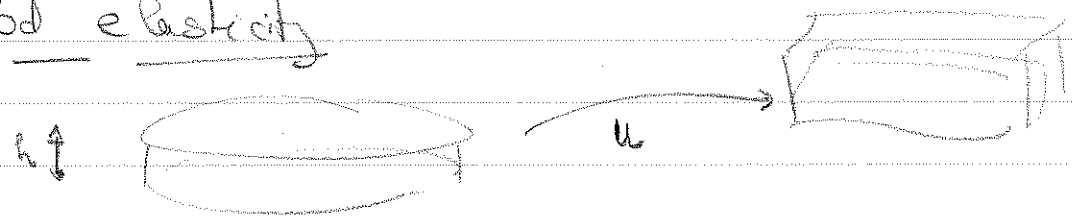
Lecture 2 Kirchhoff's plate theory

G. Friesecke, R.D. James, S.M., CPAM 55 (2002) 1461–1506

3d elasticity



$$\Omega_h = S \times \left(-\frac{h}{2}, \frac{h}{2}\right), \quad S \subset \mathbb{R}^2 \text{ bd. Lipschitz}$$

Elastic energy

$$E^h(u) = \int_{\Omega_h} W(\nabla u(z))\,dz$$

$W : \mathbb{R}^{n \times n} \to \mathbb{R} \cup \{+\infty\}$ free energy density

Assumptions (W Borel measurable)

(i) $W \geq 0$, $W(Id) = 0$ (normalization)

(ii) $W(RF) = W(F) \quad \forall R \in SO(n)$ (frame indifference)

(iii) $W(F) \geq c\, \mathrm{dist}^2(F, SO(n))$, $c > 0$ (non-degeneracy)

(iv) W is C^2 in a neighbourhood of $SO(n)$

Notation

$$Q_3(H) = \frac{\partial^2 W}{\partial F^2}(Id)(H, H) \quad \text{quadratic form}$$

$$Q_3(H) \underset{(ii)}{=} Q_3(\mathrm{sym}\, H) \underset{(iii)}{\geq} c\, |\mathrm{sym}\, H|^2, \quad c > 0$$

Rescale to fixed domain

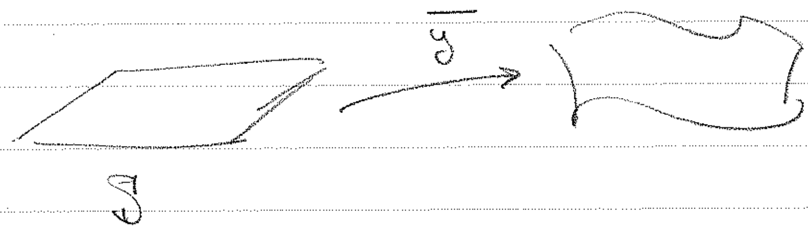
$$z = (\cancel{x}_1, \cancel{x}_2, h\cancel{x}_3) = (\cancel{x}', h\cancel{x}_3), \quad x \in \Omega = \omega \times (-\tfrac{1}{2}, \tfrac{1}{2})$$

$$u(z) = y(x)$$

$$\nabla u(z) = (\underbrace{y_{,1}(x), y_{,2}(x)}_{\nabla' y}, \tfrac{1}{h} y_{,3}(x)) = \nabla_h y$$

$$\boxed{I^h(y) := \frac{1}{h} E^h(u) = \int_\Omega W(\nabla_h y(x))\,dx}$$

2d theory (Kirchhoff, 1850)



Basic variable $\bar{y} : S \to \mathbb{R}^3$

Constraint $(\nabla' \bar{y})^T \nabla' \bar{y} = Id \quad (*)$

$\Leftrightarrow \quad \bar{y}_{,i} \cdot \bar{y}_{,j} = \delta_{ij}$

Normal $\nu = \bar{y}_{,1} \wedge \bar{y}_{,2}$

Second fundamental form $A_{ij} = -\bar{y}_{,ij} \cdot \nu$

$= \bar{y}_{,i} \cdot \nu_{,j}$.

