# The Euler Equations

$$(1)\quad \partial_t v + \mathrm{div}\,(\overbrace{v\otimes v - \tfrac{|v|^2}{n} I}^{A}) + \nabla(p + \overbrace{\tfrac{|v|^2}{n}}^{q}) = 0$$

$$\mathrm{div}\, v = 0$$

$$\left[\ \partial_t \overbrace{\tfrac{|v|^2}{2}}^{e} + \mathrm{div}\left(v\left(\tfrac{|v|^2}{2}+p\right)\right) = 0\ \right]$$

Constitutive set:

$$\left.\begin{aligned} v\otimes v - A &= \tfrac{|v|^2}{n} I \\ \tfrac{|v|^2}{2} &= e \end{aligned}\right\} \iff \boxed{v\otimes v - A = \tfrac{2e}{n} I}$$

Notation $\quad \tau = v\otimes v - A \quad$ "Reynolds stress"

Observation 1:

$$(1) \iff \mathrm{div}\, U = 0 \qquad U = \left(\begin{array}{c|c} A+qI & v \\ \hline v & 0 \end{array}\right)$$

If $U(x) = \bar U h(x\cdot\xi)$, then $\mathrm{div}\, U = \bar U \xi\, h'(x\cdot\xi)$

so $\bar u \in \Lambda \iff \det \bar U = 0$.

But actually $q$ doesn't enter in constraint,

and for any $v, A$ $\exists q$ s.t. $\det\left(\begin{array}{c|c} A+qI & v \\ \hline v & 0 \end{array}\right) = 0$.

Therefore $\boxed{K^\Lambda = K^{co}}$ $\qquad K = \{(v,A) : v\otimes v - A = \tfrac{2e}{n} I\}$

**Fact**

$$K^{co} = \left\{ (v, A) \; : \; v \otimes v - A \leq \frac{2e}{n} I \right\}$$

Proof : • easy part : $\left\{ v \otimes v - A \leq \frac{2e}{n} I \right\} \subseteq K^{co}$

• converse : Exercise

Hint prove that

$$\text{extr} \left\{ v \otimes v - A \leq \frac{2e}{n} I \right\}^{co} \subseteq K$$

by contradiction.

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Therefore, inside $K^{co}$ the function

$$(v, A) \longmapsto e - \frac{|v|^2}{2}$$

measures how far $(v, A)$ is away from $K$.

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Proposition [The perturbation property holds for $\mathcal{U} = \text{int}\, K^{co}$

Let $z_0 = (v_0, A_0, q_0) \in \mathcal{U}$ (i.e. $v_0 \otimes v_0 - A_0 < \frac{2e}{n} I$). Then there exists

$(v, A, q) \in C_c^\infty(Q)$ s.t. $Q$ = unit cylinder in $(x,t)$

① $(v_0 + v) \otimes (v_0 + v) - (A_0 + A) < \frac{2e}{n} I \quad \forall (x,t)$

② $\partial_t v + \text{div}\, A + \nabla q = 0$ & $\text{div}\, v = 0$ in $\mathbb{R}^n \times \mathbb{R}$

③ $\displaystyle\iint |v|^2 \, dx\, dt \geq c \cdot \left( e - \tfrac{1}{2} |v_0|^2 \right)^2$

## Step 1

Observation: If $z_1 = (v_1, A_1, q)$, $z_2 = (v_2, A_2, q) \in K$

then $z_1 - z_2 \in \Lambda$

i.e. the matrix

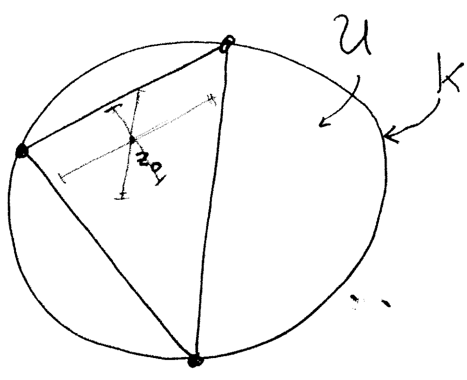
$$\begin{pmatrix} \overbrace{v_1 \otimes v_1 - \frac{|v_1|^2}{n} I + q I}^{A_1} & v_1 \\ v_1 & 0 \end{pmatrix} - \begin{pmatrix} \overbrace{v_2 \otimes v_2 - \frac{|v_2|^2}{n} I + q I}^{A_2} & v_2 \\ v_2 & 0 \end{pmatrix}$$

has determinant zero. Notice: $|v_1|^2 = |v_2|^2 = 2e$.

