HOPF ALGEBRAS AND MULTIPLE ZETA VALUES

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1. Tensor product of vector spaces

If E, F are vector spaces on a field K, we define their tensor product $E \otimes_K F$ on K. What matters is that if E has base $(e_i)_{i\in I}$ and F has base $(f_j)_{j\in J}$, then $E \otimes_K F$ has for basis the vectors denoted $e_i \otimes e_j$, $i \in I, j \in J$. For any $e \in E$ and any $f \in F$, there is a vector $e \otimes f$. The following properties hold for any $e \in E$, $f \in F$, $\alpha \in K$:

- $(e + e') \otimes f = e \otimes f + e' \otimes f$;
- $f \otimes (e + e') = f \otimes e + f \otimes e'$;
- $\alpha(e \otimes f) = (\alpha e) \otimes f = e \otimes (\alpha f)$.

With the basis described above, every vector in $E \otimes F$ can be written uniquely as a linear combination:

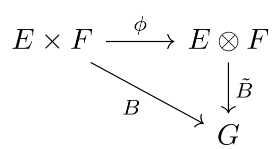
$$\sum_{\substack{i\in I\\ j\in J}} \alpha_{ij} e_i \otimes f_j, \qquad \alpha_{ij} \in K.$$

Observe that if E and F are finite dimensional over K then

$$dim_K E \otimes F = dim_K E \cdot dim_K F$$

Universal property of the tensor product

For any bilinear map $B : E \times F \to G$ there exists a unique linear map $\tilde{B} : E \otimes F \to G$ such that the following diagram commutes:



where $\phi : (e, f) \mapsto e \otimes f$. In other words, B factors via ϕ, i.e. $B = \tilde{B} \circ \phi$.

Canonical identifications

1. *Canonical identification of* $K \otimes E$ *with* E. One identifies $\alpha \otimes v$ with αv (for $\alpha \in K, v \in E$). In the same way one identifies the space $E \otimes K$ with E: $v \otimes \alpha \mapsto \alpha v$. Notice that K is of dimension 1 over K, with canonical basis $\{1\}$. The canonical identification $\alpha \otimes v \mapsto \alpha v$ is equivalent to identiying the base $(1 \otimes e_i)_{i\in I}$ of $K \otimes E$ with the basis $(e_i)_{i\in I}$ of E.

2. *Canonical identification of* $(E\otimes F)\otimes G$ *with* $E\otimes(F\otimes G)$. One identifies the vectors $(e \otimes f) \otimes g$ and $e \otimes (f \otimes g)$ and write $e \otimes f \otimes g$.

3. *Canonical identification for linear maps.* Let $\mathcal{L}(E, F)$ be the K-vector space of linear maps from E to F. Then one can identify

$$\mathcal{L}(E, F) \otimes \mathcal{L}(E', F') \to \mathcal{L}(E \otimes E', F \otimes F').$$

For $\alpha \in \mathcal{L}(E, F), \beta \in \mathcal{L}(E', F')$, define $\alpha \otimes \beta \in \mathcal{L}(E \otimes E', F \otimes F')$ as

$$\alpha \otimes \beta(e \otimes e') = \alpha(e) \otimes \beta(e').$$

Exercise 1. *Justify this identification with the universal property. (Or, simply, use a dimension argument.)*

Tensor product and duality. Let E be a K-vector space. Its *dual* is the set E^* of *linear forms*, i.e. linear maps from E to K. It is also a K vector space. Recall that if $dim_K E = n < \infty$, then $dim_K E^* = dim_K E$: if $(e_i)_{i\in[n]}$ is a basis of E, then the *canonical dual basis* $(e_i^*)_{i\in[n]}$ is defined by

$$e_i^*(c_1e_1 + \ldots + c_ne_n) = c_i \qquad i = 1, \ldots, n,$$

that is

$$e_i^*(e_j) = \delta_{ij},$$

where δ_{ij} is the so-called Kronecker delta, which is 1 if $i = j$, 0 otherwise.

