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Modeling and Complexity Reduction in PDES for Multiphysics
Heterogeneous DD

Alfio Quarteroni

EPFL, Lausanne (Switzerland)
MATHICSE Mathematics Institute of Computational Science and Engineering

POLITECNICO di MILANO (Italy)
MOX Modellistica e Calcolo Scientifico

POLITECNICO DI MILANO

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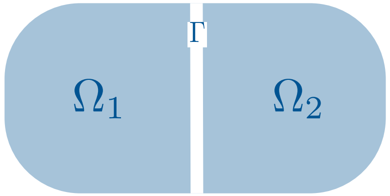
Modeling Strategies: Outlook

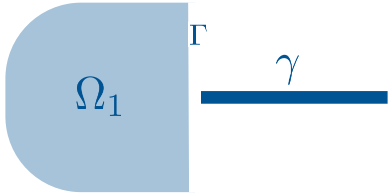
Homogeneous Domain Decomposition



$$\begin{aligned} Lu_1 &= f_1 \quad \text{in } \Omega_1 \\ Lu_2 &= f_2 \quad \text{in } \Omega_2 \end{aligned}$$

+ coupling conditions on Γ

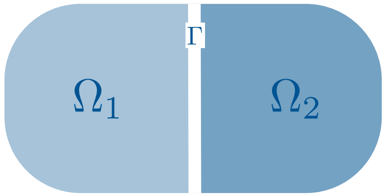
Sequential Multiscale



$$\begin{aligned} Lu_1 &= f_1 \quad \text{in } \Omega_1 \\ L_\gamma u_\gamma &= f_\gamma \text{ on } \gamma \end{aligned}$$

+ coupling conditions on Γ

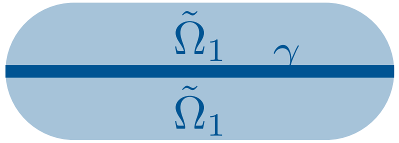
Heterogeneous Domain Decomposition



$$\begin{aligned} Lu_1 &= f_1 \quad \text{in } \Omega_1 \\ \tilde{L}u_2 &= f_2 \quad \text{in } \Omega_2 \end{aligned}$$

+ coupling conditions on Γ

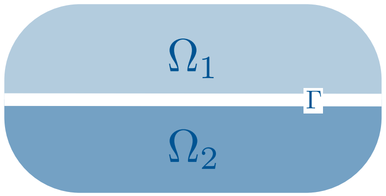
Embedded Multiscale 1



$$\begin{aligned} L_1 u_1 &= f_1 \text{ in } \tilde{\Omega}_1 \\ L_\gamma u_\gamma &= f_\gamma \text{ on } \gamma \end{aligned}$$

+ coupling conditions on γ

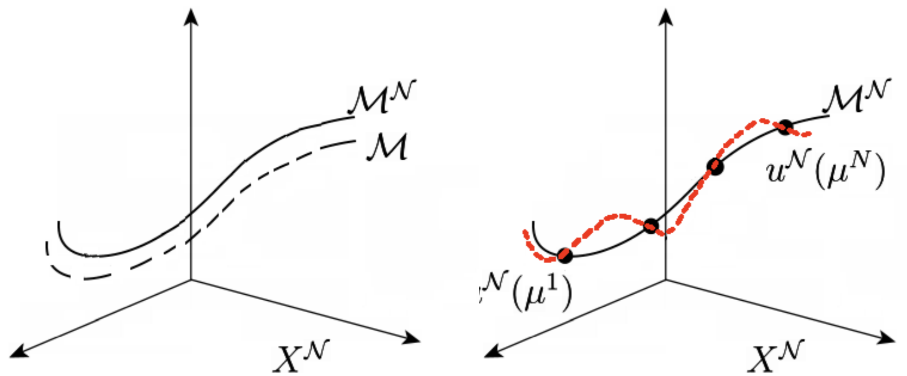
Multiphysics



$$\begin{aligned} L_1 u_1 &= f_1 \text{ in } \Omega_1 \\ L_2 u_2 &= f_2 \text{ in } \Omega_2 \end{aligned}$$

+ coupling conditions on Γ

Reduced Basis Method



Heterogeneous Domain Decomposition

Advection-Diffusion Equations

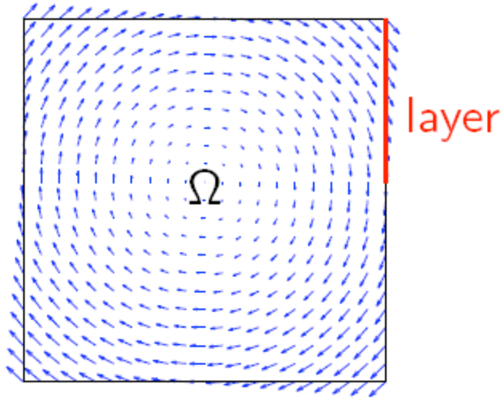
ON THE COUPLING OF HETEROGENEOUS PROBLEMS

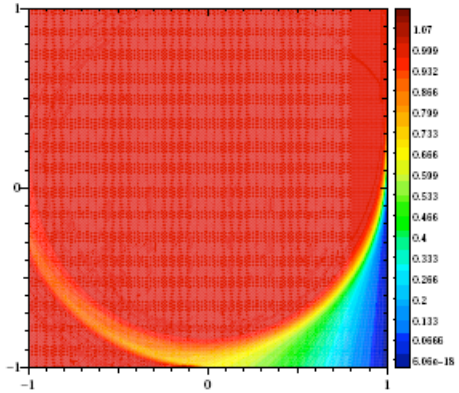
(with P. Gervasio, University of Brescia, Italy)

MOTIVATIONS

Consider a simple **advection-diffusion equation** with **boundary layer**

$$\begin{array}{ll} Au \equiv \mathrm{div}(-\nu \nabla u + \mathbf{b} u) + b_0 u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{array}$$





$$\mathbb{P}\mathrm{e}_g(\mathbf{x}) = \frac{|b(\mathbf{x})|}{2\nu} \gg 1$$

(advection-dominated case)

Let ν be a characteristic parameter: e.g.

-ν =thermal diffusivity
in heat transfer problems, or

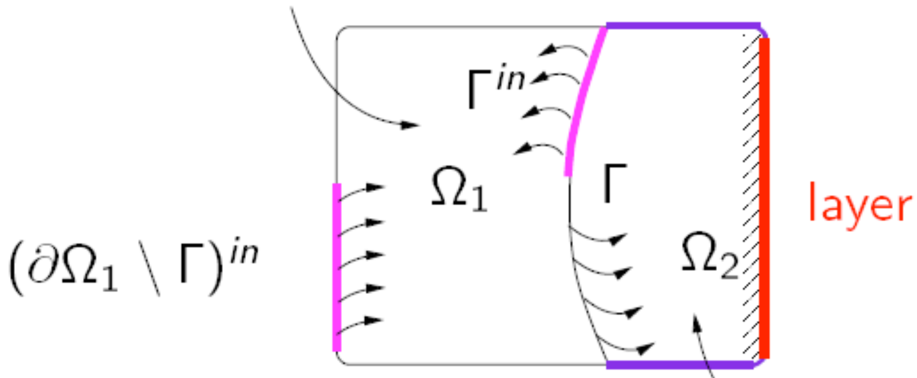
-$\nu = \dfrac{1}{Re}$ in incompressible fluid-dynamics, ...

1. Split the domain (SHARP INTERFACE) as:

$$\Omega_1,\ \Omega_2 \subset \Omega : \quad \overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2, \quad \Omega_1 \cap \Omega_2 = \emptyset, \quad \Gamma = \partial\Omega_1 \cap \partial\Omega_2$$

Solve a reduced problem as follows:

$$A_1 u = \mathrm{div}(\mathbf{b} u) + b_0 u = f$$



$$A_2 u = \mathrm{div}(-\nu \nabla u + \mathbf{b} u) + b_0 u = f$$

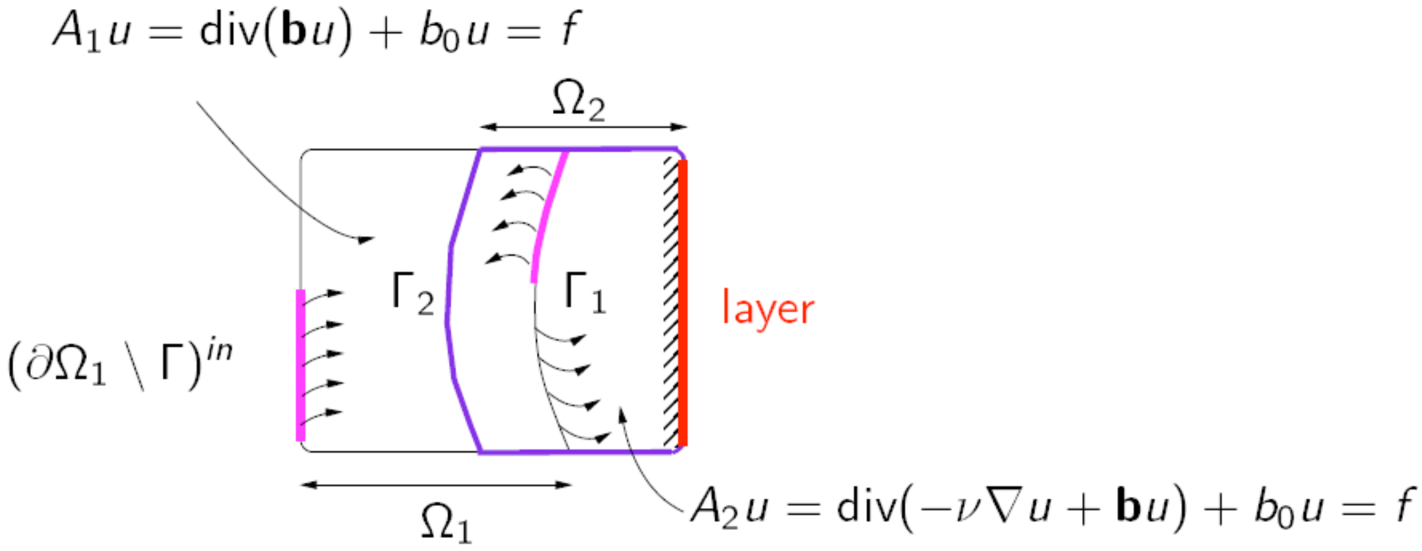
2 main steps:

1. Find interface conditions s.t.
   1. the new reduced problem is well posed
   2. its solution is “close to” the original one
2. Set up efficient solution algorithms of the reduced problem