Admissible functions

$\mathcal{A} = \{ y \in W^{2,2}(S, \mathbb{R}^3) : (*) \text{ holds} \}$

Energy

$$\bar{I}_{Ki} = \begin{cases} \frac{1}{24} \int_S Q_2(A(x))\, dx & \text{if } \bar{y} \in \mathcal{A} \\ +\infty & \text{else} \end{cases}$$

$Q_2 : \mathbb{R}^{2\times 2} \to \mathbb{R}$ quadratic form

$$Q_2(A) = \min_{a \in \mathbb{R}^3} Q_3 \left(\begin{array}{c|c} A & \begin{matrix} a_1 \\ a_2 \end{matrix} \\ \hline \begin{matrix} a_1 & a_2 \end{matrix} & a_3 \end{array} \right)$$

Thm 2.

(i) (compact-free lower bound) Suppose that $\limsup_{h\to 0} \frac{1}{h^2} I^{(h)}(y^{(h)}) \le C$

Then $\exists$ subsequence s.t.

$$y^{(h)} \longrightarrow \bar{y} \quad \text{in } W^{1,2}(\Omega; \mathbb{R}^3) \qquad (1)$$

$$\bar{y} \text{ is independent of } x_3 \qquad (2)$$

$$\bar{y} \in \mathcal{A} \qquad (3)$$

$\nabla_h y \to (\nabla' \bar{y}, b)$ L^2

[illegible]

$$\liminf_{h_j \to 0} \frac{1}{h^2} I^{(h)}(y^{(h)}) \geq I_{2d}(\bar{y})$$

(ii) (upper bound) Given $\bar{y} \in \mathcal{A}$ there exist $y^{(h)} : \Omega \to \mathbb{R}^3$ satisfying (1) – (3) and

$$\lim_{h\to 0} \frac{1}{h^2} I^{(h)}(y^{(h)}) = I_{Ki}(\bar{y})$$

Remark 1) This essentially says

$$\frac{1}{h^2} I^{(h)} \xrightarrow{\Gamma} I_{Ki}$$

Book G. Dal Maso, A. Braides

Short Intro

G. Alberti cvgmt.sns.it/papers/alberti

2) O. PANTZ if $y^{(h)}$ C^1 Diffeom.,

$+ \| \operatorname{dist}(\nabla_h y^{(h)}, SO(n)) \|_{L^\infty} < \varepsilon$

How are I_{K_i} and $\frac{1}{h^2} I^h$ related?

Consider
$$\bar{y} \in C^2 \cap \ldots$$
Define 3d def.

$$y^{(h)}(x) = \bar{y}(x') + h x_3 \nu(x') \qquad \text{Picture}$$

Compute $I^h(y^{(h)})$:

$$\nabla_h y^{(h)} = \Big(\underbrace{\bar{y}_{,1} + h x_3 \nu_{,1}}_{1^{st}\text{ column}} \Big| \bar{y}_{,2} + h x_3 \nu_{,2} \Big| \nu \Big)$$

W frame indifferent Polar decomposition

$$F = RU, \quad U = (F^T F)^{1/2} \quad \text{if } \det F > 0$$

$$W(F) = W\left((F^T F)^{1/2} \right)$$

$F = (a_1 | a_2 | a_3) \Rightarrow (F^T F)_{ij} = (a_i \cdot a_j)$

$$(\nabla_h y^{(h)})^T (\nabla_h y^{(h)}) = \begin{pmatrix} Id + 2h x_3 A & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} + O(h^2)$$

Geometric view $\quad \frac{\partial}{\partial x_3}$ metric = curvature

$$\bar{y}_{,i} \cdot \nu = 0$$
$$\nu_{,i} \cdot \nu = 0$$

$$\left((\nabla_h y^{(h)})^T \nabla_h y^{(h)} \right)^{1/2} \approx Id + h x_3 \begin{pmatrix} A & 0 \\ 0\ 0 & 0 \end{pmatrix}$$

$$W(\nabla_h y^{(h)}) \approx \frac{1}{2} Q_3 (h x_3 A)$$

$$\int W(\nabla_h y^{(h)})\, dx_3 = h^2 \Big(\underbrace{\int_{-1/2}^{1/2} \frac{1}{2} x_3^2}_{=1/24} \Big) Q_3(A)$$

Proof of Thm 2.

