Natural Hopf algebras and polynomial realizations through related alphabets

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Algebraic Combinatorics and Finite Groups III

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Objectives

Present a **polynomial realization** of some Hopf algebras constructed from operads.

Main points:

1. Combinatorial Hopf algebras.
2. Polynomial realizations.
3. Nonsymmetric operads.
4. Natural Hopf algebras of nonsymmetric operads.
5. Polynomial realization of natural Hopf algebras of free operads.
6. Polynomial realization of natural Hopf algebras of non-free operads.

Combinatorial Hopf algebras

All algebraic structures are over a field $\mathbb{K}$ of characteristic zero.

A combinatorial Hopf algebra (CHA) $\mathcal{H}$ is a graded vector space decomposing as

$$\mathcal{H} = \bigoplus_{n \in \mathbb{N}} \mathcal{H}(n)$$

such that $\dim \mathcal{H}(0) = 1$ and each $\mathcal{H}(n)$ is finite dimensional, and endowed with

- an associative unital graded **product**

$$\star : \mathcal{H}(n_1) \otimes \mathcal{H}(n_2) \to \mathcal{H}(n_1 + n_2)$$

- a coassociative counital cograded **coproduct**

$$\Delta : \mathcal{H}(n) \to \bigoplus_{n = n_1 + n_2} \mathcal{H}(n_1) \otimes \mathcal{H}(n_2)$$

such that

$$\Delta(x_1 \star x_2) = \Delta(x_1) \mathbin{\bar{\star}} \Delta(x_2).$$

Let $\mathbf{WQSym}$ be the space such that $\mathbf{WQSym}(n)$ is the linear span of $\mathcal{P}(n)$, the set of **packed words** of size n (words on $[n]$ where each letter from 1 to n appears at least once, like 13223 but not 131).

The set $\{\mathsf{M}_{\mathfrak{p}} : \mathfrak{p} \in \mathcal{P}\}$ is a basis of $\mathbf{WQSym}$.

Let $\star$ be the convolution product on $\mathbf{WQSym}$.

Example — Product of $\mathbf{WQSym}$ on the M-basis

$$\mathsf{M}_{11} \star \mathsf{M}_{121} = \mathsf{M}_{11121} + \mathsf{M}_{11232} + \mathsf{M}_{22121} + \mathsf{M}_{22131} + \mathsf{M}_{33121}$$

Let Δ be the packed unshuffling coproduct on $\mathbf{WQSym}$.

Example — Coproduct of $\mathbf{WQSym}$ on the M-basis

$$\Delta(\mathsf{M}_{2312411}) = \mathsf{M}_{\epsilon} \otimes \mathsf{M}_{2312411} + \mathsf{M}_{111} \otimes \mathsf{M}_{1213} + \mathsf{M}_{21211} \otimes \mathsf{M}_{12} + \mathsf{M}_{231211} \otimes \mathsf{M}_{1} + \mathsf{M}_{2312411} \otimes \mathsf{M}_{\epsilon}$$

$$[1, 2], [3, 4]$$

$$21211, 34$$

This is the CHA of word quasi-symmetric functions [Hivert, 1999].

Polynomial realizations

For any alphabet A, let $\mathbb{K}\langle A\rangle$ be the space of noncommutative polynomials on A having a possibly **infinite support** but a **finite degree**.

Example — Some noncommutative polynomials

Set $A_{\mathbb{N}} := \{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \ldots\}$.

- An element of $\mathbb{K}\langle A_{\mathbb{N}}\rangle$:

$$\sum_{0 \leqslant i_1 < i_2} \mathbf{a}_{i_1}\mathbf{a}_{i_2} = \mathbf{a}_0\mathbf{a}_1 + \mathbf{a}_0\mathbf{a}_2 + \cdots + \mathbf{a}_1\mathbf{a}_2 + \mathbf{a}_1\mathbf{a}_3 + \cdots$$

- An element which is not in $\mathbb{K}\langle A_{\mathbb{N}}\rangle$:

$$\sum_{n \geqslant 0} \mathbf{a}_0^n = 1 + \mathbf{a}_0 + \mathbf{a}_0^2 + \mathbf{a}_0^3 + \cdots$$

The space $\mathbb{K}\langle A\rangle$, endowed with the product of noncommutative polynomials, is a unital associative algebra.

A polynomial realization of a CHA $\mathcal{H}$ is a map

$$\mathsf{r}_A : \mathcal{H} \to \mathbb{K}\langle A \rangle$$

defined for any alphabet A of C, a class of alphabets possibly endowed with n-ary relations, such that

1. r_A is a graded unital associative **algebra morphism**;
2. there exists an alphabet $\mathbb{A}$ of C such that $\mathsf{r}_{\mathbb{A}}$ is **injective**;
3. there exists a **sum operation** $\dotplus$ on C such that for any $x \in \mathcal{H}$ and any alphabets A_1 and A_2 of C,

   $$\mathsf{r}_{A_1 \dotplus A_2}(x) = (\mathsf{r}_{A_1} \otimes \mathsf{r}_{A_2}) \circ \Delta(x),$$

   where the variables of A_1 and A_2 are considered **mutually commuting** in $\mathbb{K}\langle A_1 \dotplus A_2 \rangle$.