2. Split the domain as: (with OVERLAP)

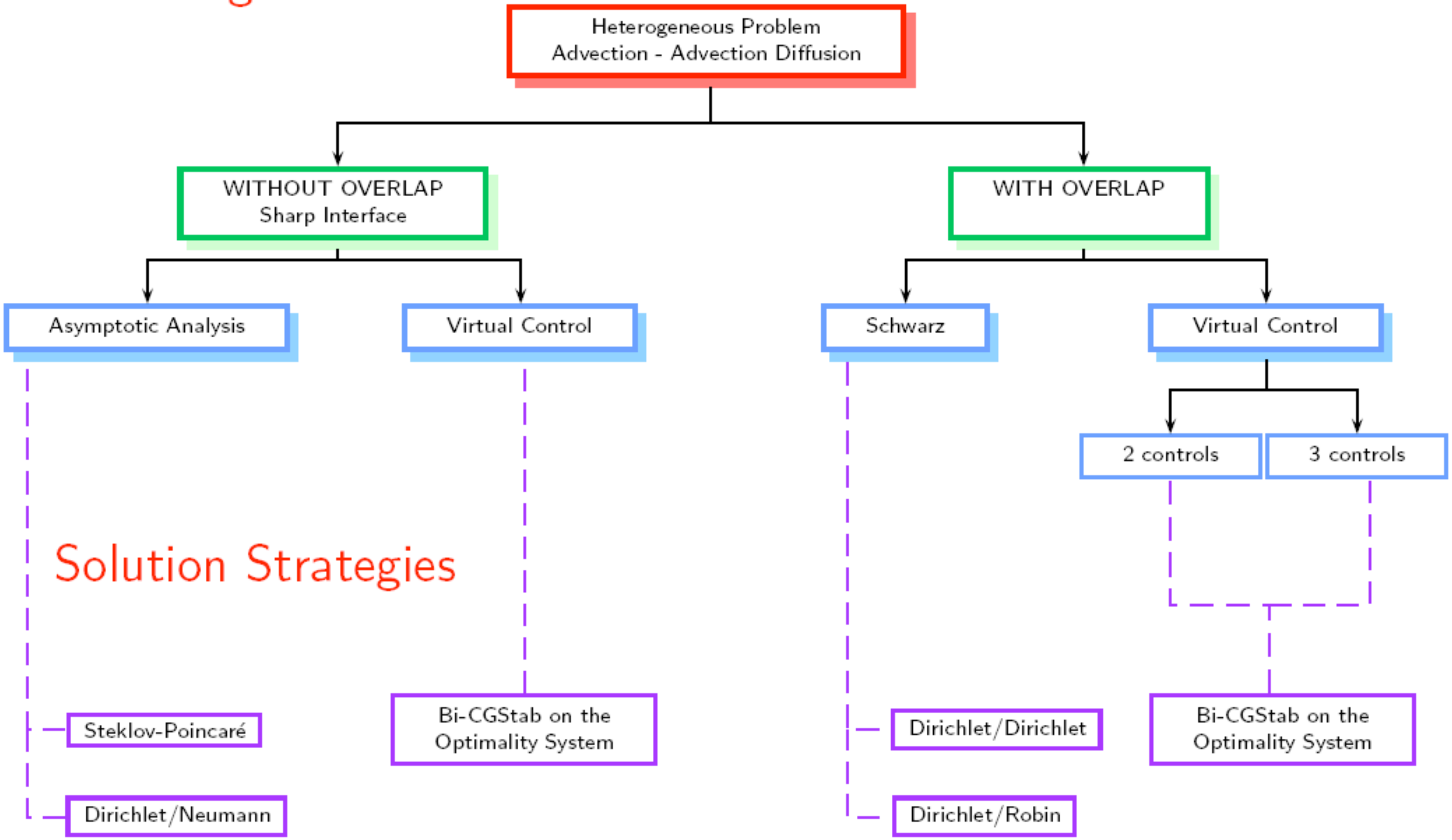
$$\Omega_1,\ \Omega_2 \subset \Omega : \quad \overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2, \quad \Omega_1 \cap \Omega_2 = \Omega_{12}, \quad \Gamma_k = \partial\Omega \setminus \partial\Omega_k$$

Solve a reduced problem as follows:

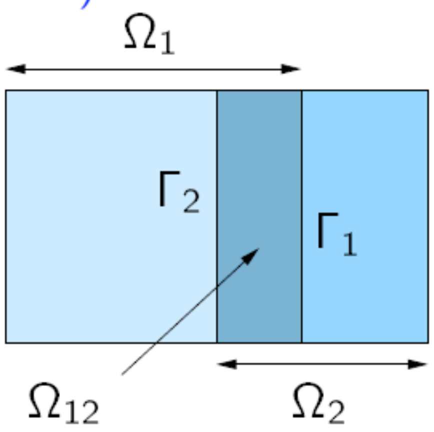


1. Find efficient solution algorithms
2. Use Dirichlet conditions at subdomain boundaries

Methodological framework



ENGINEERING PRACTICE (on overlapped subdomains)



SCHWARZ METHOD: decompose Ω in two overlapping subdomains and then iterate on the Dirichlet data on the interfaces.

The algorithm reads:
given $u_1^{(0)}$ and $u_2^{(0)}$, for $n \geq 1$ do

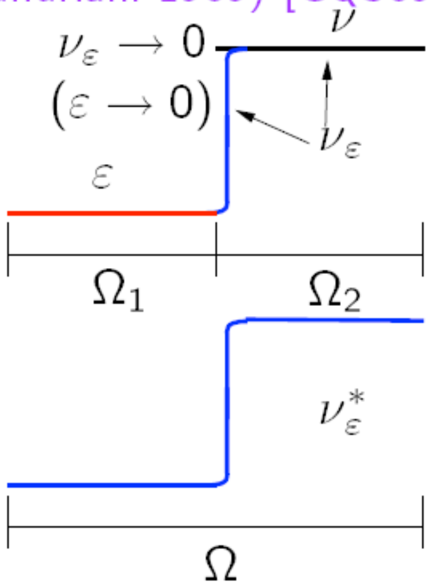
$$\begin{cases} A_1 u_1^{(n)} = f & \text{in } \Omega_1 \\ \text{b.c.} & \text{on } (\partial\Omega_1 \setminus \Gamma_1)^{in} \\ u_1^{(n)} = u_2^{(n-1)} & \text{on } \Gamma_1^{in} \end{cases} \qquad \begin{cases} A_2 u_2^{(n)} = f & \text{in } \Omega_2 \\ \text{b.c.} & \text{on } \partial\Omega_2 \setminus \Gamma_2 \\ u_2^{(n)} = u_1^{(n-1)} & \text{on } \Gamma_2 \end{cases}$$

A possible alternative consists in using Robin data instead of Dirichlet on interface Γ_2. (Houzeaux, Codina '03)

$$\begin{aligned} & u_1 = u_2 && \text{on } \Gamma_1 \\ & \nu \frac{\partial u_1}{\partial n} - \frac{1}{2}(\mathbf{b} \cdot \mathbf{n}) u_1 = \nu \frac{\partial u_2}{\partial n} - \frac{1}{2}(\mathbf{b} \cdot \mathbf{n}) u_2 && \text{on } \Gamma_2 \end{aligned}$$

ASYMPTOTIC ANALYSIS (sharp interface)

(F. Gastaldi, A. Quarteroni, G. Sacchi Landriani 1989) [GQS89]



- Define the global original problem $P_\Omega(\nu)$ and the coupled problem

$$P_\Omega(\nu_\varepsilon^*) \equiv [P_{\Omega_1}(\varepsilon)/P_{\Omega_2}(\nu_\varepsilon)]$$

Regularization on the data. Let $\varepsilon > 0$

$$\begin{array}{ll}
-\varepsilon\Delta u_{1,\varepsilon} + \mathrm{div}(\mathbf{b}u_{1,\varepsilon}) + b_0 u_{1,\varepsilon} = f & \text{in } \Omega_1 \\
\mathrm{div}(-\nu_\varepsilon \nabla u_{2,\varepsilon} + \mathbf{b}u_{2,\varepsilon}) + b_0 u_{2,\varepsilon} = f & \text{in } \Omega_2 \\
\epsilon \dfrac{\partial u_{1,\epsilon}}{\partial n_\Gamma} - \mathbf{b}\cdot\mathbf{n}_\Gamma u_{1,\epsilon} = \nu_\epsilon \dfrac{\partial u_{2,\epsilon}}{\partial n_\Gamma} - \mathbf{b}\cdot\mathbf{n}_\Gamma u_{2,\epsilon} & \text{on } \Gamma \\
u_{1,\epsilon} = u_{2,\epsilon} & \text{on } \Gamma \\
\text{boundary conditions} & \text{on } \partial\Omega
\end{array} \qquad (1)$$

ASYMPTOTIC ANALYSIS (sharp interface)

- $V_\Omega(\varepsilon)$ is the variational formulation associated to $P_\Omega(\varepsilon)$
- By asymptotic analysis on $V_{\Omega_1}(\varepsilon)$, recover the reduced problem $P_{\Omega_1}(0)$,

$$P_\Omega(\nu_\varepsilon^*) \to [P_{\Omega_1}(0)/P_{\Omega_2}(\nu)] \quad \text{when } \varepsilon \to 0$$

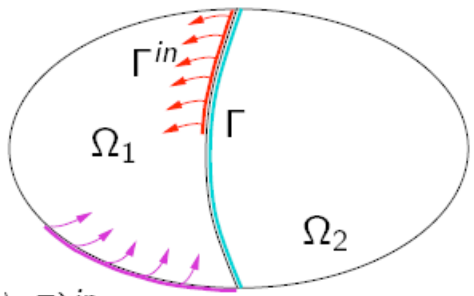
The new coupled problem $[P_{\Omega_1}(0)/P_{\Omega_2}(\nu)]$ inherits from the limit process a proper set of interface conditions

$$u_{1,\epsilon} \rightharpoonup u_1 \text{ in } L^2(\Omega_1), \quad u_{2,\epsilon} \rightharpoonup u_2 \text{ in } H^1(\Omega_2), \text{ when } \epsilon \to 0$$

The limit (u_1, u_2) satisfies the reduced coupled problem:

$$\begin{array}{ll}
\mathrm{div}(\mathbf{b}u_1) + b_0 u_1 = f & \text{in } \Omega_1 \\
\mathrm{div}(-\nu\nabla u_2 + \mathbf{b}u_2) + b_0 u_2 = f & \text{in } \Omega_2 \\
-\mathbf{b}\cdot\mathbf{n}_\Gamma u_1 = \nu\dfrac{\partial u_2}{\partial n_\Gamma} - \mathbf{b}\cdot\mathbf{n}_\Gamma u_2 & \text{on } \Gamma \\
u_1 = u_2 & \text{on } \Gamma^{in} \\
\text{b.c.} & \text{on } \partial\Omega
\end{array}$$

