# Algebraic combinatorics of parking trees

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avec Hélène Han (ENS Saclay), Matthieu Josuat-Vergès (IRIF) et Lucas Randazzo (Nomadic Labs)

Algebraic Combinatorics of the Symmetric Groups and Coxeter Groups II
Cetraro, July 2024

# Outline

1. Parking functions and Catalan objects
2. Species of parking trees
3. Parking and Tamari-parking posets

# Parking functions and Catalan objects

# Outline

# Let's drive and park !

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |

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| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |

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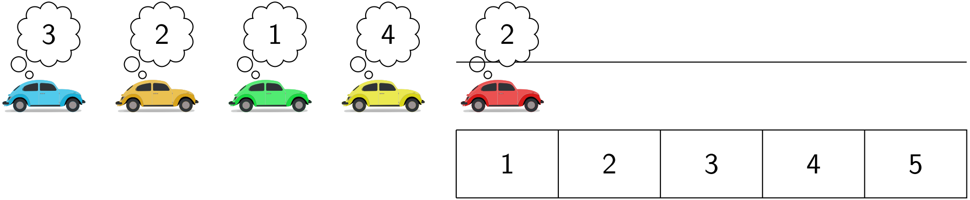

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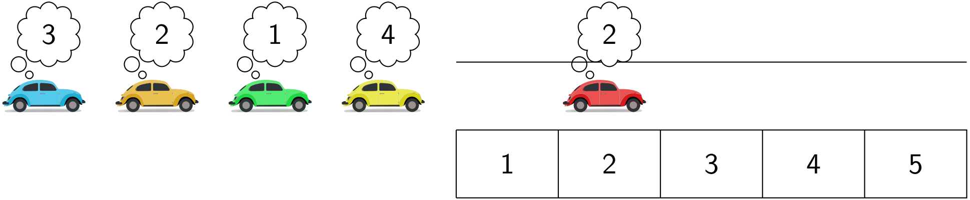

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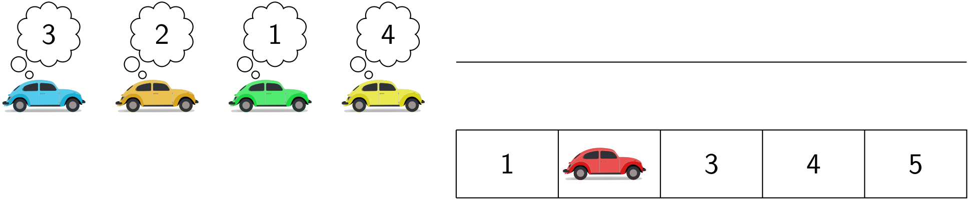

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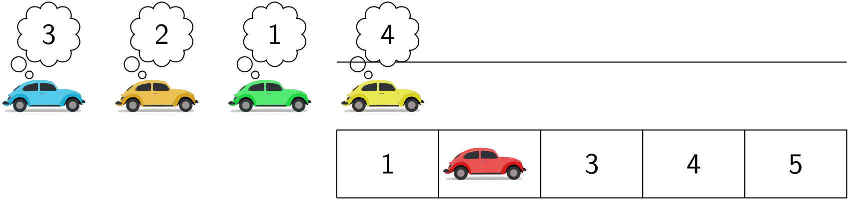

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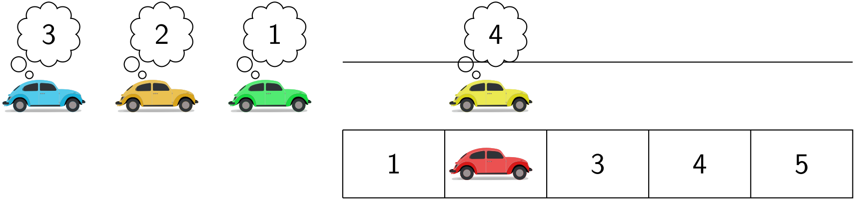

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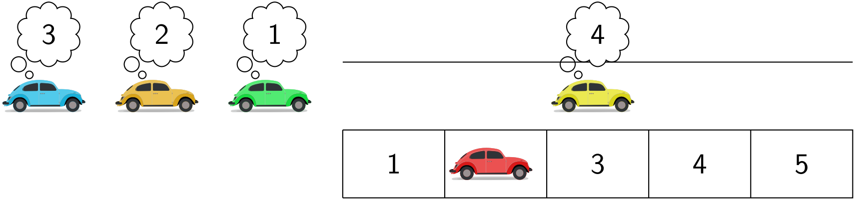

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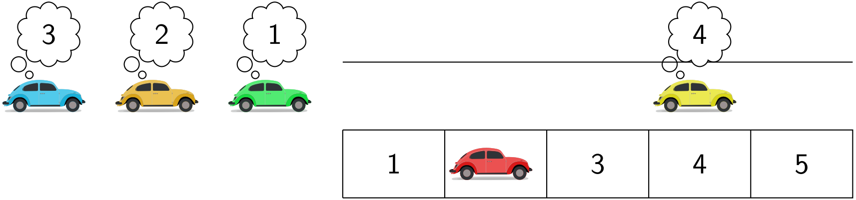

