Commutator Estimates

(3) $$\| (v \cdot w)_\varepsilon - (v_\varepsilon \cdot w_\varepsilon) \|_0 \leq \varepsilon^{2\alpha} \|v\|_\alpha \|w\|_\alpha$$

$$(vw)_\varepsilon - v_\varepsilon w_\varepsilon = \int v(x-y)\, w(x-y)\, \varphi_\varepsilon(y)\, dy - v_\varepsilon w_\varepsilon$$

$$= \int \big(v(x-y) - v(x)\big)\big(w(x-y) - w(x)\big)\, \varphi_\varepsilon(y)\, dy$$

$$+ \underbrace{v\, w_\varepsilon + w\, v_\varepsilon - vw - v_\varepsilon w_\varepsilon}_{-(v - v_\varepsilon)(w - w_\varepsilon)}$$

(3') $$\| (vw)_\varepsilon - v_\varepsilon w_\varepsilon \|_k \leq \varepsilon^{2\alpha - k} \|v\|_\alpha \|w\|_\alpha$$

$$\partial^a \big[(vw)_\varepsilon - v_\varepsilon w_\varepsilon \big] = \partial^a (vw)_\varepsilon - \sum_{b \leq a} \binom{a}{b} \partial^b v_\varepsilon\, \partial^a w_\varepsilon$$

$$= \partial^a (vw)_\varepsilon - (\partial^a v_\varepsilon)\, w_\varepsilon - v_\varepsilon (\partial^a w_\varepsilon) - \sum_{0<b<a} \binom{a}{b} \partial^b v_\varepsilon\, \partial^a w_\varepsilon$$

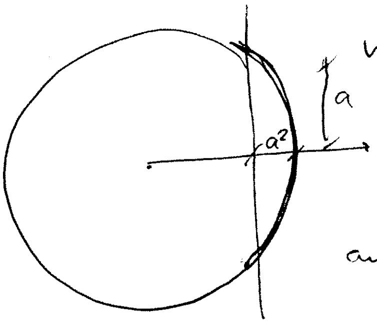
$$= \int \partial^a \varphi_\varepsilon(y) \big(v(x-y) - v(x)\big)\big(w(x-y) - w(x)\big)\, dy$$

$$- \sum_{0<b<a} \binom{a}{b} \int \partial^b \varphi_\varepsilon(y) \big(v(x-y) - v(x)\big)\, dy \int \partial^{a-b} \varphi_\varepsilon(y) \big(w(x-y) - w(x)\big)\, dy$$

Isometric Immersions $M^n \hookrightarrow \mathbb{R}^{n+1}$ in $C^{1,\alpha}$

Recall $\boxed{\text{one step}}$

$$\tilde{u}(x) = u(x) + \frac{1}{\lambda}\left\{ \gamma_1(x, \lambda x\cdot\xi)\,\tilde{\zeta}(x) + \gamma_2(x, \lambda x\cdot\xi)\,\tilde{\eta}(x) \right\}$$



where η = unit normal to $u(\Omega)$ $\qquad \tilde{\eta} = \frac{\eta}{|\zeta|}$

$\zeta = \nabla u\,(\nabla u^T \nabla u)^{-1}\xi \qquad \tilde{\zeta} = \frac{\zeta}{|\zeta|^2}$

and

$$(1+\dot{\gamma}_1)^2 + \dot{\gamma}_2^2 = 1 + |\zeta|^2 a^2$$

This time also interested in development of $\|u\|_{C^2}$ along the iteration.

