1. Minimal Surfaces

We consider mappings $X : B \to \mathbb{R}^3$ from a parameter (i.e. connected, open) domain B in $\mathbb{R}^2$ into the 3-dim. Euclidean space $(\mathbb{R}^3, |\cdot|)$:

$$X(u,v) = \left(X^1(u,v), X^2(u,v), X^3(u,v) \right), \quad w = (u,v) \in B.$$

Such a mapping will be called a surface or, more precisely, a parameter representation of a surface $\mathcal{S}$, given by

$$\mathcal{S} = [X] := \{ X \circ \tau : \tau \text{ is a homeo- or diffeomorphism from } \Omega \text{ onto } B \}.$$

This is much to clumsy, and so we stick to the notation "surface".

A surface $X \in C^1(B, \mathbb{R}^3)$ is called "regular" or "immersion" if $X_u \wedge X_v (w) \neq 0 \quad \forall w \in B$.

Then we can define the "surface normal" N by

$$N := \frac{1}{W} X_u \wedge X_v, \quad W := |X_u \wedge X_v| = \sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2}$$

$$\mathcal{E} := |X_u|^2, \ \mathcal{F} := X_u \cdot X_v, \ \mathcal{G} := |X_v|^2$$

$N : B \to S^2 \subset \mathbb{R}^3$ is the Gauss map of X, $N(B) :=$ spherical image

1st Ff $= I = |dX|^2 \quad = \mathcal{E}\,du^2 + 2\mathcal{F}\,du\,dv + \mathcal{G}\,dv^2$

2nd Ff $= II = -dX \cdot dN \quad = \mathcal{L}\,du^2 + \mathcal{M}\,du\,dv + \mathcal{N}\,dv^2$

3rd Ff $= III = |dN|^2$

κ_1, κ_2 : = principal curvatures = eigenvalues of II/I

$$H = \frac{\mathcal{L}\mathcal{G} + \mathcal{N}\mathcal{E} - 2\mathcal{M}\mathcal{F}}{2(\mathcal{E}\mathcal{G} - \mathcal{F}^2)} = \frac{1}{2}(\kappa_1 + \kappa_2) = \text{mean curvature of } X$$

$$K = \frac{\mathcal{L}\mathcal{N} - \mathcal{M}^2}{\mathcal{E}\mathcal{G} - \mathcal{F}^2} = \kappa_1 \kappa_2 = \text{Gauß curvature of } X$$

Special coordinates :

(i) Conformal parameters u, v : $\mathcal{E} = \mathcal{G}$, $\mathcal{F} = 0$

i.e. $|X_u|^2 = |X_v|^2$, $X_u \cdot X_v = 0$

(ii) Nonparametric representation (for graphs):

$X(x,y) = (x, y, z(x,y))$

Remark : (i) can always be achieved on $\mathcal{B}$ (or on $\overline{\mathcal{B}}$) for regular surfaces of class $C^{1,\alpha}(\mathcal{B}, \mathbb{R}^3)$ (or $C^{1,\alpha}(\overline{\mathcal{B}}, \mathbb{R}^3)$, $\partial\mathcal{B} \in C^{1,\alpha}$)
Then,

(+) $\boxed{2H\mathcal{E} = \mathcal{L} + \mathcal{N}}$ in conformal coordinates u, v

Surface area of $X \in C^1(\overline{\mathcal{B}}, \mathbb{R}^3)$ (or $X \in H^{1,2}(\mathcal{B}, \mathbb{R}^3)$):

$$A(X) := \int_{\mathcal{B}} |X_u \wedge X_v| \, du\, dv = \int_X dA \quad , \quad dA = \mathcal{W}\, du\, dv$$

Assumption : Let $X \in C^1(\overline{\mathcal{B}}, \mathbb{R}^3) \cap C^2(\mathcal{B}, \mathbb{R}^3)$ be regular

Consider "normal variation of X":

$$Z(\cdot,\varepsilon) := X + \varepsilon\lambda N \quad , \quad |\varepsilon| \ll 1 \ , \ \lambda \in C_c^\infty(B)$$

Then:

$$Z_u = X_u + \varepsilon\lambda_u N + \varepsilon\lambda N_u$$

$$Z_v = X_v + \varepsilon\lambda_v N + \varepsilon\lambda N_v$$

Since $N \perp X_u, X_v$ and $\mathcal{L} = -X_u \cdot N_u$, $\mathcal{N} = -X_v \cdot N_v$, $\mathcal{M} = -(X_u \cdot N_v + X_v \cdot N_u)$

$(N \perp N_u, N_v)$

$$|Z_u|^2 = \mathcal{E} - 2\varepsilon\lambda\mathcal{L} =: \tilde{\mathcal{E}}$$

$$|Z_v|^2 = \mathcal{G} - 2\varepsilon\lambda\mathcal{N} =: \tilde{\mathcal{G}}$$

$$Z_u \cdot Z_v = \mathcal{F} - 2\varepsilon\lambda\mathcal{M} =: \tilde{\mathcal{F}}$$

Therefore

$$\tilde{\mathcal{W}}^2 := \tilde{\mathcal{E}}\tilde{\mathcal{G}} - \tilde{\mathcal{F}}^2 = \mathcal{W}^2 - 2\varepsilon\lambda\,\underbrace{\left[\mathcal{E}\mathcal{N} + \mathcal{G}\mathcal{L} - 2\mathcal{F}\mathcal{M}\right]}_{=2H\mathcal{W}^2} + O(\varepsilon^2)$$

whence

$$\tilde{\mathcal{W}}^2 = \mathcal{W}^2\left[1 - 4\varepsilon\lambda H\right] \quad \text{and} \quad \tilde{\mathcal{W}}\big|_{\varepsilon=0} = \mathcal{W}$$

Differentiation by ε yields

$$2\tilde{\mathcal{W}}\tilde{\mathcal{W}}_\varepsilon = -4\lambda H\mathcal{W}^2 \quad \Rightarrow \quad \tilde{\mathcal{W}}_\varepsilon\big|_{\varepsilon=0} = -2\lambda H\mathcal{W}$$

Set

$$\delta A(X,\lambda N) := \left(\frac{d}{d\varepsilon}\int_B \tilde{\mathcal{W}}\,du\,dv\right)\Big|_{\varepsilon=0} = \int_B \tilde{\mathcal{W}}_\varepsilon(0)\,du\,dv$$

Then

$$\delta A(X,\lambda N) = -\int_B 2\lambda H\mathcal{W}\,du\,dv = \text{First variation of } X$$

This yields:

<u>Proposition 1</u>. If $\delta A(X, \lambda N) = 0 \quad \forall \lambda \in C_c^\infty(B)$ then $H = 0$.

<u>Preliminary Definition</u>:

A regular surface $X \in C^2(B, \mathbb{R}^3)$ is called <u>minimal surface</u> if $H = 0$.

<u>Proposition 2</u>. If $X \in C^2(B, \mathbb{R}^3)$ is regular and satisfies

$$(1) \qquad |X_u|^2 = |X_v|^2 \,, \quad X_u \cdot X_v = 0$$

then

$$(*) \qquad \Delta X = 2H\, X_u \wedge X_v$$

In particular, if $H = 0$ then

$$(2) \qquad \Delta X = 0$$

<u>Proof</u>. Differentiating (1) by u or v, we obtain

$$X_u \cdot X_{uu} = X_v \cdot X_{uv} \,, \quad X_{uu} \cdot X_v + X_u \cdot X_{uv} = 0$$

$$X_u \cdot X_{uv} = X_v \cdot X_{vv} \,, \quad X_{uv} \cdot X_v + X_u \cdot X_{vv} = 0$$

whence

$$\Delta X \cdot X_u = 0 \quad \text{and} \quad \Delta X \cdot X_v = 0 \,,$$

and therefore

$$\Delta X = (\Delta X \cdot N) N$$

On the other hand, since $N \cdot X_u = 0$ and $N \cdot X_v = 0$,

$$\mathcal{L} := -X_u \cdot N_u = X_{uu} \cdot N \,, \quad \mathcal{N} := -X_v \cdot N_v = X_{vv} \cdot N \,,$$

and therefore

$$\Delta X \cdot N = \mathcal{L} + \mathcal{N} \underset{(1)}{=} 2H\mathcal{E} \underset{(1)}{=} 2HW$$

Consequently,

$$\Delta X = 2HW N = 2H X_u \wedge X_v . \qquad \square$$

Definition 1

(i) A nonconstant surface $X \in C^2(B, \mathbb{R}^3)$ is called minimal surface (whether regular or not) if it satisfies (1) & (2)

(ii) X is called H-surface if it satisfies (1), $X(w) \not\equiv$ const, and

$$\Delta X = H(X)\, X_u \wedge X_v$$

with a prescribed function $H: \mathbb{R}^3 \to \mathbb{R}$.

(Note that these definitions are invariant with respect to conformal parameter changes; the second only for strictly conformal changes.)

Definition 2. Let $X: B \to \mathbb{R}^3$ be a minimal surface on a simply connected domain B. Then one can define the adjoint minimal surface $X^*: B \to \mathbb{R}^3$ by the Cauchy-Riemann equations

$$\boxed{X_u = X^*_v , \quad X_v = -X^*_u}$$

Note: X^* is clearly a minimal surface

(X^* is unique except for an additive const $\in \mathbb{R}^3$).

Then, for $w = u + iv$, the mapping $F: B \to \mathbb{C}^3$, defined by

$$F(w) := X(u,v) + i\, X^*(u,v)$$

is holomorphic on B and

$$F' = X_u + i X^*_u = X_u - i X_v = 2 X_w .$$

Furthermore,

$$(1) \iff X_w \cdot X_w = 0 \iff F' \cdot F' = 0 \quad \text{(i.e. } F \text{ is "isotropic")}$$

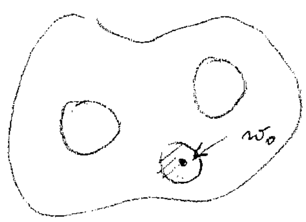
$$(2) \iff X_{w\bar{w}} = 0$$

, with $\partial_w = \frac{1}{2}(\partial_u - i\partial_v)$, $\partial_{\bar{w}} = \frac{1}{2}(\partial_u + i\partial_v)$
= Wirtinger symbols

Note that a minimal surface may have singular points $w_0 \in B$, i.e. points with $\mathcal{W}(w_0) = |X_u(w_0) \wedge X_v(w_0)| = 0$.

Proposition 3. The only singular points of a minimal surface are points $w_0 \in B$ with $\mathcal{E}(w_0) = 0$, or equivalently, with $F'(w_0) = 0$. Such points are called branch points of X.

Proof. Because of (1), we have $\mathcal{W} = \mathcal{E} = \mathcal{G}$, and in a neighbourhood of w_0 we have $F' = X_u - iX_v$.



$\square$

Proposition 4. Interior branch points $w_0 \in B$ of a minimal surface X have a finite order m and are isolated points.

Proof. Let $w_0 \in B$. Then we have the Taylor expansion

$$F(w) = \sum_{k=0}^{\infty} \frac{1}{k!} F^{(k)}(w_0)(w-w_0)^k \qquad \text{for } |w-w_0| < r \ll 1.$$

If $F^{(k)}(w_0) = 0 \ \forall k \geq 1$ then $F(w) \equiv F(w_0)$ for $w \in B_r(w_0)$
$\Rightarrow F(w) \equiv$ const on B, impossible by definition.
Hence $\exists\, m \in \mathbb{N}_0$ such that (assuming w.l.o.g. $F(w_0) = 0$)

$$F^{(k)}(w_0) = 0 \text{ for } 0 \leq k \leq m, \qquad F^{(m+1)}(w_0) \neq 0.$$

Then

$$F(w) = F(w_0) + \frac{1}{(m+1)!} F^{(m+1)}(w_0)(w-w_0)^{m+1} + \cdots$$

whence

$$(**) \qquad \boxed{F'(w) = \frac{1}{m!} F^{(m+1)}(w_0)(w-w_0)^m + \cdots .}$$

The point w_0 is $\left\{ \begin{array}{l} \text{regular if } m=0 \\ \text{singular if } m \geq 1 \end{array} \right\}$. Hence, if w_0 is a branch point then $m \geq 1$; one calls m the order of the branch point w_0 which clearly is isolated.