$\Gamma^{in} = \{\mathbf{x} \in \Gamma : \ \mathbf{b}(\mathbf{x})\cdot\mathbf{n}_\Gamma(\mathbf{x}) < 0\}$



$(\partial\Omega_1 \setminus \Gamma)^{in}$

POSSIBLE SOLUTION ALGORITHMS

The previous interface conditions are used to define a

Dirichlet/Neumann algorithm (DN)

Given $\psi^{(0)} \in L^2_{\mathbf{b}}(\Gamma^{in}) = \{v : \Gamma^{in} \to \mathbb{R} : \int_\Gamma \mathbf{b} \cdot \mathbf{n}_\Gamma v^2 < +\infty\}$, for $n \geq 0$ do

solve
$$\begin{cases} A_1 u_1^{(n+1)} = f & \text{in } \Omega_1 \\ u_1^{(n+1)} = g & \text{on } (\partial\Omega_1 \setminus \Gamma)^{in} \\ u_1^{(n+1)} = \psi^{(n)} & \text{on } \Gamma^{in} \end{cases}$$

solve
$$\begin{cases} A_2 u_2^{(n+1)} = f & \text{in } \Omega_2 \\ u_2^{(n+1)} = g & \text{on } \partial\Omega_2 \setminus \Gamma \\ -\nu \dfrac{\partial u_2^{(n+1)}}{\partial n_\Gamma} = 0 & \text{on } \Gamma^{in} \\ -\nu \dfrac{\partial u_2^{(n+1)}}{\partial n_\Gamma} + \mathbf{b} \cdot \mathbf{n}_\Gamma u_2^{(n+1)} = \mathbf{b} \cdot \mathbf{n}_\Gamma u_1^{(n+1)} & \text{on } \Gamma^{out} \end{cases}$$

compute $\psi^{(n+1)} = (1-\theta)\psi^{(n)} + \theta u_2^{(n+1)}|_{\Gamma^{in}}$

For suitable choices of the real parameter θ (depending on the data of the problem), the Dirichlet-Neumann algorithm converges to the heterogeneous solution (u_1, u_2) ([GQS89])

(C.Carlenzoli, A.Q., 1995)

Adaptive Robin/Neumann algorithm (ARN)

Given $\psi^{(0)} \in L^2_{\mathbf{b}}(\Gamma^{in}) = \{v : \Gamma^{in} \to \mathbb{R} : \int_\Gamma \mathbf{b}\cdot\mathbf{n}_\Gamma v^2 < +\infty\}$,
$\mu^{(0)} \in L^2_{\mathbf{b}}(\Gamma^{out})$, $u_2^{(0)}$ in Ω_2

for $n \geq 0$ do

solve
$$\begin{cases} A_1 u_1^{(n+1)} = f & \text{in } \Omega_1 \\ u_1^{(n+1)} = g & \text{on } (\partial\Omega_1 \setminus \Gamma)^{in} \\ -\mathbf{b}\cdot\mathbf{n}_\Gamma u_1^{(n+1)} = \nu \dfrac{\partial u_2^{(n)}}{\partial n_\Gamma} - \mathbf{b}\cdot\mathbf{n}_\Gamma \psi^{(n)} & \text{on } \Gamma^{in} \end{cases}$$

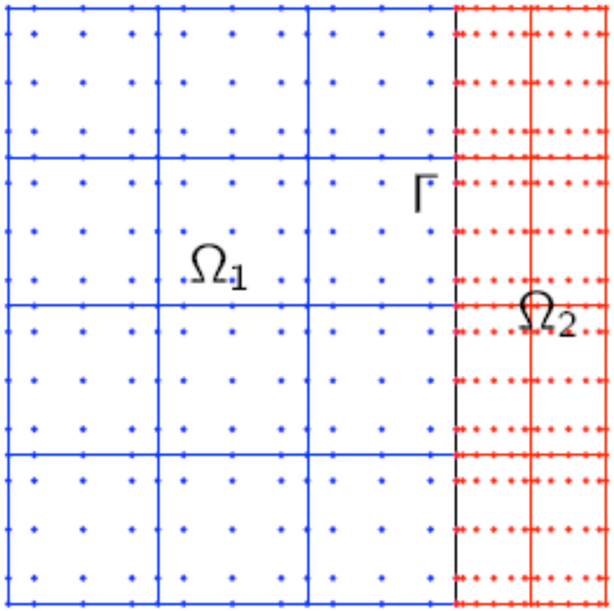
solve
$$\begin{cases} A_2 u_2^{(n+1)} = f & \text{in } \Omega_2 \\ u_2^{(n+1)} = g & \text{on } \partial\Omega_2 \setminus \Gamma \\ \nu \dfrac{\partial u_2^{(n+1)}}{\partial n_\Gamma} - \mathbf{b}\cdot\mathbf{n}_\Gamma u_2^{(n+1)} = -\mathbf{b}\cdot\mathbf{n}_\Gamma \mu^{(n)} & \text{on } \Gamma^{out} \\ \nu \dfrac{\partial u_2^{(n+1)}}{\partial n_\Gamma} = 0 & \text{on } \Gamma^{in} \end{cases}$$

compute
$$\psi^{(n+1)} = (1-\theta)\psi^{(n)} + \theta u_2^{(n+1)}|_{\Gamma^{in}}$$
$$\mu^{(n+1)} = (1-\theta)\mu^{(n)} + \theta u_1^{(n+1)}|_{\Gamma^{out}}$$

CONTINUITY ACROSS THE INTERFACE

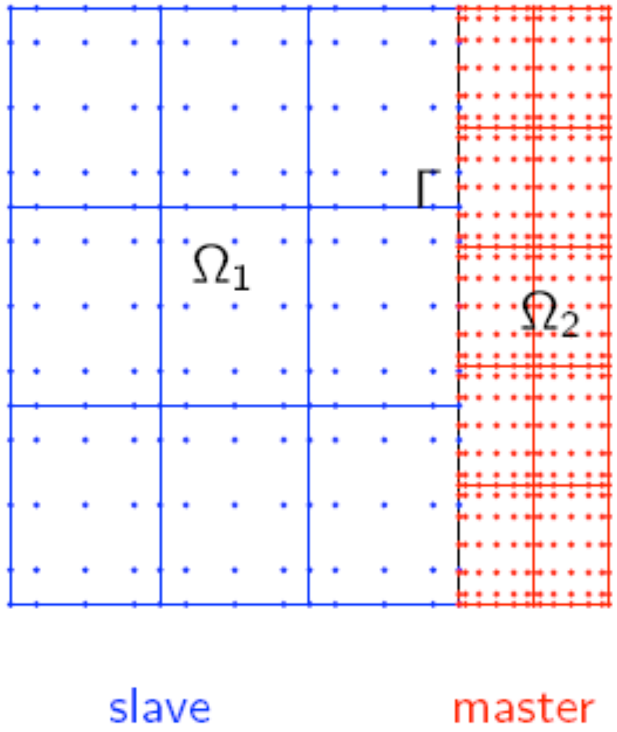
Conforming coupling

$$u_1 = u_2 = \lambda \text{ on } \Gamma$$



Mortar coupling

$$\int_\Gamma (u_1 - u_2)\psi = 0 \quad \forall \psi \in \text{constraint space}$$



The nodal values on $\Gamma \cap \mathcal{T}_2$ are the active d.o.f.
Those on $\Gamma \cap \mathcal{T}_1$ are obtained through the mortar coupling

MORTAR COUPLING

The differential problem reads:
given $f \in H^{-1}(\Omega)$, find u s.t.

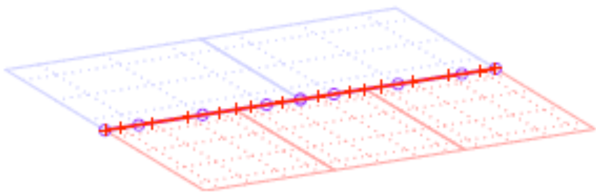
$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Variational Formulation:

$$\text{find } u \in V = H_0^1(\Omega) : \quad \int_\Omega \nabla u \cdot \nabla v d\Omega = \int_\Omega f v d\Omega \quad \forall v \in V$$

Discrete Variational Formulation

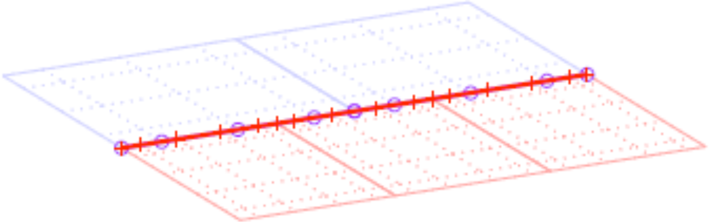
$$\text{find } u_\delta \in V_\delta : \quad \sum_{m=1}^{2} \int_{\Omega_m} \nabla u_\delta \cdot \nabla v_\delta d\Omega = \sum_{m=1}^{2} \int_{\Omega_m} f v_\delta d\Omega \quad \forall v_\delta \in V_\delta$$



$\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$, $\Omega_1 \cap \Omega_2 = \emptyset$,
$\Gamma = \partial\Omega_1 \cap \partial\Omega_2$, $u_\delta^{(m)} = u_\delta|_{\Omega_m}$
$V_\delta = \{v_\delta \in L^2(\Omega) : v_\delta|_{\Omega_m} \in C^0(\overline{\Omega}_m)$
and piecewise polynomial of degree N_m on each spectral element of Ω_m:

$$\int_\Gamma (u_\delta^{(1)} - u_\delta^{(2)})\psi = 0, \forall \psi \in \text{the constraint space}\}$$

- $V_\Gamma = \{p \in \mathcal{T}_1 \cap \Gamma : p$ is end-point of a spectral-element edge in $\mathcal{T}_1\}$ ($\mathcal{T}_1$= mesh in $\overline{\Omega}_1$)
- $\widetilde{\Lambda} = \text{span}\{\psi_\ell \in L^2(\Gamma) : \; \psi_\ell$ discontinuous piecewise polynomials of degree $N_1 - 2$ on each spectral-element edge of $\Gamma\}$, $\dim(\widetilde{\Lambda}) = \mathcal{N}_{slave} - \dim(V_\Gamma)$



In our example: $\dim(V_\Gamma) = 3$, $N_1 = 5$ and $\mathcal{N}_{slave} = 9$

Set: $\lambda^{(1)} = u_\delta^{(1)}|_\Gamma$ and $\lambda^{(2)} = u_\delta^{(2)}|_\Gamma$

Global weak continuity $\Leftrightarrow$

$$\int_\Gamma (\lambda^{(1)} - \lambda^{(2)})\psi_\ell d\Gamma = 0 \quad \forall \psi_\ell \in \widetilde{\Lambda}$$
$$\lambda^{(1)}(p) = \lambda^{(2)}(p) \quad \forall p \in V_\Gamma$$

$\mathcal{N}_{slave}$ conds.

