Theorem A (Nash–Kuiper): Let (M^n, g) be a smooth, compact mfd, and let $u : M^n \hookrightarrow \mathbb{R}^m$ for some $m \geq n+$ a smooth, strictly short embedding.
Then $\forall \varepsilon > 0$ $\exists\, \tilde{u} : M^n \hookrightarrow \mathbb{R}^m$ C^1 isometric embedding such that $\|u - \tilde{u}\|_{C^0(M)} < \varepsilon$.

Remarks 1) Locally the equation satisfied by isometric ~~embeddings~~ maps is

$$\partial_i u \cdot \partial_j u = g_{ij}$$

ie. system with $\frac{1}{2}n(n+1)$ equations
m unknowns

$\longrightarrow$ local solvability expected only if $m \geq \frac{1}{2}n(n+1)$

Janet-Cartan (1926)
$m = \frac{1}{2}n(n+1)$, g analytic $\exists\, u$ analytic

2) $n = 2$, $\frac{n(n+1)}{2} = 3$

e.g. Weyl Problem

$$(S^2, g) \hookrightarrow \mathbb{R}^3$$

$K_g > 0$

$\boxed{\text{existence + uniqueness}}$

~~Cauchy~~ Weyl, Lewy, Pogorelov, Nirenberg

$g \in C^\infty \to u \in C^\infty$

$g \in$ analytic $\to u \in$ analytic

$g \in C^{2,\alpha} \to u \in C^{2,\alpha}$

<u>uniqueness in Weyl problem</u> ($K_g > 0$)

- Cohn-Vossen ($u \in C^2$) see also Spivak Vol 5
  (1927) e.g. Herglotz integral formula

- Pogorelov, Sabitov $u \in C^1$ and $u(S^2)$ convex
  (1950s)

3) $u : M^n \hookrightarrow \mathbb{R}^m$ is an

- <u>immersion</u> if locally injective
- <u>embedding</u> if globally injective
- <u>strictly short map</u> if

$$g - du \cdot du > 0$$

i.e. the matrix

$$(g_{ij} - \partial_i u \cdot \partial_j u)$$

for some choice of coordinates is positive definite everywhere.

(easy exercise : independent of choice of coordinates)

<u>geometrically</u> being (strictly) short means that the length of curves shrinks. Equally, being isometric means that the length of curves is preserved.

Theorem B (Scheffer - Shnirelman)

There exists a nontrivial weak solution $v \in L^2(\mathbb{R}^2 \times \mathbb{R})$ of the incompressible Euler equations, which has compact support in space and time.

Remarks 1)

$$\partial_t v + \operatorname{div} v \otimes v + \nabla p = 0$$
$$\operatorname{div} v = 0$$

makes sense in $\mathcal{D}'$ without explicit appearance of the pressure p:

$$\int_{\mathbb{R}^n \times \mathbb{R}} \partial_t \varphi \, v + \nabla \varphi : v \otimes v \quad dx\,dt = 0$$

$$\forall \varphi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n)$$
with $\operatorname{div} \varphi = 0$

&

$$\int_{\mathbb{R}^n} \nabla \psi \cdot v \, dx = 0 \qquad \forall \psi \in C_c^\infty(\mathbb{R}^n) .$$

2) "Formal" conservation of energy:

multiply equation with v

$$v \cdot \operatorname{div} v \otimes v = v_i \, \partial_j (v_i v_j) = v_i v_j \partial_j v_i$$
$$= v_j \partial_j \frac{|v|^2}{2} = \partial_j \left(v_j \frac{|v|^2}{2} \right)$$
$$= \operatorname{div} \left(v \frac{|v|^2}{2} \right)$$

$$v \cdot \nabla p = \operatorname{div} (v p)$$

Hence

$$\partial_t \frac{|v|^2}{2} + \mathrm{div}\left(v\left(\frac{|v|^2}{2} + p\right)\right) = 0$$

Assuming some decay on v at ∞ and integrating in

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n} |v|^2 \, dx \qquad \underline{\text{constant}}.$$

Of course, if $v \in L^2$ only, the product

$\underline{v \cdot \mathrm{div}\, v\otimes v \quad \text{makes no sense}}$.