Now $\qquad \|\tilde{\zeta}\|_{C^{k+1}}\ ;\ \|\tilde{\eta}\|_{C^{k+1}} \sim \|u\|_{C^{k+2}}$

$$\nabla\tilde{u}^T\nabla\tilde{u} = \nabla u^T\nabla u + a^2\,\xi\otimes\xi + \underbrace{\frac{1}{\lambda}\left(|a||\nabla a| + |a||\nabla^2 u|\right) + \frac{1}{\lambda^2}\left(|\nabla a|^2 + |a|^2|D^2u|^2\right)}_{\text{metric error in } C^0}$$

dominating term

and

$$\|\tilde{u}-u\|_{C^{k+2}} \lesssim \lambda^{k+1}\|a\|_0 + \frac{1}{\lambda}\left(\|a\|_{k+2} + \|u\|_{k+3}\right)$$

$$\|\tilde{u}-u\|_{C^1} \lesssim \|a\|_0 + \frac{1}{\lambda}\left(\|a\|_1 + \|u\|_2\right)$$

$\longrightarrow$ loss of derivative

<u>One Solution : mollify at each stage</u>

Stage
$$h = g - \nabla u^T \nabla u$$

mollify ~~u~~ and g at length-scale ℓ,

and estimate all norms in terms of

$$\boxed{\delta^2 := \|h\|_0 \qquad \& \qquad \|u\|_2}$$

- $\|u_\ell\|_{k+2} \lesssim \ell^{-k} \|u\|_2$
- $\|u_\ell - u\|_1 \lesssim \ell \|u\|_2$
- $\|g_\ell - \nabla u_\ell^T \nabla u_\ell\|_k \leq \|g_\ell - (\nabla u^T \nabla u)_\ell\|_k + \|(\nabla u^T \nabla u)_\ell - \nabla u_\ell^T \nabla u_\ell\|$

commutator

$$\lesssim \ell^{-k} \delta^2 + \ell^{2-k} \|u\|_2^2$$

Forced to choose
$$\ell \|u\|_2 = \delta$$

so that

$$\left.\begin{aligned} &\bullet\ \|u_\ell - u\|_1 \lesssim \delta \\ &\bullet\ \|g_\ell - \nabla u_\ell^T \nabla u_\ell\|_k \leq \delta^2 \ell^{-k} \\ &\bullet\ \|u_\ell\|_{k+2} \lesssim \delta \ell^{-(k+1)} \end{aligned}\right\}$$

Decomposition of $h_\ell := g_\ell - Du_\ell^T Du_\ell$ into primitive metrics:

$$h_\ell = \sum_{i=1}^{n_*} a_i^2 \, \xi^i \otimes \xi^i$$

$$\Longrightarrow \quad \| a_i \|_0 \sim \| h_\ell \|_0^{1/2} = \delta$$

$$\| a_i \|_{k+1} \sim \frac{\| h_\ell \|_{k+1}}{\| h_\ell \|_0^{1/2}} = \delta \, \ell^{-(k+1)}$$

Warning: Notation is a bit sloppy. In All estimates there is a constant depending on k. So we can continue ignoring it as long as only $k \le N$ for some N is considered.

For the first step:

C^0 error in metric: $\quad \frac{1}{\lambda}\left(\delta^2 \frac{1}{\ell} \right) + \frac{1}{\lambda^2}\left(\frac{\delta^2}{\ell^2} \right)$

$$= \frac{\delta^2}{\lambda \ell}\left(1 + \frac{1}{\lambda \ell} \right)$$

Increase in C^{k+2} norm $\quad \delta \lambda^{k+1} + \frac{1}{\lambda} \delta \ell^{-(k+2)}$

So best choice is $\lambda \ge \frac{1}{\ell}$, e.g. $\lambda = \frac{K}{\ell}$ $\quad (K \ge 1)$

so that metric error $= \dfrac{\delta^2}{K}$

increase in C^{k+2} $= \delta \ell^{-(k+1)} K^{k+1}$

increase in C^1 $= \delta$.