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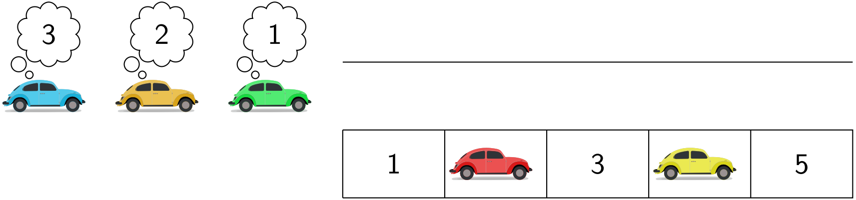

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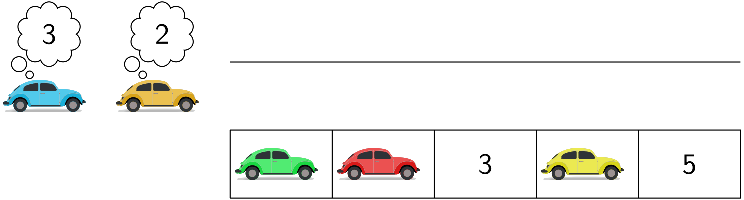

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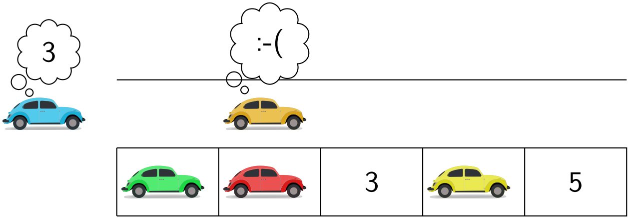

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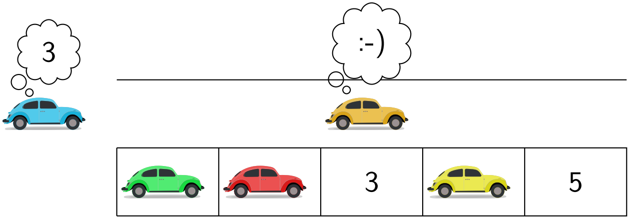

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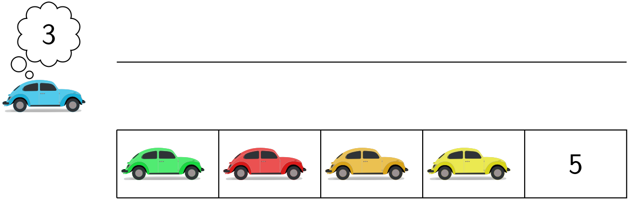

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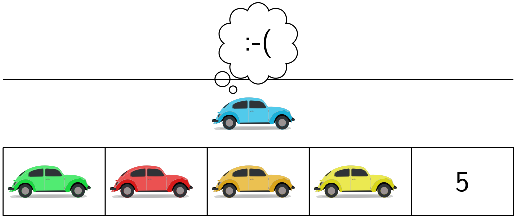

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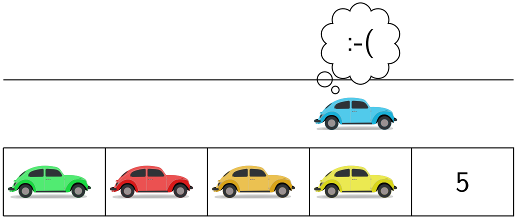

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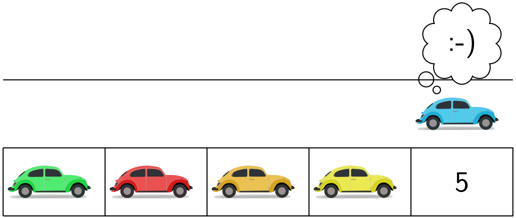

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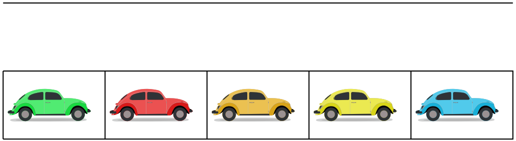

## Definition

If every car can park, the word, obtained by reading from the first car entering the street to the last one, is called a parking function.

## Examples and counter-examples:

| Parking functions of length $n$ | Other words of length $n$ |
|---|---|
| 1 | |
| 11,12,21 | 22 |
| 111, 112, 121, 211, 113, 131, 311,122, 212, 221, 123, 132, 213, 231, 312, 321 | 222, 333, 223, 232, 322, 233, 323, 332, 133, 313, 331 |

# Formal definition of parking functions [Konheim-Weiss, 1966]

## Definition

A sequence $\mathbf{a} = (a_1, \ldots, a_n)$ is a parking function of length $n$ iff

$$|\{i | 1 \leqslant a_i \leqslant j\}| \geqslant j.$$

Denoting by $a^{\uparrow}$ the non-decreasing rearrangement of $\mathbf{a}$, this is equivalent to $1 \leqslant a_j < j$ for any $1 \leqslant j \leqslant n$. We call non-decreasing parking function a parking function satisfying $a = a^{\uparrow}$.