Point 3. offers a way to compute the coproduct of $\mathcal{H}$ by expressing the realization of x on the sum of two alphabets. This is the alphabet doubling trick.

Let A be an alphabet **endowed with a total order** $\preccurlyeq$.

The packing of $u \in A^*$ is the word of positive integers $\mathrm{pck}(u)$ such that

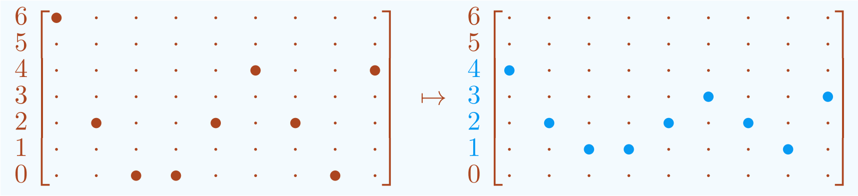
$$\mathrm{pck}(u)_i = \#\{u_j : u_j \preccurlyeq u_i\}.$$

$\mathrm{pck}(u)$ is the packed word obtained by projecting u on the segment $[1, \max(u)]$.

Example — Packing of a word

Let, on the alphabet $A_{\mathbb{N}}$, the total order relation $\preccurlyeq$ satisfying $\mathbf{a}_{i_1} \preccurlyeq \mathbf{a}_{i_2}$ iff $i_1 \leqslant i_2$.

$$\mathrm{pck}(\mathbf{a}_6\ \mathbf{a}_2\ \mathbf{a}_0\ \mathbf{a}_0\ \mathbf{a}_2\ \mathbf{a}_4\ \mathbf{a}_2\ \mathbf{a}_0\ \mathbf{a}_4) = \mathrm{pck}(6\ 2\ 0\ 0\ 2\ 4\ 2\ 0\ 4) = 4\ 2\ 1\ 1\ 2\ 3\ 2\ 1\ 3$$



Let $\mathfrak{p} \in \mathcal{P}$. A word $u \in A^*$ is $\mathfrak{p}$-compatible, denoted by $u \Vdash^A \mathfrak{p}$, if $\mathrm{pck}(u) = \mathfrak{p}$.

Let $\mathsf{r}_A : \mathbf{WQSym} \to \mathbb{K}\langle A \rangle$ be the map defined by

$$\mathsf{r}_A(\mathsf{M}_{\mathfrak{p}}) := \sum_{\substack{u \in A^* \\ u \Vdash^A \mathfrak{p}}} u.$$

Example — The polynomial of a basis element

$$\mathsf{r}_{A_{\mathbb{N}}}(\mathsf{M}_{3121}) = \sum_{\ell_1 < \ell_2 < \ell_3 \in \mathbb{N}} \mathbf{a}_{\ell_3}\mathbf{a}_{\ell_1}\mathbf{a}_{\ell_2}\mathbf{a}_{\ell_1} = \mathbf{a}_2\mathbf{a}_0\mathbf{a}_1\mathbf{a}_0 + \mathbf{a}_3\mathbf{a}_0\mathbf{a}_1\mathbf{a}_0 + \mathbf{a}_3\mathbf{a}_0\mathbf{a}_2\mathbf{a}_0 + \mathbf{a}_3\mathbf{a}_1\mathbf{a}_2\mathbf{a}_1 + \cdots$$

The sum $A_1 \mathbin{\#} A_2$ of the totally ordered alphabets A_1 and A_2 is the **disjoint ordinal sum** of A_1 and A_2.

Theorem [Novelli, Thibon, 2006]

The map r_A is a polynomial realization of $\mathbf{WQSym}$.

Example — An alphabet doubling in WQSym

$$\mathsf{r}_{A_1 \# A_2}(\mathsf{M}_{2131}) = \sum_{\substack{u \in (A_1 \# A_2)^* \\ \mathrm{pck}(u) = 2131}} u = \sum_{\substack{u_1, u_2, u_3 \in A_1 \# A_2 \\ u_1 \prec u_2 \prec u_3}} u_2 u_1 u_3 u_1$$

$$= \sum_{\substack{u_1, u_2, u_3 \in A_1 \\ u_1 \prec u_2 \prec u_3}} u_2 u_1 u_3 u_1 + \sum_{\substack{u_1, u_2 \in A_1, u_3 \in A_2 \\ u_1 \prec u_2 \prec u_3}} u_2 u_1 u_3 u_1 + \sum_{\substack{u_1, u_3 \in A_1, u_2 \in A_2 \\ u_1 \prec u_2 \prec u_3}} u_2 u_1 u_3 u_1 + \sum_{\substack{u_2, u_3 \in A_1, u_1 \in A_2 \\ u_1 \prec u_2 \prec u_3}} u_2 u_1 u_3 u_1$$