How can we get the nodal values of $\lambda^{(1)}$ in terms of those of $\lambda^{(2)}$?

that is

How to characterize the matrix Ξ such that $\boldsymbol{\lambda}^{(1)} = \Xi \boldsymbol{\lambda}^{(2)}$

We have to use basis functions of V_δ

Basis functions in V_δ

$$V_\delta = \text{span}(\{\varphi^{(1)}_{k'}\} \cup \{\varphi^{(2)}_{k''}\} \cup \{\mu_k\})$$

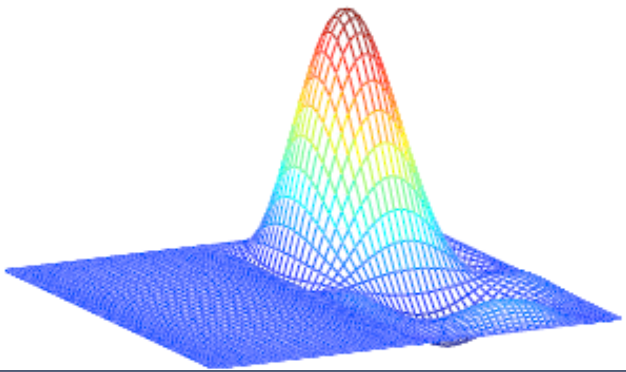
and

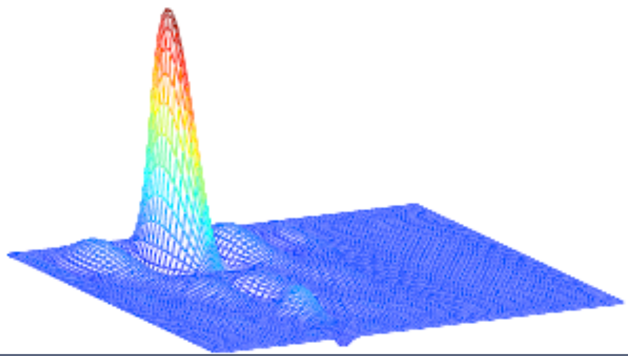
$$V_\delta \ni u_\delta(x,y) = \sum_{k'} u^{(1)}_{k'}\varphi^{(1)}_{k'}(x,y) + \sum_{k''} u^{(2)}_{k''}\varphi^{(2)}_{k''}(x,y) + \sum_{k} u^{(\Gamma)}_{k}\mu_k(x,y)$$

where:

- $\varphi^{(1)}_{k'}$ are the characteristic Lagrange functions in Ω_1, associated to the Legendre-Gauss-Lobatto (LGL) nodes of $\mathcal{T}_1 \setminus \Gamma$ (they vanish on Γ)
- $\varphi^{(2)}_{k''}$ are the characteristic Lagrange functions in Ω_2, associated to the LGL nodes of $\mathcal{T}_2 \setminus \Gamma$ (they vanish on Γ)

...





. . .

$$\mu_k(x,y) = \begin{cases} \tilde{\mu}_k^{(1)}(x,y) & \text{in } \overline{\Omega}_1 \\ \mu_k^{(2)}(x,y) & \text{in } \overline{\Omega}_2 \end{cases}$$

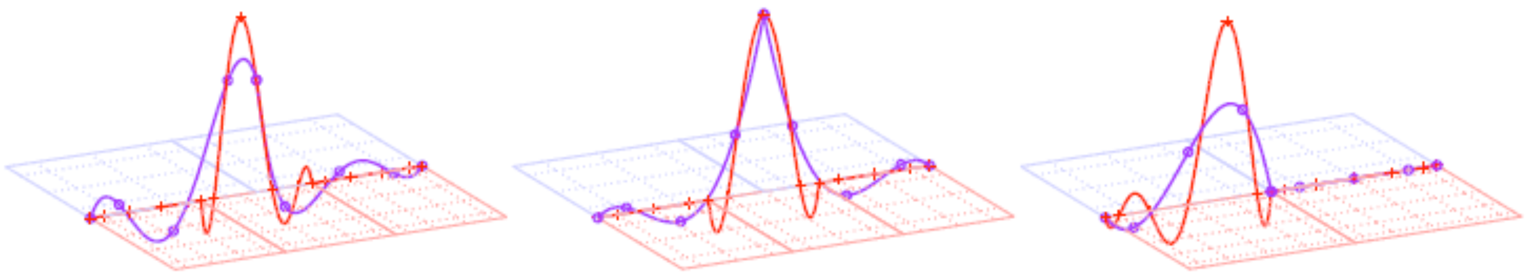
for $k = 1, \ldots, \mathcal{N}_{master}$, where:

$\tilde{\mu}_k^{(1)}$ are $\mathcal{N}_{master}$ continuous piecewise polynomials of degree N_1 on each Spectral Element (SE) of Ω_1

$\mu_k^{(2)}$ are $\mathcal{N}_{master}$ linearly independent continuous piecewise polynomials of degree N_2 on each SE of Ω_2 (characteristic Lagrange functions associated to the nodes of $\mathcal{T}_2 \cap \Gamma$), and

$$\int_\Gamma (\tilde{\mu}_k^{(1)} - \mu_k^{(2)})\psi_\ell = 0 \quad \forall \psi_\ell \in \widetilde{\Lambda} \quad \text{(weak cont. on each SE edge)}$$

$$\tilde{\mu}_k^{(1)}(p) = \mu_k^{(2)}(p) \quad \forall p \in V_\Gamma \quad \text{(strong cont. at SE edge end-points)}$$



Three different instances of μ_k

By writing $\tilde{\mu}_k^{(1)}$ as linear combinations of $\mu_j^{(1)}$, where $\mu_j^{(1)}$ are $\mathcal{N}_{slave}$ linearly independent continuous piecewise polynomials of degree N_1 on each SE of Ω_1 (characteristic Lagrange functions associated to the nodes of $\mathcal{T}_1 \cap \Gamma$):

$$\tilde{\mu}_k^{(1)}(x,y) = \sum_{j=1}^{\mathcal{N}_{slave}} \xi_{jk}\mu_j^{(1)}(x,y)$$

the $\mathcal{N}_{slave}$ constraint conditions read for $k = 1, \ldots, \mathcal{N}_{master}$:

$$\sum_{j=1}^{\mathcal{N}_{slave}} \xi_{jk} \int_\Gamma \mu_j^{(1)} \psi_\ell = \int_\Gamma \mu_k^{(2)} \psi_\ell \qquad \ell = 1, \ldots, \dim(\tilde{\Lambda})$$

$$\sum_{j=1}^{\mathcal{N}_{slave}} \xi_{jk} \mu_j^{(1)}(p) = \mu_k^{(2)}(p) \qquad \forall p \in V_\Gamma$$

$$\Updownarrow$$

for $\ell = 1, \ldots, \mathcal{N}_{slave}$, $k = 1, \ldots, \mathcal{N}_{master}$

$$\sum_{j=1}^{\mathcal{N}_{slave}} \xi_{jk} P_{\ell j} = \Phi_{\ell k} \quad \Leftrightarrow \quad \Xi = P^{-1}\Phi$$

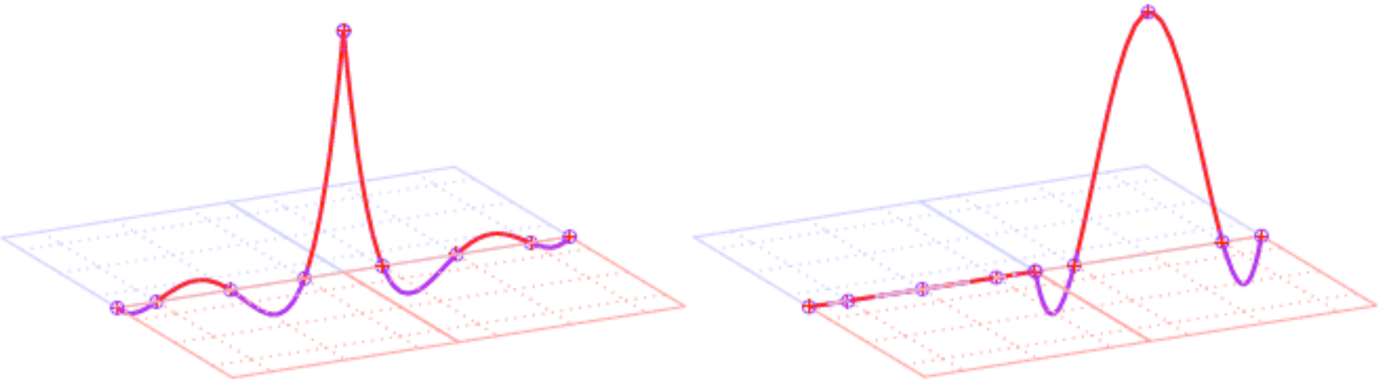
The $\mathcal{N}_{slave}$ values of $\boldsymbol{\lambda}^{(1)}$ are used to built the Dirichlet data for $u_\delta^{(1)}$ in Ω_1

When the discretization is conforming, then

$$\Xi = \text{Identity matrix}$$

and strong continuity is imposed on Γ

Here, $\mu_k^{(2)} = \tilde{\mu}_k^{(1)}$:



Algebraic formulation of Mortar Coupling

The master trace is $\lambda^{(2)}$,the slave trace is $\lambda^{(1)}$

Find $\mathbf{u}^{(1)} = [u_{k'}^{(1)}]^T$, $\mathbf{u}^{(2)} = [u_{k''}^{(2)}]^T$, $\boldsymbol{\lambda}^{(2)} = [\lambda_k^{(2)}]^T$, s.t.:

$$\begin{bmatrix} A_{11} & 0 & A_{1,\Gamma_1}\Xi \\ 0 & A_{22} & A_{2,\Gamma_2} \\ \Xi^T A_{\Gamma_1,1} & A_{\Gamma_2,2} & \Xi^T A_{\Gamma_1,\Gamma_1}\Xi + A_{\Gamma_2,\Gamma_2} \end{bmatrix} \begin{bmatrix} \mathbf{u}^{(1)} \\ \mathbf{u}^{(2)} \\ \boldsymbol{\lambda}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \Xi^T \mathbf{f}_{\Gamma_1} + \mathbf{f}_{\Gamma_2} \end{bmatrix} \quad (2)$$

A_{mm}, A_{m,Γ_m} and $A_{\Gamma_m,m}$ are blocks of the stiffness matrix on Ω_m.
Let $S_m = A_{\Gamma_m,\Gamma_m} - A_{\Gamma_m,m}A_{mm}^{-1}A_{m,\Gamma_m}$ be the Schur complement matrix with respect to the interface unknowns of Ω_m $(m = 1, 2)$.

