# Tridendriform algebras, Schröder trees and Hopf algebras.

## The free tridendriform structure

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ULCO

Cetraro Workshop
" Algebraic Combinatorics and finite groups",
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# Plan

# Plan

# The word algebra

## Definition

Let $X$ be a set. We denote $X^\star$ the set of *finite words* over $X$ :

$$X^\star := \{x_1 \dots x_k \,|\, k \in \mathbb{N}, x_1, \dots, x_k \in X\}.$$

Let $T(X)$ be the $\mathbb{K}$-vs generated by $X^\star$.

## Example

$$X = \{0,1\},$$
$$0110010001 \in X^\star,$$
$$\lambda \cdot 01011 + \mu \cdot 1101 \in T(X).$$

Université Littoral Côte d'Opale

Tridendriform algebras

Pierre CATOIRE



# Plan

# Examples of shuffles



$$x_1x_2 ⧢ x_3x_4 = x_1x_2x_3x_4 + x_1x_3x_2x_4 + x_1x_3x_4x_2 + x_3x_1x_2x_4 + x_3x_1x_4x_2 + x_3x_4x_1x_2,$$

$$x_1 ⧢ x_2x_3x_4 = x_1x_2x_3x_4 + x_2x_1x_3x_4 + x_2x_3x_1x_4 + x_2x_3x_4x_1.$$

## Definition (shuffles)

Let $m, n \in \mathbb{N}$. We call $(m, n)$-*shuffle* an element $\sigma \in S_{m+n}$ such that :

$$\sigma(1) < \cdots < \sigma(m) \text{ and } \sigma(m+1) < \cdots < \sigma(m+n).$$

We put $\mathrm{Sh}(m, n)$ the set of $(m, n)$-shuffles.

# What is a dendriform and a tridendriform algebras ?

## Shuffle definitions



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- $\mathrm{Sh}(2,2) = \{(1\,2\,3\,4),(1\,3\,2\,4),(1\,4\,2\,3),(2\,3\,1\,4),(2\,4\,1\,3),(3\,4\,1\,2)\}$.

# Action of shuffles on words



**Definition (Sweedler, 1969)**

Let $X$ be an alphabet. We give $T(X)$ a product $\sqcup\!\sqcup$ defined by :

$$x_1 \cdots x_k \sqcup\!\sqcup x_{k+1} \cdots x_{k+l} = \sum_{\sigma \in \mathrm{Sh}(k,l)} x_{\sigma^{-1}(\{1\})} \cdots x_{\sigma^{-1}(\{k+l\})}$$

The unit is the empty word 1. We call $\sqcup\!\sqcup$ the *shuffle product*.

$$x_1x_2 \sqcup\!\sqcup\, x_3x_4 = x_1x_2x_3x_4 + x_1x_3x_2x_4 + x_1x_3x_4x_2 + x_3x_1x_2x_4 + x_3x_1x_4x_2 + x_3x_4x_1x_2.$$

## Definition (products $\prec$ « left » and $\succ$ « right »)

We define for all $x, y \in T(X)$ with $x = x_1 \dots x_k$ and $x' = x_{k+1} \dots x_{k+l}$ :

$$x \prec x' = \sum_{\substack{\sigma \in \mathrm{Sh}(k,l) \\ \sigma^{-1}(1)=1}} x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(k+l)},$$

$$x \succ x' = \sum_{\substack{\sigma \in \mathrm{Sh}(k,l) \\ \sigma^{-1}(1)=k+1}} x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(k+l)}.$$

Then $\sqcup\!\sqcup = \prec + \succ$.

## Remark

- $x \prec x' \longleftrightarrow$ the first letter comes from $x$.
- $x \succ x' \longleftrightarrow$ the first letter comes from $x'$.

$$\begin{aligned} x_1x_2 \sqcup\!\sqcup\, x_3x_4 &= x_1x_2 \prec x_3x_4 + x_1x_2 \succ x_3x_4 \\ &= x_1x_2x_3x_4 + x_1x_3x_2x_4 + x_1x_3x_4x_2 \\ &+ x_3x_1x_2x_4 + x_3x_1x_4x_2 + x_3x_4x_1x_2, \end{aligned}$$

$$\begin{aligned} x_1 \sqcup\!\sqcup\, x_2x_3x_4 &= x_1 \prec x_2x_3x_4 + x_1 \succ x_2x_3x_4 \\ &= x_1x_2x_3x_4 + x_2x_1x_3x_4 + x_2x_3x_1x_4 + x_2x_3x_4x_1. \end{aligned}$$

**Definition (dendriform algebra, J.A Robinson, 1965, M. Ronco, 1999)**

Let $V$ be a $\mathbb{K}$-vectorial space with two binary operations $\prec$ and $\succ$ satisfying for all $x, y, z \in V$ :

$$\begin{aligned}(x \prec y) \prec z &= x \prec (y \star z),\\ (x \succ y) \prec z &= x \succ (y \prec z),\\ x \succ (y \succ z) &= (x \star y) \succ z,\end{aligned}$$

where $\star = \prec + \succ$ is associative.
Then, $(V, \prec, \succ)$ is called a *dendriform algebra*.

# Examples



Endow $X$ with an associative product $\cdot$

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$$x_1x_2 \,\overline{\sqcup\!\sqcup}\, x_3x_4$$
$$= x_1x_2x_3x_4 + x_1x_3x_2x_4 + x_1x_3x_4x_2 + x_3x_1x_2x_4 + x_3x_1x_4x_2$$
$$+ x_3x_4x_1x_2$$

# Examples





Endow $X$ with an associative product $\cdot$

$$\begin{aligned}
& x_1x_2 \overline{\sqcup\!\sqcup}\, x_3x_4 \\
= & x_1x_2x_3x_4 + x_1x_3x_2x_4 + x_1x_3x_4x_2 + x_3x_1x_2x_4 + x_3x_1x_4x_2 \\
+ & x_3x_4x_1x_2 + x_1(x_2\cdot x_3)x_4 + (x_1\cdot x_3)x_2x_4 + (x_1\cdot x_3)(x_2\cdot x_4) \\
+ & (x_1\cdot x_3)x_4x_2 + x_1x_3(x_2\cdot x_4) + x_3(x_1\cdot x_4)x_2 + x_3x_1(x_2\cdot x_4).
\end{aligned}$$

# Definition of the quasi-shuffles





## Definition (quasi-shuffles)

Let $m, n \in \mathbb{N}$. We call $(m,n)$-*quasi-shuffle* each $\sigma : [\![1, m+n]\!] \twoheadrightarrow [\![1, m']\!]$ with $m' \in \mathbb{N}$ and :

$$\sigma(1) < \cdots < \sigma(m) \text{ and } \sigma(m+1) < \cdots < \sigma(m+n).$$