$$= \begin{pmatrix} v_1 \otimes v_1 - v_2 \otimes v_2 & v_1 - v_2 \\ v_1 - v_2 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ c \end{pmatrix} \qquad \begin{matrix} \xi \in \mathbb{R}^n \\ c \in \mathbb{R} \end{matrix}$$

$$= \begin{pmatrix} v_1 (v_1 \cdot \xi) - v_2 (v_2 \cdot \xi) + c (v_1 - v_2) \\ v_1 \cdot \xi - v_2 \cdot \xi \end{pmatrix}$$

choose $\xi \perp (v_1 - v_2)$ & $c = -(v_1 \cdot \xi)$ $\blacksquare$




Using Carathéodory,

$$z_0 = \sum_{i=1}^{d+1} \lambda_i z_i \qquad z_i \in K$$

work at constant pressure $q$.

One of the line segments centered at $z_0$ in direction $z_i - z_j$, contained in the simplex, has to have length comparable to the distance of $z_0$ to vertices.

More precisely, assume $\lambda_1 \geq \lambda_2 \geq \dots$

Consider the line segments

$$\left[ z_0 - \frac{\lambda_j}{2}(z_j - z_1) \quad , \quad z_0 + \frac{\lambda_j}{2}(z_j - z_1) \right]$$

for $j > 1$. These are contained in the simplex, because still have a convex representation.

Length $= \lambda_j |z_j - z_1|$

Choose $j_0 > 1$ with maximum length.

Since $$z - z_1 = \sum_{j=2}^{d+1} \lambda_j (z_j - z_1)$$

we have $$|z - z_1| \leq d \max_j \lambda_j |z_j - z_1|$$

so that

$$\lambda_{j_0} |z_{j_0} - z_1| \geq \frac{1}{d} \operatorname{dist}(z, \text{vertices}) .$$

$\boxtimes$

<u>Step 2</u> Construct a compactly supported solution of

$$\partial_t v + \operatorname{div} A + \nabla q = 0$$
$$\operatorname{div} v = 0$$

which "behaves" like a plane-wave in direction $z_2 - z_1$ with $z_1, z_2 \in K$.

i.e. $$\begin{pmatrix} v_1 \otimes v_1 - v_2 \otimes v_2 & v_1 - v_2 \\ v_1 - v_2 & 0 \end{pmatrix} .$$

## Potentials

e.g. 1 $\qquad \operatorname{div} v = 0 \quad$ in $\mathbb{R}^n$

find an operator $\mathbb{P}(\nabla)$ s.t. $v = \mathbb{P}(\nabla)\phi$ div free for any $\phi \in C_c^\infty$ scalar function.

$$\hat{v}(\xi) = \underbrace{\mathbb{P}(\xi)}_{\text{polynomial}} \hat{\phi}(\xi)$$

$$\operatorname{div} \hat{v}(\xi) = \hat{\phi}(\xi)\underbrace{\mathbb{P}(\xi)\cdot\xi}_{\text{vector product}} \qquad \text{i.e.} \quad \mathbb{P}(\xi) \perp \xi$$

e.g. $\quad \mathbb{P}(\xi) = R\xi \quad$ for antisymmetric $R$

e.g. $\quad R = a \otimes b - b \otimes a$.

e.g. 2 $\qquad \operatorname{div} U = 0 \quad$ ($n \times n$ matrix) ; $\quad U^T = U \quad$ ; $\quad U_{n,n} = 0$

operator $\mathbb{P}(\nabla)$ with symbol $\mathbb{P}(\xi)$ polynomial

s.t. $\quad \mathbb{P}(\xi)\xi = 0 \quad ; \quad \mathbb{P}^T = \mathbb{P} \quad ; \quad \mathbb{P}_{n,n} = 0$

$$\mathbb{P}(\xi) = R\xi \otimes Q\xi + Q\xi \otimes R\xi$$

for antisymmetric $R, Q$.