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## Theorem (Konheim-Weiss, 1966; Pollak, 1969)

*There are $(n+1)^{n-1}$ parking functions of length $n$.*
*There are $C_n = \frac{1}{n+1}\binom{2n}{n}$ non-decreasing parking functions of length $n$.*

# Parking functions and Cayley trees

- $(n+1)^{n-1}$ is also the number of Cayley trees on $n+1$ vertices (or equivalently of forest of rooted Cayley trees on $n$ vertices)
- There are several bijections between these objects (see Yan's survey for instance) which enable to refine the enumeration of parking functions with statistics such as displacements, number of lucky cars, ...

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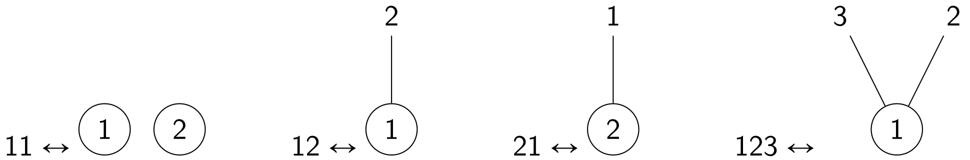
## Pollak's bijection

Consider the Cayley tree associated with the Prüfer code $(c_1, \ldots, c_{n-1})$ where

$$c_i \equiv a_{i+1} - a_i [n+1]$$



# Action of the symmetric group on parking functions and Cayley forests

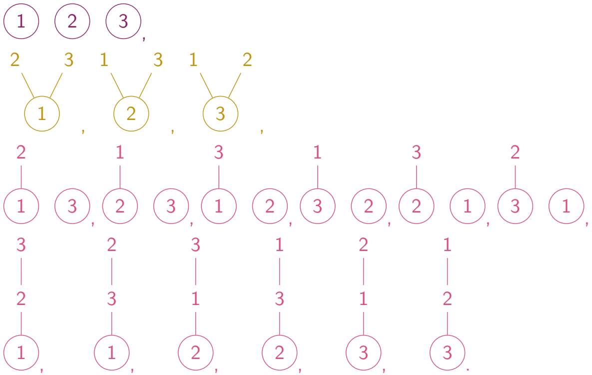
111,

112, 121, 211,

113, 131, 311,

122, 212, 221,

123, 132, 213, 231, 312, 321.



# Non-crossing partitions [Kreweras, 1972]

**Definition**

A (set) partition of $E$ is
$\pi = \{\pi_1, \dots, \pi_k\}$ s.t. :

- $\pi_k \cap \pi_l \neq \emptyset \implies k = l$
- and $\bigcup_{i=1}^{k} \pi_i = E$.

$\Pi_E$ = set of partitions of $E$

**Examples**

$\{1,2\}$, $\{1\}\{2\}$
$\{1\}\{2\}\{3\}$, $\{1,2,3\}$
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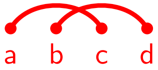
**Definition (Kreweras, 1972)**

A partition $\pi = \{\pi_1, \ldots, \pi_k\}$ of $\{1, \ldots, n\}$ is non-crossing iff

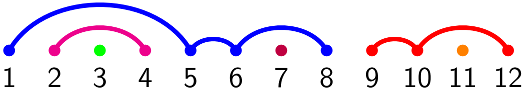
$$\begin{cases} a < b < c < d \\ a, c \in \pi_i \\ b, d \in \pi_j \end{cases} \implies i = j$$

$NC_n$ = set of non-crossing partitions of $\{1, \ldots, n\}$




→ Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$

# Parking functions and non-crossing partitions



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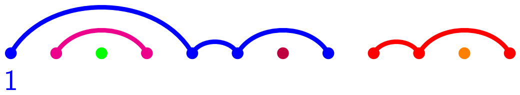


Label $i$ by $\min \pi$ for $i \in \pi$

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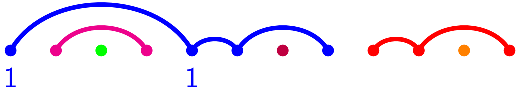


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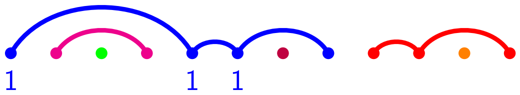


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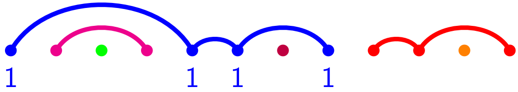


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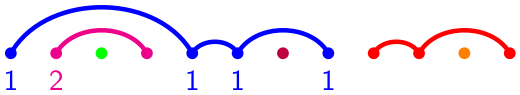


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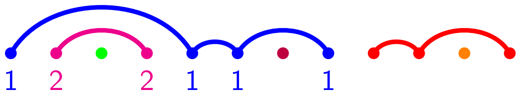


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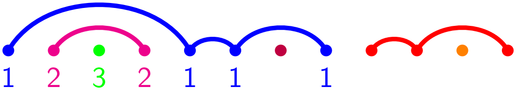


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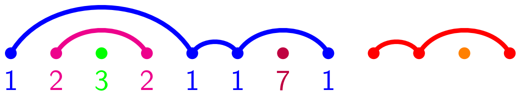


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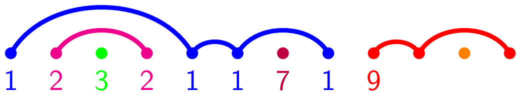


