Minimizing Interaction Energies: Nonlocal Potentials and Nonlinear Diffusions

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Outline

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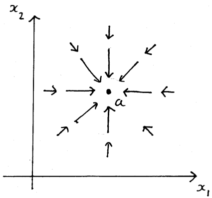
Aggregation for particles - Continuum Model

One particle attracted/repelled by a fixed location $x = a$

$$\dot{X} = -\nabla U(X - a) \qquad U(x) = U(-x), U(0) = 0\,, U \in C^1(\mathbb{R}^d/\{0\}, \mathbb{R})$$

Multiple particles attracted/repelled by one another



$$\dot{X}_i = -\sum_{j \neq i} m_j \, \nabla U(X_i - X_j)$$

$\rho(t, x) =$ density of particle at time t

$$v(x) = -\int_{\mathbb{R}^d} \nabla U(x - y)\, \rho(y) dy$$

So $\boxed{v = -\nabla U * \rho}$:

$$\begin{cases} \rho_t + \mathrm{div} \partial \rho v = 0 \\ v = -\nabla U * \rho \end{cases}$$

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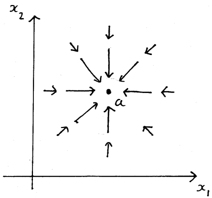

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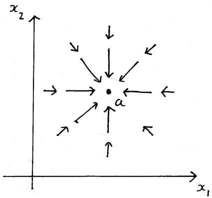

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Aggregation-Diffusion Equation

$$\begin{cases} \rho_t + \mathrm{div}\, \rho v = 0 \\ v = -\nabla U * \rho - \nabla P(\rho) \end{cases}$$

$\rho(t,x)$: density
$v(t,x)$: velocity field
$x \in \mathbb{R}^d$, $t > 0$

$U : \mathbb{R}^d \to \mathbb{R}$
"interaction potential"

$-\nabla U : \mathbb{R}^d \to \mathbb{R}^d$
"attracting/repelling field"

For which interaction repulsive/attractive potentials do we get convergence towards some nontrivial steady states?

How can we characterize these stationary states and what are their qualitative and stability properties?

If repulsion is modelled by diffusion, when does a balance between attraction and diffusion happen?

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Formal Gradient Flow

Basic Properties

1. **Conservation of the center of mass.**
2. **Liapunov Functional:** Gradient flow of

$$\mathcal{F}[\rho] = \frac{1}{2}\iint U(x-y)\,\rho(x)\,\rho(y)\,dxdy + \int_{\mathbb{R}^d} \Phi(\rho(x))\,dx$$

with respect to the Wasserstein distance W_2.
(C., McCann, Villani; RMI 2003, ARMA 2006).

The macroscopic equation can be rewritten as

$$\frac{\partial \rho}{\partial t}(t,x) = \operatorname{div}\left(\rho(t,x)\nabla\left[\frac{\delta \mathcal{F}}{\delta \rho}(t,x)\right]\right) .$$

with entropy dissipation:

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{F}[\rho(t)] = -\int_{\mathbb{R}^2} \rho(t,x)\left|\nabla \frac{\delta \mathcal{F}}{\delta \rho}(t,x)\right|^2 dx .$$

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Free Energy Minimization: Stable Steady States

Minimization Problem

We want to find local minimizers of the total interaction energy

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What is the right topology to talk about measures/densities being close?

When does a balance between attraction and repulsion (modelled either by nonlocality or diffusion) happen?

Recurrent Question in many fields:

- Statistical Mechanics & Crystallization: Typically very singular potentials at zero: Lennard-Jones.
- Semiconductors - Astrophysics - Chemotaxis: Macroscopic model obtained from Vlasov Equation under certain limits. Newtonian Potential.
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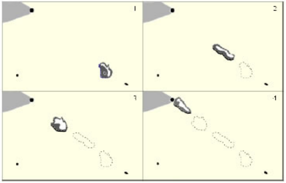
Cell/Bacteria Movement by Chemotaxis



$$
\begin{cases}
\dfrac{\partial n}{\partial t} = \Delta\Phi(n) - \chi\nabla\cdot(n\nabla c) & x\in\mathbb{R}^2\,,\ t>0\,,\\[2mm]
\dfrac{\partial c}{\partial t} - \Delta c = n - \alpha c & x\in\mathbb{R}^2\,,\ t>0\,,\\[2mm]
n(0,x) = n_0 \geq 0 & x\in\mathbb{R}^2\,.
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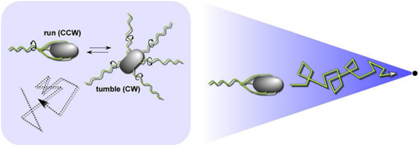
Patlak (1953), Keller-Segel (1971), Nanjundiah (1973).