Theorem A.1

[based on J. Nash Annals 1954]

- $\Omega \subset \mathbb{R}^n$ open bounded
- $g \in C^\infty(\bar\Omega)$ pos. def matrix-valued, i.e. smooth metric.
- $u : \Omega \hookrightarrow \mathbb{R}^m$ $\boxed{m \geq n+2}$

strictly short immersion, in $C^\infty(\bar\Omega)$ (up to boundary)

Then $\forall \varepsilon > 0$ $\exists$ $\tilde u : \Omega \hookrightarrow \mathbb{R}^m$

C^1 isometric immersion such that $\|u - \tilde u\|_{C^0(\bar\Omega)} < \varepsilon$

i.e.
- $\tilde u \in C^1(\bar\Omega)$
- $\nabla \tilde u^T \nabla \tilde u = g$ in Ω
- $\|u - \tilde u\|_{C^0(\bar\Omega)} < \varepsilon$.

Idea Starting with u, consider

$$u_1(x) = u(x) + \frac{a(x)}{\lambda}\Big(\sin(\lambda x\cdot\xi)\,\zeta(x) + \cos(\lambda x\cdot \xi)\,\eta(x)\Big)$$

where $\xi \in \mathbb{R}^n$; ζ, η are unit normal vectors to $u(\Omega)$,

i.e. (i) $\zeta \perp \eta$, $|\zeta| = |\eta| = 1$

(ii) $\zeta \perp \partial_i u$ & $\eta \perp \partial_i u$ $i = 1, \dots n$

i.e. $\nabla u^T \zeta = 0$

$\nabla u^T \eta = 0$

$$\nabla u_1(x) = \nabla u(x) + a(x)\left(\cos(\lambda x\cdot\xi)\,\zeta\otimes\xi - \sin(\lambda x\cdot\xi)\,\eta\otimes\xi\right) + O\left(\frac{1}{\lambda}\right)$$

$$\nabla u_1^T\nabla u_1 = \nabla u^T\nabla u + a^2(x)\,\xi\otimes\xi + O\left(\frac{1}{\lambda}\right)$$

[geometric picture (with $n=1$)

]

The following is WRONG, but philosophically correct.

Now, since u is strictly short,

$$\underbrace{g - \nabla u^T\nabla u}_{\text{pos. def.}} = \sum_{k=1}^{n} a_k^2\,\xi^k\otimes\xi^k \qquad \text{for each fixed } x.$$

diagonalizing

Therefore, repeatedly adding such spiralling perturbations we can (should be able to) achieve u_N s.t.

$$\begin{cases} \cancel{g}\ \nabla u_N^T\nabla u_N = g + O\left(\frac{1}{\lambda}\right) \\ \|\nabla u_N - \nabla u\|_{C^0} = \sum_k \|a_k\|_0 + O\left(\frac{1}{\lambda}\right) \simeq \|g - \nabla u^T\nabla u\|_{C^0}^{1/2} \\ \|u_N - u\|_{C^0} = O\left(\frac{1}{\lambda}\right) \end{cases}$$

Important point : as x varies,
a, ζ, η may vary with x, but ξ not

i.e. cannot simply take the eigenvectors of
$g(x) - \nabla u(x)^T \nabla u(x)$.

Instead, we consider a "partition of unity" of positive definite matrices:

$$\mathcal{P} = \{ n \times n \text{ positive definite matrices} \}$$

open, convex cone in $\mathbb{R}^{n\times n}_{sym} \cong \mathbb{R}^{\frac{n(n+1)}{2}}$

<u>Lemma 1</u> There exists a sequence $\{\xi^k\}$ of unit vectors and a sequence $\lambda_k \in C^\infty_c(\mathcal{P}; [0,\infty[)$ such that for any $A \in \mathcal{P}$

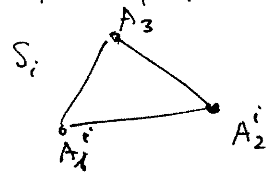
$$A = \sum_k \lambda_k(A)\, \xi^k \otimes \xi^k$$

and there exists a number $N \in \mathbb{N}$ depending only on n such that for any $A \in \mathcal{P}$ at most N of the $\lambda_k(A)$ are nonzero.