$$\mathbb{P}_{n,n} = e_n \cdot \mathbb{P} e_n = 2(R\xi \cdot e_n)(Q\xi \cdot e_n)$$

Pick range of $R \perp e_n \quad$ e.g. $R = a \otimes b - b \otimes a \qquad a, b \perp e_n$

e.g. 3 $\quad \operatorname{div} U = 0 \quad , \quad U^T = U \quad , \quad U_{n,n} = 0 \quad , \quad \operatorname{tr} U = 0$

$$p(\xi) = R\xi \otimes Q\xi + Q\xi \otimes R\xi$$

$$\operatorname{tr} p = 2\, R\xi \cdot Q\xi$$

$$R = a \otimes b - b \otimes a \qquad \text{with} \quad a, b \perp e_n$$

$$Q = Q(\xi) = \xi \otimes e_n - e_n \otimes \xi$$

then $\quad a, b \perp \operatorname{rg} Q \quad$, so $\quad \operatorname{tr} p = 0$.

$\longrightarrow$ 3rd order polynomial $p$

i.e. 3rd order operator

For Euler do the same in $\mathbb{R}^{n+1}$ (time)

<u>Fact</u> Given $a, b \in \mathbb{R}^n$ with $|a| = |b|$, $a \neq \pm b$

$\exists\ \eta \in \mathbb{R}^{n+1} \setminus \{0\}$ s.t.

$$p(\eta) = \begin{pmatrix} a \otimes a - b \otimes b & a - b \\ a - b & 0 \end{pmatrix}.$$

Then, if $\quad \phi(y) = \psi(y \cdot \eta) \qquad y = (x, t)$

then

$$p(\nabla)\,\phi = p(\eta)\,\psi'''$$

so plane-waves recovered by the potential $p$.

Moreover, $\quad \eta \not\parallel e_{n+1} \quad$ (ie. oscillation not only in time)

Finally, for the proof of the Proposition,

take $(v, A, q) \sim \frac{1}{\lambda^3}\, \rho\left(\nabla_{(x,t)}\right)\left\{ \phi(x,t)\, \sin\left(\lambda \eta \cdot (x,t)\right)\right\}$

$$= \phi(x,t)\, \rho(\eta)\, \cos\left(\lambda \eta \cdot (x,t)\right) + O\left(\frac{1}{\lambda}\right)$$

In particular, upto the cutoff and $O(\frac{1}{\lambda})$,

$v(x,t) = \bar{v} \cos\left(\lambda \eta \cdot (x,t)\right)$

where $|\bar{v}| \sim \left(e - \frac{1}{2}|v_0|^2\right)$

Next, fix

$$X_0 = \Big\{ (v, A, q) \in C_c^\infty(\Omega \times ]0,T[) : \begin{array}{l} \partial_t v + \operatorname{div} A + \nabla q = 0 \\ \operatorname{div} v = 0 \end{array} \text{ in } \mathbb{R}^n \times \mathbb{R}$$

$$\text{and} \quad v \otimes v - A < \frac{2}{n} I \quad \forall (x,t) \in \Omega \times ]0,T[ \Big\}$$

$e \equiv 1$.

with $L^\infty$ w* : induces a metric on bounded sets

$$I(v, A, q) = \iint_{\Omega \times ]0,T[} 1 - \frac{|v|^2}{2}\, dx\, dt$$

Lemma 3 $\Longrightarrow$ Proof of Theorem B,

with $v \in L^\infty(\mathbb{R}^n \times \mathbb{R})$, in fact $\frac{|v|^2}{2} = 1_{\Omega \times ]0,T[}$ a.e.