Movement and aggregation due to chemical signalling. Wikinut

J. Saragosti etal, PLoS Comput. Biol. 2010.

S. Volpe etal, PLoS One 2012.

Individual Based Models (Particle models)

Swarming = Aggregation of agents of similar size and body type generally moving in a coordinated way.

Highly developed social organization: insects (locusts, ants, bees ...), fish, birds, micro-organisms,... and artificial robots for unmanned vehicle operation.

Interaction regions between individuals[a]

[a] Aoki, Helmerijk et al., Barbaro, Birnir et al.

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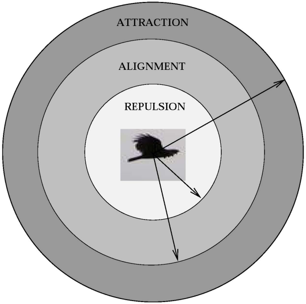

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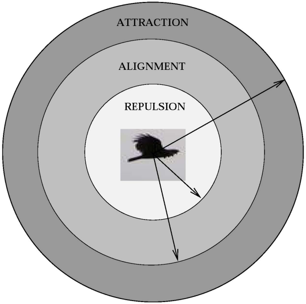

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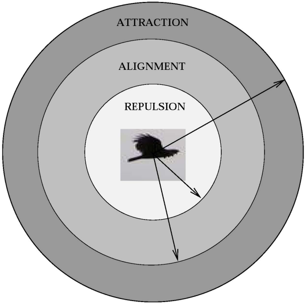
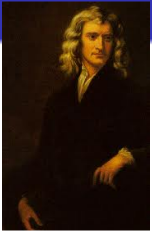


2nd Order Model: Newton's like equations



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$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ m\dfrac{dv_i}{dt} = (\alpha - \beta\,|v_i|^2)v_i - \displaystyle\sum_{j\neq i} \nabla U(|x_i - x_j|). \end{cases}$$

Model assumptions:

- Self-propulsion and friction terms determines an asymptotic speed of $\sqrt{\alpha/\beta}$.
- Attraction/Repulsion modeled by an effective pairwise potential $U(x)$.

$$U(r) = -C_A e^{-r/\ell_A} + C_R e^{-r/\ell_R}.$$

One can also use Bessel functions in 2D and 3D to produce such a potential.

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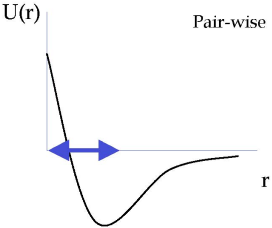

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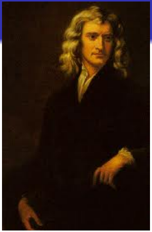

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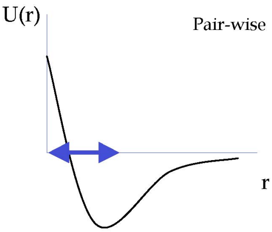

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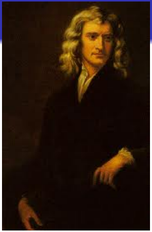

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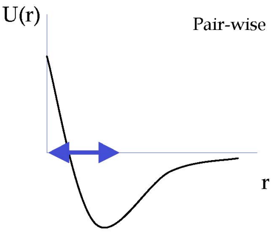

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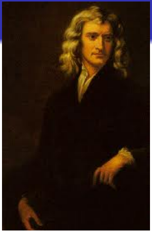

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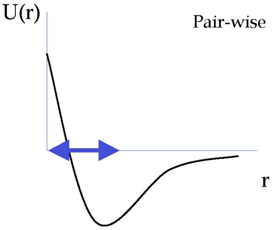
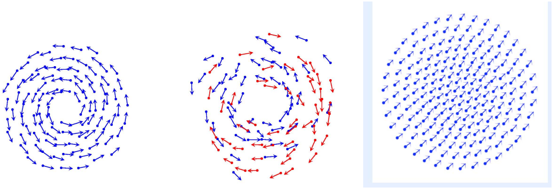
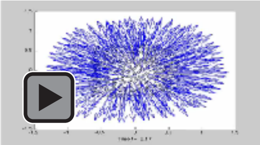
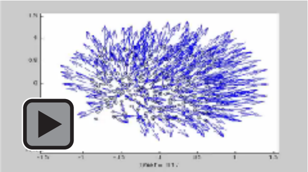


Model with an asymptotic speed

Typical patterns: milling, double milling or flocking:







Outline

1st Order Friction Model:

Edelshtein-Keshet, Mogilner (JMB 2000): Assume the variations of the velocity and speed are much smaller than spatial variations, then from Newton's equation:

$$m\frac{d^2x_i}{d^2t} + \alpha\frac{dx_i}{dt} + \sum_{j\neq i}\nabla U(|x_i - x_j|) = 0$$

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with v_0 fixed with $|v_0|^2 = \frac{\alpha}{\beta}$.