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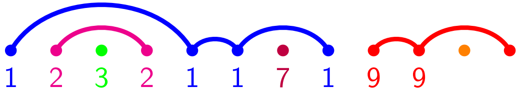


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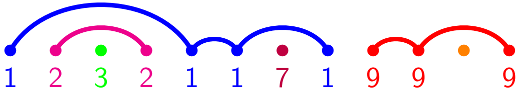


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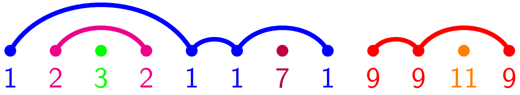


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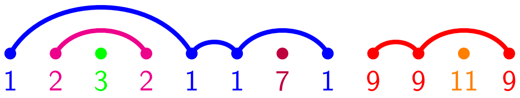


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- Gives a parking function as the label of the $j$th node is smaller or equal to $j$.

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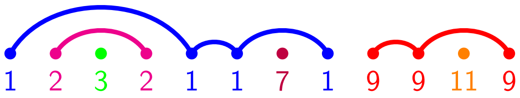


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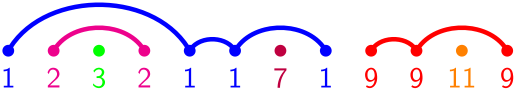
- Gives a parking function as the label of the $j$th node is smaller or equal to $j$.
- It is the unique parking function in the orbit which maximizes the number of lucky cars. Call it the lucky parking function (used by Blass and Sagan to compute the Möbius function of Tamari lattices under the name "left bracket vector").

# Non-crossing 2-partitions (aka. parking functions)

**Definition (Edelman, 1980)**

A n.c. 2-partition of size $n$ is a pair $(\pi, \sigma) \in NCP_n \times \mathfrak{S}_n$ s.t.

$$\begin{cases} \{b_1, \ldots, b_k\} \in \pi \\ b_1 < b_2 < \ldots < b_k \end{cases} \implies \sigma(b_1) < \sigma(b_2) < \ldots < \sigma(b_k).$$




2 6 5 12 9 10 7 11 3 4 1 8

in terms of parking function:

- - - - - - - - - - - -

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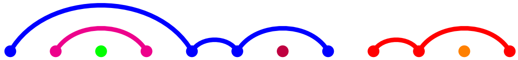

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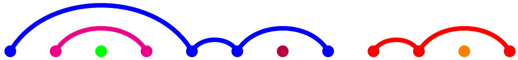

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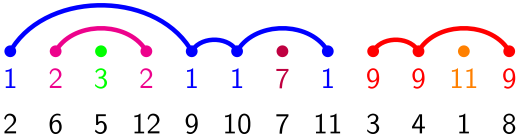


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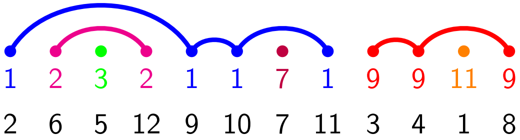


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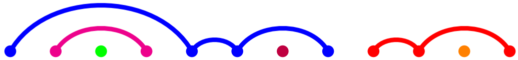

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in terms of parking function:

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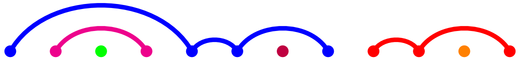

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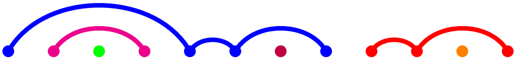


2 6 5 12 9 10 7 11 3 4 1 8

in terms of parking function:

11 1 9 9 3 2 7 9 1 1 1 2

# Parking functions also appear

- as labellings of the shi arrangement
- as labellings of maximal chains in the noncrossing partition poset
- in two posets:
  - the poset of 2-noncrossing partitions [Edelman, 80]
  - the poset of Tamari-parking, linked with the study of diagonal coinvariants [Chapuy–Bousquet-Mélou–Préville-Ratelle, 13]

# Parking functions also appear

- as labellings of the shi arrangement
- as labellings of maximal chains in the noncrossing partition poset

**Our goal for today:**

- in two posets:
  - the poset of 2-noncrossing partitions [Edelman, 80]
  - the poset of Tamari-parking, linked with the study of diagonal coinvariants [Chapuy–Bousquet-Mélou–Préville-Ratelle, 13]

# Species of parking trees

# Outline

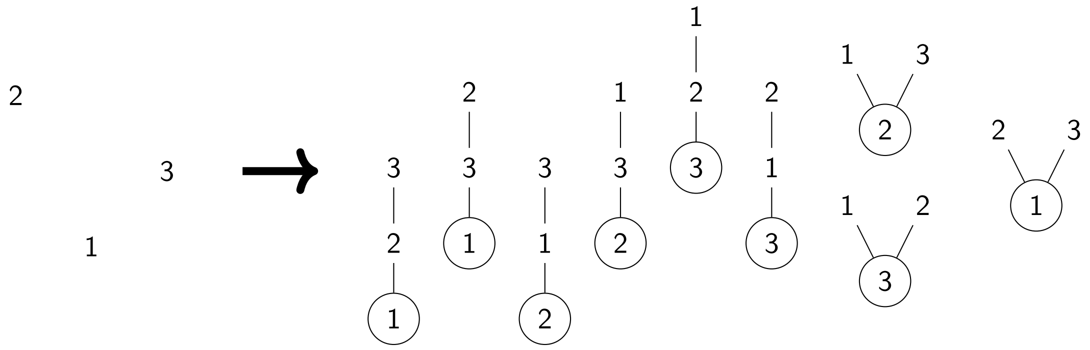
# What are species?