<u>Proof</u> Observe : $P \subseteq \mathbb{R}^{n\times n}_{sym}$ is convex.

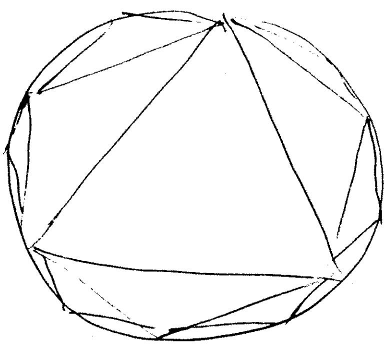
Fix a locally finite simplicial covering for P

i.e. a family $\{S_i\}$ of open simplices



such that any $A \in P$ is contained in at most $N = N(n)$ of the S_i .

e.g. start with a triangulation of P :



covers $P \setminus \{$sides of the simplices$\}$

any other triangulation of P will generically consist of simplices whose sides intersect these sides transversally , $\longrightarrow$ repeat n times to obtain $\{S_i\}$.

In each simplex $S_i = \{A_1^{(i)} \dots A^{(i)}_{\frac{n(n+1)}{2}+1}\}^{co}$

there exist smooth functions $\lambda_j^{(i)} : S_i \longrightarrow \mathbb{R}$ $j = 1 \dots \frac{n(n+1)}{2}$

s.t. $S_i \ni A = \sum_{j=1}^{\frac{n(n+1)}{2}+1} \lambda_j^{(i)}(A)\, A_j^{(i)}$, with $\lambda_j^{(i)} > 0$ in S_i.

Next, choose a partition of unity $\{\psi_i\}$ subordinate $\{S_i\}$

e.g. $$\psi_i = \frac{\exp\left\{-\sum_j \frac{1}{\lambda_j^{(i)}}\right\}}{\sum_k \exp\left\{-\sum_j \frac{1}{\lambda_j^{(k)}}\right\}}$$

$\left(0 \le \psi_i \le 1 \;,\; \psi_i \in C_c^\infty(S_i) \;,\; \sum_i \psi_i \equiv 1 \text{ in } P\right)$

Finally, each $A_j^{(i)} \in \overline{P}$ positive semidefinite,

hence $$A_j^{(i)} = \sum_{k=1}^{n} \alpha_{ijk}^2 \, \xi^{ijk} \otimes \xi^{ijk} \qquad \alpha_{ijk} \geq 0, \quad |\xi^{ijk}| = 1$$

and can define

$$\mu_{ijk} := \alpha_{ijk} \, \psi_i \, \lambda_j^{(i)} \in C_c^\infty(P)$$

so that for any $A \in P$

$$\begin{aligned} A &= \sum_i \psi_i(A) A \\ &= \sum_{i,j} \psi_i(A) \, \lambda_j^{(i)}(A) \, A_j^{(i)} \\ &= \sum_{i,j,k} \psi_i(A) \, \lambda_j^{(i)}(A) \, \alpha_{ijk} \, \xi^{ijk} \otimes \xi^{ijk} \\ &= \sum_{ijk} \mu_{ijk}(A) \, \xi^{ijk} \otimes \xi^{ijk} \end{aligned}$$

Since the sum in $j = 1 \ldots \frac{n(n+1)}{2} + 1$, $k = 1 \ldots n$

and for each fixed A ψ_i nonzero only for at most N_i

~~the sum~~ we are done. ☒

<u>general scheme</u> :

<u>a step</u> : consists of <u>adding a primitive metric</u>

ie. $\nabla u^T \nabla u \longrightarrow \nabla u^T \nabla u + a^2 \xi \otimes \xi$

<u>a Stage</u> : consists of decomposing the error into primitive metrics and adding them successively in steps.

ie. $\nabla u^T \nabla u \rightsquigarrow \nabla u^T \nabla u + h$

where $h \approx g - \nabla u^T \nabla u$.

