Isometric Maps : Equidimensional case

NOT POSSIBLE IN C^1

e.g. $g_{ij} = \delta_{ij}$ (Euclidean metric on $\mathbb{R}^n$)

$$\nabla u^T \nabla u = Id \iff \nabla u \in O(n) = \underset{\det = +1}{SO(n)} \cup \underset{\det = -1}{SO(n)_{-}}$$

2 connected components, so if $u \in C^1$,

then $\nabla u \in SO(n) \ \forall x$ or $\nabla u \in SO(n) J \ \forall x$

$\Rightarrow$ u is affine (Liouville theorem)

TWO ALTERNATIVE PROBLEM DESCRIPTIONS :

for Lipschitz maps

PDE version

$u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz

$\nabla u(x) \in O(n)$ a.e. x

or more generally

$\nabla u^T \nabla u = g$ a.e. x

Geometric version

length of all rectifiable curves is preserved

$$\int_0^\ell \underbrace{\sqrt{g(\dot\Gamma(t), \dot\Gamma(t))}}_{= |\dot\Gamma|_g} \, dt = \int_0^\ell |\nabla u(\Gamma(t)) \dot\Gamma(t)| \, dt$$

$\forall \Gamma$

The "geometric" method

The step

$$\tilde{u}(x) = u(x) + \frac{1}{\lambda}\gamma(x, \lambda x\cdot\xi)\,\tilde{\zeta}(x)$$

as before, ie. $\gamma : \Omega\times\mathbb{R} \to \mathbb{R}$

$$\zeta = \nabla u\,(\nabla u^T\nabla u)^{-1}\xi \quad , \quad \tilde{\zeta} = \frac{\zeta}{|\zeta|^2}$$

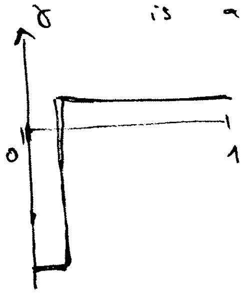
$$\nabla\tilde{u}^T\nabla\tilde{u} = \nabla u^T\nabla u + \frac{1}{|\zeta|^2}(2\dot{\gamma} + \dot{\gamma}^2)\,\xi\otimes\xi + O\left(\frac{1}{\lambda}\right)$$

So $\dot{\gamma}$ needs to satisfy

(a) $(1+\dot{\gamma})^2 = 1 + |\zeta|^2 a^2$

(b) 1-periodic with average zero.

$\dot{\gamma}$ is a step-function.



OR

replace (a) by (a1) $(1+\dot{\gamma})^2 \le 1 + |\zeta|^2 a^2$

(a2) $\int_0^1 a^2 - \frac{1}{|\zeta|^2}(2\dot{\gamma}+\dot{\gamma}^2)\,dt \le \varepsilon$

$\longrightarrow$ $\gamma\in C^\infty$ still ok.

The stage

After finite number of steps + Fubini:

(i) $$\int_\Omega (g - \nabla\tilde{u}^T\nabla\tilde{u})\,dx \leq \varepsilon$$

(ii) $$\nabla\tilde{u}^T\nabla\tilde{u} < g \qquad \forall x$$

(iii) $$\|u - \tilde{u}\|_0 \leq \varepsilon$$

(iv) $$\|\nabla\tilde{u} - \nabla u\|_{L^2} \leq C\,\|g - \nabla u^T\nabla u\|_{L^1}^{1/2}$$

Actually, (iv) ALWAYS follows from (i), (ii), (iii):

$$g - \nabla\tilde{u}^T\nabla\tilde{u} = g - \nabla u^T\nabla u - \left[\nabla u^T[\nabla\tilde{u} - \nabla u] + [\nabla\tilde{u} - \nabla u]^T\nabla u\right]$$
$$- (\nabla\tilde{u} - \nabla u)^T(\nabla\tilde{u} - \nabla u)$$

$$\operatorname{tr}(g - \nabla\tilde{u}^T\nabla\tilde{u}) = \operatorname{tr}(g - \nabla u^T\nabla u) - 2\langle\nabla u, \nabla\tilde{u} - \nabla u\rangle - |\nabla\tilde{u} - \nabla u|^2$$

$$\int_\Omega |\nabla\tilde{u} - \nabla u|^2\,dx \leq \int_\Omega \operatorname{tr}(g - \nabla u^T\nabla u) + \int_\Omega \operatorname{tr}(g - \nabla\tilde{u}^T\nabla\tilde{u})$$
$$+ C\,\|\tilde{u} - u\|_0$$

integration by parts & C^2 bound on u.