We eliminate the unknowns $\mathbf{u}^{(1)}$, $\mathbf{u}^{(2)}$ from the system (2) and set

$$S = \Xi^T S_1 \Xi + S_2, \qquad \chi = \Xi^T(\mathbf{f}_{\Gamma_1} - A_{11}^{-1}\mathbf{f}_1) + \mathbf{f}_{\Gamma_2} - A_{22}^{-1}\mathbf{f}_2$$

(2) is equivalent to the SCHUR COMPLEMENT SYSTEM

$$S\boldsymbol{\lambda}^{(2)} = \chi$$

It is understood that $\boldsymbol{\lambda}^{(1)} = \Xi\boldsymbol{\lambda}^{(2)}$

Theoretical and Computational aspects

- Error estimate: if $u_{|\Omega_m} \in H^2(\Omega_m)$ with $s_m \geq 1$, then (Bernardi, Maday, Patera, '93, '94)

$$\|u - u_\delta\|_{H^1(\Omega),broken} \leq C \sum_{m=1}^{2} N_m^{1-s_m} |u|_{H^{s_m}(\Omega_m)}$$

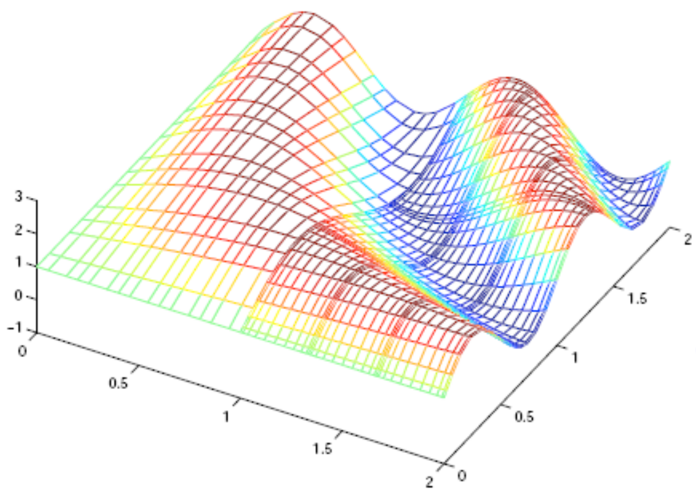
- $S\lambda^{(2)} = \chi$ can be solved, e.g., by Krylov methods and it can be preconditioned by S_2 (the Schur complement associated to the master domain)

$$\exists C_1 > 0 \text{ indep of } \delta \text{ s.t.} \qquad \mathcal{K}((S_2)^{-1}S) \leq C_1$$

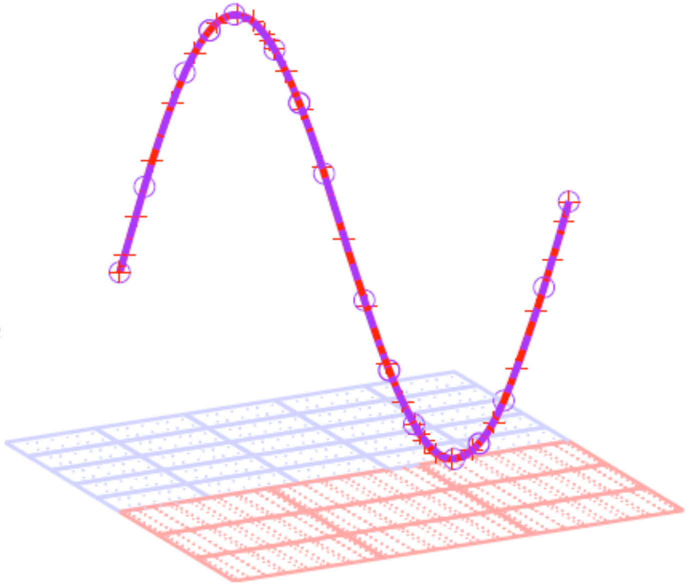
- Ξ is built once the discretization is known
- at $n-th$ iteration, with $\mathbf{x}^{(n)}$ known, the computation of the matrix-vector product $S\mathbf{x}^{(n)}$ requires the solution of one differential problem in Ω_1 and one in Ω_2, starting from the discrete traces $\Xi\mathbf{x}^{(n)}$ and $\mathbf{x}^{(n)}$, respectively

Numerical results

$-\Delta u = f$ in Ω, with Dirichlet boundary conditions
$\Omega = (0,2)^2$, $\Omega_1 = (0,1) \times (0,2)$, $\Omega_2 = (1,2) \times (0,2)$
Discretization:
in Ω_1: 5×5 spectral elements with $N_1 = 3$
in Ω_2: 3×3 spectral elements with $N_2 = 10$
Master domain: Ω_2



Numerical solution



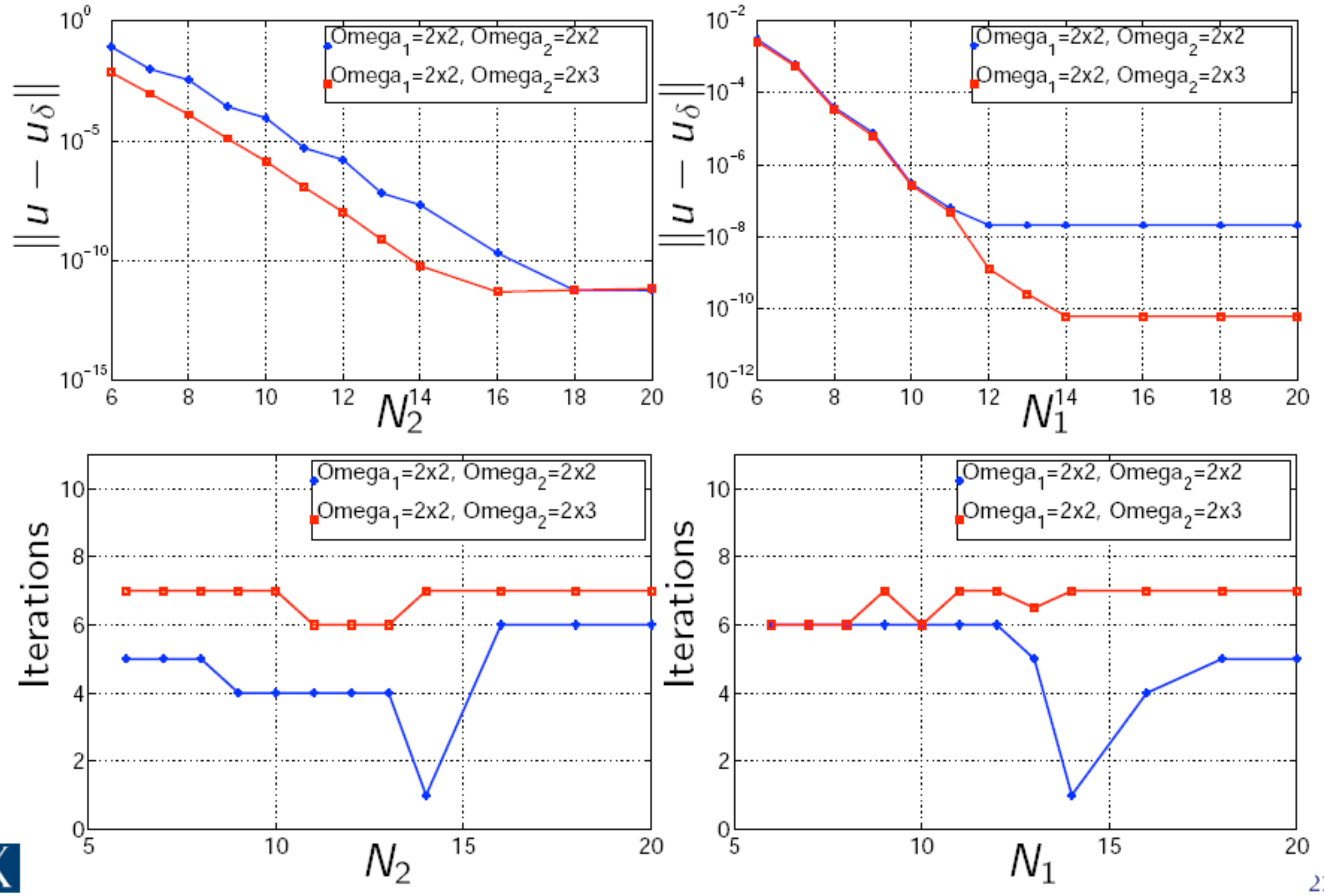
Traces on Γ

Numerical results

Krylov solver: Preconditioned Bi-CGStab (van der Vorst)

Errors in broken $H^1(\Omega)$ norm

$N_1 = 14$ fixed $\qquad\qquad$ $N_2 = 14$ fixed



THE INTERFACE HETEROGENEOUS PROBLEM

Let λ_k denote the unknown trace on Γ of the solution u_k
Write the solution (u_1, u_2) of the heterogeneous problem as

$$u_1 = u_1^{\lambda_1} + w_1^f, \quad u_2 = u_2^{\lambda_2} + w_2^f,$$

where: $\forall\lambda \in H_{00}^{1/2}(\Gamma)$, $u_1^{\lambda_1}$ and $u_2^{\lambda_2}$ are solutions of:

$$\begin{cases} A_1 u_1^{\lambda_1} = 0 & \text{in } \Omega_1 \\ u_1^{\lambda_1} = 0 & \text{on } (\partial\Omega_1 \setminus \Gamma)^{in} \\ u_1^{\lambda_1} = \lambda_{1|\Gamma^{in}} & \text{on } \Gamma^{in} \end{cases} \qquad \begin{cases} A_2 u_2^{\lambda_2} = 0 & \text{in } \Omega_2 \\ u_2^{\lambda_2} = 0 & \text{on } \partial\Omega_2 \setminus \Gamma \\ u_2^{\lambda_2} = \lambda_2 & \text{on } \Gamma. \end{cases} \tag{3}$$

while w_1^f and w_2^f are solution of

$$\begin{cases} A_1 w_1^f = f & \text{in } \Omega_1 \\ w_1^f = 0 & \text{on } \partial\Omega_1^{in} \end{cases} \qquad \begin{cases} A_2 w_2^f = f & \text{in } \Omega_2 \\ w_2^f = 0 & \text{on } \partial\Omega_2. \end{cases} \tag{4}$$