Exercise 2. *Find an example of infinite dimensional K-vector space E for which $E^* \not\simeq E$.*

Proposition 1. *if E and F are bot vector spaces of finite dimension over K, then*

$$(E \otimes F)^* \simeq E^* \otimes F^*.$$

Proof. The above spaces have the same dimension $dim_K E \cdot dim_K F$. Furthermore, for $u \in E^*, v \in F^*$, then

$$\mu_K \circ (u \otimes v)$$

is a linear form on $E \otimes F$, where $\mu_K : K \otimes K \to K, \alpha \otimes \beta \mapsto \alpha\beta$ is the multiplication of the base field K. One can omit μ_K using the canonical identification $K \otimes K \simeq K$: for $x \in E, y \in F$,

$$\mu_K \circ (u \otimes v)(x \otimes y) = u(x)v(y).$$

2. Algebras

Let K be a field. A *K-algebra* A is a ring containing K in its center. It is hence a vector space over K. Some examples are: the ring $K[x]$ of commutative polynomials in one variable, the ring $K[x, y]$ of commutative polynomials in two variables, the $n \times n$ (square) matrices $M_n(K)$ with entries in K.

The non-commutative polynomials.

Let X be a set, called *alphabet*, $X^2 = \{xy : x, y \in X\}$ the set of sequences of length 2, more generally $X^n = \{x_1 x_2 \dots x_n : x_i \in X, i = 1, 2, \dots n\}$ the sequences of length n on the letters of X. Set

$$X^* := \bigcup_{i=0}^{\infty} X^i =$$

be the set of all finite length sequences on X, or *words in the alphabet* X, including the so-called *empty word* denoted by $|$ (or sometime ϵ), so that $X^0 = \{|\}$. With respect to the *concatenation of words*, X^* is a monoid, where the identity element is the empty word. For instance, for $X = \{a, b, c\}$, $u = abbacb$ and $v = accba$, the concatenation of u and v is the word $w = abbacbaccba$. It is a non commutative product.

Now, consider the vector space on K whose basis is X^*; we denote it by $K\langle X\rangle$; an element of $K\langle X\rangle$ is called a *non commututive polynomial.* For example, for $K = \mathbb{Q}$, $X = \{x, y\}$, $1 - 2x + \frac{12}{y} + 3xy - \frac{23}{y}x + x^3$, $(x + y)^2 = x^2 + xy + yx + y^2$ are such a polynomial. With the concatenation product, extended linearly from the base of words to all polynomials, and identity 1_K, $K\langle X\rangle$ is a non-commutative K-algebra (except when $|X| = 1$) and of center K. It is called *the K-algebra of non-commutative polynomials.*

The plactic congruence. The monoid X^* is freely generated by X as monoid, and its product is non-commutative. As defined by Lascoux and Schützenberger in [?], on X^* one can define the *plactic congruence* as the smallest congruence $\sim$ containing the so-called *Knuth relations* (see Knuth [?]):

$$\text{if } x < y < z \quad \text{then} \quad yzx \sim yxz \quad \text{and} \quad zxy \sim xzy$$

$$\text{and if} \quad x < y \text{ then} \quad yxx \sim xyx \quad \text{and} \quad yyx \sim yxy.$$

The plactic class of a word $w \in X^*$ is the set of words congruent to w. A subset of words $L \subseteq X^*$ is closed under the plactic congruence if for any $w \in L$ such that $w \sim v$ then $v \in L$: in other words, L is disjoint union of plactic classes of its elements.

Exercise 3. *Let* $S_n = \{\sigma : [n] \to [n] : \sigma \text{ a bijection}\}$ *the set of permutations: here a permutation σ is seen as words, i.e. the list of successive* *images of the elements of* $[n]$*, that is* $\sigma = \sigma(1)\sigma(2)\dots\sigma(n)$*. Compute* $S_3/\sim$*,* $S_4/\sim$*, the set of congruence classes of* S_3 *and* S_4*. Compute as well the class of* $3152476 \in S_7$*.*