<u>Lemma</u>: Let $v \in L^\infty([0,T]; L^2(\mathbb{R}^n))$, $A, q \in L^1_{loc}([0,T]\times\mathbb{R}^n)$ be a solution of

$$\partial_t v + \operatorname{div}(A + q I) = 0 .$$

Then, after redefining $v$ on a set of $t$'s of measure zero

$$v \in C(]0,T[ ; L^2_w(\mathbb{R}^n)),$$

i.e. $t \longmapsto \int_{\mathbb{R}^n} v(x,t)\,\varphi(x)\,dx$ continuous for any $\varphi \in L^2$.

<u>Proof</u>: For any $\phi \in L^2(\mathbb{R}^n)$, let

$$\Phi(t) = \int_{\mathbb{R}^n} v(x,t)\,\phi(x)\,dx$$

Equation $\Longrightarrow$ $\Phi \in W^{1,1}$

$\Longrightarrow$ $\Phi$ continuous after redefining on a set of measure zero

RRT $\Longrightarrow$ can accordingly redefine $v(x,t)$.

Do this for a countable set $\{\phi_i\}$ which is dense in $L^2(\mathbb{R}^n)$

so that the $\Phi_i(t)$ cts $\forall i$. Then for any $\varphi \in L^2$,

can approximate from $\{\phi_i\}$. ☒

<u>Admissible weak solutions</u>: $(v, p)$

- $v \in C([0,T]; L^2_w(\mathbb{R}^n))$ ; $p \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$
- equations satisfied in $\mathcal{D}'$
- $E(t) = \int_{\mathbb{R}^n} \frac{|v|^2}{2}\,dx$ non-increasing
- $v(0) = v_0$

**FACT** (P.L.Lions)

Weak - strong uniqueness

If $\exists\, v \in \underbrace{C^1([0,T]\times\mathbb{R}^n)}$ classical solution to IVP,

then $v$ is the unique admissible weak solution.

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It is sufficient that $\nabla v + \nabla v^T \in L^1([0,T]; L^\infty(\mathbb{R}^n))$

**Theorem B1** (C. De Lellis - Sz)

Let $\Omega \subset \mathbb{R}^n$ open, $\bar{e} \in C(\bar{\Omega}\times]0,T[) \cap C([0,T];$

and let $(v_0, A_0, q_0) \in C^\infty(\mathbb{R}^n \times ]0,T[)$ s.t.

(i) $\begin{cases} \partial_t v_0 + \operatorname{div}(A_0 + q_0 I) = 0 & \text{in } \mathbb{R}^n \times ]0,T[ \\ \operatorname{div} v_0 = 0 \end{cases}$

(ii) $v_0 \in C([0,T]; L^2_w(\mathbb{R}^n))$

(iii) $\operatorname{supp}(v_0(t), A_0(t)) \subset\subset \Omega \quad \forall t \in ]0,T[$

(iv) $v_0 \otimes v_0 - A_0 < \dfrac{2\bar{e}}{n} I \quad \forall (x,t) \in \Omega \times ]0,T[$

Then $\exists (v,p)$ weak solutions of Euler in $\mathbb{R}^n \times ]0,T[$

s.t.

$$v \in C([0,T]; L^2_w(\mathbb{R}^n))$$

$$\frac{1}{2}|v(x,t)|^2 = \bar{e}\, \mathbb{1}_\Omega \quad \forall t \in ]0,T[, \text{ a.e. } x \in \mathbb{R}$$

$$v(0) = v_0(0) \qquad v(T) = v_0(T)$$

$$p = q_0 - \frac{1}{2}|v|^2 = q_0 - \frac{2\bar{e}}{}$$

The definition of $X_0$ and $I$ :

$$X_0 = \Big\{ v \in C^\infty(\mathbb{R}^n \times ]0,T[) \cap C([0,T]; L^2_w(\mathbb{R}$$

s.t. • $\exists\, A_{\varepsilon} \in C^\infty(\mathbb{R}^n \times ]0,T[)$ so that

(i), (ii), (iii), (iv) holds,

• $v(0) = v_0(0)$ , $v(T) = v_0(T)$ $\Big\}$

with the topology of $C([0,T]; L^2_w(\mathbb{R}^n))$.

