-wise) to insert a thin layer of air to detach the drop from the container. Similarly, if $c>1$, i.e., $\sigma_{LS}+\sigma_{LV}<\sigma_{VS}$, it is always convenient to cover the wall of the container with a thin film of liquid, and separate it from the air.
And this is exactly the point of the examples we used to show the lack of semicontinuity of F!

4.5 Final remarks

- Using finite perimeter sets one obtains easy existence result for minimizers of the capillary energy with prescribed volume. Then one should prove that these minimizers are smooth enough in order to obtain that they actually satisfy the equilibrium conditions we derived in the classical setting (indeed it is not possible to define the mean curvature H nor the contact curve γ^c and the contact angle θ for a generic finite perimeter set).

- Even though not fully satisfactory (at least without a suitable regularity theory) the existence of minimizers in a "weak" framework (finite perimeter sets here, Sobolev functions elsewhere) is relevant because it offers a validation of the physical model.
Clearly the standard model of capillarity does not need validations, but this is not true of the many variants which have been proposed even in recent times.
For instance, the model of capillarity with line tension consists in adding to the usual capillary energy an additional term proportional to the length of the contact line: however this addition makes the model ill-posed, in the sense that there is no existence theory, even in a weak setting.
More precisely, the energy is not lower semic., and its lower-semicontinuous envelope leads to minimizers and equilibrium conditions which are not those predicted by the original model.