For the <u>second step</u>: $\|a_2\|$ has same bounds

$\|u_2\|_{k+2} \sim \delta\, \ell^{-(k+1)} K^{k+1}$

Also, if estimates hold for $\|u_1\|_{k+2}$ & $k = 0, 1, \dots, N$

then for $\|u_2\|_{k+2}$ for $k = 0, 1, \dots N-1$

New metric error

$$\frac{1}{\lambda_2}\left(\frac{\delta^2}{\ell} + \frac{\delta^2}{\ell} K\right) + \frac{1}{\lambda_2^2}\left(\frac{\delta^2}{\ell^2} + \delta^4 \frac{K^2}{\ell^2}\right)$$

$$= \frac{\delta^2 K}{\lambda_2 \ell}\left(1 + \frac{1}{\lambda \ell k} + \frac{\delta^2 K}{\lambda_2 \ell}\right)$$

So choose $\lambda_2 = K^2 \ell^{-1}$ so that

C^0 metric error $= \dfrac{\delta^2}{K}$.

Then C^{k+2} increase $= \delta\, \ell^{-(k+1)} K^{2(k+1)}$

so that

$$\|u_3\|_{k+2} \sim \delta\, \ell^{-(k+1)} K^{2(k+1)}.$$

$$\|u_3 - u_2\|_1 \sim \delta.$$

Repeat now with $u_4, \dots, u_{n+1}$ with

$$\lambda_j = \ell^{-1} K^j$$

Obtain after n_* steps $\tilde{u} = u_{n_*+1}$:

$$\| \tilde{\tilde{g}}_* - \nabla\tilde{u}^T \nabla\tilde{u} \|_0 \lesssim \frac{1}{K} \delta^2$$

$$\| \tilde{u} - u \|_0 \lesssim \delta$$

$$\| \tilde{u} - u \|_2 \lesssim \|u\|_2 K^{n_*} .$$

for any $K \geq 1$.

$\boxed{\text{Iteration over stages}}$

Starting with u fix δ_0, μ_0 s.t.

$$\| \nabla u^T \nabla u - g \|_0 \leq \delta_0^2$$

$$\| u \|_2 \leq \mu_0 .$$

iterating on stages we obtain u_k s.t.

$$\| g - \nabla u_k^T \nabla u_k \|_0 \leq \delta_k^2$$

$$\| u_k \|_2 \leq \mu_k$$

$$\| u_{k+1} - u_k \|_1 \leq C \cdot \delta_k$$

where $\delta_{k+1}^2 = C \frac{\delta_k^2}{K}$ and $\mu_{k+1} = C \mu_k K^{n_*}$

$= \mu_k (CK)^{n_*}$

$$\mu_k = \mu_0 K^{k n_*}$$

$$\delta_k = \delta_0 K^{-\frac{k}{2}}$$

hence $\quad \|u_{k+1} - u_k\|_1 \leq \delta_0 K^{-k/2}$

$$\|u_{k+1} - u_k\|_2 \leq \mu_0 K^{k n_*}$$

Interpolation

$$\|u_{k+1} - u_k\|_{1,\alpha} \leq \left(\delta_0 K^{-k/2}\right)^{1-\alpha} \left(\mu_0 K^{k n_*}\right)^{\alpha}$$

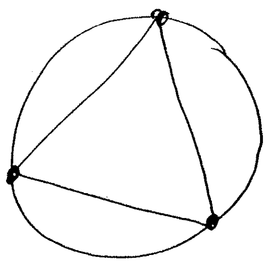
$$= \delta_0^{1-\alpha} \mu_0^{\alpha} K^{k\left(\alpha n_* - \frac{1}{2}(1-\alpha)\right)}$$

$$\alpha n_* - \tfrac{1}{2}(1-\alpha) = \alpha\left(n_* + \tfrac{1}{2}\right) - \tfrac{1}{2} < 0 \quad \text{iff} \quad \alpha < \frac{1}{1+2n_*}$$

$\longrightarrow$ Convergence in $C^{1,\alpha}$ for such α.

Number of steps ?

~~Fo~~ nhood of I can be covered by a single simplex, e.g.



with vertices which are rank-one matrices.