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$$\sigma(1) < \cdots < \sigma(m) \text{ and } \sigma(m+1) < \cdots < \sigma(m+n).$$

## Examples

- $\mathrm{QSh}(1,1) = \{(1\,2), (2\,1), (1\,1)\}$,
- $\mathrm{QSh}(1,2) = \{(1\,2\,3), (2\,1\,3), (3\,1\,2), (1\,1\,2), (2\,1\,2)\}$,
- $\mathrm{QSh}(2,1) = \{(1\,2\,3), (1\,3\,2), (2\,3\,1), (1\,2\,2), (1\,2\,1)\}$,
- $\mathrm{QSh}(2,2) =$
$$\left\{ \begin{array}{lllll} (1\,2\,3\,4), & (1\,3\,2\,4), & (1\,4\,2\,3), & (2\,3\,1\,4), & (2\,4\,1\,3), \\ (3\,4\,1\,2), & (1\,2\,2\,3), & (1\,2\,1\,3), & (1\,2\,1\,2), & (1\,3\,1\,2), \\ & (1\,3\,2\,3), & (2\,3\,1\,2), & (2\,3\,1\,3) & \end{array} \right\}.$$

# Action of the quasi-shuffle on words



## Definition

Let $(X,\cdot)$ be a semi-group. We endow $T(X)$ with the product $\overline{\sqcup\!\sqcup}$ defined by :

$$x_1\cdots x_k \,\overline{\sqcup\!\sqcup}\, x_{k+1}\cdots x_{k+l} = \sum_{\sigma\in \mathrm{QSh}(k,l)} x_{\sigma^{-1}(\{1\})}\cdots x_{\sigma^{-1}(\{\max(\sigma)\})},$$

where $x_{\{k,l\}} = x_k \cdot x_l$ for $k<l$. The unit is the empty word 1. We call $\overline{\sqcup\!\sqcup}$ the *quasi-shuffle product*.

# Action of the quasi-shuffle on words

## Example



$$\begin{aligned}
&x_1x_2 \,\overline{⧢}\, x_3x_4\\
=&x_1x_2x_3x_4 + x_1x_3x_2x_4 + x_1x_3x_4x_2 + x_3x_1x_2x_4 + x_3x_1x_4x_2\\
+&x_3x_4x_1x_2 + x_1(x_2\cdot x_3)x_4 + (x_1\cdot x_3)x_2x_4 + (x_1\cdot x_3)(x_2\cdot x_4)\\
+&(x_1\cdot x_3)x_4x_2 + x_1x_3(x_2\cdot x_4) + x_3(x_1\cdot x_4)x_2 + x_3x_1(x_2\cdot x_4).
\end{aligned}$$

## Definition (products $\prec$ « left », $\cdot$ « middle » and $\succ$ « right »)

We endow $X$ with an associative product $\cdot$. We define for all $x, y \in T(X)$ with $x = x_1 \dots x_k$ and $x' = x_{k+1} \dots x_{k+l}$ :

$$x \prec x' = \sum_{\substack{\sigma \in QSh(k,l),\\ \sigma^{-1}(\{1\})=\{1\}}} x_{\sigma^{-1}(\{1\})} \cdots x_{\sigma^{-1}(\{k+l\})},$$

$$x \succ x' = \sum_{\substack{\sigma \in QSh(k,l),\\ \sigma^{-1}(\{1\})=\{k+1\}}} x_{\sigma^{-1}(\{1\})} \cdots x_{\sigma^{-1}(\{k+l\})},$$

$$x \cdot x' = \sum_{\substack{\sigma \in QSh(k,l),\\ \sigma^{-1}(\{1\})=\{1,k+1\}}} x_{\sigma^{-1}(\{1\})} \cdots x_{\sigma^{-1}(\{k+l\})},$$

où $\overline{⧢} = \prec + \cdot + \succ$.

## Remark

- $x \prec x' \longleftrightarrow$ the first letter only comes from $x$.
- $x \succ x' \longleftrightarrow$ the first letter only comes from $x'$ .
- $x \cdot x' \longleftrightarrow$ the first letter comes from both words.

$$
\begin{aligned}
& x_1x_2 \,\overline{\sqcup\!\sqcup}\, x_3x_4 \\
=& x_1x_2 \prec x_3x_4 + x_1x_2 \cdot x_3x_4 + x_1x_2 \succ x_3x_4 \\
=& x_1x_2x_3x_4 + x_1x_3x_2x_4 + x_1x_3x_4x_2 + x_3x_1x_2x_4 + x_3x_1x_4x_2 \\
+& x_3x_4x_1x_2 + x_1(x_2 \cdot x_3)x_4 + x_1x_3(x_2 \cdot x_4) + (x_1 \cdot x_3)(x_2 \cdot x_4) \\
& + (x_1 \cdot x_3)x_4x_2 + (x_1 \cdot x_3)x_2x_4 + x_3(x_1 \cdot x_4)x_2 + x_3x_1(x_2 \cdot x_4).
\end{aligned}
$$

**Definition (tridendriform algebra, ≈2000)**

Let $V$ be a $\mathbb{K}$-vector spaces endowed with three binary operations $\prec, \cdot$ and $\succ$ satisfying $x, y, z \in V$ :

$$(x \prec y) \prec z = x \prec (y * z),$$
$$(x \succ y) \prec z = x \succ (y \prec z),$$
$$(x * y) \succ z = x \succ (y \succ z),$$
$$(x \succ y) \cdot z = x \succ (y \cdot z),$$
$$(x \prec y) \cdot z = x \cdot (y \succ z),$$
$$(x \cdot y) \prec z = x \cdot (y \prec z),$$
$$(x \cdot y) \cdot z = x \cdot (y \cdot z),$$

where $* = \prec + \cdot + \succ$ is associative.
$(V, \prec, \cdot, \succ)$ is called a *tridendriform algebra*.

We have :

**Lemma**

*Let $A$ be a $\mathbb{K}$-vector space endowed with three binary operations $\prec, \succ$ and $\cdot$. We get :*

$$(A, \prec, \cdot, \succ) \text{ is a tridendriform algebra}$$

$$\Longleftrightarrow$$

$$(A, \preceq, \succ) \text{ and } (A, \prec, \succeq) \text{ are both dendriform algebras,}$$

*where $\preceq = \prec + \cdot$ and $\succeq = \succ + \cdot$.*

# Where is the free tridendriform algebra ?

## Let's start investigations

Is the free algebra the one over words ?

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Is the free algebra the one over words ?

