Lecture 1
Introduction and a quantitative rigidity estimate

Thin is different

- Small forces → large deformations
- instability under compression (buckling, Euler column)
- large rotations (→ geometric nonlinearity)

Thin is (often) optimal

Common in nature + engineering
beams, plates, shells, helix
⇝ optimal behavior

Deformations of 1d or 2d objects are easier to visualize and to understand.

From the beginning of elasticity intense study of lower-dimensional theories.

Two approaches

1) formulate 'new' theories for lower-dimensional objects
[Euler elastica, Cosserat brothers, ...]

Ref.: S.S. Antman, Nonlinear theories of elasticity, Springer

2) Derivation: $3d \to 2d$
$\qquad\qquad\quad \searrow 1d$

(based on physics + eng. intuition)

2) Classical: make ansatz, (formal) expansion
⇝ has lead to many different lower-d theories
(celebrated work Reissner → 50 pages,
shell and rod theories)

In these lectures: Variational approach,
ansatz-free

Key ingredient (cf. F. John)

$\boxed{\text{Small energy} \Rightarrow \text{almost rigid motion}}$

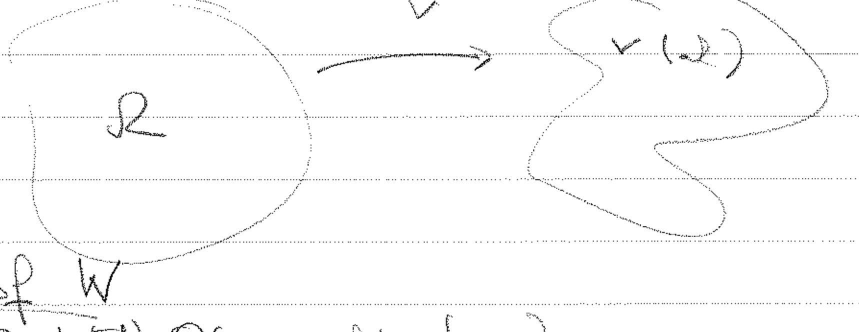
Set-up

$\Omega \subset \mathbb{R}^n$ 'reference configuration'
bounded Lipschitz domain

$v : \Omega \to \mathbb{R}^n$ deformation (in some Sobolev space)

$E(v) = \int_\Omega W(\nabla v(x))\,dx$ elastic energy

$W : \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{+\infty\}$ (free) energy density

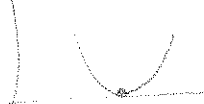


Properties of W

$W \geq 0$, $W(\mathrm{Id}) = 0$ (normalisation)

$W(RF) = W(F) \quad \forall R \in SO(n)$ (frame indifference)

$W(F) \geq c\, \mathrm{dist}^2(F, SO(n)), \quad c > 0$
(non-degeneracy)



Define $SO(n) = \{F \in \mathbb{R}^{3\times3} : F^T F = Id, \det F = 1\}$
(proper) rotation, isometries, orientation preserving !!! 1/3

(quantitative rigidity estimate)
Thm 1. There exists $C(\Omega)$ with the following property. For each $v \in W^{1,2}(\Omega; \mathbb{R}^n)$ there exists $R \in SO(n)$ such that

$$\int_\Omega |\nabla v(x) - R|^2 dx \le C(\Omega) \int_\Omega \operatorname{dist}^2(\nabla v(x), SO(n))\, dx. \quad (1)$$

L^2-dist. from a single matrix; L^2-dist from a set of matrices

Remarks: Optimal estimate, linear scaling, crucial for 3d→2d limit
Also in L^p, $1<p<\infty$, not $p=1$ or $p=\infty$

① Rigidity / "Liouville thm.":
$\nabla v(x) \in SO(n)$ a.e. $\Rightarrow$ $\nabla v = \text{const}$

② F. John (CPAM 1961) Suppose
v locally Bilipschitz, i.e.