(~~a~~) Upper bound is 'easy'. For $\bar{y} \in \mathcal{A}' \cap C^2$ use

$$y^{(h)} = \bar{y}(x') + h x_3 \nu(x') + \underbrace{\frac{h^2}{2} x_3^2 d(x')}$$

$\rightarrow$ relaxation from Q_3 to Q_2 ('Poisson effect')

*) $\bar{y} \in W^{2,\infty}$ ... done

$\bar{y} \in W^{2,2}$ careful approximation by $W^{2,\infty}$ on a large set ('truncation of gradients')

Main point: Lower bound

1 Compactness

2 Identification of limiting strain

Show: $\nabla_h y^{(h)} \approx \text{Rotation}(x)\left(Id + h x_3 A^{(h)}\right)$ 'curvature'

for an arbitrary sequence $y^{(h)}$ with $I^h(y^{(h)}) \le C h^2$.

*) One could now also use recent results that smooth isometries are dense in $W^{2,2}$ isometries

convex S : R. Pakzad, J. Diff. Geometry 66 (2004), 47–69.

piecewise C^1 S : P. Hornung, Preprint, University Duisburg-Essen

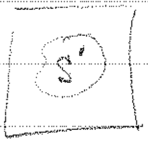
Compactness

Prop. 3 Suppose $I^{(h)}(y^{(h)}) \leq Ch^2$. Then $\exists R^{(h)} : S \subset \mathbb{R}^2 \to SO(3)$

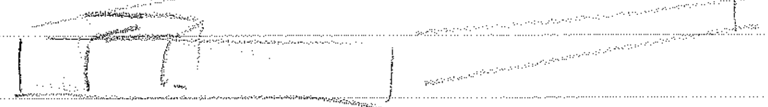
$$\| \nabla_h y^{(h)} - R^{(h)} \|_{L^2(\Omega)} \leq Ch^2 \qquad (4)$$

$$\int_{S'} | R^{(h)}(x'+\xi) - R^{(h)}(x') |^2 \leq C(|\xi|^2 + h^2) \qquad (5)$$

If $\operatorname{dist}(S', \partial S) < 2(|\xi| + h)$



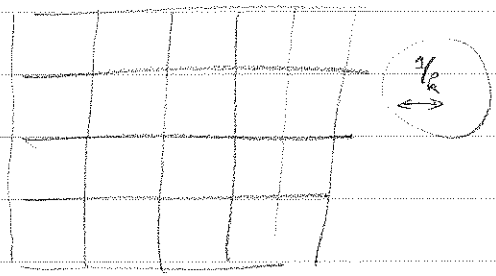
Idea:



- Divide Ω_h into cubes of size h
- Apply rigidity estimate in each cube
- Compare neighbouring cubes

To illustrate main idea suppose

$$S = (0,1)^2, \quad h = 1/k$$



$$S = \bigcup S_i \qquad S_i = a_i + (0,h)^2,$$

$$I = (-1/2, 1/2)$$

$$y(x_1, x_2, x_3) = u(x_1, x_2, hx_3)$$

$$\exists R_i \in SO(3) \quad \frac{1}{h} \int_{\underbrace{S_i \times hI}_{\text{cube}}} |\nabla u - R_i|^2 \leq \frac{C_1}{h} \int \operatorname{dist}^2(\nabla u, SO(3))$$

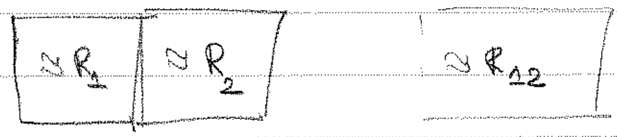
$$\leq \frac{C_1}{h} \int_{S_i \times hI} V(\nabla u)$$

$$\boxed{\int_{S_i \times I} |\nabla_h y^{(h)} - R_i|^2 \leq C_1 I^h(y^{(h)}, S_i \times I)}$$

Define $R^{(h)}(x') = R_i$ if $x' \in S_i$.

$$\Rightarrow \quad \|\nabla_h y^{(h)} - R^{(h)}\|_{L^2}^2 \le C_1 I^{(h)}(y^{(h)}) \le C_1 h^2$$

Apply rigidity estimate to two neighbouring cubes and to their union.