$\square$

Set

$$A = \frac{1}{2}(\alpha - i\beta) := \frac{1}{2\, m!} F^{(m+1)}(w_0) \quad , \quad \alpha, \beta \in \mathbb{R}^3 \setminus \{0\}$$

$$|\alpha| = |\beta| > 0 \, , \; \alpha \cdot \beta = 0$$

and note

$$X_w = \frac{1}{4}(F_u - i F_v) = \frac{1}{2} F'$$

Then we obtain:

Proposition 5. If $w_0 \in B$ is a branch point of the minimal surface $X : B \to \mathbb{R}^3$, then there is an integer $m \geq 1$ and a vector $A \in \mathbb{C}^3 \setminus \{0\}$ with $A \cdot A = 0$ such that

$$(3) \qquad X_w(w) = A\,(w - w_0)^m + o(|w - w_0|^m) \quad \text{as } w \to w_0$$

Conclusion: Set w.l.o.g. $w_0 = 0$, and let $A = \frac{1}{2}(\alpha - i\beta)$ as above

Then

$$X_u(w) - i X_v(w) = (\alpha - i\beta)\left(\operatorname{Re} w^m + i \operatorname{Im} w^m\right) + \dots$$

$$= \left(\alpha \operatorname{Re} w^m + \beta \operatorname{Im} w^m\right) - i\left(-\alpha \operatorname{Im} w^m + \beta \operatorname{Re} w^m\right) + \dots$$

and ~~therefore~~

$$(X_u \wedge X_v)(w) = (\alpha \wedge \beta)\left[\left(\operatorname{Re} w^m\right)^2 + \left(\operatorname{Im} w^m\right)^2\right] + \dots$$

Thus we obtain:

<u>Proposition 6</u>. If w_0 is an interior branch point of the minimal surface $X : B \to \mathbb{R}^3$, then

$$(X_u \wedge X_v)(w) = (\alpha \wedge \beta)\, |w - w_0|^{2m} + \ldots \qquad \text{as } w \to w_0$$

with $\alpha, \beta \in \mathbb{R}^3$, $|\alpha| = |\beta| > 0$, $\alpha \cdot \beta = 0$, and

$$N(w) \to N_0 := \frac{\alpha \wedge \beta}{|\alpha \wedge \beta|} \in S^2 .$$

<u>Remark 1.</u> Analogous statements hold for boundary branch points of minimal surfaces, and even for branch points of H-surfaces, $H \in C^{0,\alpha}$; moreover for H-surfaces in Riemannian manifolds. The proof of these facts is more difficult; one has to use Hartman - Wintner technique.

<u>Remark 2</u>. The Weingarten equations imply that the Gauß map N of a minimal surface satisfies

$$(**) \quad \Delta N + N\, |\nabla N|^2 = 0 ,$$

and for $H = 0$ one has $III = -K\, I$. This implies

$$|N_u|^2 = |N_v|^2 , \qquad N_u \cdot N_v = 0 ,$$

$$|\nabla N|^2 = -K\, |\nabla X|^2 = 2\, |N_u \wedge N_v| ,$$

$$N_u \wedge N_v = K\, X_u \wedge X_v , \quad K \le 0$$

therefore

$$\Delta N = 2\, N_u \wedge N_v ,$$ i.e. N is a surface of constant mean curvature one.

Furthermore: $(**)$ and $N \in C^0(B, \mathbb{R}^3)$ imply that $N \in C^\omega(B, \mathbb{R}^3)$.

This can also be proved by means of stereographic projection and Riemann's theorem for removing isolated singularities of holomorphic maps.

<u>Remark 3</u>. Set

$$e_1 := \frac{\alpha}{|\alpha|}, \quad e_2 := \frac{\beta}{|\beta|}, \quad e_3 := N_0 = e_1 \wedge e_2, \quad a := \frac{|\alpha|}{m+1} = \frac{|\beta|}{m+1} > 0,$$

$X_0 := X(w_0)$. Then

$$X(w) = X_0 + a\left[e_1 \operatorname{Re}(w-w_0)^{m+1} + e_2 \operatorname{Im}(w-w_0)^{m+1} \right] + o(|w-w_0|^{m+1})$$

If we introduce new Euclidean coordinates with the axes e_1, e_2, e_3, then $X(w) = (x(w), y(w), z(w))$ takes the form

$$\boxed{\begin{aligned} x(w) + i\,y(w) &= (x_0 + i y_0) + a(w-w_0)^{m+1} + o(|w-w_0|^{m+1}) \\ z(w) &= z_0 + o(|w-w_0|^{m+1}) \end{aligned}}$$

The x, y - plane is the "generalized" tangent plane.

2. Dirichlet's Integral

Minimal surfaces are conformally parametrized harmonic mappings $X : B \to \mathbb{R}^3$. Therefore they are critical points of Dirichlet's integral

$$D(X) := \frac{1}{2} \int_B \{ \underbrace{|X_u|^2 + |X_v|^2}_{= |\nabla X|^2} \} \, du\, dv$$

(= conformally invariant)

(i.e. $D(X \circ \sigma) = D(X)$ for σ = strictly conformal or anticonformal)

in a two-fold sense.

Proposition 1. Let $X \in H^{1,2}(B, \mathbb{R}^3)$, $\phi \in C_c^\infty(B, \mathbb{R}^3)$, and $Z(\cdot, \varepsilon) := X + \varepsilon \phi$. Then the first variation

$$\delta D(X, \phi) := \frac{d}{d\varepsilon} D(X, \phi) \Big|_{\varepsilon = 0}$$

is given by

$$\delta D(X, \phi) = \int_B \nabla X \cdot \nabla \phi \, du\, dv .$$

Secondly, if

$$\delta D(X, \phi) = 0 \quad \forall \phi \in C_c^\infty(B, \mathbb{R}^3)$$

then $X \in C^\infty(B, \mathbb{R}^3)$ and $\Delta X = 0$ in B; in fact, $X \in C^\omega(B, \mathbb{R}^3)$.

Proof. Well-known.

Remark 1. Analogous statements we have for

$$D(X) := \frac{1}{2} \int g_{jk}(X) \left(X_u^j X_u^k + X_v^j X_v^k \right) du\, dv$$

and

$$E(X) := D(X) + \int_B Q(X) \cdot (X_u \wedge X_v) \, du\, dv$$

Now we turn to the "variation of the independent variables.

Choose $\lambda = (\mu, \nu) \in C^1(\overline{B}, \mathbb{R}^2)$, extend to $\lambda \in C^1_c(\mathbb{R}^2, \mathbb{R}^2)$ and form the 1-parameter family of diffeomorphisms $\tau(\cdot, \varepsilon) : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\tau(w, \varepsilon) := w - \varepsilon\lambda(w) + o(\varepsilon) \quad \text{as } \varepsilon \to 0$$

with the inverses

$$\sigma(w, \varepsilon) = w + \varepsilon\lambda(w) + o(\varepsilon) \quad \text{as } \varepsilon \to 0 ,$$

and set $B^*_\varepsilon := \tau(B, \varepsilon)$ whence $B = \sigma(B^*_\varepsilon, \varepsilon)$.

<u>Proposition 2</u> (i) Let $X \in H^{1,2}(B, \mathbb{R}^3)$, $\lambda \in C^1(\overline{B}, \mathbb{R}^3)$, $\sigma(\cdot, \varepsilon) : \overline{B}^*_\varepsilon \to \overline{B}$ as above, $Z(\cdot, \varepsilon) := X \circ \sigma(\cdot, \varepsilon)$, $D(Z(\cdot,\varepsilon), B^*_\varepsilon) := \frac{1}{2}\int_{B^*_\varepsilon} |\nabla Z(\cdot, \varepsilon)|^2 \, du\, dv$. Then the <u>first inner variation</u>

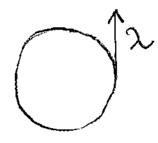
$$\partial D(X, \lambda) := \frac{d}{d\varepsilon} D(Z(\cdot,\varepsilon), B^*_\varepsilon)\Big|_{\varepsilon = 0}$$

is given by

$$\partial D(X, \lambda) = \frac{1}{2}\int_B \{ a(\mu_u - \nu_v) + b(\mu_v + \nu_u) \} \, du\, dv$$

with

$$a := |X_u|^2 - |X_v|^2 , \quad b := 2 X_u \cdot X_v$$



(ii) If we have

(1) $\quad \partial D(X, \lambda) = 0 \quad \forall \lambda \in C^1(\overline{B}, \mathbb{R}^3) \quad$ (or only: $\forall \lambda \in C^1_{\text{tang}}(\overline{B}, \mathbb{R}^3)$)

then $a = 0$ and $b = 0$ in B if $\partial B \in C^{2,\alpha}$.

<u>Proof</u>. Part (i) is standard in calculus of variations. Thus we prove (ii) for $\lambda \in C^1(\overline{B}, \mathbb{R}^2)$; the other assertion is very useful, but more difficult to prove.

<u>Step 1</u>. Apply (1) to $\lambda = S_\delta \eta$, $\eta = (\eta^1, \eta^2) \in C_c^\infty(B', \mathbb{R}^2)$ with $B' \subset\subset B$, $0 < \delta << 1$, S_δ = smoothing operator with a symmetric kernel k_δ, and set $a^\delta := S_\delta a$, $b^\delta := S_\delta b$. Then standard manipulations yield

$$0 = \int_B \{ a^\delta (\eta^1_u - \eta^2_v) + b^\delta (\eta^1_v + \eta^2_u) \} \, du\, dv$$

$$= \int_B \{ -(a^\delta_u + b^\delta_v)\eta^1 + (a^\delta_v - b^\delta_u)\eta^2 \} \, du\, dv$$

Then
$$a^\delta_u + b^\delta_v = 0 \quad \text{and} \quad a^\delta_v - b^\delta_u = 0 \quad \text{in } B' \subset\subset B$$

i.e. $\Phi^\delta(w) := a^\delta(u,v) - i b^\delta(u,v)$ is holomorphic in B'

and $\Phi^\delta \to \Phi := a - ib$ in $L^1(B')$ $\Rightarrow$ Φ is hol. in B', and then in B.

<u>Step 2</u>. Choose $\rho, \sigma \in C_c^\infty(B)$ and determine $h, k \in C^{2,\alpha}(\overline{B})$ such that

$$\Delta h = \rho \text{ in } B, \quad h = 0 \text{ on } \partial B,$$

$$\Delta k = \sigma \text{ in } B, \quad k = 0 \text{ on } \partial B.$$

Then the functions

$$\mu := h_u + k_v, \quad \nu := -h_v + k_u \qquad \text{lie in } C^{1,\alpha}(\overline{B}) \text{ and}$$

satisfy

$$\mu_u - \nu_v = \rho, \quad \mu_v + \nu_u = \sigma \quad \text{in } \overline{B}$$

Then (1) implies

$$\int_{B} \{ a\rho + b\sigma \} \, du\,dv = 0 \quad \forall \rho, \sigma \in C_c^\infty(B)$$

whence $a = 0$, $b = 0$. $\square$

<u>Note</u>: Step 1 is <u>not</u> needed if $\partial D(X, \lambda) = 0$ holds for all $\lambda \in C^1(\bar{B}, \mathbb{R}^2)$, but will be used if one only assumes $\lambda \in C^1_{tang}(\bar{B}, \mathbb{R}^2)$.

What is the connection between area A and energy D?

<u>Proposition 3</u>. For any $X \in H^{1,2}(B, \mathbb{R}^3)$ we have

$$A(X) \leq D(X),$$

and

$$A(X) = D(X) \iff |X_u|^2 = |X_v|^2 \;\&\; X_u \cdot X_v = 0 \quad (\text{a.e. in } B)$$

<u>Proof</u>. The assertion follows from Lagrange's identity

$$|p \wedge q| = \sqrt{|p|^2 |q|^2 - (p \cdot q)^2}$$

or simply from

$$|p \wedge q| \leq |p|\,|q| \leq \frac{1}{2}\left(|p|^2 + |q|^2\right)$$

$$(= \text{ iff } p \cdot q = 0) \qquad (= \text{ iff } |p| = |q|)$$

$\square$

3. Plateau's Problem

The original formulation of this problem might have been the following:
Given a closed curve Γ in space, find a surface that minimizes area among all surfaces spanning Γ.

Lagrange (about 1760) was the first to investigate this problem, but the name was coined after the Belgian physicist Plateau who, in the middle of the 19th century, devised many illustrative soap film experiments. What have soap films to do with "minimal" surfaces? The connection was established by Gauss's work on capillarity (1830) which states that, in equilibrium, any liquid surface is a minimizer of the potential energy caused by the molecular forces, and for soap films the energy is proportional to the area. In other words: Soap films can be viewed as physical models of stable minimal surfaces.

Motivated by experiments, Plateau conjectured that every closed curve (without double points) spans a surface which minimizes area, as every closed wire seemed to span some soap film. This conjecture was first proved by Jesse Douglas and, independently, by Tibor Radó in 1930/31. A simplified proof was given by R. Courant and L. Tonelli 1936, which I will use in my presentation.

First we have to give a precise definition what we mean by "Plateau's problem". Originally, minimal surface meant: "surface minimizing area". This meaning was replaced by "regular surface being a critical point of area", or else, by "regular surface with $H=0$". (A similar transformation occured for "geodesics".)