$$+ \sum_{\substack{u_1 \in A_1, u_2, u_3 \in A_2 \\ u_1 \prec u_2 \prec u_3}} u_2 u_1 u_3 u_1 + \sum_{\substack{u_2 \in A_1, u_1, u_3 \in A_2 \\ u_1 \prec u_2 \prec u_3}} u_2 u_1 u_3 u_1 + \sum_{\substack{u_3 \in A_1, u_1, u_2 \in A_2 \\ u_1 \prec u_2 \prec u_3}} u_2 u_1 u_3 u_1 + \sum_{\substack{u_1, u_2, u_3 \in A_2 \\ u_1 \prec u_2 \prec u_3}} u_2 u_1 u_3 u_1$$

$$= \mathsf{r}_{A_1}(\mathsf{M}_{2131}) \otimes \mathsf{r}_{A_2}(\mathsf{M}_{\epsilon}) + \mathsf{r}_{A_1}(\mathsf{M}_{211}) \otimes \mathsf{r}_{A_2}(\mathsf{M}_{1}) + 0 + 0$$

$$+ \mathsf{r}_{A_1}(\mathsf{M}_{1}) \otimes \mathsf{r}_{A_2}(\mathsf{M}_{12}) + 0 + 0 + \mathsf{r}_{A_1}(\mathsf{M}_{\epsilon}) \otimes \mathsf{r}_{A_2}(\mathsf{M}_{2131})$$

$$= (\mathsf{r}_{A_1} \otimes \mathsf{r}_{A_2}) \circ \Delta(\mathsf{M}_{2131})$$

There are many CHAs defined on linear spans of **various families** of combinatorial objects endowed with very **different products and coproducts**, admitting polynomials realizations (very incomplete list, sorry):

- **NCSF**, the noncommutative symmetric functions CHA [**Gelfand, Krob, Lascoux, Leclerc, Retakh, Thibon**, 1995];
- **FQSym**, the Malvenuto-Reutenauer CHA [**Malvenuto, Reutenauer**, 1995], [**Duchamp, Hivert, Thibon**, 2002];
- $\mathbf{PQSym}^{\star}$, the dual parking functions CHA [**Novelli, Thibon**, 2007];
- CK and **NCK**, the commutative and noncommutative Connes-Kreimer CHAs [**Connes, Kreimer**, 1998], [**Foissy**, 2002], [**Foissy, Novelli, Thibon**, 2014];
- $\mathbf{H}_{\mathcal{FG}}$, the CHA on Feynman graphs [**Foissy**, 2020].

Polynomials realizations are interesting at least because

1. they provide a **unified encoding** of these CHAs as spaces of polynomials;
2. they provide families of polynomials **generalizing symmetric functions**.

Nonsymmetric operads

A nonsymmetric operad (operad) is a set

$$\mathcal{O} = \bigsqcup_{n \in \mathbb{N}} \mathcal{O}(n)$$

endowed with

- a unit $\mathbb{1} \in \mathcal{O}(1)$;
- a composition map $-[-, \dots, -] : \mathcal{O}(n) \times (\mathcal{O}(m_1) \times \cdots \times \mathcal{O}(m_n)) \to \mathcal{O}(m_1 + \cdots + m_n)$

such that

$$\mathbb{1}[x] = x = x[\mathbb{1}, \dots, \mathbb{1}]$$

and

$$x[y_1, \dots, y_n][z_{1,1}, \dots, z_{1,m_1}, \dots, z_{n,1}, \dots, z_{n,m_n}] = x[y_1[z_{1,1}, \dots, z_{1,m_1}], \dots, y_n[z_{n,1}, \dots, z_{n,m_n}]].$$

The arity $\mathrm{ar}(x)$ of $x \in \mathcal{O}$ is the unique integer n such that $x \in \mathcal{O}(n)$.

Let $\mathcal{O}$ be an operad.

An element $x \in \mathcal{O}(n)$ is finitely factorizable if the set of pairs $(y, (z_1, \dots, z_n))$ satisfying

$$x = y[z_1, \dots, z_n]$$

is finite.

When all elements of $\mathcal{O}$ are finitely factorizable, by extension, $\mathcal{O}$ is finitely factorizable.

A map $\mathrm{dg} : \mathcal{O} \to \mathbb{N}$ is a grading of $\mathcal{O}$ if

- $\mathrm{dg}^{-1}(0) = \{\mathbb{1}\}$;
- for any $y \in \mathcal{O}(n)$ and $z_1, \dots, z_n \in \mathcal{O}$,

$$\mathrm{dg}(y[z_1, \dots, z_n]) = \mathrm{dg}(y) + \mathrm{dg}(z_1) + \dots + \mathrm{dg}(z_n).$$

When such a map exists, $\mathcal{O}$ is graded.