"Controlled L^∞ convergence implies strong convergence of the gradient"

cf. Müller - Šverák

MOREOVER, for any curve Γ, writing $v = \dot{\Gamma}$,

$$\int_\Gamma |\nabla \tilde{u}\, v|^2 = \int_\Gamma |\nabla u\, v|^2 + \frac{1}{|\xi|^2}\,(2\gamma + \gamma^2)\,(\xi \cdot v)^2$$

$$+ \, O\left(\frac{1}{\lambda}\right)$$

depending on the length of the curve

$$= \int_\Gamma |\nabla u\, v|^2 + a^2\,(\xi \cdot v)^2 \, ds + O\left(\frac{1}{\lambda}\right)$$

by (a2).

Now, since

$$g - \nabla u^T \nabla u = \sum_k a_k^2\, \xi^k \otimes \xi^k ,$$

in particular

$$|v|_g^2 - |\nabla u\, v|^2 = \sum_k a_k^2\, (\xi^k \cdot v)^2 ,$$

hence, after <u>a stage</u> we obtain

$$\text{(V)} \qquad \int_\Gamma |\nabla \tilde{u}\, v|^2 = \int_\Gamma |v|_g^2 + \varepsilon + O\left(\frac{1}{\lambda}\right)$$

So, iterating over the stages we obtain u_k s.t.

$$\int_\Gamma |\nabla u_k v|^2 \longrightarrow \int_\Gamma |v|_g^2 . \quad \underline{\text{and}} \quad |\nabla u_k v| < |v|_g$$

$\blacksquare$

We have proved

Theorem A2
- $\Omega \subset \mathbb{R}^n$ bounded open
- $g \in C^\infty(\bar{\Omega})$ smooth metric
- $u_0 : \Omega \to \mathbb{R}^n$ smooth strictly short

Then $\forall \varepsilon > 0$ $\exists \tilde{u} : \Omega \to \mathbb{R}^n$ Lipschitz s.t.

- $\nabla \tilde{u}^T \nabla \tilde{u} = g$ a.e. in Ω
- $\text{length}(\Gamma) = \text{length}(u(\Gamma))$ $\forall \Gamma \subset \Omega$
- $\|\tilde{u} - u\|_0 \leq \varepsilon$

REMARK: We could also obtain the estimates (i) – (iv) by taking instead a perturbation of the form

$$\underset{\substack{\uparrow \\ \text{cutoff}}}{\frac{1}{\lambda}\phi(x)}\gamma(\lambda x \cdot \xi)\zeta$$

and covering the domain Ω. (Just need to get (i), (ii), (iii))

This is actually much more flexible, because we can now adjust ξ, ζ, γ depending on the point (nhood)

General Setting (Baire Category Method)

c.f. Kirchheim, Sychev

X_0 : metric space

e.g. all smooth strictly short maps with uniform topology

$I : X_0 \to \mathbb{R}$: functional measuring how far we are from a solution,

e.g.
$$I(u) = \int_\Omega \operatorname{tr} g - |\nabla u|^2 \, dx$$

$$0 < I(u) \le \int_\Omega \operatorname{tr} g \, dx \qquad \forall u \in X_0$$

Observe : I is not continuous, but upper-semicontinuous and X_0 bounded in $W^{1,2}$ (even $W^{1,\infty}$)

Finally, let $X = \overline{X_0}$ (complete metric space)

Theorem FA 1 : If $I : X \to \mathbb{R}$ is upper-semicontinuous and takes values in a bounded interval, then it is the pointwise ~~supre~~ infimum of countably many continuous f'tals.

FA 2 : If $I : X \to \mathbb{R}$ is Baire-1 (pointwise limit of countably many continuous functionals) then the set of continuity points S is residual, i.e. the complement of a countable union of nowhere dense sets

FA 3 : In X ~~the~~ residual sets are dense.

i.e. $S = \{$ set of continuity points of $I\}$ is dense in

Lemma 2 If $\forall u \in X_0 \quad \exists u_k \in X_0$ s.t.

$$\begin{cases} u_k \to u \\ I(u_k) \to 0 \end{cases}$$

then $\{I = 0\}$ is residual (hence dense).

Proof: Let $u \in S$. By density of X_0, $\exists u_k \in X_0$ with $u_k \to u$. By assumption, $\exists u_{k,j} \in X_0$ with $u_{k,j} \xrightarrow[j\to\infty]{} u_k$; $I(u_{k,j}) \to 0$.