**Definition (Joyal, 80s; cited from Bergeron-Labelle-Leroux)**

A species F is a functor from Bij to Set. To a finite set $S$, the species F associates a finite set F($S$) such that any bijection $\sigma : S \to T$ gives rise to a map $F(\sigma) : F(S) \to F(T)$ satisfying

$$\forall \sigma : S \to T, \tau : T \to U, F(\tau \circ \sigma) = F(\tau) \circ F(\sigma), \qquad F(Id_S) = Id_{F(S)}.$$

Species = Construction plan, such that the obtained set is invariant by relabelling



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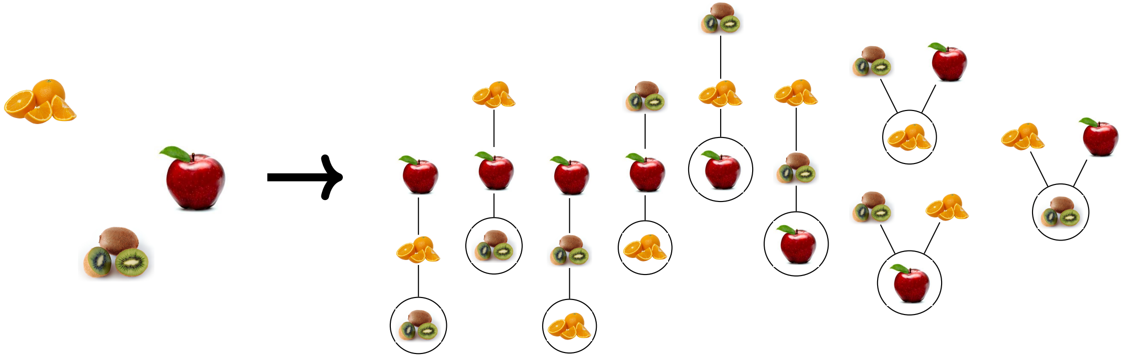
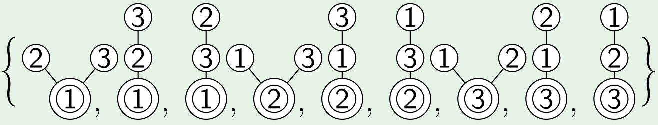
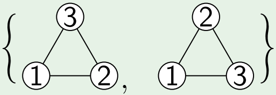


## Examples of species

- $\{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$ (Species of lists $\mathbb{L}$ on $\{1,2,3\}$)
- $\{\{1,2,3\}\}$ (Species of non-empty sets $\mathbb{E}^+$)
- $\{\{1\},\{2\},\{3\}\}$ (Species of pointed sets $\mathbb{E}^\bullet$)
-  (Species of Cayley trees $\mathbb{T}$)
-  (Species of cycles)

These sets are the image by species of the set $\{1,2,3\}$.

# Why do we need species ?

Let $F$ and $G$ be two species.

- $(F + G)(I) = F(I) \sqcup G(I)$,

- $(F \times G)(I) = \bigsqcup_{I_1 \sqcup I_2 = I} F(I_1) \times G(I_2)$.

# Cycle index series

**Definition**

Given a finite set $V$ of size $n$, the cycle type of a permutation $\sigma \in \mathfrak{S}_V$ is the tuple $(\sigma_1, \dots, \sigma_n)$ , where $\sigma_k$ is the number of cycles of type $k$ in the decomposition of $\sigma$ into disjoint cycles.

**Examples**

The cycle type of $(123)(4)(567)$ is $(1, 0, 2, 0, 0, 0, 0)$.

**Definition**

The cycle index series of a species $F$ is the formal power series

$$Z_F(p_1, \dots, p_n, \dots) = \sum_{n \geqslant 0} \sum_{\sigma} \operatorname{fix} F(\sigma) \frac{p_\sigma}{z_\sigma}, \tag{1}$$

where the sum runs over a set of representatives of each cycle type of $\mathfrak{S}_n$, $p_\sigma = p_1^{\sigma_1} \dots p_n^{\sigma_n}$ and $z_\sigma = \prod_{i \geqslant 1} i^{p_i} \times p_i!$

# Cycle index series of usual species

## Definition

The cycle index series of a species $F$ is the formal power series

$$Z_F(p_1, \dots, p_n, \dots) = \sum_{n \geqslant 0} \sum_{\sigma} \operatorname{fix} F(\sigma) \frac{p_\sigma}{z_\sigma}, \quad (2)$$

where the sum runs over a set of representatives of each cycle type of $\mathfrak{S}_n$, $p_\sigma = p_1^{\sigma_1} \dots p_n^{\sigma_n}$ and $z_\sigma = \prod_{i \geqslant 1} i^{p_i} \times p_i!$