The nonsymmetric associative operad As is the operad such that

- $\mathsf{As} := \{\alpha_n : n \in \mathbb{N}\}$ with $\mathrm{ar}(\alpha_n) := n + 1$;
- the unit is α_0;
- the composition map satisfies

$$\alpha_n[\alpha_{m_1}, \ldots, \alpha_{m_n}] = \alpha_{n+m_1+\cdots+m_n}.$$

Example — A composition in As

$$\alpha_4[\alpha_1, \alpha_0, \alpha_2, \alpha_1, \alpha_0] = \alpha_{4+1+0+2+1+0} = \alpha_8$$

The map dg defined by $\mathrm{dg}(\alpha_n) := n$ is a grading of As.

The operad As is finitely factorizable.

Natural Hopf algebras of operads

Let $\mathcal{O}$ be an operad.

The reduced $\mathrm{rd}(v)$ of $v \in \mathcal{O}^*$ is the word obtained by removing the letters $\mathbb{1}$ in v.

Example — The reduced word of a word of As^*

$$\mathrm{rd}(\alpha_1\ \alpha_1\ \alpha_0\ \alpha_3\ \alpha_0\ \alpha_0) = \alpha_1\ \alpha_1\ \alpha_3$$

The natural space $\mathbf{N}(\mathcal{O})$ of $\mathcal{O}$ is the linear span of the set of reduced elements of $\mathcal{O}^*$.

The set $\{\mathsf{E}_v : v \in \mathrm{rd}(\mathcal{O}^*)\}$ is the elementary basis of $\mathbf{N}(\mathcal{O})$.

If $\mathcal{O}$ admits a grading dg, then $\mathbf{N}(\mathcal{O})$ becomes a **graded space** by setting

$$\mathrm{dg}(\mathsf{E}_{v_1 \dots v_\ell}) := \mathrm{dg}(v_1) + \cdots + \mathrm{dg}(v_\ell).$$

Note that $\mathrm{dg}(\mathsf{E}_\epsilon) = 0$.

Let $\star$ be the **product** on $\mathbf{N}(\mathcal{O})$ defined by

$$\mathsf{E}_v \star \mathsf{E}_{v'} := \mathsf{E}_{vv'}.$$

Let Δ be the **coproduct** on $\mathbf{N}(\mathcal{O})$ defined by

$$\Delta(\mathsf{E}_x) = \sum_{n \geqslant 0} \sum_{\substack{(y,v) \in \mathcal{O}(n) \times \mathcal{O}^n \\ x = y[v_1, \dots, v_n]}} \mathsf{E}_{\mathrm{rd}(y)} \otimes \mathsf{E}_{\mathrm{rd}(v)}.$$

Theorem [van der Laan, 2004] [Méndez, Liendo, 2014]

For any finitely factorizable operad $\mathcal{O}$, $\mathbf{N}(\mathcal{O})$ is a bialgebra.

Moreover, if $\mathcal{O}$ is graded, then $\mathbf{N}(\mathcal{O})$ is a Hopf algebra.

Under these two conditions on $\mathcal{O}$, $\mathbf{N}(\mathcal{O})$ is the natural Hopf algebra of $\mathcal{O}$.

Let us apply this construction on As endowed with the grading dg satisfying $\mathrm{dg}(\alpha_n) = n$.

For any $n \geqslant 1$, $\dim \mathbf{N}(\mathsf{As})(n) = 2^{n-1}$.

Example — A product in $\mathbf{N}(\mathsf{As})$

$$\mathsf{E}_{\alpha_2\alpha_1\alpha_1\alpha_4} \star \mathsf{E}_{\alpha_3\alpha_1} = \mathsf{E}_{\alpha_2\alpha_1\alpha_1\alpha_4\alpha_3\alpha_1}$$

Example — A coproduct in $\mathbf{N}(\mathsf{As})$

$$\Delta(\mathsf{E}_{\alpha_3}) = \mathsf{E}_\epsilon \otimes \mathsf{E}_{\alpha_3} + 2\mathsf{E}_{\alpha_1} \otimes \mathsf{E}_{\alpha_2} + \mathsf{E}_{\alpha_1} \otimes \mathsf{E}_{\alpha_1\alpha_1} + 3\mathsf{E}_{\alpha_2} \otimes \mathsf{E}_{\alpha_1} + \mathsf{E}_{\alpha_3} \otimes \mathsf{E}_\epsilon.$$