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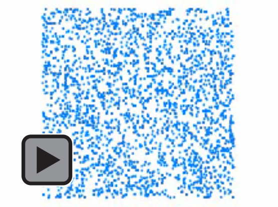
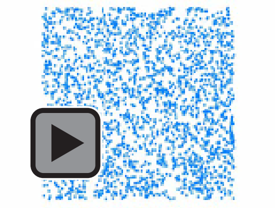
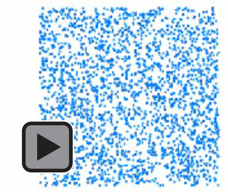
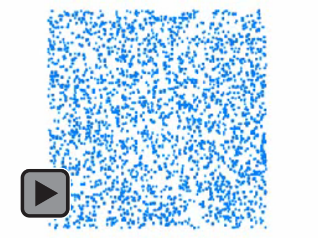
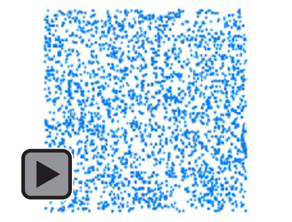
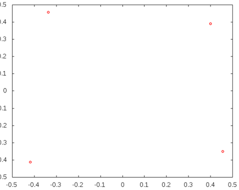
Outline

Nontrivial patterns? - Particle Simulations

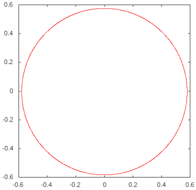






Summary: Particle Simulations $d = 2$

Potential $a = 4$, $b = 2.1$



Potential $a = 4$, $b = 1.85$



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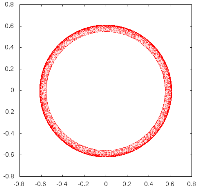

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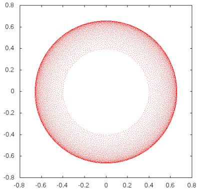
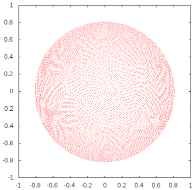


Potential $a = 4$, $b = 0.05$



$$\dot{X}_i = -\sum_{j \neq i} m_j \, \nabla U(X_i - X_j)$$

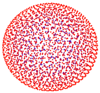
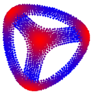
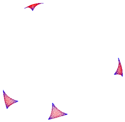
$$U(x) = \frac{|x|^a}{a} - \frac{|x|^b}{b}$$

$$2 - d \leq b < a$$

Dimensionality of the support

Some simulations with power law potentials of the form

$$W(x) = \frac{|x|^a}{a} - \frac{|x|^b}{b}, \qquad 2 - d < b < a$$

dim=3	dim=2	dim=2	dim=0
			
$b = -0.5$ $a = 5$	$b = 0.5$ $a = 23$	$b = 1.25$ $a = 15$	$b = 2.5$ $a = 5$

Local minimizers in 3D for different parameters when $b > -1$ increases. The computations were done with $n = 2,500$ particles.

Outline

W_∞-Topology

The W_∞-distance is defined as the optimal maximal mass displacement given by

$$W_\infty(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \sup_{(x,y)\in\mathrm{supp}(\pi)} |x-y|\,,$$

- It is a good topology since it is closer to linearization around equilibrium of dynamical systems.
- It is the coarser topology among Wasserstein distances since all of them are ordered.
- Then, a local minimizer in W_2 is a local minimizer in W_∞ but not viceversa.

Basic Hypotheses:
(H1) U is a bounded from below lower semi-continuous function in $L^1_{loc}(\mathbb{R}^d)$.

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Euler-Lagrange Conditions

W_∞ EL-Conditions

Assume that U satisfies (H1) and let μ be a local compactly supported minimizer of the energy $\mathcal{F}[\mu]$ in the W_∞ ball or radius ε. Then any point $x_0 \in \mathrm{supp}(\mu)$ is a local minimimum of $\psi = U * \mu$ in the sense that

$$\psi(x_0) \leq \psi(x) \text{ for a.e. } x \in B_\varepsilon(x_0).$$

Note that ε is uniform on the support of μ.

W_2 EL-Conditions

Under the same assumptions, if μ is a W_2-local minimizer of the energy, then the potential ψ satisfy

(i) $\psi(x) = (U * \mu)(x) = 2\mathcal{F}[\mu]$ μ-a.e.

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Strong Repulsive potentials

Dimension of the Support depends on $2 - d < b < 2$.

Assume that μ is a local minimizer of the interaction energy $\mathcal{F}$ with respect to W_∞ such that U is radial and $U(x)$ is γ-repulsive at the origin $0 < \gamma < d$. If μ contains s-Haussdorff dimensional connected components in its support, then $s \geq \gamma$.

(Balagué, C., Laurent, Raoul; ARMA 2013)

Strategy:

- W_∞ EL-Conditions: Pure variational approach, by contradiction we build better competitors.
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$$-\Delta^\epsilon U(x) := \frac{2(d+2)}{\epsilon^2}\left(U(x) - \fint_{B(0,\epsilon)} U(x+y)dy\right),$$

and $\fint_{B(x_0,r)} f(x)dx$ stands for the average of f over $B(x_0,r)$.