Problem :

$$\forall a, b \in T(X), a \prec b = b \succ a \,!$$

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Problem :

$$\forall a, b \in T(X), a \prec b = b \succ a \,!$$

So, this is not the free tridendriform algebra !

# Plan

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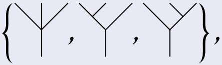
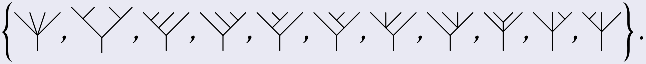
# Schröder trees

## Examples

$T_0 = \{|\}$,

$T_1 =$ 

$T_2 =$ 

$T_3 =$ 

## Definition

For all $n \in \mathbb{N}$, $T_n$ is the set of all these *planar* trees with $n+1$ leaves.

# Grafting operator

We denote by $\vee$ the *grafting operator*. We consider $k \in \mathbb{N}, k \geq 2$, for all $t_1, \ldots, t_k$ trees, we have :

$$t_1 \vee \cdots \vee t_k = \text{ the grafting of } t_1, \ldots, t_k \text{ on a common root.}$$

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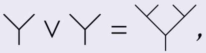

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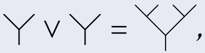

$$| \vee | \vee | =$$

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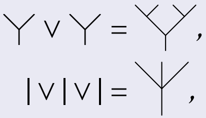

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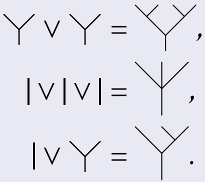


**Theorem (2002, J-L.Loday and M.Ronco)**

*The free tridendriform algebra is generated by one element is :*

$$\mathrm{Tridend}(\mathbb{K}) := \bigoplus_{n\geq 1} \mathbb{K}T_n.$$

*The generator is* $\curlyvee$. *The binary operations are given by :*

$$x \prec y = x^{(1)} \vee \cdots \vee \left(x^{(k)} * y\right),$$

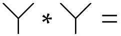
$$x \cdot y = x^{(1)} \vee \cdots \vee \left(x^{(k)} * y^{(1)}\right) \vee \cdots \vee y^{(l)},$$

$$x \succ y = \left(x * y^{(1)}\right) \vee \cdots \vee y^{(l)},$$

*where* $x = x^{(1)} \vee \cdots \vee x^{(k)}$ *and* $y = y^{(1)} \vee \cdots \vee y^{(l)}$, *putting* $| * t = t = t * |$ *for all* $t$.

# The free tridendriform algebra

## Some horrible computations



# The free tridendriform algebra

## Some horrible computations

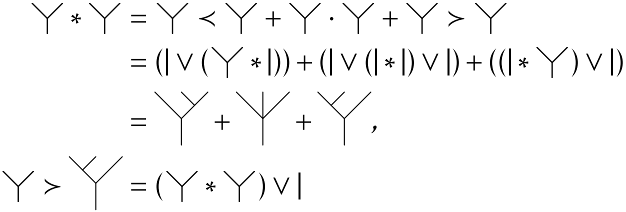


$$\mathsf{Y} * \mathsf{Y} = \mathsf{Y} \prec \mathsf{Y} + \mathsf{Y} \cdot \mathsf{Y} + \mathsf{Y} \succ \mathsf{Y}$$

# The free tridendriform algebra

## Some horrible computations

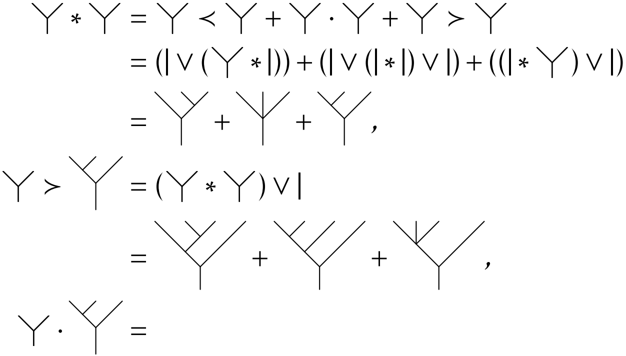
$$\begin{aligned}\mathsf{Y} * \mathsf{Y} &= \mathsf{Y} \prec \mathsf{Y} + \mathsf{Y} \cdot \mathsf{Y} + \mathsf{Y} \succ \mathsf{Y} \\ &= (| \vee (\mathsf{Y} * |)) + (| \vee (| * |) \vee |) + ((| * \mathsf{Y}) \vee |)\end{aligned}$$



# The free tridendriform algebra

## Some horrible computations





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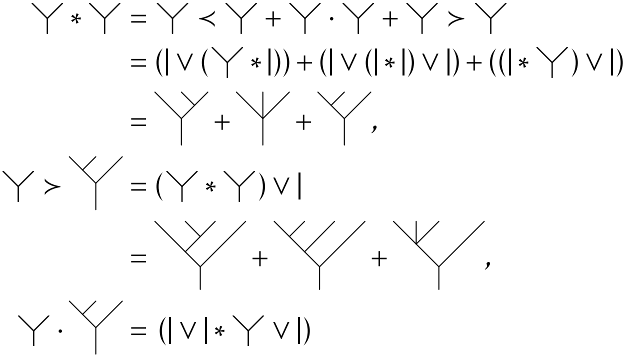
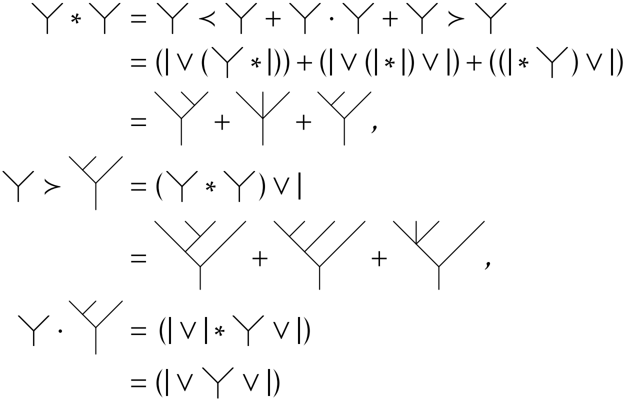