Contributions to the coefficient 2 of $\mathsf{E}_{\alpha_1} \otimes \mathsf{E}_{\alpha_2}$:

$$\alpha_3 = \alpha_1[\alpha_0, \alpha_2], \quad \alpha_3 = \alpha_1[\alpha_2, \alpha_0].$$

Contributions to the coefficient 3 of $\mathsf{E}_{\alpha_2} \otimes \mathsf{E}_{\alpha_2}$:

$$\alpha_3 = \alpha_2[\alpha_0, \alpha_0, \alpha_1], \quad \alpha_3 = \alpha_2[\alpha_0, \alpha_1, \alpha_0], \quad \alpha_3 = \alpha_2[\alpha_1, \alpha_0, \alpha_0].$$

$\mathbf{N}(\mathsf{As})$ is the **noncommutative Faà di Bruno Hopf algebra FdB** [Figueroa, Gracia-Bondía, 2005] [Foissy, 2008].

Terms and forests

A signature is a set $\mathcal{S}$ decomposing as $\mathcal{S} = \bigsqcup_{n \geqslant 0} \mathcal{S}(n)$.

An $\mathcal{S}$-term is an **ordered rooted tree** decorated on $\mathcal{S}$ such that an internal node decorated by $\mathsf{s} \in \mathcal{S}(n)$ has exactly n children.

Let $\mathfrak{T}(\mathcal{S})$ be the set of $\mathcal{S}$-terms.

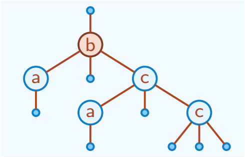
For any $\mathfrak{t} \in \mathfrak{T}(\mathcal{S})$,

- the degree $\mathrm{dg}(\mathfrak{t})$ of $\mathfrak{t}$ is the number of internal nodes of $\mathfrak{t}$;
- the arity $\mathrm{ar}(\mathfrak{t})$ of $\mathfrak{t}$ is the number of leaves of $\mathfrak{t}$.

Example — An $\mathcal{S}$-term

Let the signature $\mathcal{S} := \mathcal{S}(1) \sqcup \mathcal{S}(3)$ with $\mathcal{S}(1) := \{\mathsf{a}\}$ and $\mathcal{S}(3) := \{\mathsf{b}, \mathsf{c}\}$.

This $\mathcal{S}$-term has degree 5 and arity 7.

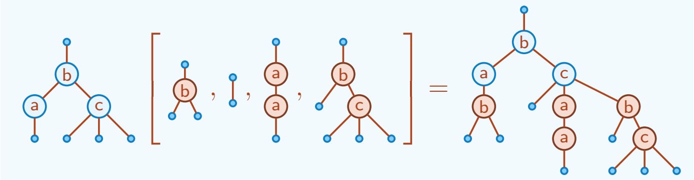


Let $\mathcal{S}$ be a signature.

The free operad on $\mathcal{S}$ is the set $\mathfrak{T}(\mathcal{S})$ such that

- $\mathfrak{T}(\mathcal{S})(n)$ is the set of $\mathcal{S}$-terms of arity n;
- the unit is the $\mathcal{S}$-term containing exactly one leaf ;
- the composition map is such that $\mathfrak{t}[\mathfrak{t}_1, \dots, \mathfrak{t}_n]$ is the $\mathcal{S}$-term obtained by grafting simultaneously each $\mathfrak{t}_i$ on the i-th leaf of $\mathfrak{t}$.

Example — A composition in a free operad



The map dg is a grading of $\mathfrak{T}(\mathcal{S})$ and this operad is finitely factorizable.

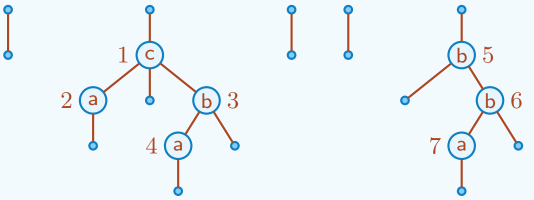
Let $\mathcal{S}$ be a signature.

An $\mathcal{S}$-forest is a word on $\mathfrak{T}(\mathcal{S})$. Let $\mathfrak{F}(\mathcal{S})$ be the set of $\mathcal{S}$-forests.

The internal nodes of an $\mathcal{S}$-forest $\mathfrak{f}$ are identified by their positions during the **preorder traversal**.

Let $\xrightarrow{\mathfrak{f}}_j$ be the binary relation on the set of internal nodes of $\mathfrak{f}$ such that $i_1 \xrightarrow{\mathfrak{f}}_j i_2$ if i_1 is the j-th child of i_2 in $\mathfrak{f}$.

Example — An $\mathcal{S}$-forest

$\mathfrak{f} :=$ 

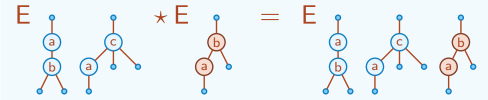
For instance, $1 \xrightarrow{\mathfrak{f}}_1 2$, $1 \xrightarrow{\mathfrak{f}}_3 3$, and $5 \xrightarrow{\mathfrak{f}}_2 6$.