Generalized Laplacian

Suppose $U : \mathbb{R}^d \to (-\infty, +\infty]$ is locally integrable. U is said to be γ-repulsive at the origin if there exists $\epsilon > 0$ and $C > 0$ such that

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Th. 8.7, Mattila, *Geometry of sets and measures*

Let A be a Borel subset of $\mathbb{R}^d$, and $s \geq 0$. If there exists a probability measure $\mu \in \mathcal{P}(\mathbb{R}^d)$ supported on A such that

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then $dim_H A \geq s$, with dim_H being the Hausdorff dimension of A.

- Let A be the support of μ.
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Th. 8.7, Mattila, *Geometry of sets and measures*

Let A be a Borel subset of $\mathbb{R}^d$, and $s \geq 0$. If there exists a probability measure $\mu \in \mathcal{P}(\mathbb{R}^d)$ supported on A such that

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then $dim_H A \geq s$, with dim_H being the Hausdorff dimension of A.

- Let A be the support of μ.
- Choose ϵ small enough by the hypothesis of W being γ-repulsive, choose $x_0 \in A$ and define
$$\mu_0(B) = \mu(B \cap B(x_0, \epsilon/2)).$$
- Write $\mu = \mu_0 + \mu_1$, where μ_0 and μ_1 are nonnegative measures with mass m_0 and m_1 respectively, and μ_0 supported in $A \cap B(x_0, \epsilon/2)$.
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Outline

Mild Repulsive potentials

Support is on finite number of points for $b > 2$.

Let $U \in C^2(\mathbb{R}^N)$ be a radially symmetric potential which behaves like $-|x|^b/b$ in a neighborhood of the origin with $b > 2$.
Then a local minimizer of the interaction energy $\mathcal{F}$ with respect to W_∞ is supported in a finite number of points.

(C., Figalli, Patacchini; preprint)

- $U \in C^2(\mathbb{R}^d)$ and U is radially symmetric.
- U is bounded from below and $U(0) = 0$.
- There exists $R > 0$ with $U(x) < 0$ for all $|x| < R$ and $U(x) \geq 0$ for all $|x| \geq R$.
- Fix $b > 2$. We write $\widetilde{U}(|x|) := U(x)$ and $\widetilde{U}_p(r) := \frac{\widetilde{U}(pr)}{p^b}$ for any $p > 0$ and $r \geq 0$. There exists a constant $C > 0$ such that

$$\begin{cases} \widetilde{U}_p(r) \to -Cr^b \\ \widetilde{U}_p'(r) \to -Cbr^{b-1} \end{cases} \quad \text{as } p \to 0 \text{ for all } r \geq 0.$$

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Second Variation

Let μ be a W_∞-local minimizer of $\mathcal{F}$ with $\mathcal{F}(\mu) < +\infty$. There exists $\delta > 0$ such that for all $x_0 \in \operatorname{supp}\mu$ we have

$$\boxed{\int_{\mathbb{R}^d \times \mathbb{R}^d} U(x-y)\,\mathrm{d}\nu(x)\,\mathrm{d}\nu(y) \geq 0}$$

for any signed measure ν with $\operatorname{supp}\nu \subset \operatorname{supp}\mu \cap B(x_0, \delta)$ and $\nu(\mathbb{R}^d) = 0$.

As a consequence, we obtain that **global minimizers** of the interaction energy are compactly supported with $\operatorname{diam}(\operatorname{supp}\mu) \leq R$.

Given $x, y \in \operatorname{supp}\mu$, take $\nu = \delta_x - \delta_y$ as signed measure and we deduce that $W(x-y) \leq 0$. Then, by assumptions on U we obtain the estimate on the support.

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Qualitative Properties: Dimensionality of Support

Theorem.

Let μ be a W_∞-local minimizer of $\mathcal{F}$. Then **each point of** $\operatorname{supp}\mu$ **is isolated**; in particular μ is atomic.

Steps of proof.

- Suppose $0, x_1, -x_2 \in \operatorname{supp}\mu \cap B(0,\delta)$. Choose $\nu_\lambda = -\delta_0 + \lambda\delta_{x_1} + (1-\lambda)\delta_{-x_2}$ in place of ν in the second variation and get, for an appropriate choice of λ,

$$\sqrt{-U(x_1)} + \sqrt{-U(x_2)} \geq \sqrt{-U(x_1+x_2)}.$$

- Assume, by homogeneity, that $x_1 + x_2 = pe_1$, where e_1 is the first unit vector of the orthonormal base of $\mathbb{R}^d$, and $p > 0$ is a small rescaling parameter. From the above inequality, get