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e.g. $\{(e_i + e_j) \otimes (e_i + e_j) \; : \; i \leq j\}$ = ~~$\{f_k\}$~~

is a basis for $\mathbb{R}^{n \times n}_{sym}$ $\frac{n(n+1)}{2}$ matrices

So best hope $\boxed{\text{locally}}$: $n_* = \frac{n(n+1)}{2}$

$$\Rightarrow \quad \alpha < \frac{1}{1+n+n^2} .$$

$$n = 2 \; : \quad \alpha < \frac{1}{7} .$$

$\boxed{\text{Globally}}$: need to cover M^n with charts.

no of overlaps $= n+1$

$\Rightarrow$ no of steps increases by factor $n+1$.

$\boxed{n=2}$ alternatively, at each stage change into conformal coordinates, i.e.

$$h = g - Du^T Du = a^2(x)\left(\xi^1 \otimes \xi^1 + \xi^2 \otimes \xi^2\right)$$

then $n_* = 2$, so

$$\alpha < \frac{1}{5} .$$

Rigidity $M^2 \hookrightarrow \mathbb{R}^3$ | Bounded Extrinsic Curvature

for a (smooth) surface with smooth isometric embedding,

$$K_g \, dA = N^* d\sigma$$

$d\sigma$ ← area form on S^2.

N ← unit normal

K_g ← Gauss curvature

dA ← Area form $= \sqrt{\det g}\, dx$

Definition: Let $\Omega \subset \mathbb{R}^2$ be open and $u \in C^1(\Omega; \mathbb{R}^3)$ an immersion. The surface $u(\Omega)$ has

bounded extrinsic curvature

if there exists a constant s.t.

$$\sum_{i=1}^{k} |N(E_i)| \leq C$$

for any finite collection $\{E_i\}$ of pairwise disjoint closed subsets of Ω.

Theorem (Pogorelov)

Closed C^1 surfaces with positive Gauss curvature and bounded extrinsic curvature are convex.

Area formula:

$$\text{(A)} \qquad \int_V f(N(x))\, K_g(x)\, dA(x) = \int_{S^2} f(y) \deg(y, V, N)\, d\sigma(y)$$

for $V^{\text{open}} \subset\subset M$ and $f \in C^1(S^2)$.

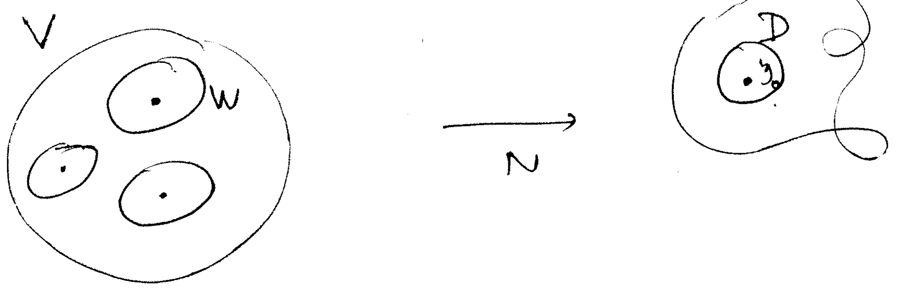
If (M^2, g) smooth but $u: M^2 \hookrightarrow \mathbb{R}^3$ only C^1, LHS & RHS of (A) still make sense, but don't have to be equal.

Assuming that (A) holds, ~~fixed~~ and $K_g > 0$ everywhere,

(i) $\deg(y, V, N) \geq 0 \quad \forall y \in S^2$

(ii) $\deg(y, V, N) \geq 1 \quad \forall y \in N(V) \setminus N(\partial V)$

(ii) If not, $\exists\, y_0 \in N(V) \setminus N(\partial V)$ s.t. $\deg(y_0, V, N) = 0$