Set

$$\chi_1 = \begin{cases} -\mathbf{b} \cdot \mathbf{n}_\Gamma w_1^f & \text{on } \Gamma^{out} \\ 0 & \text{on } \Gamma^{in}, \end{cases} \qquad \chi_2 = -\nu \frac{\partial w_2^f}{\partial n_\Gamma} \quad \text{on } \Gamma, \qquad \boxed{\chi = \chi_1 + \chi_2}$$

STEKLOV-POINCARE' equation

Define the Steklov-Poincaré operators (Dirichlet to Neumann maps):

$$\mathcal{S}_1\lambda_1 = \begin{cases} 0 & \text{on } \Gamma^{in} \\ -\mathbf{b}\cdot\mathbf{n}_\Gamma u_1^{\lambda_1} & \text{on } \Gamma^{out} \end{cases}, \quad \mathcal{S}_2\lambda_2 = \begin{cases} -\nu \dfrac{\partial u_2^{\lambda_2}}{\partial n_\Gamma} & \text{on } \Gamma^{in} \\ -\left(\nu \dfrac{\partial u_2^{\lambda_2}}{\partial n_\Gamma} - \mathbf{b}\cdot\mathbf{n}_\Gamma u_2^{\lambda_2}\right) & \text{on } \Gamma^{out} \end{cases}$$

At continuous level:

$$u_1 = u_2, \quad \nu\frac{\partial u_2}{\partial n_\Gamma} = 0 \quad \text{on } \Gamma^{in},$$
$$-\mathbf{b}\cdot\mathbf{n}_\Gamma u_1 = \nu\frac{\partial u_2}{\partial n_\Gamma} - \mathbf{b}\cdot\mathbf{n}_\Gamma u_2 \quad \text{on } \Gamma^{out}$$

$$\Updownarrow$$

$$\mathcal{S}_1\lambda_1 + \mathcal{S}_2\lambda_2 = \chi_1 + \chi_2$$

Mortar Steklov-Poincaré equation for A-AD coupling

Continuity is enforced only on Γ^{in}:
the matrix Ξ is built to transform $\lambda_2^{in} = \lambda_{2|\Gamma^{in}}$ in $\lambda_1^{in} = \lambda_{1|\Gamma^{in}}$

On Γ^{out} no continuity is enforced on the traces, but only on fluxes through the SP equation and the interpolation matrix Q from $\mathcal{T}_1 \cap \Gamma^{out}$ to $\mathcal{T}_2 \cap \Gamma^{out}$,

The mortar Steklov-Poincaré equation reads:

$$\lambda_1^{in} = \Xi \lambda_2^{in}$$

$$\underbrace{\left(\begin{bmatrix} 0 & 0 \\ DS_1^{out}\Xi & 0 \end{bmatrix} + S_2 \right)}_{S} \begin{bmatrix} \lambda_2^{in} \\ \lambda_2^{out} \end{bmatrix} = \underbrace{\begin{bmatrix} \chi_2^{in} \\ D\chi_1^{out} + \chi_2^{out} \end{bmatrix}}_{\boldsymbol{\chi}}$$

where: $D = M_{\delta_2}^{out} Q (M_{\delta_1}^{out})^{-1}$,
$M_{\delta_m}^{out}$= one-dimensional mass matrix on $\Gamma^{out} \cap \mathcal{T}_m$

If $Q = I$, $\Xi = I$, we recover the conforming coupling

PRECODITIONING of SP equations

SP equation $S\boldsymbol{\lambda}_2 = \boldsymbol{\chi}$ can be solved by preconditioned Bi-CGStab method.

If either $\Gamma^{in} = \emptyset$ or $\Gamma^{out} = \emptyset$, then the convergence is achieved in 2 iterations. Otherwise #it grows with #d.o.f.

- $\mathcal{S}_2$ is coercive $\Rightarrow$ S_2 can be used to precondition SP equations

Conforming coupling ($\Xi = I$):

(F. Gastaldi, A. Quarteroni, G. Sacchi Landriani, 1989)

$\exists C_1 > 0$ indep of $\delta,\ \nu,\ \mathbf{b}$ s.t. $\quad \mathcal{K}((S_2)^{-1}S) \leq C_1$

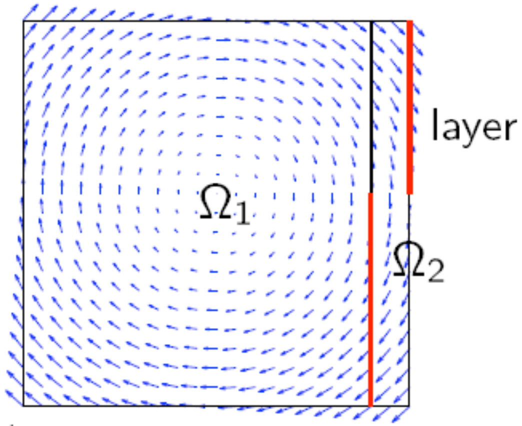
Mortar coupling:

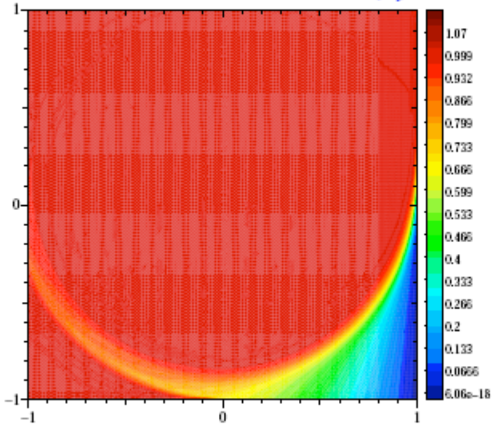
(experimentally)

$$\exists C_2 > 0 \text{ indep of } \delta,\ \nu,\ \mathbf{b} \text{ s.t.} \qquad \mathcal{K}((S_2)^{-1}S) \leq C_2$$

PBi-CGstab iterations on SP equation

$\mathbf{b} = [y - 0.1, -x]^T$





Stopping test: $||\mathbf{r}^{(k)}||/||\mathbf{r}^{(0)}|| \leq 10^{-14}$

3-4 iterations

for both conforming and mortar coupling

versus the polynomial degree N:
$N_1 = N_2 = 6, 8, \ldots, 16$, except $N_{2,x} = 64$
close to the right vertical side.
5×5 spectral elements in each Ω_k, $\nu = 10^{-2}$

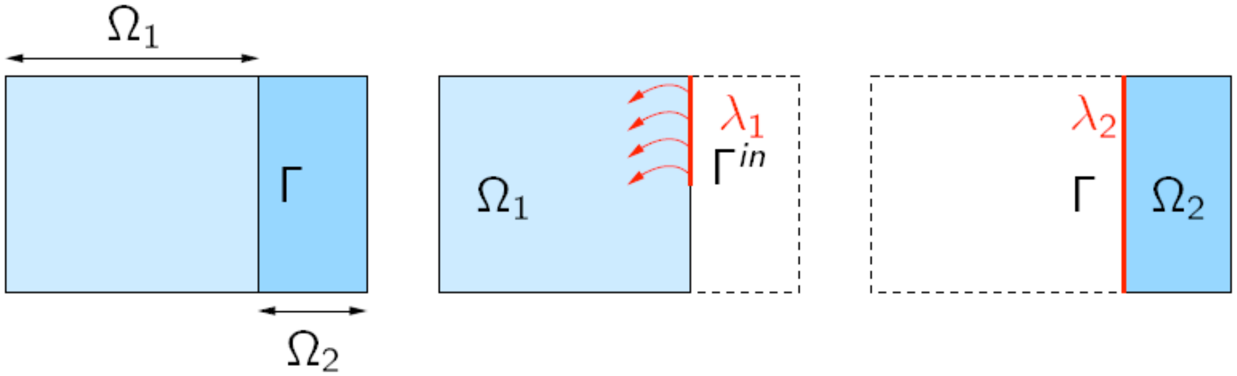
versus # of elem: $M_x = M_y = 3, 5, \ldots, 13$
and $N_1 = N_2 = 8$, $\nu = 10^{-2}$

versus viscosity: $\nu = 10^{-1},\ 10^{-2}, \ldots, 10^{-4}$

VIRTUAL CONTROL APPROACH (without overlap)

Introduce two functions λ_1 and λ_2 to be used as unknown Dirichlet data on Γ:

$$\begin{cases} A_1 u_1 = f & \text{in } \Omega_1 \\ b.c. & \text{on } (\partial\Omega_1 \setminus \Gamma)^{in} \\ u_1 = \lambda_1 & \text{on } \Gamma^{in} \end{cases} \qquad \begin{cases} A_2 u_2 = f & \text{in } \Omega_2 \\ b.c. & \text{on } \partial\Omega_2 \setminus \Gamma \\ u_2 = \lambda_2 & \text{on } \Gamma \end{cases}$$



λ_1, λ_2 solutions of the MINIMIZATION PROBLEM $\quad \inf_{\lambda_1,\lambda_2} J(\lambda_1, \lambda_2)$

with

$$J(\lambda_1, \lambda_2) = \frac{1}{2}\|u_1 - u_2\|^2_{L^2_{\mathbf{b}}(\Gamma^{in})} + \frac{1}{2}\left\|\mathbf{b}\cdot\mathbf{n}_\Gamma u_1 + \left(\nu\frac{\partial u_2}{\partial \mathbf{n}_\Gamma} - \mathbf{b}\cdot\mathbf{n}_\Gamma u_2\right)\right\|^2_{H^{-1/2}(\Gamma)}$$

We have to solve a control problem with both control and observation on the boundary (the interface).

THEOREM (P.Gervasio, J.-L. Lions, A.Q., 2001)
There exists a unique solution of the minimum problem $\inf_{\lambda_1,\lambda_2} J(\lambda_1,\lambda_2)$

We can define the inner product in $H^{-1/2}(\Gamma)$ as

$$(\phi,\psi)_{H^{-1/2}(\Gamma)} = \int_\Gamma ((-\Delta_\Gamma)^{-1/4}\phi)((-\Delta_\Gamma)^{-1/4}\psi)d\Gamma = \int_\Gamma ((-\Delta_\Gamma)^{-1/2}\phi)\psi d\Gamma$$

where $-\Delta_\Gamma$ is the Laplace-Beltrami operator on Γ.
The operator $(-\Delta_\Gamma)^{-1/2}$ could be replaced by any isomorphism

$$\mathcal{P} : (H_{00}^{1/2}(\Gamma))' \to H_{00}^{1/2}(\Gamma)$$

e.g.:

1. a Dirichlet to Neumann map,
2. the inverse of the Steklov-Poicaré S_2 on Γ.