" The only difficulty in all this is in forming a clear picture "

J. Nash

<u>Proposition</u> (Stage)

Let
- $\Omega \subset \mathbb{R}^n$ open bounded
- $g \in C^\infty(\bar{\Omega})$ metric
- $u : \Omega \hookrightarrow \mathbb{R}^m$ $\quad m \geq n+2$ smooth strictly short upto the boundary

Then $\forall \varepsilon > 0 \quad \exists \tilde{u} : \Omega \hookrightarrow \mathbb{R}^m$ s.t.
- $\tilde{u} \in C^\infty(\bar{\Omega})$, strictly short upto the boundary
- $\| g - \nabla \tilde{u}^T \nabla \tilde{u} \|_{C^0(\bar{\Omega})} < \varepsilon$
- $\| \nabla u - \nabla \tilde{u} \|_{C^0(\bar{\Omega})} < C \cdot \| g - \nabla u^T \nabla u \|_{C^0(\bar{\Omega})}^{1/2} \quad (+\,\varepsilon)$
- $\| u - \tilde{u} \|_{C^0(\bar{\Omega})} < \varepsilon$

Proof

Let $h = g - \nabla u^T \nabla u : \bar{\Omega} \to P$.

Lemma $\Rightarrow$
$$h = \sum_k \lambda_k(h)\, \xi^k \otimes \xi^k = \sum_k a_k^2\, \xi^k \otimes \xi^k$$

[If $f \in C_c^\infty(\bar{\Omega})$ with $f \geq 0$ then $\sqrt{f} \in C^\infty(\bar{\Omega})$?
consider Taylor expansion at zeros of f.]

Observe : 1) Since $h(\bar{\Omega}) \subset\subset P$ compact, the sum is finite

2) Moreover for each $x \in \bar{\Omega}$ at most N of the $a_k(x)$ are non-zero.

Add successively the primitive metrics $a_k^2 \xi^k \otimes \xi^k$ using the spiralling construction.
Actually, add $(1-\delta) a_k^2\, \xi^k \otimes \xi^k$, i.e. $\tilde{a}_k = (1-\delta)^{1/2} a_k$

i.e. $u_1 = u$,
$$u_{k+1} = u_k + \frac{\tilde{a}_k}{\lambda_k}\left(\sin(\lambda_k x \cdot \xi^k)\, \zeta^k + \cos(\lambda_k x \cdot \xi^k)\, \eta^k\right)$$

$$\to \quad |u_{k+1} - u_k| = O\left(\frac{1}{\lambda_k}\right)$$
$$|\nabla u_{k+1} - \nabla u_k| = |\tilde{a}_k| + O\left(\frac{1}{\lambda_k}\right)$$
$$\nabla u_{k+1}^T \nabla u_{k+1} = \nabla u_k^T \nabla u_k + \tilde{a}_k^2\, \xi^k \otimes \xi^k + O\left(\frac{1}{\lambda_k}\right)$$

After finitely many steps we obtain $\tilde{u}$ s.t.

$$\| \tilde{u} - u \|_0 < \varepsilon$$

$$| \nabla \tilde{u}(x) - \nabla u(x) | \leq \sum_k{}' |a_k(x)| + \varepsilon$$

$$\nabla \tilde{u}^T \nabla \tilde{u} = \nabla u^T \nabla u + (1-\delta) \sum a_k^2 \, \xi^k \otimes \xi^k + O(\ldots)$$

Finally, observe that

$$\operatorname{tr} h = \sum_k a_k^2 \geq a_\ell^2 \quad \forall \ell$$

Therefore

$$\| \nabla \tilde{u} - \nabla u \|_0 \leq N \left(\| \operatorname{tr} h \|_0 \right)^{1/2} + \varepsilon$$

$\blacksquare$

Extensions

1) $m = n+1$ (Kuiper 55)

2) M^n general manifold

3) embeddings

The case $m = n+1$ (modify the "steps")

Key issue : replace spirals by corrugations, since there is only one normal vector η.

for a parametrized curve $\gamma : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}^2$, 1-periodic in t,
$(x, t) \longmapsto (\gamma_1, \gamma_2)$

consider

$$\tilde{u}(x) = u(x) + \frac{1}{\lambda}\Big(\gamma_1(x, \lambda x\cdot\xi)\,\zeta(x) + \gamma_2(x, \lambda x\cdot\xi)\,\eta(x)\Big)$$

where η = unit normal to $u(\Omega)$ i.e. $\nabla u^T \eta = 0$

and ζ still to be chosen.