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- Write $x_1 = p(te_1 + y)$, where $y \in \mathbb{R}^d$ with zero first coordinate, and, by homogeneity, $t \in [0,1]$. Then, using that $|x_1| \le pt + p|y|$ and $|pe_1 - x_1| \le p(1-t) + p|y|$, and that, for any $x \in \mathbb{R}^d$ and p small enough, $\sqrt{-U(px)}$ is radially non-decreasing as a function of $x \in \mathbb{R}^d$, get

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- Divide the inequality above by $p^{\alpha/2}$ and obtain

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$$s_{\mathrm{b}}(s,z) = (s+|z|)^{\mathrm{b}/2} + ((1-s)+|z|)^{\mathrm{b}/2} - 1$$

  and define, for any two distinct points $v, v' \in \mathbb{R}^d$, the open set

$$S_{\mathrm{b}}(v,v') := \left\{ w \in \mathbb{R}^d \mid s_{\mathrm{b}}\left(\frac{|\pi w - v|}{|v-v'|}, \pi w - w\right) < 0 \right\},$$

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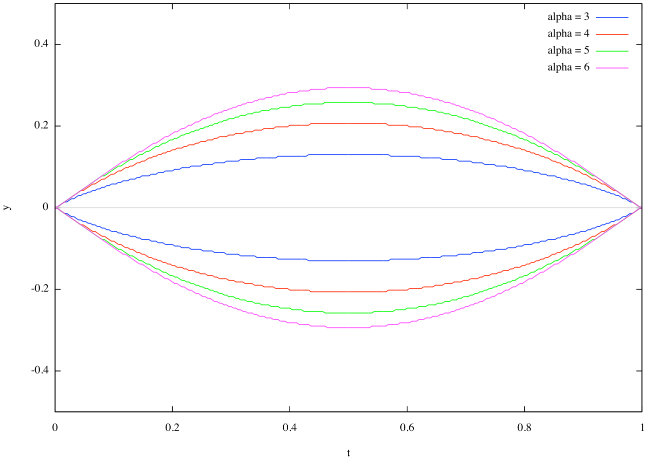

Qualitative Properties: Dimensionality of Support

- For any two distinct points $v, v' \in \mathbb{R}^d$, define the open "double cone" with opening $\tau > 0$ by

$$C_\tau(v, v') := \left\{ w \in \mathbb{R}^d \mid \frac{\mathrm{dist}(w, [v, v'])}{\min\{|\pi w - v|, |\pi w - v'|\}} < \tau \right\},$$

where $[v, v']$ denotes the segment joining v to v' and π denotes the orthogonal projection on the segment $[v, v']$.

- Since $\alpha > 2$, $r \mapsto r^{\alpha/2}$ is a convex function on $[0, +\infty)$, and so $S_\alpha(y_0, y_1)$ is a convex set. Therefore we can fit a double cone generated by y_0 and y_1 inside it.
- We can actually compute the opening $\gamma(\alpha)$ of the cone that fits in $S_\alpha(y_0, y_1)$ with maximal volume:
$$\gamma(\alpha) = \frac{1}{2^{\alpha/2-1}} - 1.$$
- **Finish the proof by contradiction**. Suppose y_0 is not an isolated point, then it can be approached by a sequence of points in suppμ in some direction. Therefore, using we know that, close enough to y_0, one can find two points belonging to this sequence, say x_k and x_{k+1}, such that $x_{k+1} \in C_{\gamma(\alpha)}(y_0, x_k)$.

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Existence Global Minimizers

Non-HStable: Energy at infinity cost more than near the origin, i.e., the potential U satisfies

(H2) There exists $\mu \in \mathcal{P}(\mathbb{R}^d)$ compactly supported such that

$$\mathcal{F}[\mu] < 0 \leq \lim_{|x|\to\infty} U(x).$$

Main Theorem

Assume that the radial potential U satisfies Hypotheses (H1), (H2), and is increasing outside a large ball. Then there exists a global minimiser for the energy $\mathcal{F}$. Furthermore, any such global minimiser has compact support.

(Cañizo, C., Patacchini; preprint 2014)
Main idea: Use the W_2 EL-Conditions to show a uniform repartition of the mass over the support.

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Key Estimates

By (H1) for R large enough:

$$E_R := \min \left\{ \mathcal{F}[\mu] \mid \mu \in \mathcal{P}_R(\mathbb{R}^d) \right\} \le E_* < 0$$

Euler-Lagrange: for ρ_R-almost all $z \in \operatorname{supp}(\rho_R)$ we have

$$\frac{1}{2} \int_{\mathbb{R}} U(z-x)\, \mathrm{d}\rho_R(x) = E_R.$$

Choose $A \in \mathbb{R}$ with $\frac{1}{2} U_{\min} \le E_* < A < 0$ and $r' > 0$ with $U(x) \ge 2A$ for $|x| \ge r'$.