The operator $\mathcal{P}$ plays the role of the preconditioner for the dual state problems.

RECOVERING INTERFACE CONDITIONS

If $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ is the solution of the minimization problem

$$\inf_{\lambda_1, \lambda_2} J(\lambda_1, \lambda_2)$$

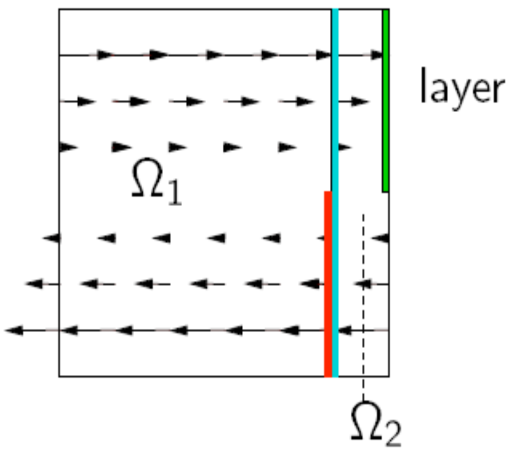
then the state solutions u_1 and u_2 satisfy the interface conditions

$$\begin{aligned} -\mathbf{b} \cdot \mathbf{n}_\Gamma u_1 &= \nu \frac{\partial u_2}{\partial n_\Gamma} - \mathbf{b} \cdot \mathbf{n}_\Gamma u_2 && \text{on } \Gamma \\ u_1 &= u_2 && \text{on } \Gamma^{in} \end{aligned}$$

and the Euler-Lagrange equation can be written in terms of SP operators:

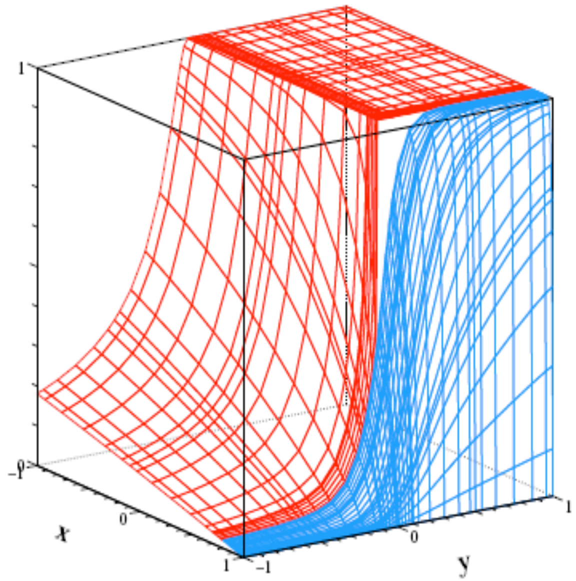
$$J'(\boldsymbol{\lambda}) = 0 \iff S^* S_2^{-1} S \boldsymbol{\lambda} = \hat{\chi}$$

S^* is the Steklov-Poincaré operator associated to the dual problems
$P = S_2^*$ is an optimal preconditioner for $S^* S_2^{-1} S$

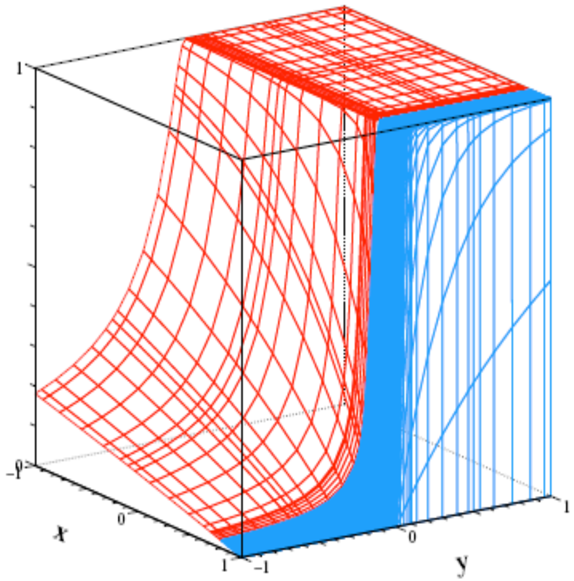


$\Omega = (-1,1)^2$, $\Omega_1 = (-1,0.8) \times (-1,1)$, $\Omega_2 = (0.8,1) \times (-1,1)$.
$\mathbf{b} = (10y,0)^T$, $b_0 = 1$, $f = 0$.
Neumann b.c. on horizontal sides of Ω_2
$u_2 = 0$ on the right vertical side of Ω_2
$u_1 = 1$ on the inflow left side of Ω_1

$\nu = 0.1$



$\nu = 0.001$



Number of iterations

D/N: Dirichlet Neumann; SP: PBi-CGStab on Steklov-Poincaré eq. with $P = S_2$;
VC: PBi-CGStab on Virtual-Control Optimality System with $P = S_2$

$\nu = 0.1$

N	D/N	SP	VC	
	#it	#it	#it	inf J
6	2	1	4	4.14e-09
8	2	1	4	1.69e-10
10	2	1	4	6.56e-12
12	2	1	4	2.08e-13
14	2	1	4	4.95e-15
16	2	1	4	8.61e-17

$\nu = 0.01$

N	D/N	SP	VC	
	#it	#it	#it	inf J
6	2	1	4	1.27e-09
8	2	1	2	4.70e-11
10	2	1	2	1.73e-12
12	2	1	2	5.28e-14
14	2	1	2	1.24e-15
16	2	1	2	2.12e-17

ν	D/N	SP	VC	
	#it	#it	#it	inf J
$5 \cdot 10^{-2}$	2	1	3	2.79e-09
$1 \cdot 10^{-2}$	2	1	2	1.27e-09
$5 \cdot 10^{-3}$	2	1	2	9.67e-10
$1 \cdot 10^{-3}$	2	1	2	5.44e-10
$5 \cdot 10^{-4}$	2	1	1	2.36e-10

Stopping test:
$\|\lambda^{(k+1)} - \lambda^{(k)}\| \leq 10^{-10}$ for DN
$\frac{\|r^{(k+1)}\|}{\|r^{(0)}\|} \leq 10^{-10}$ for SP and VC

VIRTUAL CONTROL (without overlap)

REMARKS

- The Virtual Control approach with

$$J(\lambda_1, \lambda_2) = \frac{1}{2} \|u_1 - u_2\|^2_{L^2_{\mathbf{b}}(\Gamma^{in})} + \frac{1}{2} \left\| \mathbf{b} \cdot \mathbf{n}_\Gamma u_1 + \left(\nu \frac{\partial u_2}{\partial \mathbf{n}_\Gamma} - \mathbf{b} \cdot \mathbf{n}_\Gamma u_2 \right) \right\|^2_{H^{-1/2}(\Gamma)}$$

is formally equivalent to the heterogeneous problem by asymptotic analysis

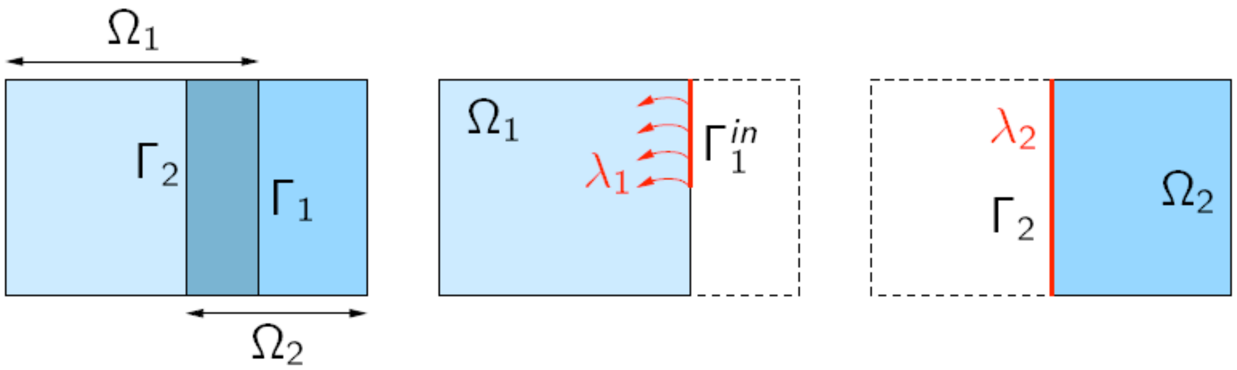
- Virtual Control approach provides a numerical algorithm (through the solution of the optimality system) alternative to the solution of the Steklov-Poincaré equation

- The cost functional is set up starting from known interface conditions, it is problem dependent and it requires a-priori knowledge of the problem. Alternative: Virtual Control with overlap: the a-priori knowledge of the interface problem is NOT required

VIRTUAL CONTROL APPROACH (with overlap)

(Glowinski et al. '80, '90, J.-L. Lions et al. 2000)

$\Omega_1,\ \Omega_2 \subset \Omega,\ \ \Omega_{12} = \Omega_1 \cap \Omega_2 \neq \emptyset,\ \ \Gamma_k = \partial\Omega_k \setminus \partial\Omega,\ k = 1, 2.$



$$\begin{cases} A_1 u_1 = f & \text{in } \Omega_1 \\ b.c. & \text{on } (\partial\Omega_1 \setminus \Gamma_1)^{in} \\ u_1 = \lambda_1 & \text{on } \Gamma_1^{in} \end{cases} \qquad \begin{cases} A_2 u_2 = f & \text{in } \Omega_2 \\ b.c. & \text{on } \partial\Omega_2 \setminus \Gamma_2 \\ u_2 = \lambda_2 & \text{on } \Gamma_2 \end{cases}$$

$\lambda_1,\ \lambda_2$, solutions of $\boxed{\inf_{\lambda_1,\lambda_2} J_{\Omega_{12}}(\lambda_1, \lambda_2)}$ with