Natural Hopf algebras of free operads

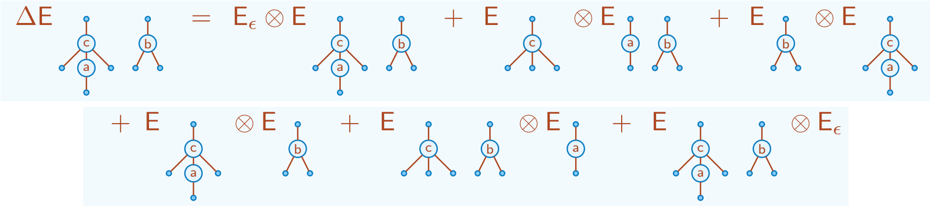
Let $\mathcal{S}$ be a signature.

The bases of $\mathbf{N}(\mathfrak{T}(\mathcal{S}))$ are indexed by the set of **reduced $\mathcal{S}$-forests**.

Example — A product in $\mathbf{N}(\mathfrak{T}(\mathcal{S}))$



Example — A coproduct in $\mathbf{N}(\mathfrak{T}(\mathcal{S}))$



Polynomial realization

Let $\mathcal{S}$ be a signature.

The class of $\mathcal{S}$-forest-like alphabets is the class of alphabets A endowed with relations R, D_{s}, and $\prec_j$ such that

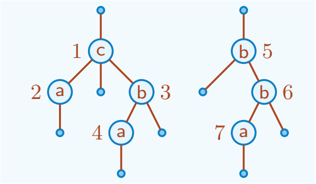
1. R is a unary relation called root relation;
2. for any $\mathsf{s} \in \mathcal{S}$, D_{s} is a unary relation called s-decoration relation;
3. for any $j \geqslant 1$, $\prec_j$ is a binary relation called j-edge relation.

Let $\mathcal{S}$ be a signature, A be an $\mathcal{S}$-forest-like alphabet, and $\mathfrak{f}$ be a reduced $\mathcal{S}$-forest.

A word $u \in A^*$ is $\mathfrak{f}$-compatible, denoted by $u \Vdash^A \mathfrak{f}$, if

1. $\ell(u) = \mathrm{dg}(\mathfrak{f})$;
2. if i is a root of $\mathfrak{f}$, then $u_i \in \mathrm{R}$;
3. if i is decorated by $\mathsf{s} \in \mathcal{S}$ in $\mathfrak{f}$, then $u_i \in \mathrm{D}_{\mathsf{s}}$;
4. if $i \xrightarrow{\mathfrak{f}}_j i'$, then $u_i \prec_j u_{i'}$.

Example — An $\mathfrak{f}$-compatible word



Considering this reduced forest $\mathfrak{f}$, any $\mathfrak{f}$-compatible word $u \in A^*$ satisfies

- ☐ $\ell(u) = 7$;
- ☐ $u_1, u_5 \in \mathrm{R}$;
- ☐ $u_2, u_4, u_7 \in \mathrm{D}_{\mathsf{a}}, \quad u_3, u_5, u_6 \in \mathrm{D}_{\mathsf{b}}, \quad u_1 \in \mathrm{D}_{\mathsf{c}}$;
- ☐ $u_1 \prec_1 u_2, \quad u_1 \prec_3 u_3, \quad u_3 \prec_1 u_4, \quad u_5 \prec_2 u_6, \quad u_6 \prec_1 u_7$.

Let $\mathcal{S}$ be a signature and A be an $\mathcal{S}$-forest-like alphabet.

Let $\mathsf{r}_A : \mathbf{N}(\mathfrak{T}(\mathcal{S})) \to \mathbb{K}\langle A \rangle$ be the linear map defined for any $\mathfrak{f} \in \mathrm{rd}(\mathfrak{F}(\mathcal{S}))$ by

$$\mathsf{r}_A(\mathsf{E}_{\mathfrak{f}}) := \sum_{\substack{u \in A^* \\ u \Vdash^{A} \mathfrak{f}}} u.$$

This polynomial is the A-realization of $\mathfrak{f}$.

Lemma

For any signature $\mathcal{S}$ and any $\mathcal{S}$-forest-like alphabet A, r_A is a graded unital associative algebra morphism.

Let $\mathcal{S}$ be a signature, and A_1 and A_2 be to $\mathcal{S}$-forest-like alphabets.