$W = N^{-1}(D) \cap V$

$$\deg(y, W, N) = \text{const. for } y \in D$$
$$= \deg(y_0, W, N) = \deg(y_0, V, N) = 0$$

hence
$$0 = \int_{S^2} \deg(y, W, N)\, d\sigma = \int_W K_g \, dA > 0$$

$\lightning$

Now, given $\{E_i\}$ closed, cover with $\{V_i\}$ open, smooth ∂V_i,

$$\sum_i |N(E_i)| \leq \sum |N(V_i) \setminus N(\partial V_i)| + |N(\partial V_i)| \quad = 0 \text{ if } N \in C^{1/2}$$
$$\leq \sum \int_{S^2} \deg(y, V_i, N) = \sum_i \int_{V_i} K_g \leq \int_M K_g < \infty$$

Remarks to understand when (A) can hold.

Claim : (A) holds if $u \in C^{1,2/3+}$.

Proof : Again mollify $u \longrightarrow u_\varepsilon$, N_ε , dA^ε

$N_\varepsilon \longrightarrow N$ uniformly, so RHS converges

$\boxed{LHS}_\varepsilon$

$$\int_V f(N^\varepsilon)\, K^\varepsilon\, dA^\varepsilon = \int_V f(N^\varepsilon(x))\, K^\varepsilon(x) \sqrt{\det g^\varepsilon(x)}\, dx$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{im} \left(\partial_k g_{jm} + \partial_j g_{mk} - \partial_m g_{kj} \right)$$

$$R_{iljk} = g_{lm} \left(\partial_k \Gamma^m_{ij} - \partial_j \Gamma^m_{ik} + \Gamma^p_{ij} \Gamma^m_{kp} - \Gamma^p_{ik} \Gamma^m_{jp} \right)$$

$$K = \frac{R_{1212}}{\det g}$$

$$= \frac{1}{\det g} \left(c_0\, \partial^2 g + c_1(g)\, \partial g \cdot \partial g \right)$$

$$\boxed{u \in C^{1,\alpha} \qquad \alpha > 2/3}$$

$$\|u_\varepsilon - u\|_1 \leq \varepsilon^\alpha \|u\|_{1,\alpha}$$

$$\| g_\varepsilon - (g)_\varepsilon \|_k \leq \varepsilon^{2\alpha - k} \|u\|^2_{1,\alpha}$$

$$\Longrightarrow \quad \|g_\varepsilon - g\|_k \leq \varepsilon^{2\alpha-k} \|u\|^2_{1,\alpha} \qquad (g \in C^\infty)$$

$$\Longrightarrow \quad \|\Gamma^\varepsilon - \Gamma\|_0 \lesssim \varepsilon^{2\alpha-1}$$

i.e. converges uniformly!

The only issue is with $\partial^2 g$, i.e.

$$\int_V f(N^\varepsilon(x))\, \partial_{k\ell}\, g^\varepsilon_{ij}\, (\det g^\varepsilon)^{-1/2}\, dx$$

$$= \int_V \psi^\varepsilon\, \partial_{k\ell}\, g^\varepsilon_{ij}$$

$\psi^\varepsilon = f(N^\varepsilon)(\det g^\varepsilon)^{-1/2} \in C_c^\infty$

and $\|\psi^\varepsilon\|_k \lesssim \varepsilon^{\alpha-k}$

$$= -\int_V \partial_k \psi^\varepsilon\, \partial_\ell\, g^\varepsilon_{ij}$$

$$= -\int_V \partial_k \psi^\varepsilon\, (\partial_\ell g^\varepsilon - \partial_\ell g) - \int_V \partial_k \psi^\varepsilon\, \partial_\ell g$$

$$= -\underbrace{\int_V \underset{\varepsilon^{\alpha-1}}{\partial_k \psi^\varepsilon}\, \underset{\varepsilon^{2\alpha-1}}{(\partial_\ell g^\varepsilon - \partial_\ell g)}}_{\varepsilon^{3\alpha-2}} + \underbrace{\int_V \psi^\varepsilon\, \partial_{k\ell}\, g}_{\longrightarrow 0}$$

(g smooth!)