# The free tridendriform algebra

## Some horrible computations

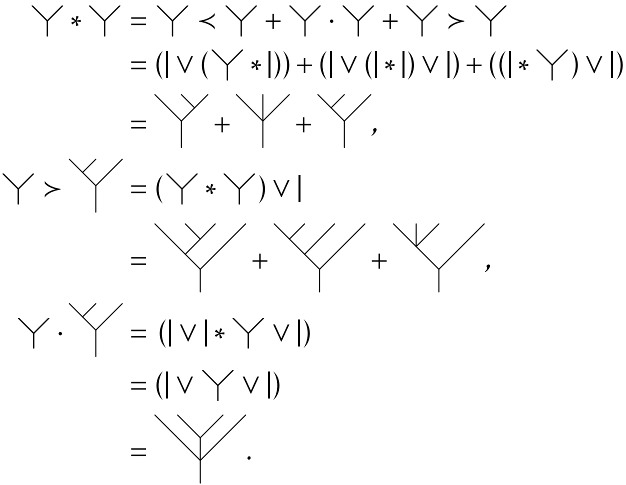




# The free tridendriform algebra

## Some horrible computations





We add to $\mathrm{Tridend}(\mathbb{K})$ a unit for the product $*$ that we will denote by $|$. This is the tree with one leaf, so an element of $T_0$. We define :

$$\mathcal{A} = \mathbb{K}| \oplus \bigoplus_{n \geq 1} \mathbb{K}T_n.$$

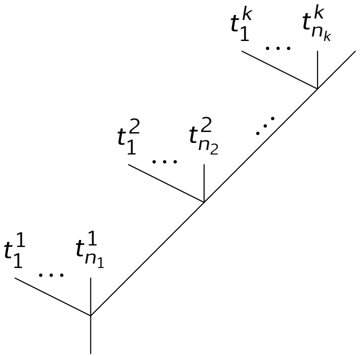
# Plan

# Comb representation of a tree





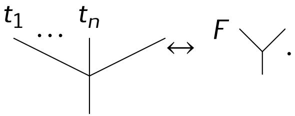
Let $t$ be a tree. We will notice that $t$ can be seen as a « right » comb or a « left » comb :



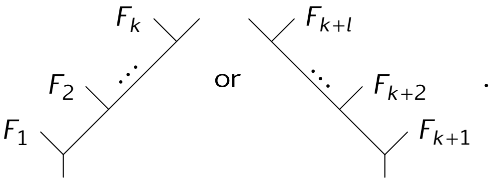

where $k \in \mathbb{N}$.

# Representation of trees as a comb

**Notation :** let $F = t_1 \cdots t_n$ a forest composed of $n$ trees. We identify :




$\Longrightarrow$ All tree $t$ can be seen as :



# Examples of the comb representation

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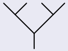
- 

# Examples of the comb representation

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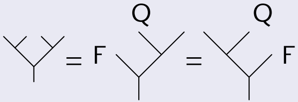
- 

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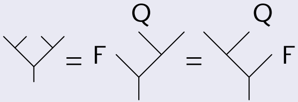
- 

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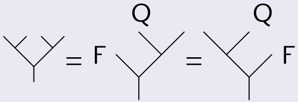
- 

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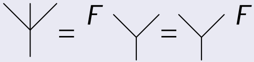
- 

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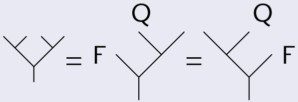
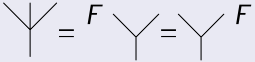
-  where $F =$  and $Q = |$.
-  where $F = ||$.

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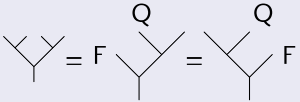
- 

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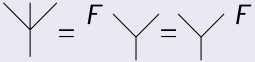
- 

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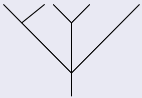
- 

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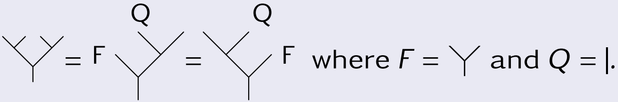
- 

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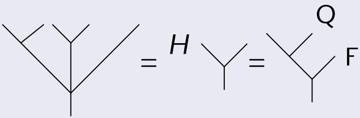
- 

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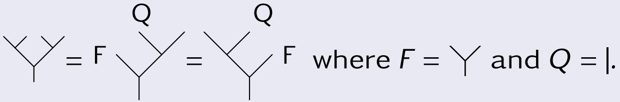
- 

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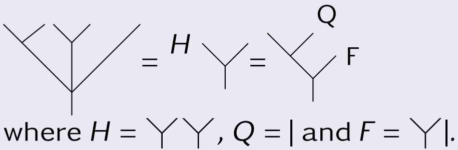
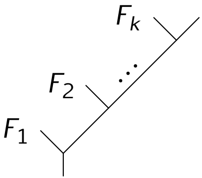
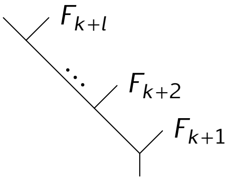
- 

# Action of the quasi-shuffles

Let $t, s$ two trees different from $|$ :

$t =$ 


et

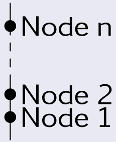
$s =$ 


where for all $i \in [\![1, k+l]\!], F_i$ is a forest.

## Definition

Let $\sigma \in \mathrm{QSh}(k, l)$ such that $\mathrm{Im}(\sigma) = [\![1, n]\!]$ :

1. We start from the following tree :



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## Shuffling trees



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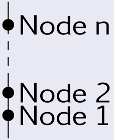


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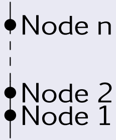


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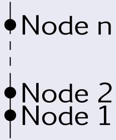


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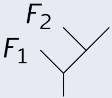
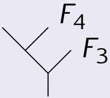
We obtain the tree $\sigma(t,s)$.

# Examples of tree shuffles

## Example

Consider $\sigma = (1\,3\,2\,3) \in \mathrm{QSh}(2,2)$.

Take $t =$ 

and $s =$ 

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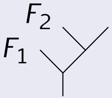
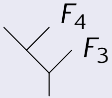
Take $t =$ 

and $s =$ 

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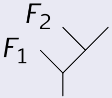
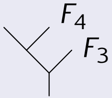
Take $t =$ 

and $s =$ 

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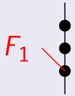
$\sigma(t,s) =$ 

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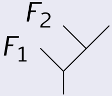
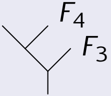
Take $t =$ 

and $s =$ 

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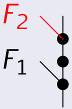
$\sigma(t,s) =$ 

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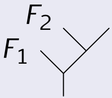
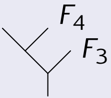
Take $t =$ 

and $s =$ 

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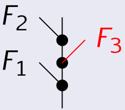
$\sigma(t,s) =$ 

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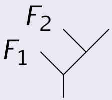
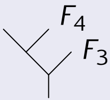
Take $t =$ 

and $s =$ 

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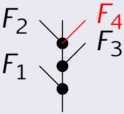
$\sigma(t,s) =$ 