The sum $A_1 \mathbin{+\!\!\!+} A_2$ of A_1 and A_2 is the $\mathcal{S}$-forest-like alphabet

$$A := A_1 \sqcup A_2$$

endowed with the relations R, D_{s}, and $\prec_j$ such that

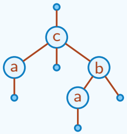
1. $\mathrm{R} := \mathrm{R}^{(1)} \sqcup \mathrm{R}^{(2)}$;
2. $\mathrm{D}_{\mathsf{s}} := \mathrm{D}_{\mathsf{s}}^{(1)} \sqcup \mathrm{D}_{\mathsf{s}}^{(2)}$;
3. $a \prec_j a'$ holds if one of the three following conditions hold:
   - $a \in A_1$, $a' \in A_1$, and $a \prec_j{}^{(1)} a'$;
   - $a \in A_2$, $a' \in A_2$, and $a \prec_j{}^{(2)} a'$;
   - $a \in A_1$, $a' \in A_2$, and $a' \in \mathrm{R}^{(2)}$.

Lemma

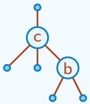
For any signature $\mathcal{S}$, any $\mathcal{S}$-forest-like alphabets A_1 and A_2, and any $\mathcal{S}$-forest $\mathfrak{f}$,

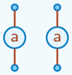
$$\mathsf{r}_{A_1 \# A_2}(\mathsf{E}_{\mathfrak{f}}) = (\mathsf{r}_{A_1} \otimes \mathsf{r}_{A_2}) \circ \Delta(\mathsf{E}_{\mathfrak{f}}).$$

Example — An alphabet doubling in $\mathbf{N}(\mathfrak{T}(\mathcal{S}))$

$\mathsf{r}_{A_1 \# A_2} \mathsf{E}$


$$= \sum_{\substack{u_1,u_2,u_3,u_4 \in A_1 \# A_2 \\ u_1 \in \mathrm{R} \\ u_2,u_4 \in \mathrm{D}_{\mathsf{a}},\ u_1 \in \mathrm{D}_{\mathsf{c}},\ u_3 \in \mathrm{D}_{\mathsf{b}} \\ u_1 \prec_1 u_2,\ u_1 \prec_3 u_3,\ u_3 \prec_1 u_4}} u_1 u_2 u_3 u_4 = \cdots + \sum_{\substack{u_1,u_3 \in A_1,\ u_2,u_4 \in A_2 \\ u_1,u_2,u_4 \in \mathrm{R} \\ u_2,u_4 \in \mathrm{D}_{\mathsf{a}},\ u_1 \in \mathrm{D}_{\mathsf{c}},\ u_3 \in \mathrm{D}_{\mathsf{b}} \\ u_1 \prec_3 u_3}} u_1 u_2 u_3 u_4 + \cdots$$

$= \cdots + \mathsf{r}_{A_1} \mathsf{E}$


$\otimes\ \mathsf{r}_{A_2} \mathsf{E}$


$+ \cdots$

The $\mathcal{S}$-forest-like alphabet of positions is the $\mathcal{S}$-forest-like alphabet

$$\mathbb{A}_{\mathrm{p}}(\mathcal{S}) := \{\mathbf{a}_v^{\mathsf{s}} : \mathsf{s} \in \mathcal{S} \text{ and } v \in \mathbb{N}^*\}$$

such that

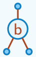
1. the root relation is defined by $\mathrm{R} := \left\{\mathbf{a}_{0^\ell}^{\mathsf{s}} \in \mathbb{A}_{\mathrm{p}}(\mathcal{S}) : \ell \geqslant 0\right\}$;
2. the s-decoration relation D_{s} is defined by $\mathrm{D}_{\mathsf{s}} := \left\{\mathbf{a}_v^{\mathsf{s}'} \in \mathbb{A}_{\mathrm{p}}(\mathcal{S}) : \mathsf{s}' = \mathsf{s}\right\}$;
3. the j-edge relation $\prec_j$ is defined by $\mathbf{a}_v^{\mathsf{s}} \prec_j \mathbf{a}_{v\,j\,0^\ell}^{\mathsf{s}'}$ where $\ell \geqslant 0$.

Example — An alphabet $\mathbb{A}_{\mathrm{p}}(\mathcal{S})$

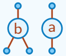
Let the signature $\mathcal{S} := \mathcal{S}(1) \sqcup \mathcal{S}(3)$ such that $\mathcal{S}(1) = \{\mathsf{a}, \mathsf{b}\}$ and $\mathcal{S}(3) = \{\mathsf{c}\}$. For instance,

- $\mathbf{a}_{000}^{\mathsf{a}} \in \mathrm{R}$, $\mathbf{a}_{10021}^{\mathsf{b}} \notin \mathrm{R}$;
- $\mathbf{a}_{1706001}^{\mathsf{c}} \in \mathrm{D}_{\mathsf{c}}$, $\mathbf{a}_{0211}^{\mathsf{b}} \notin \mathrm{D}_{\mathsf{c}}$;
- $\mathbf{a}_{103}^{\mathsf{a}} \prec_1 \mathbf{a}_{103\,1\,00}^{\mathsf{b}}$, $\mathbf{a}_{1}^{\mathsf{c}} \prec_2 \mathbf{a}_{1\,2}^{\mathsf{a}}$.