Observe : Since $\bar{e} \in C([0,T]; L^1(\Omega))$ and (iv) holds,

actually $X_0 \subset C([0,T]; Y)$

where $Y$ is a bounded subset of $L^2$ with weak topology

WLOG $Y$ compact metric space

$\Rightarrow$ $X_0$ metrizable ; $X = \overline{X_0}$.

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$$I(v) = \sup_t \int_\Omega \bar{e}(x,t) - \frac{1}{2}|v(x,t)|^2 \, dx$$

$$I_{\varepsilon,\Omega_0}(v) = \sup_{\varepsilon \le t \le T-\varepsilon} \int_{\Omega_0} \bar{e}(x,t) - \frac{1}{2}|v(x,t)|^2 \, dx$$

The aim is to use lemma 3 again, i.e. to prove

$\forall v \in X_0$ with $\underline{I}(v) \geq \alpha > 0$

$\exists v_k \in X_0$ with $\begin{cases} v_k \longrightarrow v \\ \underline{I}(v_k) \leq \underline{I}(v) - \beta \end{cases}$

Proposition

$z_0 = (v_0, A_0, q_0)$ with $v_0 \otimes v_0 - A_0 < \frac{2e}{n} I$.

Then $\exists (v_k, A_k) \in C_c^\infty(Q)$ s.t. $Q = ]0,1[ \times \Omega$

① $(v_0 + v_k) \otimes (v_0 + v_k) - (A_0 + A_k) < \frac{2e}{n} I \quad \forall (x,t)$

② $\partial_t v_k + \operatorname{div}(A_k + q_0 I) = 0 \quad ; \quad \operatorname{div} v_k = 0$

③ $\inf\limits_{\frac{1}{4} < t < \frac{3}{4}} \int_\Omega |v_k|^2 \, dx \geq c \cdot \left(e - \frac{1}{2}|v_0|^2\right)^2$

④ $v_k \longrightarrow 0$ in $C L^2_w$

i.e. $v_k(\cdot, t) \rightharpoonup 0$ in $L^2$ uniformly in $t$.

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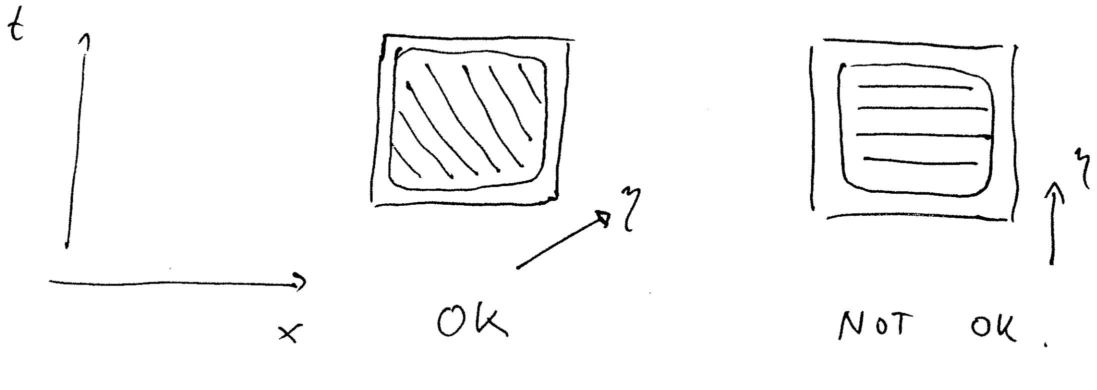
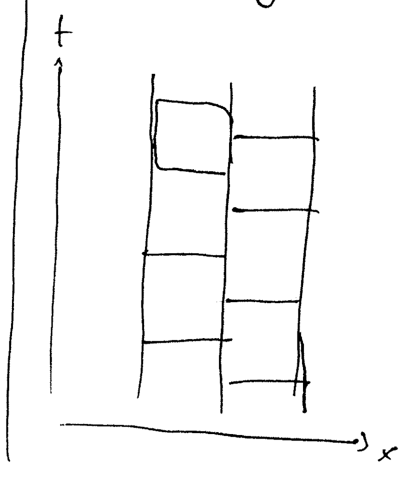
Main difference to previous:




covering $[0,T] \times \Omega$



# Beyond $L^\infty$ (or $CL^2_w$) ?

Recall energy conservation:

$$\partial_t v + \operatorname{div} v \otimes v + \nabla p = 0$$

$$\partial_t v_\varepsilon + \operatorname{div} v_\varepsilon \otimes v_\varepsilon + \nabla p_\varepsilon = \operatorname{div} (v_\varepsilon \otimes v_\varepsilon - (v \otimes v)_\varepsilon$$

$$\partial_t \frac{|v_\varepsilon|^2}{2} + \operatorname{div} \left( v_\varepsilon \left( \frac{|v_\varepsilon|^2}{2} + p_\varepsilon \right) \right) = \underbrace{v_\varepsilon \cdot \operatorname{div} (v_\varepsilon \otimes v_\varepsilon - (v \otimes v)_\varepsilon}_{\xrightarrow[\varepsilon \to 0]{} 0 \ ?}$$

<u>Estimates in Hölder spaces</u> (forget time)

Assume $v \in C^\alpha$

Mollify with $\varphi \in C_c^\infty(\mathbb{R}^n)$ symmetric, $\int \varphi = 1$.

Then

(1) $\quad \| v_\varepsilon - v \|_{C^0} \lesssim \varepsilon^\alpha \| v \|_{C^\alpha}$

~~and more generally~~

$$\cancel{\| v_\varepsilon - v \|_{C^k} \lesssim \varepsilon^{\alpha - k} \| v \|_{C^\alpha}}$$

$$v * \varphi_\varepsilon(x) - v(x) = \int (v(x-y) - v(x)) \, \varphi_\varepsilon(y) \, dy$$

(2) $\quad \| \nabla v_\varepsilon \|_0 \lesssim \varepsilon^{\alpha - 1} \| v \|_\alpha$

$$v * \nabla \varphi_\varepsilon = \int (v(x-y) - v(x)) \, \nabla \varphi_\varepsilon(y) \, dy$$

(3) $\| (v\cdot w)_\varepsilon - v_\varepsilon \cdot w_\varepsilon \|_0 \le \varepsilon^{2\alpha} \|v\|_\alpha \|w\|_\alpha$

$$(v\,w)_\varepsilon - v_\varepsilon w_\varepsilon = \int v(x-y)\, w(x-y)\, \varphi_\varepsilon(y)\,dy - v_\varepsilon w_\varepsilon$$

$$= \int \big(v(x-y) - v(x)\big)\big(w(x-y) - w(x)\big)\, \varphi_\varepsilon(y)\,dy$$

$$+ v\, w_\varepsilon + w\, v_\varepsilon - v w - v_\varepsilon w_\varepsilon$$

$$= \int \big(v(x-y) - v(x)\big)\big(w(x-y) - w(x)\big)\, \varphi_\varepsilon(y)\,dy$$

$$- (v - v_\varepsilon)(w - w_\varepsilon)$$

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So $$\partial_t \int |v_\varepsilon|^2\,dx = - \int \underset{\varepsilon^{\alpha-1}}{\nabla v_\varepsilon} \cdot \underbrace{(v_\varepsilon \otimes v_\varepsilon - (v\otimes v)_\varepsilon)}_{\varepsilon^{2\alpha}}\,dx$$

$\leadsto$ converges to zero if $\alpha > 1/3$.

$\boxed{\text{Onsager}}$

Same exponent from Kolmogorov:

hypothesis: energy cascade ($\approx$ self-similarity)

$$\partial_t v + \mathrm{div}(v\otimes v) + \nabla p = \nu \Delta v$$

on length-scale $L$, time-scale $T$ ($v \sim \frac{L}{T}$) $|\partial_t v| \sim |\mathrm{div}\, v\otimes v| \sim \frac{L}{T^2}$

$\leadsto$ $\nu \sim \frac{L^2}{T}$. Energy dissipation $= \nu |\nabla v|^2 \sim \frac{L^2}{T^3} = \underline{\text{const}}$. $\Rightarrow v \sim L^{1/3}$