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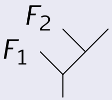
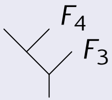
Take $t =$ 

and $s =$ 

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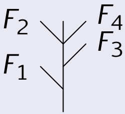
$\sigma(t,s) =$ 

**Theorem (P.C., 2022)**

*Let $t, s$ be two trees different from $|$. Then :*

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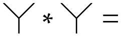

# Description of the products

## The shuffle product for trees

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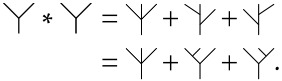

## Corollary

*Let $t, s$ be two trees different from $|$. Then :*

$$t \prec s = \sum_{\substack{\sigma \in QSh(k,l) \\ \sigma^{-1}(\{1\})=\{1\}}} \sigma(t,s),$$

$$t \succ s = \sum_{\substack{\sigma \in QSh(k,l) \\ \sigma^{-1}(\{1\})=\{k+1\}}} \sigma(t,s),$$

$$t \cdot s = \sum_{\substack{\sigma \in QSh(k,l) \\ \sigma^{-1}(\{1\})=\{1,k+1\}}} \sigma(t,s).$$

# Plan

Université Littoral Côte d'Opale

# Plan

# Admissible cuts

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Let $t$ be a tree.

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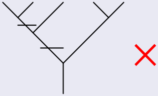
- *admissible cut* : 

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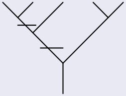
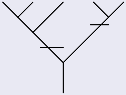
- *admissible cut* :  ✗ ,  ✓ .

We denote $\mathrm{Adm}(t)$ the set of all admissible cuts of $t$.

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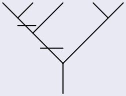
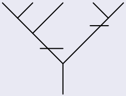
- *admissible cut* :  ✗ ,  ✓ .
- The component of $t$ which owns the root of $t$ is denoted by $R^c(t)$.

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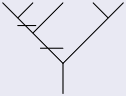
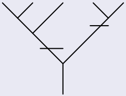
- *admissible cut* :  ×,  ✓.
- The component of $t$ which owns the root of $t$ is denoted by $R^c(t)$.
- Others are denoted by :

$$G_1^c(t), \dots, G_l^c(t),$$

naturally ordered from left to right.

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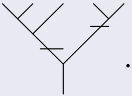
# Example of cut



## Example

Let $t =$  .

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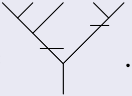
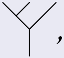
Let $t =$  .

$G_1^c(t) =$ , $G_2^c(t) =$ , $R^c(t) =$ .

# Coproduct description



**Theorem (P.C., 2022)**

*The coproduct of $\mathcal{A}$ such that $(\mathcal{A}, *, |, \Delta, \varepsilon)$ is a graded connected bialgebra preserving the tridendriform structure is given by the following formula for all t a tree :*

$$\Delta(t) = \sum_{c \in \mathrm{Adm}(t)} G^c(t) \otimes R^c(t) + | \otimes t + t \otimes |,$$

$$\Delta(|) = | \otimes |,$$

*where* $G^c(t) = G^c_1(t) * \cdots * G^c_k(t)$.

# Examples of coproducts

## Examples

1. $\Delta\Big($  $\Big) =$

# Examples of coproducts

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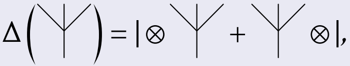
1. 

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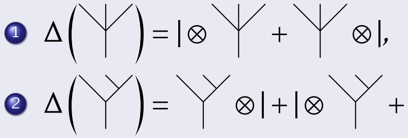

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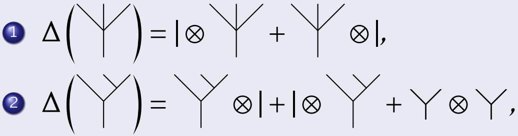

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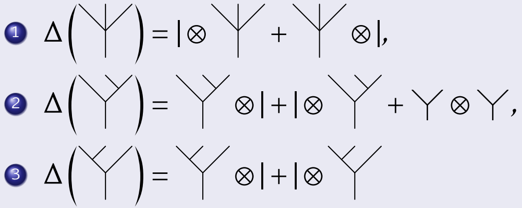
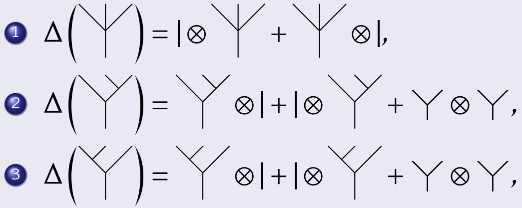


# Examples of coproducts

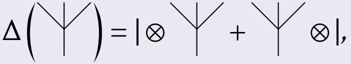
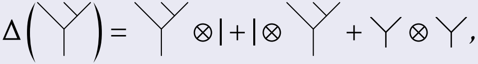
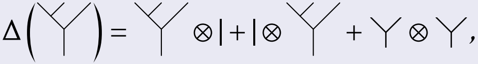
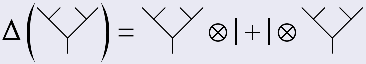
## Examples



# Examples of coproducts



## Examples

1. 
2. 
3. 
   
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# Examples of coproducts

## Examples

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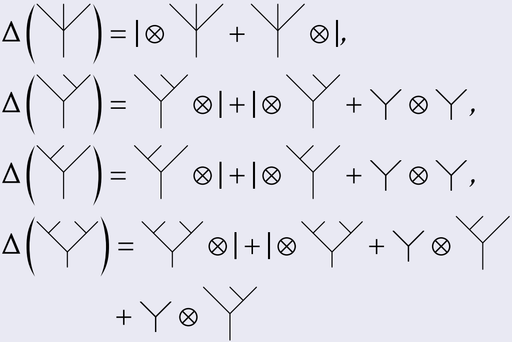

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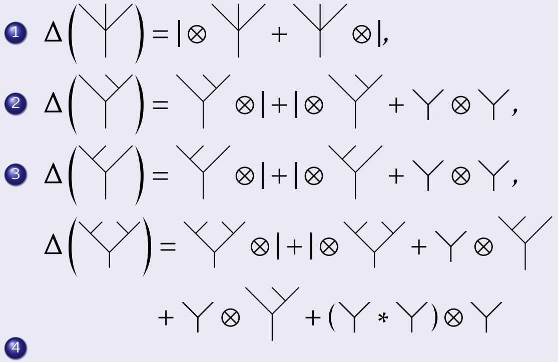