Furthermore, soap film experiments show that solutions of the "naive Plateau problem" (formulated above) often are surfaces of higher genus, or even nonorientable surfaces, i.e. the parameter domain B of $X: B \to \mathbb{R}^3$ might be a 2-dim. Riemannian manifold with boundary, orientable or not. In connection with Plateau's problem, one nowadays considers surfaces $X: B \to \mathbb{R}^3$ on simply connected domains B which map ∂B onto Γ. Since, by Riemann's mapping theorem, essentially every such B can be mapped strictly conformally onto the unit disk $B_1(0)$, we use "disk-type" mappings $X: B \to \mathbb{R}^3$ with

$$B = \{ w = (u,v) : |w| < 1 \}.$$

Secondly, the boundary contour Γ is supposed to be a rectifiable, closed Jordan curve Γ in $\mathbb{R}^3$, i.e. Γ is the image $\gamma(C)$ of $C = S^1$ under a homeomorphism $\gamma: C \to \Gamma$, which has finite length. Fixing γ, the curve Γ will be oriented.

Now we define the class $\mathcal{C}(\Gamma)$ of admissible surfaces $X: B \to \mathbb{R}^3$ by:

$$\mathcal{C}(\Gamma) := \{ X \in H^{1,2}(B,\mathbb{R}^3) \cap C^0(\partial B, \mathbb{R}^3) : \text{The trace } X|_C \text{ of } X \text{ on } C := \partial B \text{ is a weakly monotonic, continuous mapping of } \partial B \text{ onto } \Gamma \}$$

This means : If $\Gamma = \gamma(C)$, $\gamma : C \to \Gamma$ a fixed homeomorphism then there is a weakly increasing real-valued function $\tau \in C^0(\mathbb{R})$ with $X(e^{i\theta}) = \gamma(e^{i\tau(\theta)})$ for all $\theta \in \mathbb{R}$ and

$$\tau(\theta + 2\pi) = \tau(\theta) + 2\pi$$

(in other words : $X|_C$ maps C monotonically onto Γ, keeping the orientation, but there can be intervals on C where X is constant).

<u>Three-point condition</u>. Fix three distinct points $w_1, w_2, w_3 \in C = \partial B$ and $Q_1, Q_2, Q_3 \in \Gamma$ with $\gamma(w_k) = Q_k$, $1 \le k \le 3$, and set

$$\mathcal{C}^*(\Gamma) := \{ X \in \mathcal{C}(\Gamma) : \underbrace{X(w_k) = Q_k \quad , k = 1,2,3}_{\text{"Three-point condition"}} \}$$

This <u>normalization</u> can be achieved by strictly conformal (i.e. biholomorphic) mappings of $\overline{B}$ onto itself which leave D and A invariant.

Note that <u>$\mathcal{C}(\Gamma)$ is nonempty</u> if Γ is assumed to be rectifiable ; it would be sufficient to assume that

$$\gamma \in H^{1/2,2}(\partial B, \mathbb{R}^3) \cap C^0(\partial B, \mathbb{R}^3).$$

Therefore we have :

<u>Lemma 1</u>. If $\mathcal{C}(\Gamma) \neq \emptyset$ then $\mathcal{C}^*(\Gamma) = \emptyset$ and

$$\boxed{\underbrace{\inf_{\mathcal{C}(\Gamma)} D}_{=: e(\Gamma)} = \underbrace{\inf_{\mathcal{C}^*(\Gamma)} D}_{= e^*(\Gamma)}} \quad , \quad \boxed{\inf_{\mathcal{C}(\Gamma)} A = \inf_{\mathcal{C}^*(\Gamma)} A} \; .$$

(I) Plateau problem (General version):

Given a closed, (rectifiable) Jordan curve $\Gamma \in \mathbb{R}^3$, find a (disk-type) minimal surface $X \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ which is bounded by Γ in the following sense;

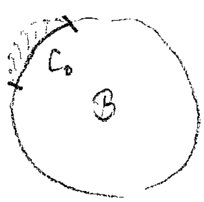
(1) $\quad X|_{\partial B}$ maps ∂B homeomorphically onto Γ, (keeping the orientation of Γ).

Lemma 2. For a minimal surface $X \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ Condition (1) is equivalent to

(2) $\quad X \in \mathcal{C}_0(\Gamma) := \{ Z \in C^0(\overline{B}, \mathbb{R}^3) : Z|_{\partial B}$ maps ∂B weakly monotonically onto $\Gamma \}$.

Proof. We only have to show: $(2) \Rightarrow (1)$.
Let $X \in \mathcal{C}_0(\Gamma)$, and suppose that

$$X(w) \equiv \text{const} \quad \text{for } w \in C_0 = \{ e^{i\theta} : \theta_1 < \theta < \theta_2 \}.$$



By Schwarz's reflection principle one can X extend as a minimal surface across C_0. Then $X_\theta \equiv 0$ on C_0, and by conformality $X_w(w) \equiv 0$ on C_0 whence $X_w(w) \equiv 0$ in $\overline{B}$, and therefore $X(w) \equiv \text{const}$ on $\overline{B}$ $\Rightarrow$ contradiction to (2). $\square$

(II) Problem of least Dirichlet integral $\mathcal{D}$ (= "energy")

(i) Given Γ, find $X \in \mathcal{C}(\Gamma)$ such that

$$\mathcal{D}(X) = \inf_{\mathcal{C}(\Gamma)} \mathcal{D} =: e(\Gamma)$$

(ii) Is X a minimal surface of class $C^0(\overline{B}, \mathbb{R}^3)$?

(III) Problem of least area A

(i) Given Γ, find $X \in \mathcal{C}(\Gamma)$ such that

$$A(X) = \inf_{\mathcal{C}(\Gamma)} A =: a(\Gamma) .$$

(ii) Is X a minimal surface of class $C^0(\overline{B}, \mathbb{R}^3)$?

(IV) Simultaneous Problem

Given Γ, find $X \in \mathcal{C}(\Gamma)$ such that

$$e(\Gamma) = D(X) = A(X) = a(\Gamma) .$$

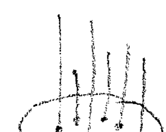
General remarks concerning the strategy solving (I – IV)

1°. Solve (II), (i), and show that the answer to (II), (ii) is "yes". Hence (I) is solved.

2°. It is rather difficult to approach (III) in the context of mappings ($\rightsquigarrow$ work of Cesari and others) because

(α) A is a bad functional : $\delta A(X, \phi)$ does not exist on $H^{1,2}$

(β) A has bad minimizers, e.g. surfaces with "hairs" :



$\Gamma = S^1$

3°. By proving the relation $a(\Gamma) = e(\Gamma)$, one shows that a solution of (II) is a solution of (III) and (IV). However the proof of this fact is rather difficult and requires Korn / Lichtenstein thm + uniformization, or quasiconformal mappings. We shall avoid this, replacing it by a simple argument.

4°. We shall proceed as follows:

Step 1 : Solve ($\overline{II}$).

Step 2 : Set $A^\varepsilon := (1-\varepsilon)A + \varepsilon D$ for $0 < \varepsilon \le 1$ and solve by the same method

($\overline{V}$) Given Γ, find $X^\varepsilon \in \mathcal{C}(\Gamma)$ with

$$A^\varepsilon(X^\varepsilon) = \inf_{\mathcal{C}(\Gamma)} A^\varepsilon =: a(\Gamma, \varepsilon)$$

Step 3 : Show that

$$a(\Gamma, \varepsilon) \equiv a(\Gamma) = e(\Gamma) \quad \text{for all } \varepsilon \in (0,1].$$

Then (I) – ($\overline{IV}$) are solved in one stroke, and each minimizer X^ε is a minimal surface bounded by Γ.

To carry out this program, we need the following result which will be proved later:

Proposition 1. $\mathcal{C}^*(\Gamma)$ is weakly sequentially closed in $H^{1,2}(B, \mathbb{R}^n)$.

Step 1. Since Γ is rectifiable we have $\mathcal{C}^*(\Gamma) \neq \emptyset$, and so there is a sequence $\{X_n\} \subset \mathcal{C}^*(\Gamma)$ with $D(X_n) \to e^*(\Gamma) = e(\Gamma)$, whence $D(X_n) \le M$ $\forall n \in \mathbb{N}$ and some $M > 0$. Furthermore, since $X_n(C) = \Gamma$, $\|X_n\|_{L^2(C, \mathbb{R}^3)} \le M'$ $\forall n \in \mathbb{N}$ and some $M' > 0$. A Poincaré inequality yields:

$$\|X_n\|_{H^{1,2}(B, \mathbb{R}^3)} \le M'' \quad \forall n \in \mathbb{N} \text{ and some } M'' > 0$$

Choosing a subsequence and renaming it, we may assume that

$$X_n \rightharpoonup X \quad \text{in } H^{1,2}(B, \mathbb{R}^3) \quad \text{as } n \to \infty.$$

Hence $X \in \mathscr{C}^*(\Gamma)$ on account of Prop. 1, and so

$$e(\Gamma) \leq D(X)$$

On the other hand, D is sequentially w.l.s. in $H^{1,2}(B,\mathbb{R}^3)$:

$$\liminf D(X_n) \geq D(X)$$

since

$$D(X_n) = D(X - X_n) + 2D(X, X_n) - D(X)$$

$$\geq 2D(X, X_n) - D(X) \to D(X)$$

and we even have $\lim D(X_n) = e(\Gamma)$. Thus

$$e(\Gamma) \leq D(X) \leq e(\Gamma)$$

whence

$$D(X) = e(\Gamma) := \inf_{\mathscr{C}(\Gamma)} D$$

i.e. X is a minimizer of D in $\mathscr{C}(\Gamma)$. Then we obtain

$$\delta D(X, \phi) = 0 \qquad \forall \phi \in C_c^\infty(B, \mathbb{R}^3)$$

and §2, Prop. 1 yields: $X \in C^\omega(B, \mathbb{R}^3)$ and

$$\Delta X = 0 \quad \text{in } B.$$

Furthermore, since $X|_{\partial B} \in C^0(\partial B, \mathbb{R}^3)$, it follows that $X \in C^0(\overline{B}, \mathbb{R}^3)$.

Next, let $\sigma(\cdot, \varepsilon) : \overline{B_\varepsilon^*} \to \overline{B}$ as in Prop. 2 of §2. Since B_ε^* is simply connected, there is a homeomorphism $\kappa^\varepsilon : \overline{B} \to \overline{B}_\varepsilon^*$ which maps B conformally onto B_ε^*. Then $Y^\varepsilon := Z^\varepsilon \circ \kappa^\varepsilon \in \mathscr{C}(\Gamma)$ for $Z^\varepsilon := X \circ \sigma(\cdot, \varepsilon)$, and therefore

$$\mathcal{D}(X) \leq \mathcal{D}(Y^\varepsilon) = \mathcal{D}(Z^\varepsilon, B_\varepsilon^*) =: \Phi(\varepsilon) \text{ for } |\varepsilon| \ll 1$$

Thus,

$$\Phi(0) \leq \Phi(\varepsilon) \text{ for } |\varepsilon| \ll 1 \Rightarrow \Phi'(0) = 0,$$

whence

$$\delta\, \mathcal{D}(X, \lambda) = 0 \quad \forall \lambda \in C^1(\overline{B}, \mathbb{R}^2)$$

(without the use of Riemann's mapping theorem we merely obtain

$$\delta \mathcal{D}(X, \lambda) = 0 \quad \forall \lambda \in C^1_{tang}(\overline{B}, \mathbb{R}^2),$$

but this suffices for the next conclusion!).

By Prop. 2 of §2 we obtain

$$|X_u|^2 = |X_v|^2, \quad X_u \cdot X_v = 0.$$

Hence the minimizer X of $\mathcal{D}$ in $\mathcal{C}(\Gamma)$ is a minimal surface of class $C^0(\overline{B}, \mathbb{R}^3)$, and we have solved (II), and therefore also (I), for rectifiable Γ.

□

Remarks:

1) If Γ is nonrectifiable, $\mathcal{C}(\Gamma)$ may be empty. Nevertheless, one can solve (I) by approximating Γ by rectifiable Γ_n in the Fréchet sense, solving (I) by $X_n \in \mathcal{C}(\Gamma_n)$ and showing that $X_n \to X$ in some sense, and that X solves (I) for Γ.

This result has not yet been extended to minimal surfaces in a Riemannian manifold, and only partially extended to H-surfaces.