If A is a K-algebra, the map

$$\begin{array}{ccc} A \times A & \to & A \\ (a,b) & \mapsto & ab \end{array}$$

is bilinear on K. For the universal property of tensor product there exists a unique linear map $\mu : A \otimes A \to A$ such that $\mu(a \otimes b) = ab$.
Denote by η_A the K-linear map

$$\begin{array}{ccc} K & \xrightarrow{\eta_A} & A \\ \alpha & \mapsto & \alpha 1: \end{array}$$

in other words, it is the canonical injection $K \hookrightarrow A$, since we suppose that K is contained in the center of A. To be more precise: $\eta_A(1_K) = 1_A$ is the identity element of the product of the algebra A. Hence, the following diagrams are commutative:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{id_A \otimes \mu} & A \otimes A \\ {\scriptstyle \mu \otimes id_A}\downarrow & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\mu} & A \end{array}$$

$$\begin{array}{ccc} K \otimes A & \xrightarrow{\eta \otimes id_A} & A \otimes A \\ & \searrow^{\simeq} & \downarrow{\scriptstyle \mu} \\ & & A \end{array} \qquad \begin{array}{ccc} A \otimes K & \xrightarrow{id_A \otimes \eta} & A \otimes A \\ & \searrow^{\simeq} & \downarrow{\scriptstyle \mu} \\ & & A \end{array}$$

The first diagram indicate exactly that the product is associative. The two others, that 1_A is the left and right identity element with respect to the multiplication. In the second of these last ones, for example, for $a \in A$:

$$\begin{array}{ccc} 1_K \otimes a & \longrightarrow & 1_A \otimes a \\ & \searrow & \downarrow \\ & & a = 1_A \cdot a \end{array}$$

Tensor product of algebras. Let A and B be two K-algebras, we define the K-algebra $A \otimes B$. It is already a vestor space over K. The product is defined "componentwise", that is by

$$(a \otimes b)(a' \otimes b') = (aa') \otimes (bb').$$

The identity element is $1_A \otimes 1_B$. In other words, $\mu_{A \otimes B} = id_A \otimes \tau_{A,B} \otimes id_B$ and $\eta_{A \otimes B} = \eta_A \otimes \eta_B$.

Example. The commutative polynomials in two variables:

$$\begin{array}{ccc} K[x] \times K[x] & \simeq & K[x,y] \\ x^i \otimes x^j & \mapsto & x^i y^j \end{array}$$

and $(x^i \otimes x^j)(x^{i'} \otimes x^{j'}) = (x^{i+i'} \otimes x^{j+j'})$, $(x^i y^j)(x^{i'} x^{j'}) = (x^{i+i'} x^{j+j'})$.

Example. $M_n(K) \otimes M_p(K) \simeq M_{np}(K)$. We generalize the notion of matrix as follows. Let $K^{I \times J}$ = the set of $I \times J$ matrices: the rows are indexed by I, the columns by J (for I, J finite sets). Then $K^{I \times J} \otimes K^{I' \times J'} \simeq K^{(I \times I') \times (J \times J')}$.

Homomorphisms of algebras. Let A_1, A_2 two K algebras; a homomorphism $f : A_1 \to A_2$ is a K-linear map such that

$$\forall x, y \in A \quad f(xy) = f(x)f(y) \quad \text{and} \quad f(1_{A_1}) = f_{A_2}.$$

This is equivalent to say that f is K-linear and that the following diagrams are commutative:

$$\begin{array}{ccc} A_1 \otimes A_1 & \xrightarrow{f \otimes f} & A_1 \otimes A_1 \\ \mu_1 \downarrow & & \downarrow \mu_2 \\ A_1 & \xrightarrow{\quad f \quad} & A_2 \end{array}$$

and

$$\begin{array}{ccc} K & \xrightarrow{\eta_1} & A_1 \\ & \eta_2 \searrow & \downarrow f \\ & & A_2. \end{array}$$

In other words: $f \circ \mu_1 = \mu_2 \circ (f \otimes f)$ and $f \circ \eta_{A_1} = \eta_{A_2}$.