Then for a diagonal subsequence

$$u_{k,j(k)} \to u$$

$$I(u_{k,j(k)}) \to 0 = I(u) \quad \text{by continuity at } u.$$

Hence $S \subseteq \{I = 0\}$. $\blacksquare$

e.g. choosing $X_0 = \{u \in C^\infty(\bar\Omega) : \text{strictly short} ; \ u = u_0 \text{ on } \partial\Omega\}$ with $\|\cdot\|_{C^0}$

we obtain:

Theorem A3
- $\Omega \subset \mathbb{R}^n$ open bounded
- $g \in C^\infty(\bar\Omega)$ smooth metric
- $u_0 : \Omega \to \mathbb{R}^n$ smooth strictly short map,

then $\forall \varepsilon > 0 \quad \exists \tilde u : \Omega \to \mathbb{R}^n$ Lipschitz such that

$$\nabla \tilde u^T \nabla \tilde u = g \quad \text{a.e. in } \Omega$$

$$\tilde u = u_0 \quad \text{on } \partial\Omega$$

$$\|\tilde u - u_0\| \le \varepsilon$$

<u>Exercise</u>: prove directly that in this example, i.e. with $X_0 = \{ u$ smooth, strictly short, prescribed on $\partial\Omega \}$ with $\|\cdot\|_{C^0}$

$$I(u) = \int_\Omega \operatorname{tr} g - |\nabla u|^2 \, dx$$

$I : X_0 \to \mathbb{R}$ is Baire - 1.

<u>Hint</u>: use difference-quotients <u>or</u> mollification.

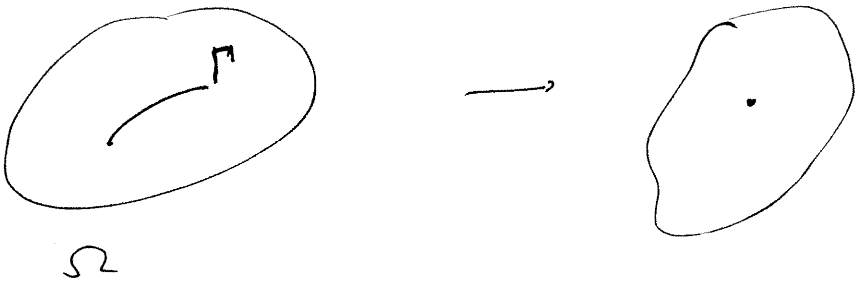
<u>Example</u> Take $\Omega \subset \mathbb{R}^n, g,$ and $\Gamma \subset\subset \Omega$ with $|\Gamma| = 0$ and let $u_0 : \Omega \to \mathbb{R}^m$ be a smooth strictly short map such that $u_0(\Gamma)$ = single point.

Then, with $X_0 = \{ u :$ smooth, strictly short; $u = u_0$ on Γ

we obtain Lipschitz maps

$$u : \Omega \to \mathbb{R}^m$$

with
$$\begin{cases} \nabla u^T \nabla u = g & \text{a.e. } \Omega \\ u(\Gamma) = \text{single point} \end{cases}.$$



<u>In contrast</u>: Nash construction preserves the length of curves

$$\tilde{u}(x) = u(x) + \frac{1}{\lambda} \gamma(x, \lambda x \cdot \xi) \zeta(x) \qquad \zeta = \nabla u (\nabla u^T \nabla u)^{-1} \xi$$

$\dot{\gamma}$ big in C^0, small in L^1 $\qquad \int (\nabla \tilde{u} - \nabla u) \nu \, ds = \int^{\ell} \dot{\gamma}(\Gamma, \lambda \Gamma \cdot \xi) \zeta(\Gamma) \xi \cdot \dot{\Gamma} \, dt$

Differential Inclusions

Problem: Given $\Omega \subset \mathbb{R}^n$; $K \subset \mathbb{R}^{m\times n}$ compact, ~~u_0~~, find

$$\boxed{u : \Omega \to \mathbb{R}^m \quad \text{Lipschitz s.t.} \quad \nabla u(x) \in K \quad \text{a.e.}}$$

e.g. coupled with Dirichlet boundary condition $u = u_0 \big|_{\partial\Omega}$

Strategy Find an open set $\mathcal{U} \subset \mathbb{R}^{m\times n}$ s.t.

$$(\mathcal{A}) \left[\begin{array}{l} \forall A \in \mathcal{U} \quad \exists\, u_k \in C_c^\infty(Q) : \qquad Q \subset \mathbb{R}^n \text{ unit cube} \\ \quad ① \quad A + \nabla u_k(x) \in \mathcal{U} \qquad \forall x, \forall k \\ \quad ② \quad \displaystyle\int_Q \operatorname{dist}(A + \nabla u_k, K)\, dx \longrightarrow 0 \qquad k \to \infty \end{array} \right.$$