2) Absolute minimizers of area have

- no interior branch points (Osserman, Gulliver, Alt), Gulliver - Osserman - Royden
- no boundary branch points if Γ is real analytic (Gulliver - Lesley, White, Gulliver - Osserman - Royden)

$\boxed{\text{Problem: Is the assumption } \Gamma \in C^\omega \text{ superfluous?}}$

Therefore we need the statement " $a(\Gamma) = e(\Gamma)$, but we avoid the "classical complications", although Courant thought that they are unavoidable. So we come to

Step 2. Fix some ε with $0 < \varepsilon \leq 1$ and set

$$A^\varepsilon := (1-\varepsilon)A + \varepsilon D.$$

Choose a sequence $\{X_n\} \subset \mathcal{C}^*(\Gamma)$ with

$$A^\varepsilon(X_n) \to \overset{*}{a}(\varepsilon, \Gamma) := \inf_{\mathcal{C}^*(\Gamma)} A^\varepsilon$$
$$= a(\varepsilon, \Gamma) := \inf_{\mathcal{C}(\Gamma)} A^\varepsilon.$$

Then

$$A^\varepsilon(X_n) \leq M(\varepsilon) \quad \forall n \in \mathbb{N} \quad \text{and some } M(\varepsilon) > 0$$

whence

$$D(X_n) \leq M(\varepsilon)/\varepsilon \quad \forall n \in \mathbb{N}$$

and

$$\|X_n\|_{L^2(C)} \leq M' \quad \forall n \in \mathbb{N} \text{ and some } M' > 0.$$

By Poincaré,

$$\|X_n\|_{H^{1,2}(B,\mathbb{R}^n)} \leq M''(\varepsilon) \quad \forall n \in \mathbb{N} \text{ and some } M''(\varepsilon) > 0.$$

Thus, as in Step 1, we may assume that

$$X_n \rightharpoonup X^\varepsilon \text{ in } H^{1,2}(B,\mathbb{R}^3) \quad \text{as } n \to \infty.$$

Since $\mathcal{C}^*(\Gamma)$ is weakly sequentially closed in $H^{1,2}(B,\mathbb{R}^3)$, it follows that $X^\varepsilon \in \mathcal{C}^*(\Gamma)$, and so

$$a(\Gamma,\varepsilon) \leq A^\varepsilon(X^\varepsilon).$$

On the other hand, by Acerbi - Fusco (1984) A^ε is sequentially w. l. s., whence

$$A^\varepsilon(X^\varepsilon) \leq \liminf_{n\to\infty} A^\varepsilon(X_n)$$

and we have

$$A^\varepsilon(X_n) \to a(\Gamma,\varepsilon).$$

Thus,

$$a(\Gamma,\varepsilon) \leq A^\varepsilon(X^\varepsilon) \leq a(\Gamma,\varepsilon)$$

and consequently

(3) $$a(\Gamma,\varepsilon) = A^\varepsilon(X^\varepsilon) \quad \text{for } 0<\varepsilon\leq 1.$$

This completes the proof of Step 2. □

Step 3. Note that we are not able to derive

$$\delta A^\varepsilon(X^\varepsilon,\phi) = 0 \quad \forall \phi \in C_c^\infty(\Omega,\mathbb{R}^3)$$

from the minimum property (3) since A^ε is not differentiable. However, since A is parameter invariant we obtain

$$0 + \varepsilon\,\partial D(X^\varepsilon,\lambda) = 0 \quad \forall \lambda \in C^1(\overline{B},\mathbb{R}^2)$$

whence

$$\partial D(X^\varepsilon,\lambda) = 0 \quad \forall \lambda\in C^1(\overline{B},\mathbb{R}^2) \quad (\text{or } \forall \lambda \in C^1_{tang}(\overline{B},\mathbb{R}^2)$$

if Riemann's mapping theorem

Then Prop. 2 of § 2 implies

(4) $$|X_u^\varepsilon| = |X_v^\varepsilon|^2 \quad , \quad X_u^\varepsilon \cdot X_v^\varepsilon = 0 \, .$$

Therefore

(5) $$A(X^\varepsilon) = D(X^\varepsilon) = A^\varepsilon(X^\varepsilon) \qquad \forall \varepsilon \in (0,1] \, .$$

Furthermore, $A \leq D$ implies

(6) $$A \leq A^\varepsilon \leq D \, .$$

We obtain for any $Z \in \mathcal{C}(\Gamma)$ and $0 < \varepsilon \leq 1$:

$$D(X^\varepsilon) \underset{(5)}{=} A^\varepsilon(X^\varepsilon) \underset{(3)}{\leq} A^\varepsilon(Z) \leq D(Z)$$

Let $\varepsilon, \varepsilon' \in (0,1]$ and choose $Z = X^{\varepsilon'}$; then

$$D(X^\varepsilon) \leq D(X^{\varepsilon'}) \qquad \forall \varepsilon, \varepsilon' \in (0,1] \, ,$$

whence

$$D(X^\varepsilon) = D(X^{\varepsilon'}) \qquad \forall \varepsilon, \varepsilon' \in (0,1] \, .$$

That means,

(7) $$D(X^\varepsilon) \overset{!}{\equiv} \mathrm{const} =: c \underset{(5)}{=} A(X^\varepsilon) \underset{(5)}{=} A^\varepsilon(X^\varepsilon) \quad \forall \varepsilon \in (0,1] \, .$$

Furthermore, we have for any $Z \in \mathcal{C}(\Gamma)$ and $\varepsilon, \varepsilon' \in (0,1]$:

(8) $$a(\Gamma) \underset{\substack{\uparrow \\ \text{by def. of } a(\Gamma)}}{\leq} A(X^\varepsilon) \underset{(7)}{=} A^\varepsilon(X^\varepsilon) \underset{(7)}{=} A^{\varepsilon'}(X^{\varepsilon'}) \underset{(3)}{\leq} A^{\varepsilon'}(Z)$$

and

(9) $$e(\Gamma) \underset{\substack{\uparrow \\ \text{by def. of } e(\Gamma)}}{\leq} D(X^\varepsilon) \underset{(7)}{=} A^\varepsilon(X^\varepsilon) \underset{(3)}{\leq} A^\varepsilon(Z) \leq D(Z) \, .$$

From (8) we infer for $\varepsilon' \to 0$ (because of $A^{\varepsilon'}(Z) \to A(Z)$) that

$$a(\Gamma) \le A(X^{\varepsilon}) \le A(Z) \quad \forall Z \in \mathcal{C}(\Gamma) \text{ and } \forall \varepsilon \in (0,1],$$

and (9) implies

$$e(\Gamma) \le D(X^{\varepsilon}) \le D(Z) \quad \forall Z \in \mathcal{C}(\Gamma) \text{ and } \forall \varepsilon \in (0,1)$$

Hence we obtain

$$a(\Gamma) \le A(X^{\varepsilon}) \le a(\Gamma)$$

and

$$e(\Gamma) \le D(X^{\varepsilon}) \le e(\Gamma) .$$

In conjunction with (5) and (3)

$$(10) \qquad a(\Gamma) = A(X^{\varepsilon}) = D(X^{\varepsilon}) = e(\Gamma) = a(\Gamma, \varepsilon) \quad \forall \varepsilon \in (0,1].$$

Therefore, every X^{ε} is a solution of the "simultaneous problem" ($\overline{IV}$) for $0 < \varepsilon \le 1$, in particular a solution of ($\overline{II}$) and ($\overline{I}$).

$\square$

Thus we have found:

<u>Theorem 1</u>. (i) <u>For any rectifiable curve Γ in $\mathbb{R}^3$ we have $a(\Gamma) = e(\Gamma)$, and there is a minimal surface $X \in C^0(\overline{B}, \mathbb{R}^3)$ which solves ($\overline{I}$) as well as the simultaneous problem ($\overline{IV}$).</u>

(ii) <u>Any minimizer of D in $\mathcal{C}(\Gamma)$ is a conformally parametrized minimizer of A in $\mathcal{C}(\Gamma)$.</u>

<u>Question</u> : Does a corresponding statement hold for relative minimizers ?

Let $X \in \mathcal{C}(\Gamma)$ be a minimal surface bounded by Γ. If X is a relative minimizer for A in $\mathcal{C}(\Gamma)$, then also for D in $\mathcal{C}(\Gamma)$, since

$$A(X) \le A(Y) \quad \forall Y \in \mathcal{C}(\Gamma) \text{ with } \|X - Y\| < \varepsilon << 1$$

implies

$$D(X) = A(X) \le A(Y) \le D(Y) \quad \forall Y \in \mathcal{C}(\Gamma) \text{ with } \|X-Y\| < \varepsilon << 1$$

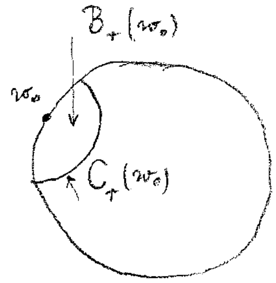
for any reasonable choice of the norm $\|\cdot\|$.

A general converse is unknown, but recently it was shown by <u>Hildebrandt</u> & <u>Sauvigny</u> ;

Let $\Gamma \in C^{1,\mu}$, and suppose that $X \in C^{1,\mu}(\bar{B}, \mathbb{R}^3) \cap \mathcal{C}(\Gamma)$ an <u>immersed</u> minimal surface which is a relative minimizer of D with respect to $\|\cdot\|_{C^1(\bar{B}, \mathbb{R}^3)}$. Then X is also a relative minimizer of D, but merely with respect to the norm $\|\cdot\|_{C^{1,\mu}(\bar{B}, \mathbb{R}}$.

The proof is quite involved, and I will not discuss it here.

In the next paragraphs I shall discuss various <u>qualitative</u> <u>and</u> <u>quantitative</u> <u>properties</u> of solutions to Plateau's problem. <u>But we still have to prove Proposition 1.</u>



For $w_0 \in \bar{B}$ and $r > 0$ we set

$$S_r(w_0) := B \cap B_r(w_0)\ , \quad C_r(w_0) := \bar{B} \cap \partial B_r(w_0)$$

If $w_0 \in C = \partial B$ and $0 < r < 2$ then

$$C_r(z_0) = \{ w = w_0 + re^{i\theta} : \theta_1(r) < \theta < \theta_2(r) \}\ , \quad 0 < \theta_2(r) - \theta_1(r) < \pi$$

Lemma 1

Suppose that $X \in H^{1,2}(B, \mathbb{R}^3)$ and $w_0 \in C = \partial B$.

Then there is a representative $Z(\rho, \theta)$ of $X(w_0 + \rho e^{i\theta})$ such that

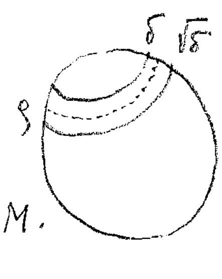
(i) $Z(\rho, \cdot)$ is AC (= absolutely continuous) on $[\theta_1(\rho), \theta_2(\rho)]$ for almost all $\rho \in (0,2)$;

(ii) $Z(\cdot, \theta)$ is AC on $[\varepsilon, r]$ with $0 < \varepsilon < r$ for a.a. $\theta \in (\theta_1(r), \theta_2(r))$;

(iii)
$$2\,\mathcal{D}(X, S_r(w_0)) := \int_{S_r(w_0)} \left(|X_u|^2 + |X_v|^2 \right) du\, dv$$
$$= \int_0^r \int_{\theta_1(\rho)}^{\theta_2(\rho)} \left\{ |Z_\rho(\rho,\theta)|^2 + \frac{1}{\rho^2} |Z_\theta(\rho,\theta)|^2 \right\} \rho\, d\theta\, d\rho$$

Proof: See e.g. Morrey's book

Lemma 2 (Courant – Lebesgue Lemma)



Let X, w_0, Z be as in Lemma 1, and $2\mathcal{D}(X) \le M$.

Claim: For any $\delta \in (0,1)$ there is a set $J \subset (\delta, \sqrt{\delta})$ with $\mathcal{L}^1(J) > 0$ such that for $r \in J$ and $\theta_1(r) \le \theta \le \theta' \le \theta_2(r)$ we have: $Z(r, \cdot)$ is AC and

$$|Z(r,\theta) - Z(r,\theta')| \le \int_\theta^{\theta'} |Z_\theta(r,\theta)|\, d\theta \le \eta(\theta,\delta) \cdot |\theta - \theta'|^{1/2}$$

with

$$\eta(\theta,\delta) := \left\{ \frac{4}{\log 1/\delta}\ \mathcal{D}(X, S_{\sqrt{\delta}}(w_0)) \right\}^{1/2} \le \left\{ \frac{2M}{\log 1/\delta} \right\}^{1/2}.$$

Proof. For almost all $\rho \in (\delta, \sqrt{\delta})$ the function

$$p(\rho) := \int_{\theta_1(\rho)}^{\theta_2(\rho)} |Z_\theta(\rho, \theta)|^2 \, d\theta$$

is well-defined, and by (iii) of Lemma 1 we have

$$(*) \quad \int_\delta^{\sqrt{\delta}} \frac{1}{\rho} \, p(\rho) \, d\rho \leq M(\delta) := 2 D(X, S_{\sqrt{\delta}}(w_0))$$

Set

$$J := \{ r \in (\delta, \sqrt{\delta}) : p(r) \int_\delta^{\sqrt{\delta}} \frac{d\rho}{\rho} \leq M(\delta) \}$$

Then we have

$$\mathcal{L}^1(J) > 0$$

because otherwise we have $\mathcal{L}^1(J) = 0$ and therefore

$$p(r) \int_\delta^{\sqrt{\delta}} \frac{d\rho}{\rho} > M(\delta) \qquad \text{for a.a. } r \in (\delta, \sqrt{\delta})$$

i.e.,

$$p(r) > \frac{M(\delta)}{\int_\delta^{\sqrt{\delta}} \frac{d\rho}{\rho}} \qquad \text{for a.a. } r \in (\delta, \sqrt{\delta})$$

Integrating both sides with respect to r over $(\delta, \sqrt{\delta})$ we obtain

$$\int_\delta^{\sqrt{\delta}} p(r) \, dr > M(\delta)$$

a contradiction to $(*)$.