$$\frac{1}{1+\varepsilon}|x-y| \le |v(x)-v(y)| \le (1+\varepsilon)|x-y| \quad \forall y \text{ near } x \quad (2)$$

$$\Rightarrow \exists R_Q \quad \int_Q |\nabla v - R_Q|^2 \le C|Q|\varepsilon \quad \forall \text{ cubes } Q \subset \Omega.$$

volume

Note: If $\det \nabla v(x) > 0$ then

$$(2) \Rightarrow \operatorname{dist}(\nabla v, SO(n)) \le C\varepsilon$$

③ Y. G. Reshetnyak / almost conformal maps

$K = \{F : F = \lambda R, R \in SO(n)\}$ conformal matrices

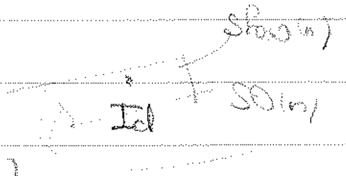
$$\begin{cases} \nabla v^{(j)} \rightharpoonup \nabla v & \text{weakly in } W^{1,n} \\ \operatorname{dist}(\nabla v^{(j)}, K) \to 0 & \text{in } L^n \end{cases}$$

$\Rightarrow \nabla v^{(j)} \to \nabla v$ strongly in L^n, $\nabla v \in K$ a.e.

③ + ① $\rightsquigarrow$ $\operatorname{dist}(\nabla v^{(j)}, SO(n)) \to 0$ L^n

$\Rightarrow \nabla v^{(j)} \to R = \text{const}$ in L^n (subsequence)

④ <u>Korn's inequality</u>



<u>Linearize</u>: $T_{Id}\, SO(n) = \text{skew}(n)$

$\text{skew}(n) = \{F \in \mathbb{R}^{n\times n} : F^T = -F\}$

$v(x) = x + \varepsilon u(x)$

$R \approx Id + \varepsilon A + \ldots, \quad A \in \text{skew}(n)$

$\operatorname{dist}(\nabla v, SO(n)) \approx \operatorname{dist}(\varepsilon\nabla, \text{skew}(n))$

$= \varepsilon \operatorname{sym}(\nabla u) = \varepsilon \dfrac{(\nabla u)^T + \nabla u}{2}$.

<u>Thm 1</u> $\Rightarrow$ (Korn's ineq) $\displaystyle\int |\nabla u - A|^2 \le C(\Omega) \int |\text{sym}\, \nabla u|^2$

<u>Warm-up</u> for <u>proof</u>. Show

$$\nabla v(x) \in SO(n) \quad \text{a.e.} \Rightarrow \nabla v = \text{const}$$

<u>Proof</u>. (Y.G. Reshetnyak, Kinderlehrer, ...)

<u>Two parts</u>

(1) for $v \in W^{1,\infty}(\Omega;\mathbb{R}^n)$

$$\operatorname{div} \operatorname{cof} \nabla v = 0 \quad \text{in the sense of distributions}$$

$$(\operatorname{cof} F)^i_j = (-1)^{i+j} \det(\ldots) \quad \text{delete row } i, \text{ delete column } j$$

$$F^T \operatorname{cof} F = (\det F)\, \mathrm{Id}$$

$$\left[\operatorname{div}(\operatorname{cof}\nabla v)\right]^i = \sum_j (\operatorname{cof}\nabla v)^i_{j,j} \qquad f_{,j} = \frac{\partial}{\partial x_j} f$$

(see e.g. L.C. Evans PDE, Section on Null-Lagrangians; J.M. Ball ARMA 63 (1977)).

(2) $F \in SO(n) \Rightarrow \operatorname{cof} F = F$

(1) + (2) $\Rightarrow \Delta v = \operatorname{div} \nabla v = 0$

Weyl's lemma $\Rightarrow v \in C^\infty$

$$0 = \underbrace{\Delta |\nabla v|^2}_{=n} = 2(\underbrace{\Delta \nabla v}_{=0} : \nabla v) + 2\underbrace{|\nabla^2 v|^2}_{\sum_{i,j,l} (v^i_{,jl})^2}$$

$$\Rightarrow \nabla^2 v = 0 \Rightarrow \nabla v = \text{const}.$$

Proof of Thm 1. Only do interior estimate

$\Omega = $ cube Q

Estimate in concentric cube Q'

Observation: For large F we have

$$|F - R| \leq 2\,\mathrm{dist}(F, SO(n)) \quad \forall R \in SO(n)$$

Step 0. It suffices to prove Thm 1. under the additional hypothesis that

$$|\nabla v| \leq M,$$

where M is a big constant (depending on n).