Then, define X_0, I as

$$X_0 = \left\{ u \in C^\infty(\overline{\Omega}) : \nabla u(x) \in \mathcal{U} \text{ and } u\big|_{\partial\Omega} = u_0 \right\}$$

with uniform topology.

$$I(u) = \int_\Omega \operatorname{dist}(\nabla u(x), K)\, dx .$$

Then
- $\mathcal{U}$ bounded (in fact $\mathcal{U} \subseteq K^{co}$), hence X_0 bounded in $W^{1,\infty}$ (exercise)
- I is Baire-1 (exercise: consider $I_\varepsilon(u) = I(u * \rho_\varepsilon)$)
- conditions of lemma 2 fulfilled (exercise: covering)

In the literature $(\mathcal{A})$ is called

$\boxed{\text{" } \mathcal{U} \text{ has the relaxation property wrt } K \text{ "}}$

Dacorogna – Marcellini

$\boxed{\text{" } \mathcal{U} \text{ can be reduced to } K \text{ "}}$

Müller – Sychev

In the example $\nabla u^T \nabla u = I$ ~~(...)~~

$$K = O(n,m)$$

$$\mathcal{U} = \operatorname{int} K^{co}.$$

<u>Boundary condition</u>: Need to verify that $X_0 \neq \emptyset$, so need to have smooth extension $u_0 : \Omega \to \mathbb{R}^m$ s.t. $\nabla u_0(x) \in \mathcal{U} \quad \forall x$.

<u>Remark</u>: alternatively to $C^\infty(\bar{\Omega})$, could work with <u>piecewise affine Lipschitz functions</u>.

<u>WARNING</u>: In general $\mathcal{U} \neq \operatorname{int} K^{co}$. $\mathcal{U}$ needs to be chosen to satisfy $(\mathcal{A})$. e.g. if $K = SO(n)$, the only possibility is $\mathcal{U} \subseteq K$, not open.

(Exercise: WHY?)

Recall that for the problem $\nabla u(x) \in O(n,m) = K$ we would prove $(\mathcal{A})$ for $\mathcal{U} = \operatorname{int} K^{\infty}$ by "adding" successively n primitive metrics. In fact, the knowledge of being able to add 1 suffices. This would correspond to the following property:

$$
(\mathcal{P})\quad
\begin{cases}
\forall \varepsilon > 0 \;\; \exists \delta_\varepsilon > 0 \;\; \text{s.t.} \\
\forall A \in \mathcal{U} \;\; \text{with} \;\; \operatorname{dist}(A, K) > \varepsilon \quad \exists u \in C_c^\infty(Q): \\
\quad ① \quad A + \nabla u(x) \in \mathcal{U} \quad \forall x \\
\quad ② \quad \int_Q |\nabla u|^2 \, dx > \delta_\varepsilon .
\end{cases}
$$

"gradients in $\mathcal{U}$ are stable only near K"

Kirchheim

The corresponding statement in the general setting is

<u>Lemma</u> 3 if $\forall u \in X_0$ with $I(u) \geq \alpha > 0$ $\exists u_k \in X_0$ s.t.

$$u_k \longrightarrow u$$

$$I(u_k) \leq I(u) - \beta$$

where $0 < \beta = \beta(\alpha)$, then $\{I = 0\}$ residual.

<u>Proof</u>: exactly as Lemma 2.

More general systems

$$z : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^d \qquad \text{state variable}$$

subject to

$$(1) \qquad \sum_{i=1}^{n} A_i \, \partial_i z = 0 \quad \text{in } \mathcal{D}' \qquad A_i \in \mathbb{R}^{N \times d}$$

N conservation laws

$$(2) \qquad z(x) \in K \quad \text{a.e. } x \in \Omega$$

constitutive set.

1D plane-wave solutions of (1) $\quad z(x) = \hat{z}\, h(x \cdot \xi)$

$$\Big(\sum_i \xi_i A_i\Big) \hat{z} = 0$$

wave cone $\quad \Lambda = \left\{ \hat{z} \in \mathbb{R}^d : \exists \xi \neq 0 : \Big(\sum_i \xi_i A_i\Big)\hat{z} = 0 \right\}$

Differential Inclusions:

(1) : $\operatorname{curl} z = 0$

(2) : $z \in K$

$z = \nabla u$

$\Lambda =$ rank-one matrices

Try the above method with $U = \operatorname{int} K^{\Lambda}$