## Examples

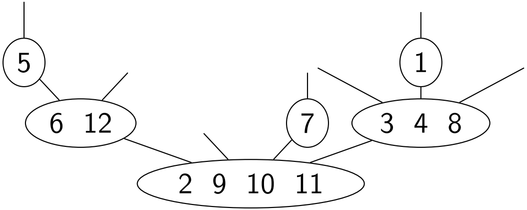
$Z_{\mathbb{L}} = \frac{1}{1-p_1}$,
$Z_{\mathbb{E}} = \exp(\sum_{i \geqslant 1} \frac{p_i}{i})$
$Z_{\mathbb{E}^\bullet} = p_1 \exp(\sum_{i \geqslant 1} \frac{p_i}{i})$
$Z_{\mathbb{T}} = p_1 \exp(Z_{\mathbb{T}})$

# Parking trees

**Definition**

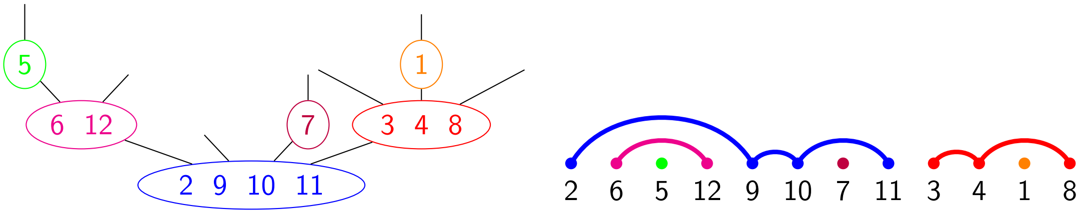
A parking tree on a set $L$ is a rooted plane tree $T = (V, E, r)$ such that:

- $V \in \Pi_L$,
- $v \in V$ has $|v|$ children.

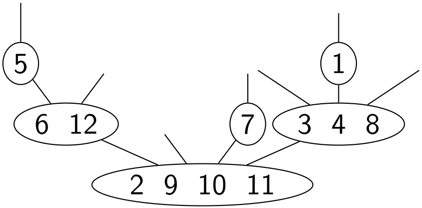



Why parking ?

# Bijection between 2-noncrossing partition and parking trees



# Functional equation parking trees




**Proposition (DO, Josuat-Vergès, Randazzo, 21)**

$$\mathcal{P}_f = \sum_{p \geqslant 1} \mathbb{E}_p \times (1 + \mathcal{P}_f)^p$$

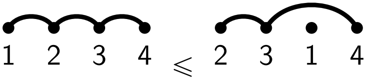
# Parking and Tamari-parking posets

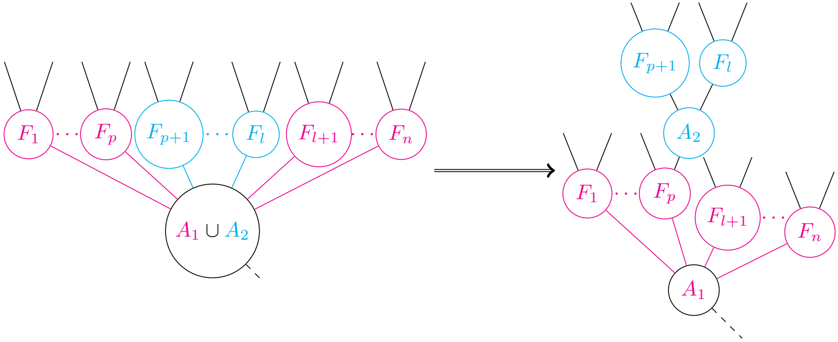
# Outline

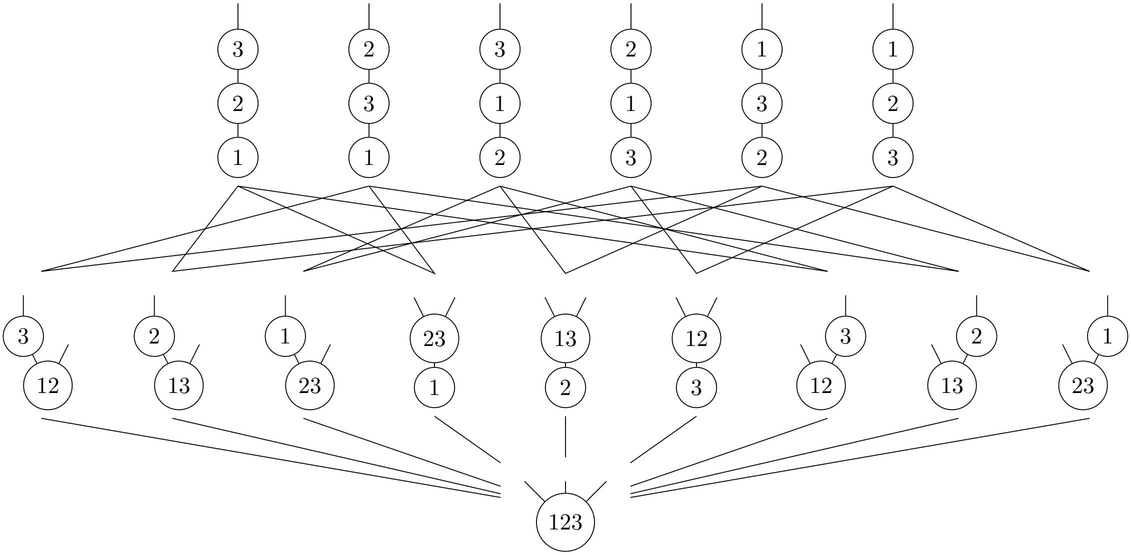
# 2-noncrossing partitions poset

Covering relation in $\Pi^2$ : merge parts and rearrange labels to respect the increasing condition