Choose $r \in J$ such that $Z(r, \cdot)$ is AC on $[\theta_1(r), \theta_2(r)]$

Then $$Z(r, \theta') - Z(r, \theta) = \int_\theta^{\theta'} Z_\theta(r, \theta) \, d\theta ,$$

whence

$$|Z(r,\theta') - Z(r,\theta)| \leq \int_\theta^{\theta'} |Z_\theta(r,\theta)|\, d\theta$$

$$\leq \Big\{ \underbrace{\int_\theta^{\theta'} |Z_\theta(r,\theta)|^2 d\theta}_{= p(r) \text{ by definition}} \Big\}^{1/2} |\theta' - \theta|^{1/2}$$

i.e.

$$|Z(r,\theta') - Z(r,\theta)| \leq \sqrt{p(r)\,|\theta'-\theta|}$$

Since $r \in J$ we have

$$p(r) \leq \frac{M(\delta)}{\int_\delta^{\sqrt{\delta}} \frac{d\rho}{\rho}} = \frac{M(\delta)}{\log \frac{\sqrt{\delta}}{\delta}} = \frac{M(\delta)}{\log \sqrt{\frac{1}{\delta}}} = \frac{2M(\delta)}{\log \frac{1}{\delta}} .$$

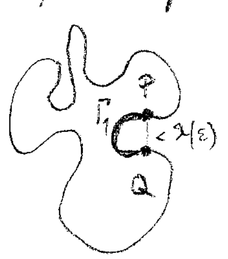
Therefore,

$$|Z(r,\theta') - Z(r,\theta)| \leq \left\{ \frac{2M(\delta)}{\log 1/\delta} |\theta'-\theta| \right\}^{1/2} ,$$

and $M(\delta) \leq M$.

□

Lemma 3. Let Γ be a closed Jordan curve. Then, for any $\varepsilon > 0$, there is a $\lambda(\varepsilon) > 0$ with the following property: Any pair of points $P, Q \in \Gamma$ with



$$0 < |P - Q| < \lambda(\varepsilon)$$

decomposes Γ into two arcs such that one of them, called $\Gamma'(P,Q)$, satisfies

$$\operatorname{diam} \Gamma'(P,Q) < \varepsilon .$$

Proof. Follows from the fact that Γ is the topological image of $C = \partial B$.

□

For $M > 0$ we set

$$\mathscr{C}^*_M(\Gamma) := \{ X \in \mathscr{C}^*(\Gamma) : D(X) \le M \}.$$

<u>Lemma 4</u>. The set $\{ X|_C : X \in \mathscr{C}^*_M(\Gamma) \}$ is uniformly bounded and consists of equicontinuous Mappings.

<u>Proof.</u> (i) The uniform boundedness follows from $X(C) = \Gamma$.
(ii) Let $X \in \mathscr{C}^*_M(\Gamma)$, and choose $\varepsilon > 0$ with

$$(11) \qquad \varepsilon < \varepsilon_0 := \min \{ |Q_j - Q_k| : j \neq k \}$$

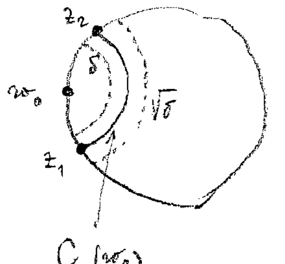
where $X(w_k) = Q_k$, $1 \le k \le 3$ is the 3-point condition $*$.
Then $\Gamma'(P, Q)$ with $|P - Q| < \lambda(\varepsilon)$ can contain at most one of the points Q_1, Q_2, Q_3.
Choose $\delta_0 \in (0,1)$ with

$$(12) \qquad 2\sqrt{\delta_0} < \min \{ |w_j - w_k| : j \neq k \}$$

and $\delta = \delta(\varepsilon) \in (0, \delta_0)$ with

$$(13) \qquad \left\{ \frac{4\pi M}{\log 1/\delta} \right\}^{1/2} < \lambda(\varepsilon)$$

By Lemma 2 (Courant – Lebesgue) there is $\tau \in (\delta, \sqrt{\delta})$ such that the images $P := X(z_1)$, $Q := X(z_2)$ of the two intersection points z_1, z_2 of C with $\partial B_\tau(w_0)$, $w_0 \in C$, satisfy



$$(14) \qquad |P - Q| \underset{\text{Lemma 2}}{\le} \left\{ \frac{4\pi M}{\log 1/\delta} \right\}^{1/2} \underset{(13)}{<} \lambda(\varepsilon)$$

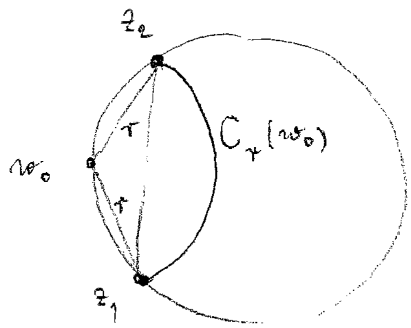
Because of Lemma 3, it follows

(15) $$\operatorname{diam} \Gamma'(P, Q) < \varepsilon ,$$

and by (11), the arc $\Gamma'(P,Q)$ can obtain at most one of the points Q_1, Q_2, Q_3.

On the other hand $0 < \delta < \delta_0$ and (12) imply

(16) $$2\sqrt{\delta} < \min \{ |w_j - w_k| : j \neq k ,\ 1 \le j, k \le 3 \}$$



$|z_1 - z_2| \le 2r < 2\sqrt{\delta}$

Hence $C \cap \overline{B}_r(w_0)$ contains at most one of the points w_1, w_2, w_3, and so its weakly monotonic image $X(C \cap \overline{B}_r(w_0))$ contains at most one of the points $Q_j = X(w_j)$, $j = 1, 2, 3$; hence we infer

(17) $$\Gamma'(P, Q) = X (C \cap \overline{B}_r(w_0)) .$$

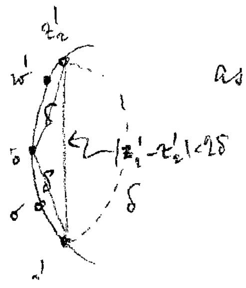
From (15) and (17) it follows

$$\operatorname{diam} X (C \cap \overline{B}_r(w_0)) < \varepsilon ,$$

which means that

(18) $$| X(w) - X(w') | < \varepsilon \quad \forall\, w, w' \in C \cap \overline{B}_r(w_0),$$

hence also $\forall w, w' \in C \cap B_\delta(w_0)$,



as $\delta < r < \sqrt{\delta}$. Since $|w - w'| < \delta$ yields $w, w' \in C \cap B_\delta(w_0)$ for a suitable choice of $w_0 \in C$, it follows certainly

$$|X(w) - X(w')| < \varepsilon \quad \forall w, w' \in C \text{ with } |w-w'| < \delta(\varepsilon).$$

This is the equicontinuity of $\{X|_C : X \in \mathcal{C}^*_M(\Gamma)\}$. $\square$

<u>Proof</u> <u>of</u> <u>Proposition 1</u>. Suppose that $\{X_n\} \in \mathcal{C}^*(\Gamma)$ satisfies $X_n \rightharpoonup X$ in $H^{1,2}(B, \mathbb{R}^3)$. Then by a Rellich-type theorem

$$(19) \qquad X_n|_C \to X|_C \quad \text{in } L^2(C, \mathbb{R}^3),$$

and the uniform boundedness principle yields

$$\|X_n\|_{H^{1,2}(B, \mathbb{R}^3)} \leq M$$

for some $M > 0$, and therefore $\{X_n|_C\} \subset \mathcal{C}^*_\Gamma(M)$.

By Lemma 4 and the theorem of Arzelà-Ascoli, using (19) and a standard reasoning, we infer that

$$X_n|_C \to X|_C \quad \text{in } C^0(C, \mathbb{R}^3)$$

Thus we obtain: $X \in \mathcal{C}^*(\Gamma)$, i.e. $\mathcal{C}^*(\Gamma)$ is weakly closed in $H^{1,2}(B, \mathbb{R}^3)$.

$\square$

4. Unstable Solutions of Plateau's Problem

For geodesics one has a well developed Morse theory. In 1941, Morse and Tompkins tried to set up such a theory; however, it is completely useless. In fact, the index theorem of Böhme & Tromba indicates that there is no Morse theory for minimal surfaces in $\mathbb{R}^n$ if $n = 3$; only $n \geq 4$ is possible, and here it was set up by M. Struwe. However, there is a result which would follow from a Morse theory, namely the Theorem of the Wall, due to M. Shiffman and, independently, Morse – Tompkins. I shall sketch Courant's proof of this result.

For the following we assume that Γ is a rectifiable closed Jordan curve in $\mathbb{R}^3$ satisfying a global chord-arc condition: For $x_1, x_2 \in \Gamma$ we have $l(x_1, x_2) \leq \mu |x_1 - x_2|$, where $l(x_1, x_2)$ = length of the shorter one of the two subarcs of Γ bounded by x_1 and x_2.

Set

$$\bar{\mathscr{C}}^*(\Gamma) := \mathscr{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3)$$

equipped with the metric d_0 which is defined by

$$d_0(X, Y) := \|X - Y\|_{C^0(\bar{B}, \mathbb{R}^3)}$$

A path P in $(\bar{\mathscr{C}}^*(\Gamma), d_0)$ is a nonempty, closed, connected subset. By $\mathbb{P}(X_1, X_2)$ we denote the set of all paths P with $X_1, X_2 \in P$.

<u>Theorem of the Wall</u>. Let $X_1, X_2 \in \bar{\mathcal{C}}^*(\Gamma)$ be two minimal surfaces satisfying

("Wall") $\quad \sup_{P} D > \max\{D(X_1), D(X_2)\} \qquad \forall P \in \mathbb{P}(X_1, X_2)$

Then there is a third minimal surface $X_3 \in \bar{\mathcal{C}}^*(\Gamma)$ which is D-unstable (and therefore also A-unstable) in the following sense:
For any $\varepsilon > 0$ there is an $Y \in \bar{\mathcal{C}}^*(\Gamma)$ with $d_0(Y, X_3) < \varepsilon$ and $D(Y) < D(X_3)$.

<u>Remark</u>. The "Wall-Condition" is satisfied $X_1, X_2 \in \bar{\mathcal{C}}^*(\Gamma)$ are strict relative minimizer of D in $(\bar{\mathcal{C}}^*(\Gamma), d_0)$.
This will be used for Nitsche's uniqueness theorem

Courant's proof of the above theorem rests on the following:

1.) Prove the theorem for polygons.

2.) Approximate Γ in a suitable way by polygons Γ_j keeping the three-point condition, apply 1.), and pass to the limit, using some profound theorems, due to Morse-Tompkins, on sequences of harmonic mappings.