Sum over all cubes

$$\Rightarrow \quad \| R^{(h)}(\cdot \pm h e_i) - R^{(h)}(\cdot)\|_{L^2(S')}^2 \le C_1 h^2$$

Iterate

$$\Rightarrow \quad \| R^{(h)}(\cdot + \xi) - R^{(h)}(\cdot)\|_{L^2(S')} \le C_1'(|\xi| + h)$$

$\boxed{}$

Consequences:

(5) + Frechet-Kolmogorov $\Rightarrow$

$$R^{(h)} \to \bar{R} \quad \text{in } L^2(S'; \mathbb{R}^{3\times3})$$

$$\|\nabla' \bar{R}\|_{L^2} \le C_1 \quad \leftarrow \text{independent of } S'$$

$$\|R^{(h)}\|_{L^2(S\setminus S')}^2 \le |S \setminus S'| \; \underbrace{\|R^{(h)}\|_{L^\infty}^2}_{\le 3}$$

$$\Rightarrow \quad R^{(h)} \to \bar{R} \quad L^2(S; \mathbb{R}^{3\times3}), \quad \nabla\bar{R} \in L^2$$

Thus

$$\nabla_h y^{(h)} \to \bar{R} \quad L^2, \quad y^{(h)} \to \bar{y} \text{ in } W^{1,2} \qquad (1)/W$$

$$\frac{1}{h} y^{(h)}_{,3} \to R e_3 \quad L^2 \Rightarrow \bar{y}_{,3} = 0 \qquad (2)$$

$$\bar{y}_{,1} = \bar{R} e_1, \quad \bar{y}_{,2} = \bar{R} e_2 \quad \rightsquigarrow \bar{y} \in \mathcal{A} \qquad (3)$$

$1 \le p < \infty$

Frechet-Kolmogorov ($\{f_k\}$ "pre compact" in $L^p(\mathbb{R}^n)$)

$\Leftrightarrow$ (i) $\sup_k \|f_k\|_{L^p} \le C_1$

(ii) $\lim_{\varepsilon \to 0} \sup_k \|f_k(\cdot + \delta) - f_k(\cdot)\|_{L^p} \Rightarrow 0$

(iii) $\lim_{R \to \infty} \sup_k \int_{\mathbb{R}^n \setminus B_R} |f_k|^p = 0$.

Rem. p = ∞ use C^0 instead of L^∞.

Proof $\Leftarrow$ $\quad f_k \longrightarrow f \quad L^p \quad \begin{matrix} \xrightarrow{*} \mathcal{M} & \text{if } p=1 \\ \xrightarrow{*} L^\infty & \text{if } p=\infty \end{matrix}$

~~$\varphi_\varepsilon * f \ (\ \ R<\infty)$~~

$$\varphi_\varepsilon * f_k \longrightarrow \varphi_\varepsilon * f \qquad L^p(B_R)\ \forall R.$$

$$\|\varphi_\varepsilon * f_k(x) - f_k(x)\|_{L^p} = \left\| \int \varphi_\varepsilon(y) \left[f_k(x-y) - f_k(x) \right] dy \right\|_{L^p}$$

$$\underset{\text{convexity of norm}}{\le} \int \varphi_\varepsilon(y) \underbrace{\| f_k(x-y) - f_k(x) \|_{L^p}}_{\le \omega(\varepsilon)} \le \omega(\varepsilon)$$

$$\limsup_{k \to \infty} \|f_k - f\|_{L^p(B_R)}$$

$$\le \omega(\varepsilon) + \|\varphi_\varepsilon * f - f\|_{L^p} \to 0 \text{ as } \varepsilon \to 0.$$