$$\nabla\tilde{u}^T\nabla\tilde{u} = \nabla u^T\nabla u + \dot{\gamma}_1\left(\nabla u^T\zeta\otimes\xi + \xi\otimes\nabla u^T\zeta\right)$$
$$+ \left(\dot{\gamma}_1^2|\zeta|^2 + \dot{\gamma}_2^2\right)\xi\otimes\xi + O\left(\frac{1}{\lambda}\right)$$

choose ζ so that $\nabla u^T\zeta = \xi$:

$$\zeta = \nabla u\left(\nabla u^T\nabla u\right)^{-1}\xi .$$

$$\rightsquigarrow \quad \left(2\dot{\gamma}_1 + |\zeta|^2\dot{\gamma}_1^2 + \dot{\gamma}_2^2\right)\xi\otimes\xi .$$

Slightly more clever choice of vectors,

$$\tilde{\zeta} = \frac{\zeta}{|\zeta|^2} \quad ; \quad \tilde{\eta} = \frac{\eta}{|\zeta|}$$

Then

$$\nabla \tilde{u}^T \nabla \tilde{u} = \nabla u^T \nabla u + \frac{1}{|\xi|^2}\left(2\dot{\gamma}_1 + \dot{\gamma}_1^2 + \dot{\gamma}_2^2 \right) \xi \otimes \xi + O($$

So need to choose γ so that

(i) $(1+\dot{\gamma}_1)^2 + \dot{\gamma}_2^2 = |\xi|^2 a^2 + 1$

(ii) $t \longmapsto \gamma(x,t)$ 1-periodic

For fixed x, can directly solve for $\dot{\gamma}$ and replace (ii) by

(ii)' $t \longmapsto \dot{\gamma}(x,t)$ 1-periodic with average 0.

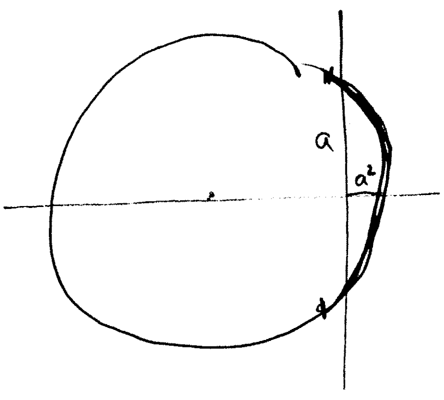
$\longrightarrow$ "CONVEX INTEGRATION" c.f. Gromov

<u>Estimates for</u> γ :

along the iteration a small and $|\xi| \sim 1$

hence $\sqrt{1 + |\xi|^2 a^2} \sim 1 + a^2$

<u>picture</u>



$$|\dot{\gamma}| \leq C \cdot a$$
$$\Downarrow$$
$$|\nabla \tilde{u} - \nabla u| \leq Ca + O\left(\tfrac{1}{\lambda}\right)$$

Immersions of a general (compact) manifold (modify the "stages")

Fix a covering by coordinate charts $M \subseteq \bigcup_p \mathcal{U}_p$

with associated partition of unity $\{\phi_p\}$.

At a <u>stage</u>, decompose the metric error

$$h = g - \nabla u^T \nabla u$$

into primitive metrics in the different charts:

$$h = \sum_{k,p} \phi_p \, a_k^2 \, \xi^k \otimes \xi^k .$$

This time the primitive metrics $\phi_p a_k^2 \xi^k \otimes \xi^k$ need to be added successively over k <u>and</u> p.

Since the decomposition is locally finite, estimates still ok

Embeddings

Need to ensure at each <u>step</u> that no self-intersections are produced <u>locally</u> or <u>globally</u>

easy to control by $\|u - \tilde{u}\|_0$

<u>$m \geq n+2$</u> : easy, since perturbation is <u>normal</u> to manifold, therefore we stay in a normal neighborhood of $u(M)$.

In other words (u, ξ, η) induces a local diffeomorphism $\mathbb{R}^{n+2} \longrightarrow$ normal nhood of M.

<u>$m = n+1$</u>, as above, but a bit more complicated.