3. Coalgebras

Let C be a K-vector space. A *coproduct* is a linear map $\delta : C \to C \otimes C$. A *coalgebra* is a vector space C with a coproduct $\delta : C \to C \otimes C$ which is *coassociative* and endowed with a *counity map*, that is:

(i) $(\delta \otimes id_C) \circ \delta = (id_C \otimes \delta) \circ \delta$ (coassociativity of δ);
(ii) there exists a linear map $\epsilon : C \to K$ such that

$$(\epsilon \otimes id_C) \circ \delta = id_C = (id_C \otimes \epsilon) \circ \delta.$$

Example. $K[x]$ with the following coproduct Δ and counity map ϵ is a coalgebra:

(i) Define the coproduct on the linear base elements of $K[x]$:

$$\begin{array}{rccl} \Delta : & K[x] & \to & K[x] \otimes K[x] \\ & x^n & \mapsto & x^n \otimes 1 + x^{n-1} \otimes x + x^{n-2} \otimes x^2 + \ldots + 1 \otimes x^n. \end{array}$$

We prove the coassociativity of Δ.

$$\begin{aligned}(\Delta\otimes id)\circ\Delta(x^n) &= (\Delta\otimes id)(\sum_{i+j=n} x^i\otimes x^j)\\ &= \sum_{i+j=n}\Delta(x^i)\otimes x^j\\ &= \sum_{i+j=n}(\sum_{a+b=i} x^a\otimes x^b)\otimes x^j\\ &= \sum_{\substack{i+j=n\\ a+b=i}} x^a\otimes x^b\otimes x^j\\ &= \sum_{s+t+u=n} x^s\otimes x^t\otimes x^u\end{aligned}$$

We obtain the same result when computing $(id\otimes\Delta)\circ\Delta(x^n)$.

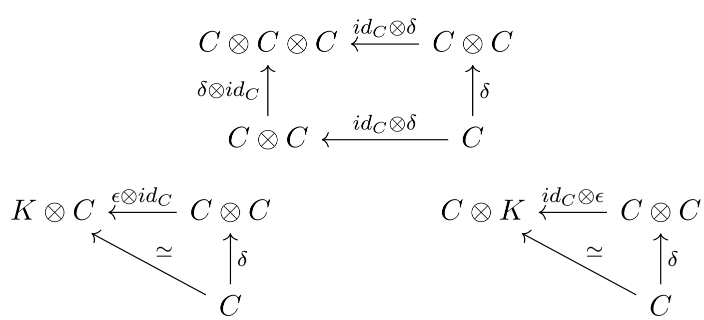
(ii) Define the counity map as:

$$\begin{array}{rccl}\epsilon: & K[x] & \to & \mathrm{K}\\ & x^n & \mapsto & \delta_{n,0}.\end{array}$$

In other words, if $P\in K[x]$, then $\epsilon(P)$ is the constant term of the polynomial P. Then:

$$\begin{aligned}(\epsilon\otimes id)\circ\Delta(x^n) &= (\epsilon\otimes id)(\sum_{i+j=n} x^i\otimes x^j)\\ &= \sum_{i+j=n}\epsilon(x^i)\otimes id(x^j)\\ &= \sum_{i+j=n}(\delta_{i,0}\otimes x^j)\\ &= 1\otimes x^n = x^n \quad \text{(identification)}.\end{aligned}$$

The axioms (i) and (ii) in the definition of a coalgebra are equivalent to the commutativity of the following diagrams:



We observe that the diagrams above are the same diagrams expressing the associative law and the unit law for algebras, except that the arrow are inverted. This fact has consequences that we will see later for the duality.

Example. (This example generalizes the preceding one.) Let X be an alphabet, and $K\langle X\rangle$ the non-commutative polynomials in X. Define a coproduct by:

$$\begin{array}{rccc} \Delta: & K\langle X\rangle & \to & K\langle X\rangle \otimes K\langle X\rangle \\ & w \in X^* & \mapsto & \sum\limits_{\substack{w=uv \\ u,v\in X^*}} u \otimes v \end{array}$$

For example, $\Delta(xyz) = 1 \otimes xyz + x \otimes yz + xy \otimes z + xyz \otimes 1$, for $x, y, z \in X$. This coproduct is called *deconcatenation.* We verify the co-associativity by showing that

$$\begin{array}{rcl} (\Delta \otimes 1) \circ \Delta(w) & = & (1 \otimes \Delta) \circ \Delta \\ & = & \sum\limits_{\substack{w=abc \\ a,b,c\in X^*}} a \otimes b \otimes c \text{ (we wrote 1 instead of id).} \end{array}$$