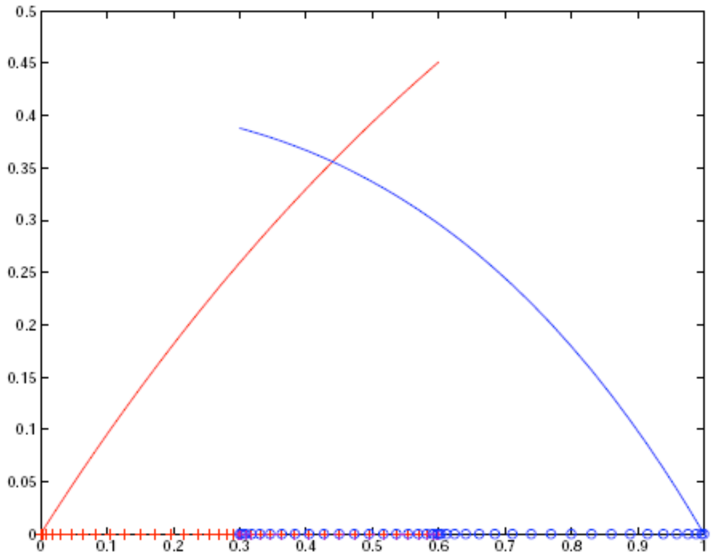
$$\boxed{J_{\Omega_{12}}(\lambda_1, \lambda_2) = \frac{1}{2} \int_{\Omega_{12}} (u_1 - u_2)^2}$$

DISTRIBUTED OBSERVATION ON THE OVERLAP Ω_{12}

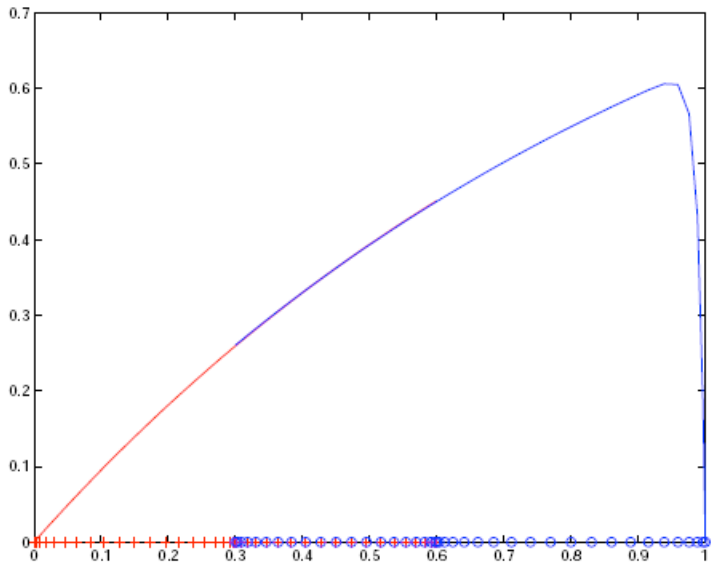
VIRTUAL CONTROL (with overlap)

THEOREM (P.Gervasio, J.-L. Lions, A.Q., 2001)
Under suitable assumptions on the data,
there exists a unique solution of this minimum problem.
Moreover $\phi(\nu) = \inf J_{\Omega_{12}}(\lambda_1, \lambda_2) \to 0$ when $\nu \to 0$

- In general $u_1 \neq u_2$ in Ω_{12}



$\nu = 1$



$\nu = 0.01$

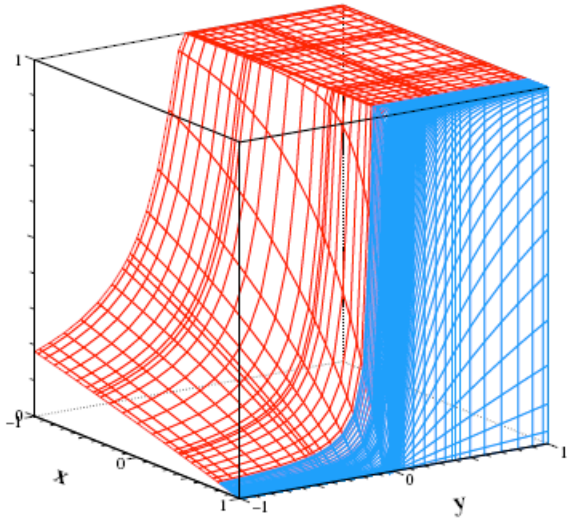
VIRTUAL CONTROL (distributed observation on the overlap)

ν	#it
10^{-1}	24
10^{-2}	19
10^{-3}	19
$5 \cdot 10^{-4}$	18

δ	#it
0.1	20
0.04	21
0.02	25
0.01	70

N	#it
8	20
12	31
16	45
20	50

N = polynomial degree, $\delta = x_{\Gamma_1} - x_{\Gamma_2} = meas_x(\Omega_{12})$.
Number of BiCG-Stab iterations needed to satisfy the stopping criterion on the residual with tolerance $\varepsilon = 10^{-6}$

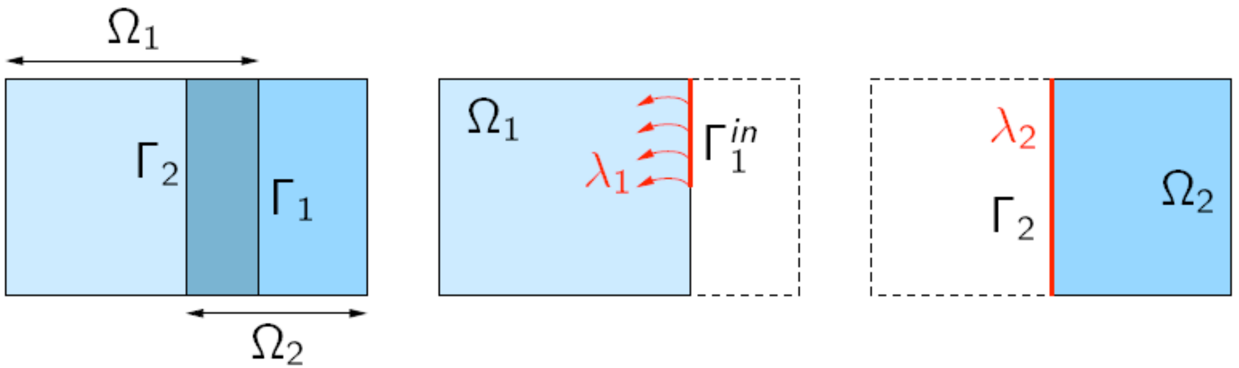


$\nu = 5 \cdot 10^{-4}$, $\mathbf{b} = [10y, 0]^T$, $b_0 = 1$, $f = 1$

#it grows w.r.t. the polynomial degree N and $1/\delta$

OBSERVATION ON THE INTERFACES $\Gamma_1 \cup \Gamma_2$

$$\Omega_1,\ \Omega_2 \subset \Omega,\ \ \Omega_{12} = \Omega_1 \cap \Omega_2 \neq \emptyset,\ \ \Gamma_k = \partial\Omega_k \setminus \partial\Omega,\ k = 1, 2.$$

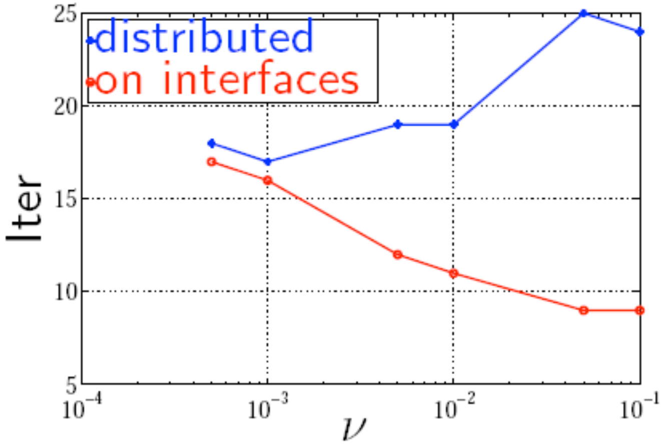


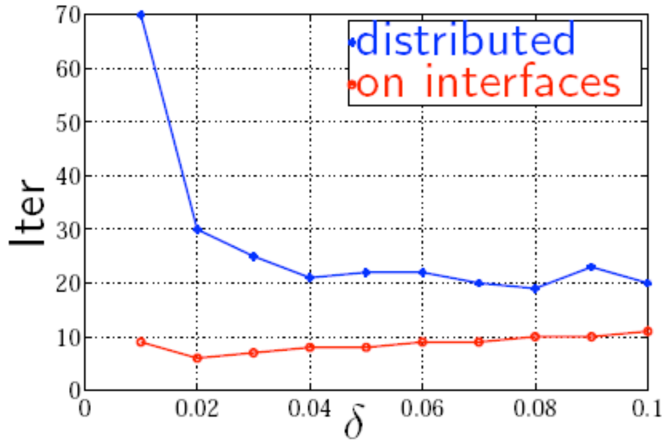
$$\begin{cases} A_1 u_1 = f & \text{in } \Omega_1 \\ b.c. & \text{on } (\partial\Omega_1 \setminus \Gamma_1)^{in} \\ u_1 = \lambda_1 & \text{on } \Gamma_1^{in} \end{cases} \qquad \begin{cases} A_2 u_2 = f & \text{in } \Omega_2 \\ b.c. & \text{on } \partial\Omega_2 \setminus \Gamma_2 \\ u_2 = \lambda_2 & \text{on } \Gamma_2 \end{cases}$$

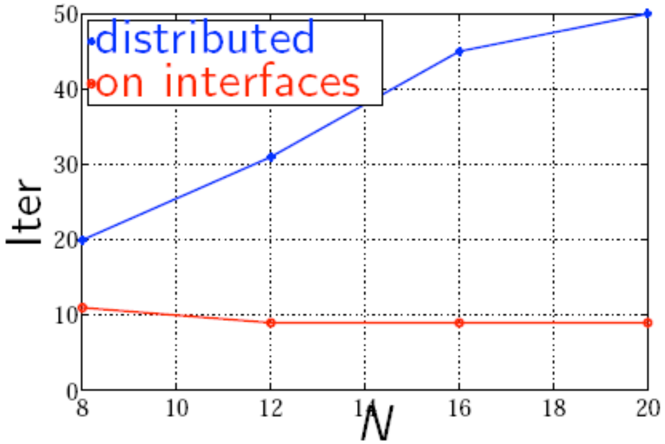
$\lambda_1,\ \lambda_2$, solutions of $\displaystyle \inf_{\lambda_1,\lambda_2} J_{\Gamma_{12}}(\lambda_1, \lambda_2)$ with

$$J_{\Gamma_{12}}(\lambda_1, \lambda_2) = \frac{1}{2} \int_{\Gamma_1^{in} \cup \Gamma_2} (u_1 - u_2)^2$$

Comparison between distributed and interface observation







- Virtual control with overlap does not require a-priori knowledge of interface conditions
- Interface observation performs better than distributed observation on the overlap, mainly for small overlap and high polynomial degree