# Examples of coproducts

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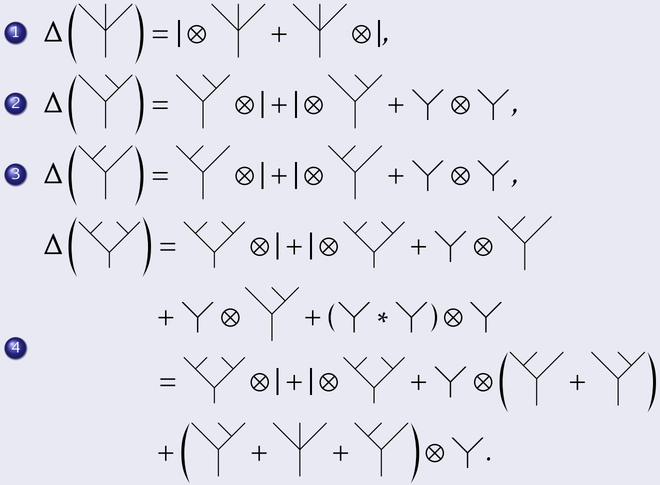
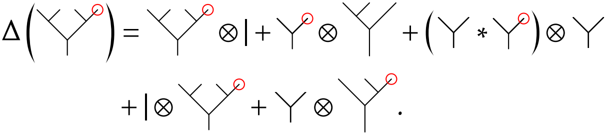


# Plan

# Splitting the coproduct



## Proposition

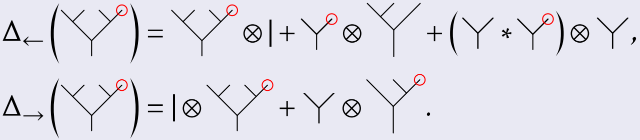
*We endow the tridendriform algebra $(A, \prec, \cdot, \succ)$ with the following half-coproducts :*

$$\Delta_{\leftarrow}(t) = \sum_{c \in \mathrm{Adm}_r(t)} G^c(t) \otimes R^c(t),$$

$$\Delta_{\rightarrow}(t) = \sum_{c \in \mathrm{Adm}_l(t)(t)} G^c(t) \otimes R^c(t).$$

# Examples

## Example



## Definition (Codendriform coalgebra, L.Foissy, 2007)

A *codendriform coalgebra* is a family $(C, \Delta_{\leftarrow}, \Delta_{\rightarrow})$ such that putting $\tilde{\Delta} = \Delta_{\rightarrow} + \Delta_{\leftarrow}$, those applications verify :

$$(\Delta_{\leftarrow} \otimes \mathrm{Id}) \circ \Delta_{\leftarrow} = \left(\mathrm{Id} \otimes \tilde{\Delta}\right) \circ \Delta_{\leftarrow}$$
$$(\Delta_{\rightarrow} \otimes \mathrm{Id}) \circ \Delta_{\leftarrow} = (\mathrm{Id} \otimes \Delta_{\leftarrow}) \circ \Delta_{\rightarrow}$$
$$(\tilde{\Delta} \otimes \mathrm{Id}) \circ \Delta_{\rightarrow} = (\mathrm{Id} \otimes \Delta_{\rightarrow}) \circ \Delta_{\rightarrow}.$$

## Remark

$(\mathcal{A}^+, \Delta_{\rightarrow}, \Delta_{\leftarrow})$ is a codendriform coalgebra.

# Example of $(3,2)$-dendriform bialgebra



## Proposition

$$(H, \prec, \cdot, \succ, |, \Delta_{\leftarrow}, \Delta_{\rightarrow}, \varepsilon)$$

*is a* $(3,2)$*-dendriform bialgebra*

$$\Longleftrightarrow$$

$(H \preceq, \succ, \Delta_{\leftarrow}, \Delta_{\rightarrow})$ *and* $(H \prec, \succeq, \Delta_{\leftarrow}, \Delta_{\rightarrow})$

*are* $(2,2)$*-dendriform bialgebras.*

Moreover :

## Proposition (P.C., 2022)

$(\mathcal{A}, \prec, \cdot, \succ, |, \Delta_{\leftarrow}, \Delta_{\rightarrow}, \varepsilon)$ *is a* $(3,2)$*-dendriform bialgebra.*

# Some technical details



In a $(3,2)$-dendriform algebra :

$$\Delta_{\leftarrow}(a \prec b) = a' * b'_{\leftarrow} \otimes a'' \prec b''_{\leftarrow} + a' * b \otimes a'' + b'_{\leftarrow} \otimes a \prec b''_{\leftarrow} + b \otimes a.$$

# Plan

If $A$ is a bialgebra of finite dimension, then $A^*$ is also a bialgebra of finite dimension.

# Studying the graded dual

## What is it ?



If $A$ is a bialgebra of finite dimension, then $A^*$ is also a bialgebra of finite dimension.
**Issue :** this is false in infinite dimension.

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**Solution :** the *graded dual*.

# Studying the graded dual

## What is it ?

If $A$ is a bialgebra of finite dimension, then $A^*$ is also a bialgebra of finite dimension.

**Issue :** this is false in infinite dimension.

**Solution :** the *graded dual*.

### Definition

If $A = \bigoplus_{n \in \mathbb{N}} A_n$ where $A_n$ is of finite dimensions, we put :

$$A^{\circledast} = \bigoplus_{n \in \mathbb{N}} A_n^*.$$

# Why primitive elements are important ?

## Definition

### Definition (Primitives)

Let $(H, m, 1, \Delta, \varepsilon)$ is a bialgebra. We define for all $x \in H$ :

$$\tilde{\Delta}(x) = \Delta(x) - 1 \otimes x - x \otimes 1.$$

We denote $\mathrm{Prim}(H)$ the following set :

$$\mathrm{Prim}(H) := \left\{ x \in H \,|\, \tilde{\Delta}(x) = 0 \right\}.$$

### Example

 $\in \mathrm{Prim}(\mathcal{A})$ and  $-$  $\in \mathrm{Prim}(\mathcal{A})$.

**Theorem (Cartier-Quillen-Milnor-Moore, 1965)**

*Suppose that $\mathbb{K}$ is a field such that $\mathrm{car}(\mathbb{K}) = 0$. Let $H$ be graded, connected and cocommutative Hopf algebra. Then :*

$$H \approx \mathcal{U}(\mathrm{Prim}(H)).$$

**Remark**

If theorem is false in characteristic $p$.

# Summary

- Studying the graded dual.
- Studying the primitives.
- Studying a quotient space.