Example :

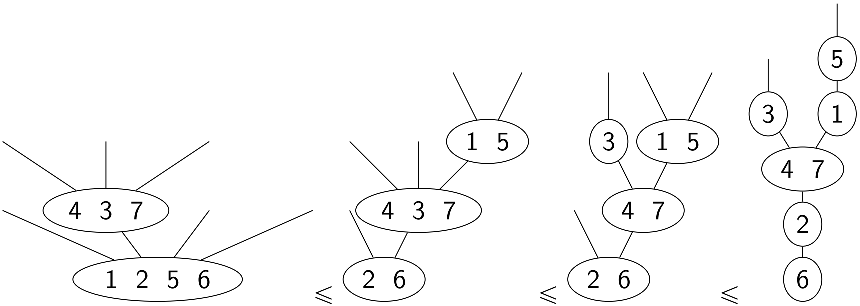


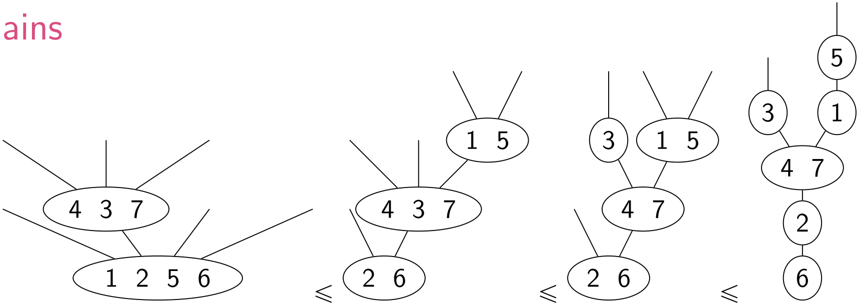

3 2 3 2 1 1
2 3 1 1 3 2
1 1 2 3 2 3
3 2 1 23 13 12 3 2 1
12 13 23 1 2 3 12 13 23
123

# Results

- This poset is a lattice
- When restricting to right combs, get the face poset of the permutohedron
- New criterion to prove shellability !
- Enumeration of (weak) k-chains



# $k$-weak chains




**Proposition (DO, Josuat-Vergès, Randazzo, 21)**

$$\mathcal{C}^{l}_{k,t} = \sum_{p \geqslant 1} \mathcal{C}^{l,p}_{k-1,t} \times \left(t\mathcal{C}^{l}_{k,t} + 1\right)^{p}$$

*Chains $\phi_1 \leqslant \cdots \leqslant \phi_k$ in $\Pi^2{}_n$ are in bijection with $k$-parking trees.*
*The number of chains $\phi_1 \leqslant \cdots \leqslant \phi_k$ in $\Pi^2{}_n$ where $\mathrm{rk}(\phi_k) = \ell$ is:*

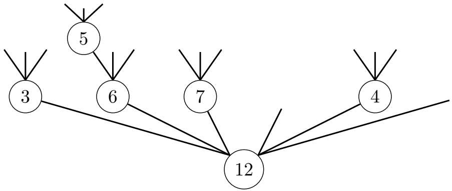
$$\ell! \binom{kn}{\ell} S_2(n, \ell + 1).$$

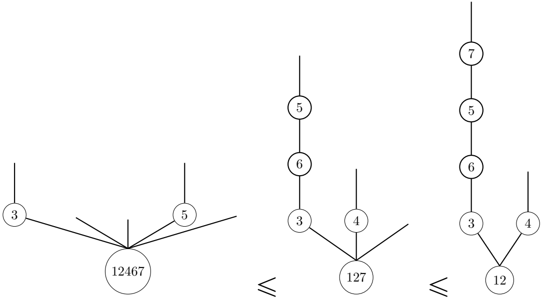
# $k$-parking tree

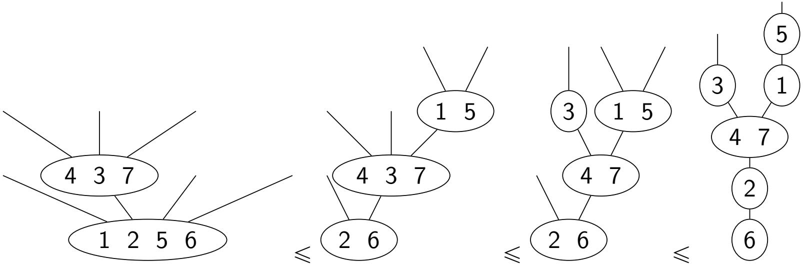
## Definition

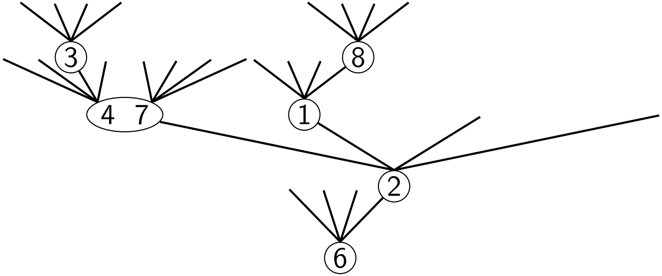
A $k$-parking tree on a set $L$ is a rooted plane tree $T = (V, E, r)$ such that:

- $V \in \Pi_L$
- $v \in V$ has $k|v|$ children.