Then for ρ_R-almost every z we have

$$\begin{aligned} 2E_R &= \int_{\mathbb{R}} U(z-x)\, \mathrm{d}\rho_R(x) \\ &= \int_{B(z,r')} U(z-x)\, \mathrm{d}\rho_R(x) + \int_{\mathbb{R}^d \setminus B(z,r')} U(z-x)\, \mathrm{d}\rho_R(x) \\ &\ge U_{\min} \int_{B(z,r')} \mathrm{d}\rho_R(x) + 2A \int_{\mathbb{R}^d \setminus B(z,r')} \mathrm{d}\rho_R(x) \\ &= (U_{\min} - 2A) \int_{B(z,r')} \mathrm{d}\rho_R(x) + 2A, \end{aligned}$$

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$$E_R := \min \left\{ \mathcal{F}[\mu] \mid \mu \in \mathcal{P}_R(\mathbb{R}^d) \right\} \leq E_* < 0$$

Euler-Lagrange: for ρ_R-almost all $z \in \operatorname{supp}(\rho_R)$ we have

$$\frac{1}{2} \int_{\mathbb{R}} U(z-x) \,\mathrm{d}\rho_R(x) = E_R.$$

Choose $A \in \mathbb{R}$ with $\frac{1}{2} U_{\min} \leq E_* < A < 0$ and $r' > 0$ with $U(x) \geq 2A$ for $|x| \geq r'$.

Then for ρ_R-almost every z we have

$$\begin{aligned} 2E_R &= \int_{\mathbb{R}} U(z-x) \,\mathrm{d}\rho_R(x) \\ &= \int_{B(z,r')} U(z-x) \,\mathrm{d}\rho_R(x) + \int_{\mathbb{R}^d \setminus B(z,r')} U(z-x) \,\mathrm{d}\rho_R(x) \\ &\geq U_{\min} \int_{B(z,r')} \mathrm{d}\rho_R(x) + 2A \int_{\mathbb{R}^d \setminus B(z,r')} \mathrm{d}\rho_R(x) \\ &= (U_{\min} - 2A) \int_{B(z,r')} \mathrm{d}\rho_R(x) + 2A, \end{aligned}$$

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$$\int_{B(z,r')} \mathrm{d}\rho_R(x) \geq \frac{A - E_R}{A - \frac{1}{2}U_{\min}} \geq \frac{A - E_*}{A - \frac{1}{2}U_{\min}} =: m.$$

- When the potential is increasing outside a large ball, one can show by competing minimizers that gaps in the support in each component cannot be too large by sliding down a bit the mass of the density to the right of the gap. All together it gives a uniform bound $\mathcal{K}$ on the diameter of the support of global minimizers.
- Since the bound on the diameter of the support of local minimizers is independent of R being the radius of the ball where we look for local minimizers, one can show that global minimizers for $\mathcal{P}_{R_0}(\mathbb{R}^d)$ are global minimizers for $\mathcal{P}_R(\mathbb{R}^d)$ with $R \geq R_0$ if $R_0 \geq 2\mathcal{K}$.

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Examples

Power-laws & Morse Potentials

Consider the following potentials for all $x \in \mathbb{R}^d$ and $C_A, C_R, \ell_A, \ell_R > 0$:

(i) *(Power-law potential)* $U(x) = \frac{|x|^a}{a} - \frac{|x|^b}{b}$ with $-d < b < a$,

(ii) *(Morse potential)* $U(x) = C_A e^{-\frac{|x|}{\ell_A}} - C_R e^{-\frac{|x|}{\ell_R}}$ with $\ell_A < \ell_R$ and $\frac{C_A}{C_R} < \left(\frac{\ell_R}{\ell_A}\right)^d$,

with the convention $\frac{|x|^0}{0} = \log|x|$.

Sufficient Condition for not H-stable

Let U be a potential satisfying (H1), and assume furthermore that $U_\infty := \lim_{|x|\to\infty} U(x)$ exists (being possibly equal to $+\infty$).

(i) If $U_\infty = +\infty$, then U is unstable.

(ii) If $U_\infty < +\infty$, call $\tilde{U} := U - U_\infty$. If $\tilde{U}_+$ is integrable and $\int_{\mathbb{R}} \tilde{U} < 0$ (being possibly equal to $-\infty$), then U is unstable.

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Discrete To Continuum: Power-law Case

(C., Chipot, Huang; PRSA 2014)

Discrete Setting: Find

$$I_N = \inf_{x\in(\mathbb{R}^d)^N} \mathcal{F}_N(x)\,,$$

with

$$\mathcal{F}_N(x_1,\cdots,x_N) = \sum_{i\neq j}^{N}\left(\frac{|x_i-x_j|^a}{a} - \frac{|x_i-x_j|^b}{b}\right).$$

Uniform Control of the support

Suppose that $1 \le b < a$. Then the diameter of any global minimizer of $\mathcal{F}_N$ achieving the infimum I_N is bounded independently of N.