Step 1. Perform Reshetnyak's argument.

Set

$$\varepsilon = \|\mathrm{dist}(\nabla v, SO(n))\|_{L^2}$$

(We may assume

$$\varepsilon \leq 1\,)$$

$$|F - \mathrm{cof}\, F| \leq C\,\mathrm{dist}(F, SO(n))$$

$$\forall F,\ |F| \leq M.$$

Set

$$f = \nabla v - \mathrm{cof}\,\nabla v$$

Then

$$\Delta v = \mathrm{div}\,\nabla v = \underbrace{\cancel{\mathrm{div}\,\mathrm{cof}\,\nabla v}}_{=0} + \mathrm{div}\, f.$$

Write

$$v = w + z,$$

with

$$\Delta z = \operatorname{div} f \qquad z = 0 \text{ on } \partial\Omega$$

$$\Rightarrow \quad \Delta w = 0$$

and

$$\int_\Omega |\nabla z|^2 \leq \int_\Omega |f|^2 \leq C' \varepsilon^2 .$$

$\rightarrow$ Only need to estimate w

Note: $\operatorname{dist}(\nabla w, SO(n)) \leq \operatorname{dist}(\nabla v, SO(n)) + |\nabla z|$

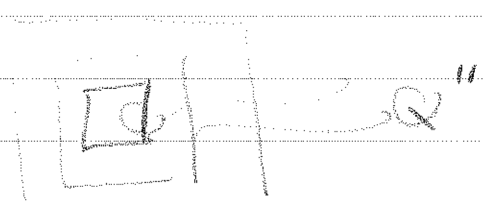
$$\| \operatorname{dist}(\nabla w, SO(n)) \|_{L^2} \leq C \varepsilon . \qquad (3)$$

Step 2. Suboptimal estimate for w

$$\Delta\left(|\nabla w|^2 - n\right) = 2 \underbrace{\left(\Delta \nabla w : \nabla w\right)}_{=0} + 2 |\nabla^2 w|^2$$

$$|\nabla w|^2 - n = C \operatorname{dist}(\nabla w, SO(n)) + C \operatorname{dist}^2(\nabla w, SO(n))$$

Choose cut-off function $\varphi \in C_0^\infty$, $\varphi = 1$ on Q''



$$\int_{Q''} |\nabla^2 w|^2 \leq \int_Q \varphi |\nabla^2 w|^2 = \int_Q \left(|\nabla w|^2 - n\right) \Delta \varphi$$

$$\leq C \int \operatorname{dist}(\nabla w, SO(n)) + \operatorname{dist}^2(\nabla w, SO(n))$$

$$\leq C'(\varepsilon + \varepsilon^2) \leq C\varepsilon.$$

$$\boxed{\|\nabla^2 w\|_{L^2(Q'')} \leq C\sqrt{\varepsilon}.} \quad (4)$$

$$\Rightarrow \quad \|\nabla w - Q\|_{L^2(Q'')} \leq C\sqrt{\varepsilon}$$

for some Q. By (3) we may take $Q \in SO(n)$.

Ravello Lect.

Step 3. Optimal estimate for w.

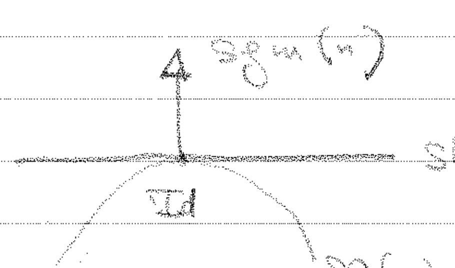
Key point: L^∞ estimate + linearization

Wlog: $Q = \mathrm{Id}$ (otherwise consider $\tilde{w} = Q^{-1} w$)

w harmonic $\Rightarrow$ $\nabla w - \mathrm{Id}$ harmonic

$$\Rightarrow \quad \|\nabla w - \mathrm{Id}\|_{L^\infty(Q''')} \leq C \|\nabla w - \mathrm{Id}\|_{L^2(Q'')} \leq C\sqrt{\varepsilon} \quad (5)$$

$$Q' \subset Q''' \subset Q'' \subset Q.$$



$$\operatorname{dist}(\nabla w, SO(n)) = \operatorname{dist}(\nabla w, \operatorname{skew}(n)) + \underbrace{O(|\nabla w - Id|^2)}_{\leq C\varepsilon}$$