# Plan

Tridendriform algebras

Pierre CATOIRE

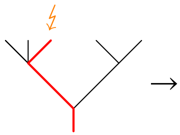


# Plan

# Lightning decomposition of a tree

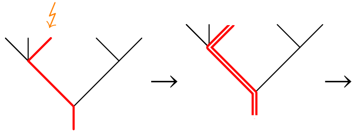
## Examples





# Lightning decomposition of a tree

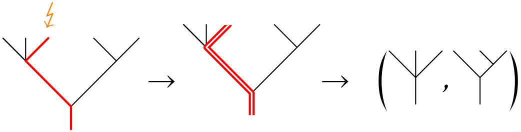
## Examples



# Lightning decomposition of a tree

## Examples





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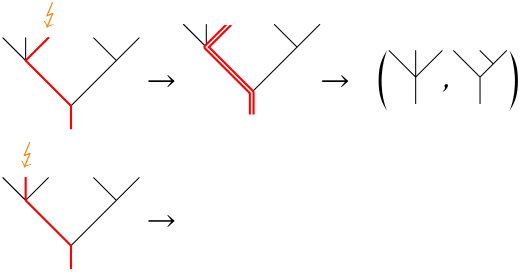
## Examples





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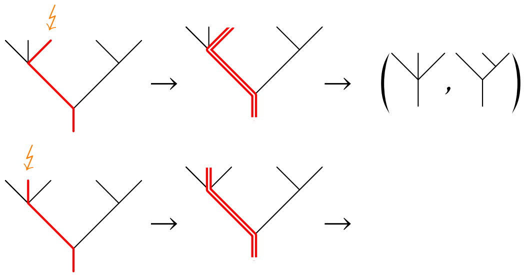
## Examples



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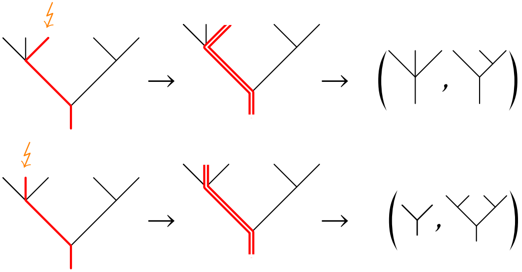

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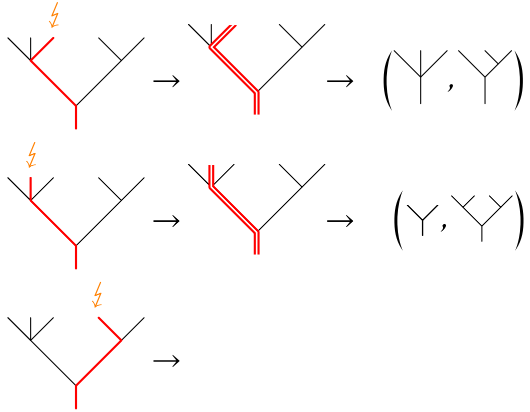
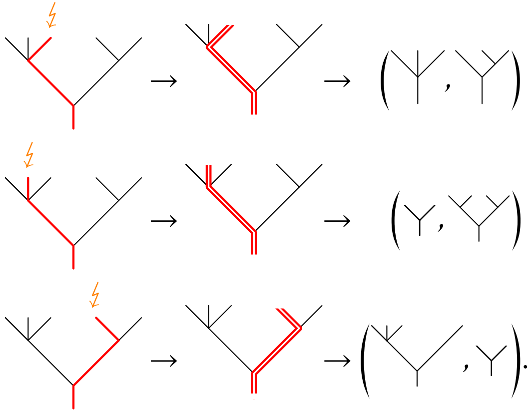
## Examples



# Lightning decomposition of a tree

## Examples



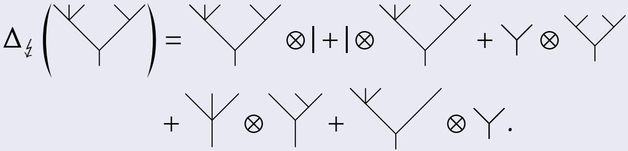


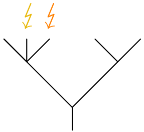
## Definition

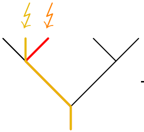
We put for any $t$ a tree, $\mathrm{nl}(t)$ the number of leaves of $t$ :

$$\Delta_{\lightning}(t) = \sum_{i=1}^{\mathrm{nl}(t)} \lightning_i(t)^{(1)} \otimes \lightning_i(t)^{(2)}.$$

## Example



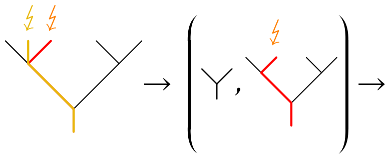




$\rightarrow$

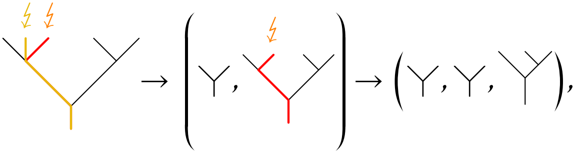
# Successive lightnings
## Coassociativity





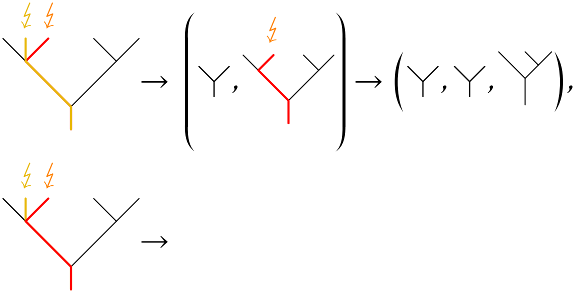
# Successive lightnings

## Coassociativity



# Successive lightnings

## Coassociativity



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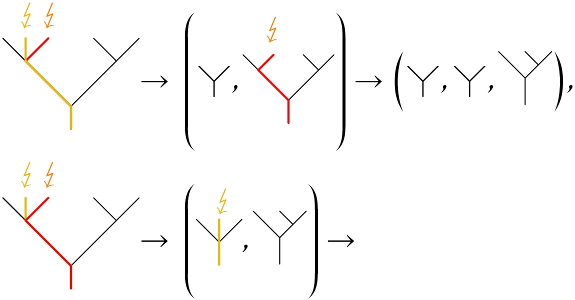
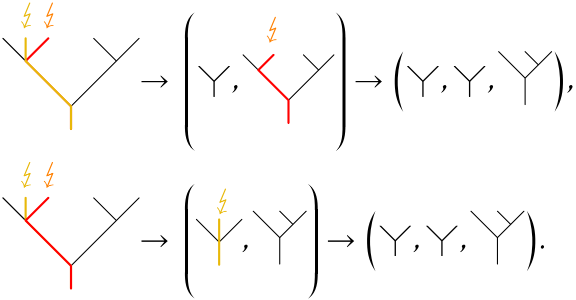


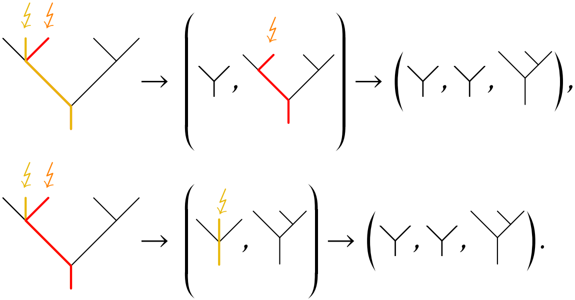
# Successive lightnings

## Coassociativity



# Successive lightnings

## Coassociativity



The operator $\Delta_{\lightning}$ is coassociative.