Key Idea: use Euler-Lagrange and a convexity argument for the repulsive potential to estimate the distance between the two particles the furthest away.

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Regularity of Local Minimizers

(H3) The function $U_a(x) := U(x) - V(x)$ with V being the Newtonian potential in dimension d satisfies:

$$\Delta U_a \in L^p_{loc}(\mathbb{R}^d) \quad \text{for some } p \in (d, \infty]$$

with ΔU_a bounded below.

Main Theorem

Assume that the potential U satisfies Hypotheses (H1) and (H3). Then any μ compactly supported W_∞ local minimizer of the energy $\mathcal{F}$ is bounded uniformly, i.e., $\mu = \rho(x) d\mathcal{L}^d$ with $\rho \in L^\infty(\mathbb{R}^d)$.

Example: $U(x) = V + |x|^a$ with $a > 0$.

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Obstacle Problem

Continuity of the potential

Assume that the potential U satisfies Hypotheses (H1) and (H3). Let μ be a W_∞ local minimizer of E. Then the potential $\psi(x) := U * \mu(x)$ associated to μ is a continuous function in $\mathbb{R}^N$.

Implicit Obstacle Problem

For all $x_0 \in \mathrm{supp}(\mu)$, the potential function ψ is equal, in $B_\varepsilon(x_0)$, to the unique solution of the obstacle problem

$$\begin{cases} \varphi & \geq C_0, & \text{in } B_\varepsilon(x_0) \\ -\Delta\varphi & \geq -F(x), & \text{in } B_\varepsilon(x_0) \\ -\Delta\varphi & = -F(x), & \text{in } B_\varepsilon(x_0) \cap \{\varphi > C_0\} \\ \varphi & = \psi, & \text{on } \partial B_\varepsilon(x_0), \end{cases}$$

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Fractional Obstacle Problem

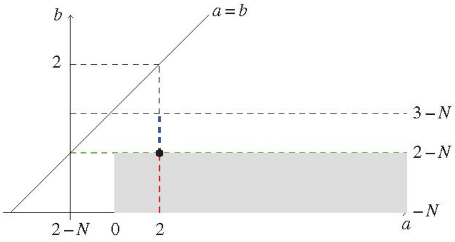
(H3s) The function $U_a(x) := U(x) - V_s(x)$ with V_s being the fundamental solution in dimension d of $(-\Delta)^s$, $s \in (0,1)$ satisfies:

$$(-\Delta)^s U_a \in L^p_{loc}(\mathbb{R}^d) \quad \text{for some } p \in (d/2s, \infty]$$

with $(-\Delta)^s U_a$ bounded above.

For $s \in (0,1)$, it is then well known that

$$V_s(x) = \frac{c_{\mathrm{d},s}}{|x|^{\mathrm{d}-2s}}$$



Fractional Obstacle Problem 2

Implicit Obstacle Problem

Assume that the potential U satisfies Hypotheses (H1) and (H3s). For all $x_0 \in \mathrm{supp}(\mu)$, the potential function ψ is continuous and is equal, in $B_\varepsilon(x_0)$, to the unique solution of the fractional obstacle problem

$$\begin{cases} \varphi & \geq C_0, & \text{in } B_\varepsilon(x_0) \\ (-\Delta)^s\varphi & \geq -F(x), & \text{in } B_\varepsilon(x_0) \\ (-\Delta)^s\varphi & = -F(x), & \text{in } B_\varepsilon(x_0) \cap \{\varphi > C_0\} \\ \varphi & = \psi, & \text{on } \partial B_\varepsilon(x_0), \end{cases}$$

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Main Theorem

Assume that the potential U satisfies Hypotheses (H1) and (H3s) + a bit of regularity. Then any μ compactly supported W_∞ local minimizer of the energy $\mathcal{F}$ is Hölder-continuous, i.e., $\mu = \rho(x)d\mathcal{L}^d$ with $\rho \in C^\alpha(\mathbb{R}^d)$ for all $0 < \alpha < 1 - s$.

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Basic Relation

Integral Equation

$$\int_{-R}^{R} |x-y|^{-\nu}\rho(y)dy = f_0(x)$$

with $\nu \in (0,1)$ has solution

$$\rho(x) = \frac{\sin \pi\nu}{2\pi}\frac{d}{dx}\int_{-R}^{x}\frac{f_0(y)}{(x-y)^{1-\nu}}\,dy$$
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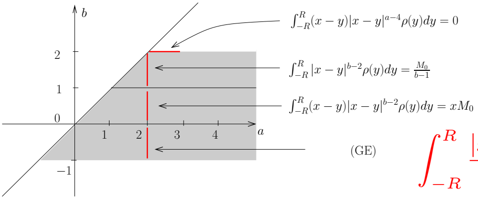
One dimensional Case



$$\int_{-R}^{R}\frac{|x-y|^a}{a}\rho(y)dy-\int_{-R}^{R}\frac{|x-y|^b}{b}\rho(y)dy=E \qquad \text{(GE)}$$