$= \operatorname{sym}(\nabla w - Id)$

$$\Rightarrow \|\operatorname{sym}(\nabla w - Id)\|_{L^2(Q''')} \leq C\varepsilon . \qquad (6)$$

optimal power

By Korn: $\exists A \in \operatorname{skew}(n)$

$$\|\nabla w - Id - A\|_{L^2(Q''')} \leq C\varepsilon$$

By (5) $\quad |A| \leq C\sqrt{\varepsilon} \qquad (\Rightarrow |e^A - A - Id| \leq C\varepsilon$

$$\Rightarrow \|\nabla w - e^A\|_{L^2} \leq C\varepsilon \qquad e^A \in SO(n)$$ ☐

Proof without Korn: $\|\nabla \operatorname{sym}(\nabla \tilde{w})\|_{L^\infty(Q')} \leq C\varepsilon$, $\nabla \tilde{w} = \nabla w - Id$

Main point: $\nabla^2 \tilde{w}$ = linear combination of $\nabla \underbrace{\operatorname{sym} \nabla \tilde{w}}_{e}$

$w^i_{,jk} + w^j_{,ik}$

$w^i_{,kj} + w^k_{,ij}$

$$\left[\quad \tilde{w}^i_{,jk} = \underline{e^i_{j,k}} + \underline{e^i_{k,j}} - e^k_{j,i} \right]$$

$$\Rightarrow \|\nabla^2 \tilde{w}\|_{L^\infty} \leq C\varepsilon.$$

$$\Rightarrow \|\nabla \tilde{w} - A\|_{L^\infty} \leq C\varepsilon$$

By (6) ~~wlog~~ $A \in \operatorname{skew}(n)$, by (5), $|A| \leq C\sqrt{\varepsilon}$

$$\Rightarrow \|\nabla w - e^A\|_{L^\infty(Q')} \leq C\varepsilon + |A|^2 \leq C\varepsilon.$$ ☐

Do global splitting $v = w + t$, $\|\nabla t\|_{L^2} \le C_1 \varepsilon$, $\Rightarrow \|\mathrm{dist}(\nabla w, SO(n))\|_{L^2} \le C\varepsilon$. $\ll 1/10$

Estimate up to the boundary (only discussed very briefly)

There exist $\lambda > 1$ and $N \in \mathbb{N}$ and cubes $Q_i = Q(a_i, r_i)$ s.t.

(i) $\bigcup Q_i = \Omega$

(ii) $\tilde{Q}_i = Q(a_i, \lambda r_i) \subset \Omega$

(iii) Each $x \in \Omega$ is contained in at most N cubes.

Interior estimate

$$c\, r_i^2 \int_{Q_i} |\nabla^2 w|^2 \le \int_{\tilde{Q}_i} \mathrm{dist}^2(\cancel{\nabla v}\ \nabla w, SO(n))$$

$$\vee$$

$$c \int_{Q_i} \mathrm{dist}^2(x, \partial\Omega) |\nabla^2 w|^2$$

$$\sum_i \Rightarrow \int_\Omega \mathrm{dist}^2(x, \partial\Omega) |\nabla^2 w|^2 \le C_N \varepsilon^2$$

Weighted Poincaré $\ne$ (

$$\Rightarrow \int_\Omega |\nabla w - F|^2 \le C_2 \varepsilon^2$$

(Use $\int g^2 \le C_3 \int_\Omega \mathrm{dist}^2 (|g|^2 + |\nabla g|^2)$ and

$$\int_{\mathrm{dist} > \delta} |f - a|^2 \le C_\delta \int_{\mathrm{dist} > \delta} |\nabla f|^2 \le \frac{C_\delta}{\delta^2} \int |\nabla f|^2 \mathrm{dist}^2,$$

$$C_n \int_{\mathrm{dist} \le \delta} |f - a|^2 \mathrm{dist}^2 \le \frac{1}{2} \int_{\mathrm{dist} \le \delta} |f - a|^2 \;)$$