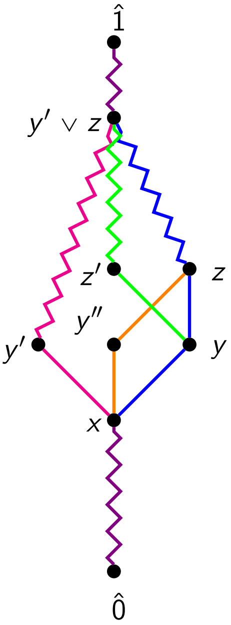



4 3 7
1 2 5 6
≤
1 5
4 3 7
2 6
≤
3
1 5
4 7
2 6
≤
5
3
1
4 7
2
6


3
8
4 7
1
2
6

# Shelling




**Lemma**

*Let $x, y, y', z \in \Pi^2{}_n$ such that $x \lessdot y \lessdot z$, $x \lessdot y'$, and $y' \prec_x y$. Then:*

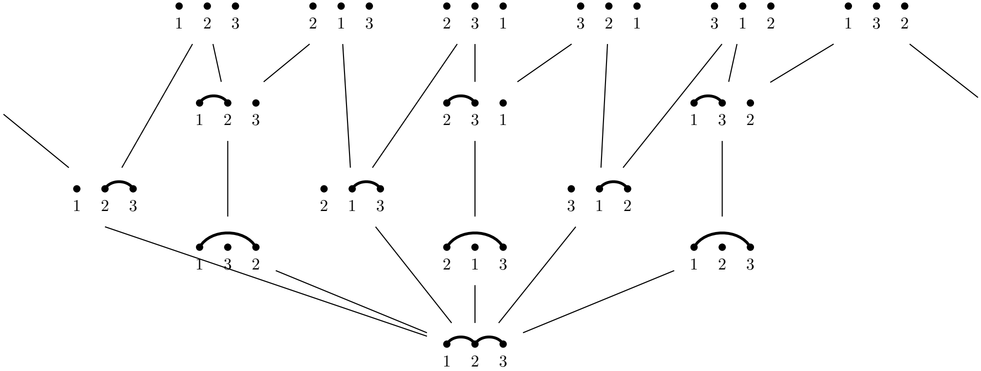
- *either there exists $y'' \in \Pi^2{}_n$ such that $x \lessdot y'' \lessdot z$ and $y'' \prec_x y$,*
- *or there exists $z' \in \Pi^2{}_n$ such that $y \lessdot z' \leqslant y' \vee z$ and $z' \prec_y z$.*

# Tamari-parking poset

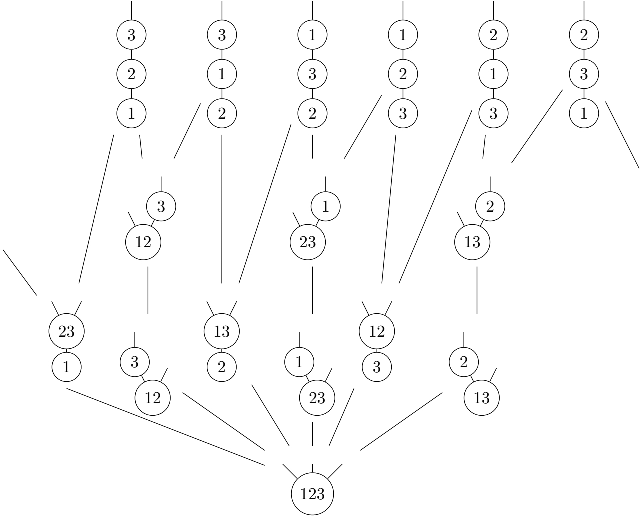
Covering relations given by:

- Moving a block to an arch to the left in the same part 

- Merging parts if there leftmost elements are adjacents 

# Tamari-parking poset



# Known results and open questions

**Proposition (Chapuy–Bousquet-Mélou–Préville-Ratelle, 12)**

*Say that two intervals in Tamari-parking posets are isomorphic if they have the same minimum element and maximal elements of the same shape.*
*The number of class of isomorphisms of intervals is given by:*

$$2^n(n+1)^{n-2}.$$

The action of the symmetric group on these isomorphisms class of intervals are likely to be the same as the one on the space of diagonal coinvariants in three sets of $n$ variables.

**Conjecture (DO)**

Augmented Tamari-parking posets are homotopic to a sphere.

**Proposition (H. Han)**

*Tamari-parking posets are lattices. They are neither EL-shellable nor CL-shellable.*

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Thank you for your attention !