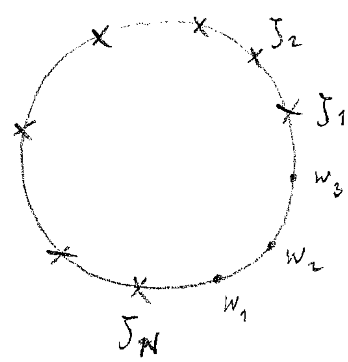
We only sketch 1.):
Take a polygon with $N+3$ consecutive vertices

$$A_1, A_2, \dots, A_N, Q_1, Q_2, Q_3$$

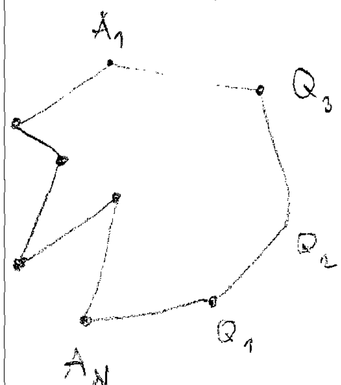
and set $T := \{ t = (t^1, \dots, t^N) \in \mathbb{R}^N : 0 < t^1 < \dots < t^N < \psi_1 < \psi_2 < \psi_3 = 2\pi \}$

$= $ bounded, open, convex subset of $\mathbb{R}^N$,

$$\mathcal{U}(t) := \{ X \in \overline{\mathcal{C}}^*(\Gamma) : X(\zeta_k) = A_k \,, \; X(w_l) = Q_l \}$$



$\zeta_k := e^{it^k}$, $w_l := e^{i\psi_l}$



Courant function $\Theta : T \to \mathbb{R}$ is defined by

$$\Theta(t) := \inf \{ D(X) : X \in \mathcal{U}(t) \}$$

$\exists_1 \; Z(t) \in \mathcal{U}(t)$ such that

$$\Theta(t) = D(Z(t))$$

and $Z(t) \in \overline{\mathcal{H}}^*(\Gamma)$

Courant mapping: $\quad Z : T \to \overline{\mathcal{C}}^*(\Gamma)$

Facts : (i) $\Theta \in C^1(T)$ and $\Theta(t) \to \infty$ if $t \to \partial T$;

(ii) If $X \in \overline{\mathcal{C}}^*(\Gamma)$ is a minimal surface, then $\exists_1 t \in T$ with

$$Z(t) = X$$

(iii) $Z(t)$ is minimal surface iff $\nabla \Theta(t) = 0$

Hence : $\quad \mathcal{M}^*(\Gamma) \xleftrightarrow{1-1} K(\Theta) := \{ t \in T : \nabla \Theta(t) = 0 \}$

Now one uses Courant's mountain pass lemma, applied to $f = \Theta$.

Theorem. Let Ω be a bounded domain in $\mathbb{R}^n$, and $f \in C^1(\Omega)$ satisfy

(i) $f(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$ for $x \in \Omega$;

(ii) $\exists\, x_1, x_2 \in \Omega$ such that

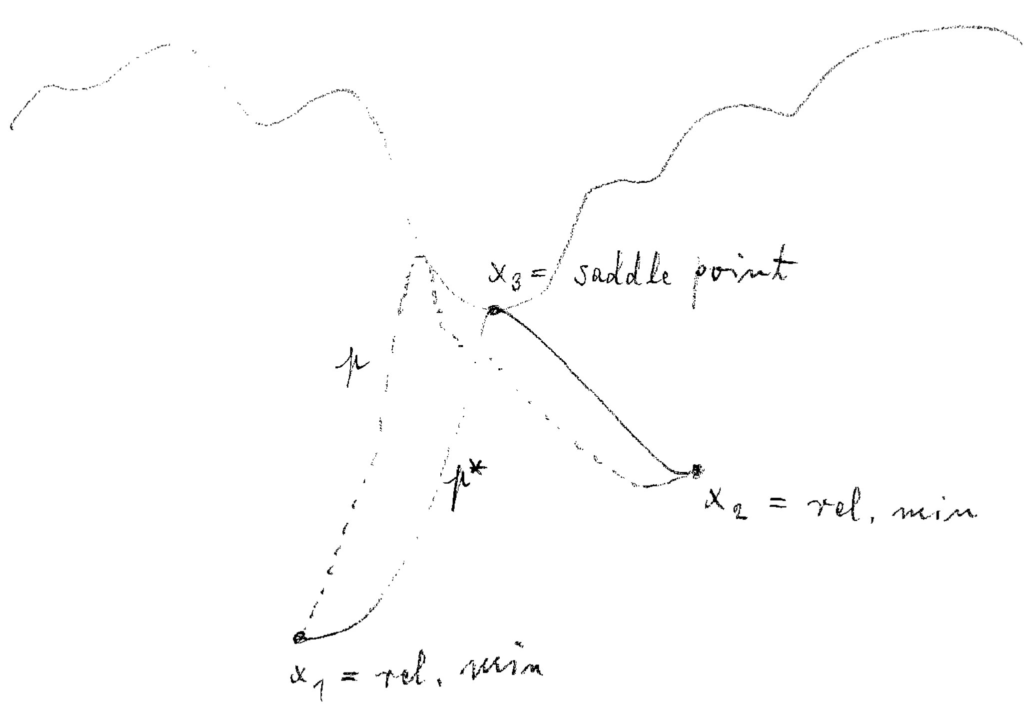
$$\max_{p} f > \max\{f(x_1), f(x_2)\} \quad \forall p \in \mathcal{P}$$

where $\mathcal{P}$ is the set of all paths in Ω containing x_1 and x_2.

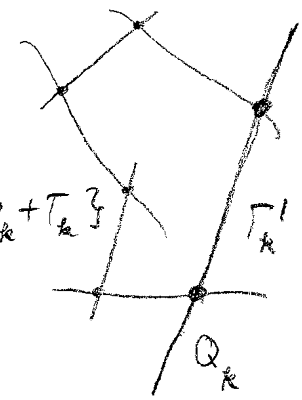
(e.g.: $\exists$ two distinct strict relative minimizers of f)

Then f possesses a minimal path $p^* \in \mathcal{P}$ and an unstable critical point x_3 (i.e. is not a relative min.) with $x_3 \in p^*$ and $f(x_3) = \max_{p^*} f$ such that

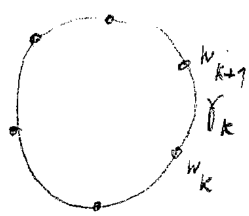
$$f(x_3) = \inf_{p \in \mathcal{P}} \max_{x \in p} f(x).$$



<u>Remark</u>. Overshooting the sides:



$$\mathcal{U}^*(t) := \{ X \in \bar{H}^{1,2}(B, \mathbb{R}^3) : X(\gamma_k) \subset \Gamma_k' := Q_k + T_k \}$$
$$k = 1, \dots, N+3$$



$\exists_1 \, X \in \mathcal{U}^*(t)$ with

$$D(X) = \inf_{\mathcal{U}^*(t)} D$$

Set $Z^*(t) = X$ =: Marx - Shiffman <u>map</u>

$$\Theta^*(t) = D(Z^*(t)) = \inf_{\mathcal{U}^*}(t)$$

=: Marx - Shiffman <u>function</u>

Any map $Z^*(t)$ is called a <u>quasi - minimal surface</u>

We have:

(i) $\Theta^*(t) \le \Theta(t)$, and $\Theta^*(t) = \Theta(t)$ if $\nabla \Theta(t) = 0$.

(ii) $\Theta^*(t) < \Theta(t)$ for some $t \in T$ is possible (Lewerenz example).

(iii) Θ^* is real analytic on T.

<u>Question 1</u>: Is $\Theta \in C^2(T)$, or better? If so, one could try to build a Morse theory for polygons.

<u>Question 2</u>: For which polygons Γ does one have $\Theta = \Theta^*$? (Then one could use Heinz's results for quasi - minimal surfaces.)

5. Properties of Minimal Surfaces

A.) First we discuss the boundary behaviour of a minimal surface $X \in C^0(B \cup \gamma, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$, which maps an open subarc γ of ∂B into the interior of a closed regular Jordan arc $\Gamma_0 \in C^1$.

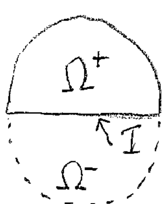
Theorem 1. (i) Then $X \in C^{0,\alpha}(B \cup \gamma, \mathbb{R}^3)$ for some $\alpha \in (0,1)$.

(ii) If $\Gamma_0 \in C^{s,\mu}$, $s \geq 1$, $\mu \in (0,1)$ then $X \in C^{s,\mu}(B \cup \gamma, \mathbb{R}^3)$
(Hildebrandt, Heinz-Tomi, Heinz, Nitsche, Kinderlehrer, Warschawski)

(iii) If Γ_0 is real analytic (i.e. $\Gamma_0 \in C^\omega$) then $X \in C^\omega(B \cup \gamma, \mathbb{R}^3)$
(H. Lewy)

(iv) If Γ_0 is a straight arc then X can be extended by the Reflection Principle as a minimal surface across γ.
(H.A. Schwarz)

Proof of (iv). By conformal mapping we can assume that B is replaced by the halfdisk Ω^+ and γ by the interval $I = (-1, 1)$ on the real axis. Let Ω^- be the mirror image of Ω^+ and $\Omega = \Omega^+ \cup I \cup \Omega^-$. Write $X(w) = (x(w), y(w), z(w))$; we can also assume that $\Gamma_0 \subset z$-axis. Then



$$(1) \qquad x(w) = 0 \quad \text{and} \quad y(w) = 0 \quad \text{on } I$$

By Schwarz's reflection principle we extend $x(w)$ and $y(w)$ harmonically to all of Ω by setting

$$(2) \qquad x(w) := -x(\bar{w}) \quad , \quad y(w) := -y(\bar{w}) \quad \text{for } w \in \Omega^-.$$

We will show that $z \in C^1(\Omega^+ \cup I)$ and

(3) $$z_v(w) = 0 \quad \text{on } I,$$

and that $z(w)$ can harmonically be extended to all of Ω by setting

(4) $$z(w) = z(\bar{w}) \quad \text{for } w \in \Omega^- .$$

Clearly the extension satisfies

$$\Delta X = 0 \;,\; |X_u|^2 = |X_v|^2 \;,\; X_u \cdot X_v = 0 \quad \text{on } \Omega .$$

To prove (3), we note that (1) implies $\{w = u+iv \in \Omega\}$:

(5) $$x_u(w) = 0 \;,\; y_u(w) = 0 \quad \text{for } w \in I .$$

Let $\{w_n\} \subset \Omega^+$ and $w_n \to w_0 \in I$. Then

$$z_u^2(w_n) = |X_u(w_n)|^2 - |x_u(w_n)|^2 - |y_u(w_n)|^2$$

$$= |X_v(w_n)|^2 - |x_u(w_n)|^2 - |y_u(w_n)|^2$$

whence

(6) $$z_u^2(w_n) \geq |z_v(w_n)|^2 - \underbrace{\{|x_u(w_n)| + |y_u(w_n)|^2\}}_{\text{by}(5):\ \to 0 \text{ as } n\to\infty}$$

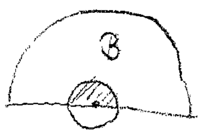
and $X_u \cdot X_v = 0$ in Ω^+ and (5) imply

(7) $$z_u(w_n)\, z_v(w_n) \to 0 \quad \text{as } n \to \infty .$$

From (6) and (7) we infer: $z_v \in C^0(\Omega^+ \cup I)$ and $z_v = 0$ on I, and from here we obtain the assertion stated above.

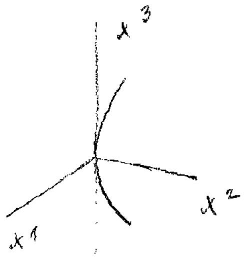
□

Proof of (iii). Assertion (iii) is a famous result due to H. Lewy [1950] and can be viewed as an extension of (iv). Let Γ_0 be given by $x = g(t)$ with:
= real analytic



$$x^1 = g^1(t), \quad x^2 = g^2(t), \quad x^3 = t$$

for $|t| \ll 1$ with



$$g^1(0) = g^2(0) = \dot g^1(0) = \dot g^2(0) = 0,$$

and form

$$f(w) := X(w) + i X^*(w) \text{ on } B, \quad X^* = \text{adjoint of } X$$

$X \in C^1(B \cup I, \mathbb{R}^3)$

$f(0) = 0$ w.l.o.g.

Since $X(I) \subset \Gamma_0$ we have locally

$$X(u) = g(X^3(u)) \Rightarrow X_u(u) = g'(X^3(u))\, X^3_u(u)$$

Then

$$X^3_u \langle g'(x^3), f' \rangle = \langle X_u, f' \rangle = \langle X_u, X_u + i X^*_u \rangle$$

$$= \langle X_u, X_u - i X_v \rangle = |X_u|^2 = \langle g'(X^3), g'(X^3) \rangle |X^3_u|^2$$

whence for $X^3_u(u) \neq 0$:

$$(8) \qquad X^3_u(u) = \frac{\langle g'(X^3), f' \rangle}{\langle g', g' \rangle}(u)$$

and the same holds for $X^3_u(u) = 0$ since this implies

$$X_u(u) = 0 \Rightarrow X_v(u) = 0 \Rightarrow X^*_u(u) = 0 \Rightarrow f'(u) = 0$$

Now we use a complex extension $g(\zeta)$ of g:

$\zeta \mapsto g(\zeta) \in \mathbb{C}^3$, $\zeta \in \mathbb{C}$ with $|\zeta| \ll 1$.