# The grafting operator over leaves

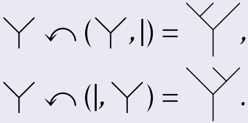
## Definition (grafting operator on leaves)

Let $t$ be a Schröder tree with $k$ leaves and $t_1, \ldots, t_k$ be a family of $k$ trees. We denote $\curvearrowleft$ the *grafting operator over leaves* defined by :

$$t \curvearrowleft (t_1, \ldots, t_k)$$

it is the tree $t$ where we have grafted each $t_i$ on the $i$-th leaf of $t$.

## Examples

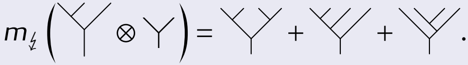


## Definition (grafting lightning product)

Let $s$ and $t$ be two Schröder trees. We define :

$$m_{\lightning}(t \otimes s) = \sum_{(t_1,\ldots,t_{\mathrm{nl}(s)}) \in \lightning^{\circ \mathrm{nl}(s)-1}(t)} s \curvearrowleft (t_1,\ldots,t_{\mathrm{nl}(s)}).$$

## Example



# The lightning bialgebra and its dual

by N.Bergeron et al.



**Proposition (N.Bergeron, S. Rafael González D'Leon, S. X. Li, C.Y. Amy Pang, Y.Vargas, 2021)**

$(\mathcal{A}, m_{\lightning}, |, \Delta_{\lightning}, \varepsilon)$ *is a graded connected bialgebras. This algebra is denoted* TSym.

**Theorem (P.C, 2022)**

$$\mathcal{A}^{\circledast} \approx \mathrm{TSym}^{\mathrm{op}}.$$

# Plan

# Primitives

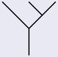
## Left and right primitives



In the case where $\tilde{\Delta} = \Delta_{\leftarrow} + \Delta_{\rightarrow}$ decomposes codendriformly. We define :

$$\mathrm{Prim}_{\leftarrow}(\mathcal{A}) := \left\{ t \in \mathcal{A} \,|\, \tilde{\Delta}_{\leftarrow}(t) = 0 \right\},$$
$$\mathrm{Prim}_{\rightarrow}(\mathcal{A}) := \left\{ t \in \mathcal{A} \,|\, \tilde{\Delta}_{\rightarrow}(t) = 0 \right\},$$
$$\mathrm{Prim}_{\mathrm{Codend}}(\mathcal{A}) := \mathrm{Prim}_{\leftarrow}(\mathcal{A}) \cap \mathrm{Prim}_{\rightarrow}(\mathcal{A}),$$
$$\mathrm{Prim}_{\mathrm{Coass}}(\mathcal{A}) := \left\{ t \in \mathcal{A} \,|\, \tilde{\Delta}(t) = 0 \right\}.$$

### Examples

 $\in \mathrm{Prim}_{\leftarrow}(\mathcal{A})$,  $\in \mathrm{Prim}_{\rightarrow}(\mathcal{A})$,

 $-$  $\in \mathrm{Prim}_{\mathrm{Coass}}(\mathcal{A})$,  $\in \mathrm{Prim}_{\mathrm{Codend}}(\mathcal{A})$.

## Definition

The $n$-th *small Schröder's number*, denoted $a_n$, counts the number of Schröder trees with $n+1$ leaves.
The sequence $(A_n)$ of *large Schröder's numbers* is given for all $n \in \mathbb{N}$ by :

$$A_0 = 0, \qquad A_1 = 1 = A_2, \qquad A_n = 2a_{n-2}.$$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $a_n$ | 0 | 1 | 3 | 11 | 45 | 197 | 903 | 4279 | 20793 | 103049 | 518859 |
| $A_n$ | 0 | 1 | 1 | 2 | 6 | 22 | 90 | 394 | 1806 | 8558 | 41586 |

# Dimensions

## using results of L.Foissy, 2005

### Proposition

*We have for all* $n \in \mathbb{N} \setminus \{0\}$ :

$$\dim(\mathcal{A}_n) = a_n,$$
$$\dim(\mathrm{Prim}_{\mathrm{Codend}}(\mathcal{A})_n) = A_n,$$
$$\dim(\mathrm{Prim}_{\mathrm{Coass}}(\mathcal{A})_n) = A_{n+1}.$$

# Link between the primitives

**Theorem (P.C., 2022)**

*For all $n \in \mathbb{N}$, we put :*

$$\theta_n : \begin{cases} \mathrm{Prim}_{\mathrm{Coass}}(A)_n \otimes \langle \curlyvee \rangle & \to & \mathrm{Prim}_{\mathrm{Codend}}(A)_{n+1} \\ a \otimes \curlyvee & \mapsto & a \cdot \curlyvee . \end{cases}$$

*Then, for all $n \in \mathbb{N}, \theta_n$ is an isomorphism of vector spaces.*

# Generating the coassociative primitives



- Suppose we know $\mathrm{Prim}_{\mathrm{Coass}}(\mathcal{A})_n$.
- We compute $\mathrm{Prim}_{\mathrm{Codend}}(\mathcal{A})_{n+1}$ with the previous theorem.
- A brace algebra computes with $\mathrm{Prim}_{\mathrm{Codend}}(\mathcal{A})_{n+1}$ the elements of $\mathrm{Prim}_{\mathrm{Coass}}(\mathcal{A})_{n+1}$.
[see this paper, M.Ronco, 2001]

# Plan

# Quotient by $\cdot$



We consider :

$$I = \langle x \,|\, \exists s, t \in \mathcal{A}, x = s \cdot t \rangle_{(\prec, \succ, \cdot)}.$$

This space is compatible with the tridendriform and codendriform structures. Then, we can divide.

# Loday-Ronco algebra

**Proposition (P.C, 2022)**

*The tridendriform algebra over $A/_{I}$ satisfy $\cdot = 0$. Moreover, $A/_{I}$ is the Loday-Ronco algebra. (1998)*

# Plan

# Bibliography

See the bibliography of the article.

Number Arxiv= arxiv :[2207.03839].

# The end

Thank you for your attention!

# The end

Thank you for your attention!

Do you have questions?