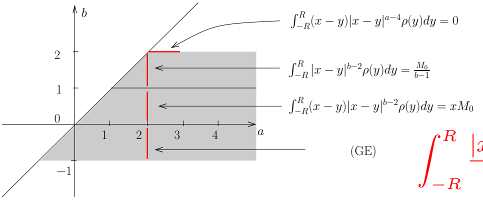
$a=2$, $b\in(1,2)$:

$$\rho(x)=\frac{M_0}{b-1}\frac{\cos\frac{\pi(2-b)}{2}}{\pi}(R^2-x^2)^{\frac{1-b}{2}}.$$

Using the definition of the total mass, one determines the radius of the support:

$$M_0=\int_{-R}^{R}\rho(x)dx=\frac{M_0}{b-1}\frac{\cos\frac{\pi(2-b)}{2}}{\pi}B\left(\frac{1}{2},\frac{3-b}{2}\right)R^{2-b}.$$

One dimensional Case



$$\int_{-R}^{R} \frac{|x-y|^a}{a}\rho(y)dy - \int_{-R}^{R} \frac{|x-y|^b}{b}\rho(y)dy = E \qquad \text{(GE)}$$

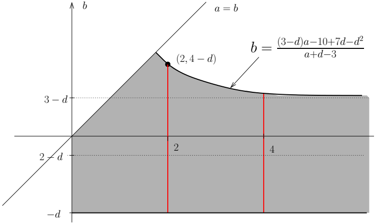
$a = 2$, $b \in (1, 2)$:

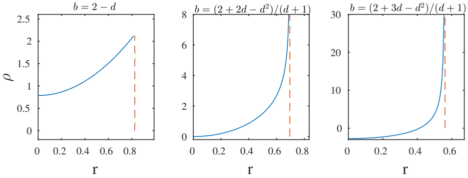
$$\rho(x) = \frac{M_0}{b-1}\,\frac{\cos\frac{\pi(2-b)}{2}}{\pi}\,(R^2 - x^2)^{\frac{1-b}{2}}.$$

Using the definition of the total mass, one determines the radius of the support:

$$M_0 = \int_{-R}^{R} \rho(x)dx = \frac{M_0}{b-1}\,\frac{\cos\frac{\pi(2-b)}{2}}{\pi} B\left(\frac{1}{2}, \frac{3-b}{2}\right) R^{2-b}.$$

Higher Dimensions





Key identity:

$$\int_{B_R} (R^2 - |y|^2)^{-\frac{b+d}{2}} |x-y|^b dy = \frac{\pi^{\frac{d}{2}+1}}{\Gamma(\frac{d}{2}) \sin \frac{(b+d)\pi}{2}}, \quad b \in (-d, 2-d)$$

$a = 2$, $b \in (-d, 4-d)$:

$$\rho(x) = -\frac{dM_0 \Gamma(\frac{d}{2}) \sin \frac{(b+d)\pi}{2}}{(b+d-2)\pi^{\frac{d}{2}+1}} (R^2 - |x|^2)^{1-\frac{b+d}{2}},$$

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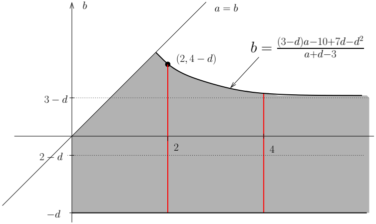

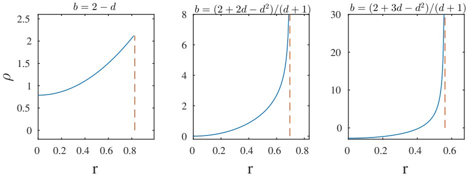

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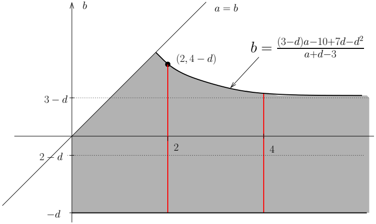

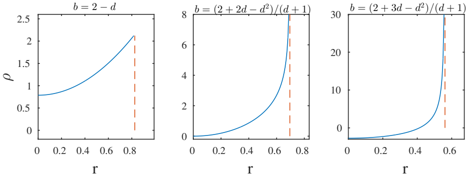

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Conclusions & Open Problems

- The strength of the repulsion at the origin determines the qualitative properties of the minimizers. As the repulsion gets stronger the regularity gets better.
- What are the implications of these properties on the long time asymptotics? Can we prove convergence towards stationary states for potentials less singular than Newtonian? What are the properties close to Newtonian singularity?
- Existence of global minimizers if adding a small diffusion either linear or nonlinear? Is H-Stability related? What are the implications for the evolution problem?
- Break of symmetry and uniqueness of minimizers?
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1. Balagué-C.-Laurent-Raoul (Physica D & ARMA 2013).
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