Set

$$(9) \qquad F(w,\zeta) := \frac{\langle g'(\zeta), f'(w) \rangle}{\langle g'(\zeta), g'(\zeta) \rangle} \qquad \text{for } |W|, |\zeta| \ll 1, \ \operatorname{Im} w \geq 0$$

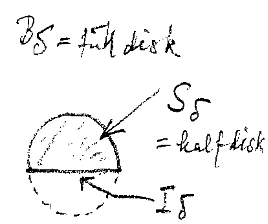
Then $\zeta(u) := x^3(u)$ is a solution of the integral equation

$$(10) \qquad \zeta(u) = \int_0^u F(\underline{u}, \zeta(\underline{u}))\, d\underline{u}$$

with

$$|F(w,\zeta) - F(w,\zeta')| \leq M|\zeta - \zeta'|$$

By Picard iteration,



$$(11) \qquad z(w) = \int_0^w F(\underline{w}, z(\underline{w}))\, d\underline{w}$$

has exactly one complex-valued solution for $\operatorname{Im} w \geq 0$, $|w| \ll 1$, which is holomorphic for $\operatorname{Im} w > 0$, continuous for $\operatorname{Im} w \geq 0$, and (9) has exactly one real-valued solution $\zeta(u) = x^3(u)$ for $u \in \mathbb{R}$, $|u| \ll 1$, whence $z(u) = x^3(u) \in \mathbb{R}$ for $u \in \mathbb{R}$, $|u| \ll 1$. By Schwarz, $z(w)$ can be extended to a holomorphic function for $|w| \ll 1$. Set

$$(12) \qquad \Phi(w) := f(w) - g(z(w)) = \text{hol. in } S_\delta, \text{ contin. on } S_\delta \cup I_\delta$$

For $u \in I_\delta$: $\quad \Phi(u) = f(u) - g(z(u)) = [X(u) + iX^*(u)] - X(u) = iX(u)$

= purely imaginary.

By Schwarz, $\phi(w)$ can be extended to a holomorphic mapping $\phi : B_\delta \to \mathbb{C}^3$. Set

$$(13) \qquad \tilde{f}(w) := \phi(w) + g(z(w)) \qquad \text{for } w \in B_\delta$$

Then $\tilde{f} : B_\delta \to \mathbb{C}^3$ is holomorphic, and

$$f(w) = \tilde{f}(w) \quad \text{for } w \in S_\delta \cup I_\delta \quad \text{by (12).}$$

This completes the proof of (iii). $\square$

<u>Proof of (i)</u> : is omitted

<u>Proof of (ii)</u> : Here we have to restrict ourselves to the case $\Gamma_0 \in C^{2,\mu}$, sketching a few ideas

Describe Γ_0 (small) as in (iii) by $x = g(t)$ with

$$x^1 = g^1(t) \,, \quad x^2 = g^2(t) \,, \quad x^3 = t \,, \quad |t| \ll 1 \,,$$

where $g \in C^{2,\mu}$, $g^k(0) = \dot{g}^k(0) = 0$ for $k = 1,2$

Set

$$Y(w) = (Y^1(w), Y^2(w))$$

with

$$Y^k(w) := X^k(w) - g^k(X^3(w)) \,, \quad k = 1,2$$

Then

$$\Delta Y^k = \Delta X^k - \dot{g}^k(X^3)\,\Delta X^3 - \ddot{g}^k(X^3)\,|\nabla X^3|^2$$

whence

$$(14) \qquad |\Delta Y| \leq \alpha\, |\nabla X|^2$$

(The same holds if

$$\Delta X = 2H(X)\, X_u \wedge X_v \quad)$$

Moreover,

(15) $$X^k_w = Y^k_w + g^k(X^3)\, X^3_w \quad , \quad k = 1,2 \; ,$$

whence

$$|X^1_w|^2 + |X^2_w|^2 \le 2\,|Y_w|^2 + \underbrace{2|g^k(X^3)|^2}_{\ll 1,\ \text{say},\ < 1/4} |X^3_w|^2 ,$$

and

(16) $X_w \cdot X_w = 0$ yields

$$|X^3_w|^2 < |X^1_w|^2 + |X^2_w|^2$$

hence

$$\frac{1}{2}|X_w|^2 \le |X^1_w|^2 + |X^2_w|^2$$

and

$$\frac{1}{4}|X_w|^2 \le 2\,|Y_w|^2 \; ;$$

therefore

(17) $$|\nabla X|^2 \le 8\,|\nabla Y|^2 .$$

Then, by (14) and (17),

(19) $$\begin{cases} |\Delta Y| \le 8\alpha\, |\nabla Y|^2 & \text{on } S_\delta \\ Y = 0 & \text{on } I_\delta \end{cases}$$

Heinz's theory of differential inequalities yields : $Y \in C^{1,\beta}(S_\varepsilon \cup I_\varepsilon, \mathbb{R}^2)$ for $0 < \varepsilon < \delta$ and any $\beta \in (0,1)$.

From (15) and (16) follows

$$(20)\qquad \left[X^3_w + \sum_{k=1}^{2} p^k(X^3)Y^k_w\right]^2 = \left\{\sum_{k=1}^{2} p^k(X^3)\,Y^k_w\right\}^2 - \frac{1}{q(X^3)}\sum_{k=1}^{2}\left(Y^k_w\right)^2$$

with $p^k := g^k/q$, $q := 1+\sum_{k=1}^{2}|g^k|^2$

The rhs of (20) is continuous on $\bar{S}_\varepsilon$ $\Rightarrow$ lhs $\in C^0(\bar{S}_\varepsilon, \mathbb{C})$

$\Rightarrow X^3_w \in C^0(\bar{S}_\varepsilon, \mathbb{C}) \underset{(15)}{\Rightarrow} X \in C^1(\bar{S}_\varepsilon, \mathbb{R}^3)$

Now we have;

$Y \in C^{1,\beta}$ and $X \in C^1$ on $\bar{S}_\varepsilon$

as well as

(21') $|\Delta X^3| + |\nabla X^3| \leq$ const on $\bar{S}_\varepsilon$,

and multiplying (20) by $-w^2$, one infers

(21'') $X^3_w \in C^{0,\beta}(I)$.

By potential theory and the Korn - Privalov - theorem one infers $X^3 \in C^{1,\beta}(\bar{S}_\varepsilon)$ and thus $X \in C^{1,\beta}(\bar{S}_\varepsilon, \mathbb{R}^3)$ $\forall \beta \in (0,1)$.

This was the essential step; then $X \in C^{2,\mu}(\bar{S}_\varepsilon, \mathbb{R}^3)$ by bootstrapping using a Dirichlet problem for Y and a Neumann problem for X^3. $\square$

B.) Now we turn to the question whether there exist regular (i.e. immersed) solutions of Plateau's problem

<u>Theorem 2.</u> (i) Let $X \in \mathcal{C}(\Gamma)$ be a solution of Plateau's problem which minimizes area. Then X has no interior branch points.

(ii) If Γ is real analytic then X has no boundary branch points.

Proof of (i) Osserman, Gulliver, Alt ; Gulliver - Osserman - Royden

(ii) Gulliver - Lesley, B. White, Gulliver - Osserman - Royden

<u>Problem</u>: It is unknown whether or not an area minimizing solution of Plateau's problem may have boundary branch points if Γ is merely smooth.

<u>Note</u>: Thm 2 does not hold in $\mathbb{R}^n$ with $n \geq 4$ (Federer's example)

<u>Theorem 3.</u> If $\Gamma \in C^2$ and $X \in \mathcal{C}(\Gamma)$ a solution of Plateau's problem with the boundary branch point $w_0 \in \partial B$, then

(22) $X_w(w) = A(w-w_0)^\nu + o(|w-w_0|^\nu)$ as $w \to w_0$, $w \in \overline{B}$,

with $A \in \mathbb{C}^3 \setminus \{0\}$ and $A \cdot A = 0$; moreover, ν is even, $\nu \in \mathbb{N}$.

<u>Proof.</u> Hartman - Wintner & Heinz

Corollary 1. If $X \in \mathcal{C}(\Gamma)$ is a minimal surface with a smooth contour Γ then X has at most finitely many branch points in $\overline{B}$.

Let Σ_0 be the set of branch points of X in $\overline{B}$, and $\nu(w)$ be the order of the branch point w. Then the Gauss-Bonnet theorem yields (κ_g = geodesic curvature of Γ on X, K = Gauss curv. of X)

$$(23)\quad \int_X K\,dA + \int_\Gamma \kappa_g\,ds = 2\pi + 2\pi \underbrace{\sum_{w\in\Sigma_0\cap B}}_{=:\Sigma_0'} \nu(w) + \pi \underbrace{\sum_{w\in\Sigma_0\cap\partial B}}_{=:\Sigma_0''} \nu(w)$$

Since $K \leq 0$ and and $\kappa_g \leq \kappa$ ($\kappa \geq 0$ is the curvature of Γ) the left-hand side is bounded from above by $\int_\Gamma \kappa\,ds$, and so

$$(24)\quad 1 + \sum_{w\in\Sigma_0'} \nu(w) + \frac{1}{2}\sum_{w\in\Sigma_0''} \nu(w) \leq \frac{1}{2\pi}\int_\Gamma \kappa\,ds$$

Note that $\int_\Gamma \kappa\,ds \geq 2\pi$, and $\int_\Gamma \kappa\,ds \geq 4\pi$ if Γ is knotted.
We infer from (24) that X has no branch points in $\overline{B}$ if $\int_\Gamma \kappa\,ds < 4\pi$.

If B is replaced by an m-fold connected domain Ω in $\mathbb{R}^n$, then

$$\int_X K\,dA + \int_{\partial\Omega} \kappa_g |dX| + 2\pi(m-2) = 2\pi \sum_{w\in\Sigma_0\cap\Omega} \nu(w) + \pi \sum_{w\in\Sigma_0\cap\partial\Omega} \nu(w)$$

(or: $+\,4\pi(g-1) + 2\pi m$ if $X: M \to \mathbb{R}^3$, g = genus M.)

C.) Isoperimetric Inequality

Theorem 4 (i) Let $X : B \to \mathbb{R}^3$ be a minimal surface of class $C^0(\bar{B}, \mathbb{R}^3)$ with $\int_{\partial B} |dX| < \infty$. Then $X \in H^{1,2}(B, \mathbb{R}^3)$ and

$$A(X) = \mathcal{D}(X) \leq \frac{1}{4\pi} \left(\int_{\partial B} |dX| \right)^2$$

(ii) If, in particular, $X \in \mathcal{C}(\Gamma)$ and $\Gamma \in C^{1,\alpha}$ then

$$A(X) = \mathcal{D}(X) \leq \frac{1}{4\pi} \; L^2(\Gamma)$$

where $L(\Gamma) :=$ length of Γ. Equality holds iff X is a disk.

Proof. We shall only verify (ii) noting that $X \in C^{1,\alpha}(\bar{B}, \mathbb{R}^3)$. Note that $A(X) = \mathcal{D}(X)$; choose a constant $P \in \mathbb{R}$. Then

$$2\mathcal{D}(X) = \int_B \nabla X \cdot \nabla (X - P) \, du\, dv \qquad (\text{note: } \Delta X = 0)$$

$$\underset{\text{Gauss}}{=} \int_{C = \partial B} X_r \cdot (X - P) \, d\theta$$

$$\leq \int_C |X_r| \, |X - P| \, d\theta \underset{\text{Conformality}}{=} \int_0^{2\pi} |X_\theta| \, |X - P| \, d\theta$$

Let us introduce the arc length parameter s by

$$s = \sigma(\theta) := \int_0^\theta |X_\theta(1, \theta)| \, d\theta$$

with the inverse

$$\theta = \vartheta(s), \qquad 0 \leq s \leq L := L(\Gamma),$$

and set $Z(s) := X(1, \vartheta(s))$. Then

$$|\dot{Z}(s)| = 1 \quad \text{a.e.}$$

and

$$2\mathcal{D}(X) \leq \int_0^{2\pi} |X_\theta||X - P|\, d\theta = \int_0^L 1 \cdot |Z(s) - P|\, ds$$

whence

$$\mathcal{D}(X) \leq \frac{1}{2}\sqrt{L} \left\{ \int_0^L |Z(s) - P|^2\, ds \right\}^{1/2}$$

Choose

$$P := \frac{1}{L}\int_0^L Z(s)\, ds$$

Wirtinger's inequality:

$$\int_0^L |Z(s) - P|^2\, ds \leq \left(\frac{L}{2\pi}\right)^2 \int_0^L \underbrace{|\dot{Z}(s)|^2}_{=1}\, ds$$

(equality iff $Z(s) = P + A_1 \cos\frac{2\pi}{L}s + B_1 \sin\frac{2\pi}{L}s$)

Then

$$\int_0^L |Z(s) - P|^2\, ds \leq \frac{L^3}{(2\pi)^2}$$

whence

$$\mathcal{D}(X) \leq \frac{1}{2}\sqrt{L} \cdot \frac{\sqrt{L^3}}{2\pi} = \frac{L^2}{4\pi}.$$

We skip the proof of the rest. □

<u>Problem.</u> Let $X : M \to \mathbb{R}^3$ be a minimal surface on an orientable, compact 2-manifold with boundary, and let $\partial X = X(\partial M)$, where $\overline{X} : \partial M \to \partial X$ is 1-1. Can one prove

$$(*) \qquad A(X) \leq \frac{1}{4\pi} L^2(X)$$

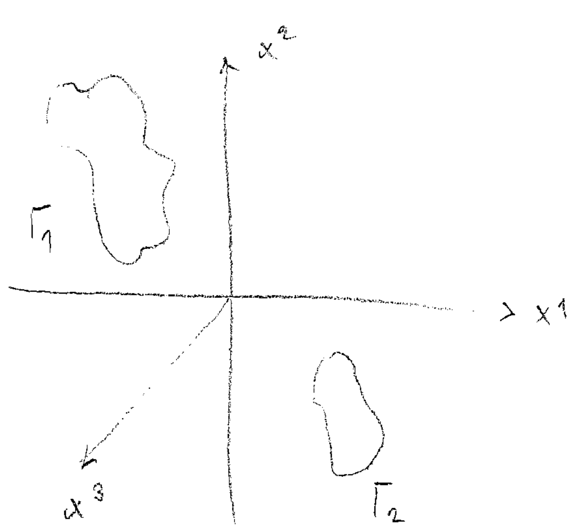
where $L(X) := $ length of ∂X ?