Observe that for a word w of length n there are $n+1$ terms in $\Delta(w)$. Define now the counity map for this coproduct as being:

$$\begin{array}{rccc} \epsilon: & K\langle X\rangle & \to & K \\ & | & \mapsto & 1 \\ & w \neq | & \mapsto & 0. \end{array}$$

The axiom of counity is trivial:

$$\begin{array}{rcl} (\epsilon \otimes 1) \circ \Delta(w) & = & (\epsilon \otimes 1)(\sum\limits_{w=uv} u \otimes v) \\ & = & \sum\limits_{w=uv} \epsilon(u) \otimes v \\ & = & \sum\limits_{w=uv} \epsilon(u) v = w. \end{array}$$

We say that a coalgebra (C, δ, ϵ) is *cocommutative* if $\tau_{C,C} \circ \delta = \delta$, where

$$\begin{array}{rccc} \tau_{C,C}: & C \otimes C & \to & C \otimes C \\ & x \otimes y & \mapsto & y \otimes x. \end{array}$$

The deconcatenation is not cocommutative if $|X| \geq 2$.

Example. (A new coproduct on the non-commutative polynomials.) Let $\delta : K\langle X\rangle \to K\langle X\rangle \otimes K\langle X\rangle$ be defined on the linear basis of words $w = x_1 \dots x_n \in X^*$ by the following:

$$\delta(w) = \sum w_I \otimes w_J,$$

where the sum is extended to pairs (I, J) with $I \cup J = [n]$, $I \cap J = \emptyset$, with the notation $w_I = x_{i_1} x_{i_2} \dots x_{i_p}$ if $I = \{i_1 < i_1 \dots i_p\}$. We call this coproduct the *unshuffling*. For example for $x, y, z \in X$, $\delta(xyz) = 1 \otimes xyz + x \otimes yx + y \otimes x^2 + x \otimes xy + xy \otimes x + x^2 \otimes y + yx \otimes x + xyz \otimes 1$. Observe that in $\delta(w)$ there are 2^n terms if $|w| = n$.

It is easy to see that if $|W| = n$, then $(\delta \otimes 1) \circ \delta(w) = \sum_{(I,J,K)} w_I \otimes w_J \otimes w_K$, with $[n] = I \cup J \cup K$, I, J, K disjoint. From this one deduces the coassociativity of the coproduct.

Exercise 4. *Show that the coalgebra $(K\langle X\rangle, \delta, \epsilon)$ is cocommutative.*

Sweedler notation. For a coassociative coproduct δ, it is useful sometimes to indicate the coproduct of a specific element using this notation:

$$\delta(x) = \sum_{(x)} x' \otimes x'',$$

or

$$\delta(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)}.$$

More generally, when applying the coproduct twice:

$$\delta^{(3)} = (\delta \otimes 1) \circ \delta(x) = \sum_{(x)} x' \otimes x'' \otimes x'''$$

or

$$\delta^{(3)} = (\delta \otimes 1) \circ \delta(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes x^{(3)}.$$

Finally, one defines by recurrence

$$\delta^{(n)}(x) = (\delta \otimes 1) \circ \delta^{(n-1)}(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes \ldots \otimes x^{(n)}.$$

Tensor product of coalgebras.

Let $(C_1, \delta_1, \epsilon_1)$ and $(C_2, \delta_2, \epsilon_2)$ two coalgebras on K. Then the K-linear space $C_1 \otimes C_2$ becomes a coalgebra with coproduct $\delta : C_1 \otimes C_2 \to C_1 \otimes C_2 \otimes C_1 \otimes C_2$ defined by:

$$\delta = (1 \otimes \tau_{C_1,C_2} \otimes 1) \circ (\delta_1 \otimes \delta_2),$$

that is

$$\delta(x_1 \otimes x_2) = \sum_{(x_1)(x_2)} x_1' \otimes x_2' \otimes x_1'' \otimes x_2'',$$

and counit $\epsilon : C_1 \otimes C_2 \to K$ given by:

$$\epsilon = \mu_K \circ (\epsilon_1 \otimes \epsilon_2),$$

in other words, $\epsilon(x_1 \otimes x_2) = \epsilon_1(x_1)\epsilon_2(x_2)$.

Homomorphism of coalgebras.

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