This is unknown (but known for larger constants than $\frac{1}{4\pi}$; what is the smallest ?).

It is known if ∂X is <u>weakly connected</u>, i.e., if there are Euclidian coordinates x^1, x^2, x^3 in $\mathbb{R}^3$ such that none of the three hyperplanes

$$H_j := \{x^j = \text{const}\}, \quad j = 1, 2, 3,$$

separates ∂X :



Theorem 5 (Morse - Tompkins)

For any $H \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ with $\Delta H = 0$ in B (i.e. $H \in \overline{\mathcal{H}}(B, \mathbb{R}^3)$) we have

$$A(H) \leq \frac{1}{4} \left(\int_{\partial B} |dH| \right)^2$$

Problem. Does such an inequality hold for harmonic 2-dim. mappings $H : M \to N$ where M and N are Riemannian manifolds with $\dim M = 2$, $\dim N \geq 2$, $K_N \leq 0$ (K_N = sectional curvature of N)?

We obtain the following convergence result:

Theorem 6. (i) If $\{H_j\} \subset \overline{\mathcal{H}}(B, \mathbb{R}^3)$ satisfies $\|H_j - H\|_{C^0(\overline{B}, \mathbb{R}^3)} \to 0$ and $\int_{\partial B} |dH_j| \to \int_{\partial B} |dH|$, then $H_j \in \overline{\mathcal{H}}(B, \mathbb{R}^3)$ and

$$\lim_{j \to \infty} A(H_j) = A(H)$$

(ii) If, in addition, H_j are conformal, then

$$\lim_{j \to \infty} D(H_j) = D(H)$$

and $X_j \to X$ in $H^{1,2}(B, \mathbb{R}^3)$.

Problem. Is there a Riemannian analog?

D.) Number of solutions to Plateau's problem

Set

$$\mathcal{M}(\Gamma) := \{ X \in \mathcal{C}(\Gamma) : X \text{ is a disk-type minimal surface } B \to \mathbb{R}^3 \},$$

$$\mathcal{M}^*(\Gamma) := \mathcal{M}(\Gamma) \cap \mathcal{C}^*(\Gamma)$$

Theorem 7 (T. Radó) : If Γ has a 1-1 parallel or central projection onto a planar convex curve γ, then

$$\# \mathcal{M}^*(\Gamma) = 1 .$$

Theorem 8 (J.C.C. Nitsche) : If Γ is real analytic and $\int_\Gamma \varkappa \, ds \leq 4\pi$, then $\# \mathcal{M}^*(\Gamma) = 1$

Böhme's example. For any $N \in \mathbb{N}$ and any $\varepsilon > 0$ there is a $\Gamma \in C^\omega$ with $\int_\Gamma \varkappa \, ds < 4\pi + \varepsilon$ such that

$$\# \mathcal{M}^*(\Gamma) \geq N .$$

Theorem 9 (Böhme - Tromba). Generically we have

$$\# \mathcal{M}^*(\Gamma) < \infty .$$

Theorem 10 (F. Tomi). If Γ is real analytic (i.e. $\in C^\omega$) then

$$\# \{ X \in \mathcal{M}^*(\Gamma) : A(X) = \min_{\mathcal{C}(\Gamma)} A \} < \infty$$

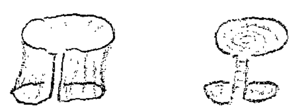
Theorem 11 (Nitsche). If the regular contour Γ is extreme and $\in C^{3,\alpha}$, or else real analytic, and if

$$\int_\Gamma \kappa \, ds < 6\pi ,$$

then $\#\{X \in \mathcal{M}^*(\Gamma) : X \text{ is an immersion of } \overline{B} \text{ into } \mathbb{R}^3\} < \infty$.

Heuristic example of Courant / P. Levy. There exist rectifiable Jordan curves Γ (such that $\Gamma' := \Gamma \setminus \{\underset{=P}{\text{one point}}\}$ is regular and C^∞) which bound nondenumerably many minimal surfaces $\in \mathcal{C}^*(\Gamma)$.

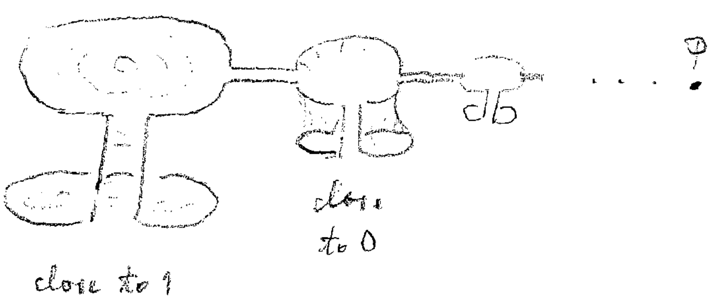
Sketch :



Type 0 Type 1

$\Gamma :=$ bounds two stable surfaces $\in \mathcal{C}^*(\Gamma)$, Type 0 & 1

Make a new contour Γ :



Construct stable solutions of Type $z_1 z_2 \dots z_k \dots$
with $z_k = 0$ or 1 $\rightsquigarrow$ $\exists$ nondenumerably many $X \in \mathcal{M}^*(\Gamma)$

Problem $\left\{\right.$ For a "proof" one needs a "bridge theorem". It might be possible to work it out with B. White's bridge theorem (other versions are either wrong, or at least doubtful, or not strong enough)

Idea for the <u>proof of Thm 8</u> (Nitsche's uniqueness thm):

Suppose that there are two solutions $X_1, X_2 \in \mathcal{M}^*(\Gamma)$. One can assume that Γ is nonplanar. One concludes that X_1 and X_2 have no branch points in $\overline{B}$, even if $\int_\Gamma \varkappa\, ds \le 4\pi$. Then, for $X = X_1$ or X_2,

$$-\int_B K\Lambda\, du\, dv = \int_\Gamma \varkappa_g\, ds - 2\pi, \qquad \Lambda := |X_u|^2,$$

and since Γ is nonplanar, one proves

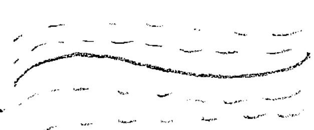
$$\int_\Gamma \varkappa_g\, ds < 4\pi$$

whence

$$-\int_B K\Lambda < 2\pi = \text{area of a hemisphere on } S^1.$$

By a result of do Carmo (which is a strengthening of work of H. A. Schwarz) one obtains that both X_1 and X_2 are <u>strictly stable</u> (i.e.,

$$\int_B |\nabla\varphi|^2\, du\, dv \ge \mu \int_B \Lambda |K| \varphi^2\, du\, dv \quad \forall \varphi \in C_c^\infty(B) \Big)$$

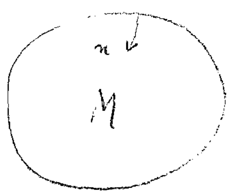
X_k 

Then one proves (Tomi / Nitsche) via "field immersion" that both X_1 and X_2 are strict relative minimizer of D in $\mathcal{C}^*(\Gamma)$, equipped with $\|\cdot\|_{C^0}$.
By the "theorem of the wall" there is a third $X_3 \in \mathcal{M}^*(\Gamma)$ which is unstable (i.e. in every ε-neighbourhood of X_3 there is a Y with $D(Y) < D(X_3)$); but this is impossible since also X_3 has to be a strict relative minimizer of D, since we can reason as for X_1 and X_2. $\square$

<u>Theorem 12</u> (Eckholm, White, & Wienholtz)
Under the assumptions of Theorem 8, the uniquely determined $X \in \mathcal{M}^*(\Gamma)$ is <u>embedded</u>.

One also has (Meeks – Yau; earlier versions by Tomi – Tromba and Almgren – Simon):

<u>Theorem 13</u>. Let M be compact, $\partial M \in C^2$, Γ = closed, rectifiable curve on ∂M, and M convex or at least $H_M \geq 0$. Then any $X \in \mathcal{M}(\Gamma)$ with $A(X) = \inf_{\mathcal{C}(\Gamma)} A$ satisfies $X(\overline{B}) \subset M$ and X is embedded



<u>Concerning the proofs of Theorems 10 and 11:</u>

1.) <u>Thm 10</u> : Suppose that $\# \{ X \in \mathcal{M}(\Gamma) : A(X) = \min_{\mathcal{C}(\Gamma)} A \} = \infty$.

Then, by a priori estimates from A.), there is $\{X_j\} \subset \mathcal{C}^*(\Gamma)$ with $X_j \to X \in \mathcal{M}^*(\Gamma)$ in $C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ if $\Gamma \in C^{3,\alpha}$, and $A(X) = \inf_{\mathcal{C}(\Gamma)} A$. Then X cannot be strictly stable (see proof of Thm 8), but it is stable, i.e. $\delta^2 A(X, \varphi N) \geq 0$ $\forall \varphi \in C_c^\infty(B)$.

Then
$$L\zeta = \lambda \zeta \text{ in } B, \quad \zeta = 0 \text{ on } \partial B$$

has the smallest eigenvalue $\lambda_1 = 0$ (which is simple).

Then one can find a <u>conjugate field for X</u>, i.e. a real analytic family of immersions

$$Y(\cdot,t) := X + \zeta(\cdot,t)\, N \quad , \ |t| \le t_0 ,$$

from $\overline{B}$ into $\mathbb{R}^3$ of zero mean curvature with

$$Y(w,t) = X(w) \quad \text{for } w \in \partial B ,$$

$$|Y_t(w,t)| = |\zeta_t(w)| > 0 \quad \text{for } w \in B ,$$

$$A(Y(\cdot,t)) \equiv \text{const} \quad \text{for } |t| \le t_0$$

Let

$$\mathcal{M}^*_c(\Gamma) := \{ X \in \mathcal{M}(\Gamma) : A(X) = D(X) = c \} , \quad c := a(\Gamma)$$

be equipped with the $C^2(\overline{B},\mathbb{R}^3)$-norm, and set

$$\mathcal{K}_c := \text{closed, connected component of } \mathcal{M}^*_c(\Gamma) \text{ containing } X .$$
$$=: \text{"block"}$$

Through every $X_0 \in \mathcal{K}_c$ we can find such a family $\{Y(\cdot,t)\}_{|t|\le t_0}$ with $Y(\cdot,0) = X_0$.

Consider

$$V(X) := \frac{1}{3} \int_B X \cdot (X_u \wedge X_v)\, du\, dv$$

on $\mathcal{K}_c$; V is continuous on $\mathcal{K}_c$. Choose $X_0 \in \mathcal{K}_c$ by

$$(*) \qquad V(X_0) = \max_{\mathcal{K}_c} V ,$$

and let $Y(\cdot,t) : [t_0, t_0] \to \mathcal{K}_c$ be the arc through X_0 as described above : $Y = X_0 + \zeta N_0$, N_0 = normal of X_0.

Then $Y_t(\cdot,0) = \xi N_0$ with $\xi := \zeta_t(\cdot,0)$.

Set $\Lambda_0 := |X_{0,u}|^2$, and note : $\frac{1}{3} \operatorname{div} x = 1$.

It follows

$$\frac{d}{dt} V(Y(\cdot,t))\Big|_{t=0} \overset{!}{=} \int_B \Lambda_0 \, \xi \, du\, dv > 0 ,$$

while (*) implies

$$\frac{d}{dt} V(Y(\cdot,t))\Big|_{t=0} = 0 ,$$

contradiction. □

2.) For the <u>proof of Thm 11</u> one combines ideas from the proofs of Thms 8 and 10. □

6. Further Topics

- Partially free and free boundary value problems : Existence, Regularity
- Edge crawling
- Bernstein theorem in a wedge
- Douglas Problem