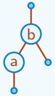
Example — $\mathbb{A}_p(\mathcal{S})$-realizations of some reduced forests

$$r_{\mathbb{A}_p(\mathcal{S})} E$$


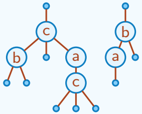
$$= \sum_{\ell_1 \in \mathbb{N}} \mathbf{a}^{b}_{0^{\ell_1}}$$

$$r_{\mathbb{A}_p(\mathcal{S})} E$$


$$= \sum_{\ell_1, \ell_2 \in \mathbb{N}} \mathbf{a}^{b}_{0^{\ell_1}} \; \mathbf{a}^{a}_{0^{\ell_2}}$$

$$r_{\mathbb{A}_p(\mathcal{S})} E$$


$$= \sum_{\ell_1, \ell_2 \in \mathbb{N}} \mathbf{a}^{b}_{0^{\ell_1}} \; \mathbf{a}^{a}_{0^{\ell_1} 1 0^{\ell_2}}$$

$$r_{\mathbb{A}_p(\mathcal{S})} E$$


$$= \sum_{\ell_1, \dots, \ell_6 \in \mathbb{N}} \mathbf{a}^{c}_{0^{\ell_1}} \; \mathbf{a}^{b}_{0^{\ell_1} 1 0^{\ell_2}} \; \mathbf{a}^{a}_{0^{\ell_1} 3 0^{\ell_3}} \; \mathbf{a}^{c}_{0^{\ell_1} 3 0^{\ell_3} 1 0^{\ell_4}} \; \mathbf{a}^{b}_{0^{\ell_5}} \; \mathbf{a}^{a}_{0^{\ell_5} 1^{\ell_6}}$$

Let $\mathfrak{f}$ be an $\mathcal{S}$-forest.

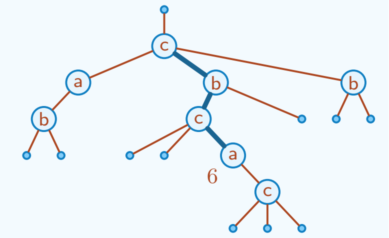
The decoration $\mathrm{dec}_i(\mathfrak{f})$ of a node i of $\mathfrak{f}$ is the element of $\mathcal{S}$ decorating it.

The address $\mathrm{adr}_i(\mathfrak{f})$ of a node i of $\mathfrak{f}$ is the word specifying the positions of the edges to reach i from the root.

If $\mathfrak{f}$ has n internal nodes, let the monomial

$$\mathrm{m}(\mathfrak{f}) := \mathbf{a}_{\mathrm{adr}_1(\mathfrak{f})}^{\mathrm{dec}_1(\mathfrak{f})} \dots \mathbf{a}_{\mathrm{adr}_n(\mathfrak{f})}^{\mathrm{dec}_n(\mathfrak{f})}.$$

Example — Decorations and addresses of a node in an $\mathcal{S}$-forest



For this $\mathcal{S}$-forest $\mathfrak{f}$, we have $\mathrm{dec}_6(\mathfrak{f}) = \mathsf{a}$ and $\mathrm{adr}_6(\mathfrak{f}) = 213$.

We have also

$$\mathrm{m}(\mathfrak{f}) = \mathbf{a}_0^{\mathsf{c}}\, \mathbf{a}_1^{\mathsf{a}}\, \mathbf{a}_{11}^{\mathsf{b}}\, \mathbf{a}_2^{\mathsf{b}}\, \mathbf{a}_{21}^{\mathsf{c}}\, \mathbf{a}_{213}^{\mathsf{a}}\, \mathbf{a}_{2131}^{\mathsf{c}}\, \mathbf{a}_3^{\mathsf{b}}.$$

The weight $\mathrm{wt}(u)$ of a monomial $u = \mathbf{a}_{v_1}^{\mathsf{s}_1} \dots \mathbf{a}_{v_n}^{\mathsf{s}_n}$ is $\ell(v_1) + \dots + \ell(v_n)$.

Lemma

For any signature $\mathcal{S}$ and any reduced $\mathcal{S}$-forest $\mathfrak{f}$,

$$\mathsf{r}_{\mathbb{A}_{\mathrm{p}}(\mathcal{S})}(\mathsf{E}_{\mathfrak{f}}) = \mathrm{m}(\mathfrak{f}) + \sum_{\substack{u \in \mathbb{A}_{\mathrm{p}}(\mathcal{S})^* \\ u \Vdash^{\mathbb{A}_{\mathrm{p}}(\mathcal{S})} \mathfrak{f} \\ \mathrm{wt}(u) > \mathrm{wt}(\mathrm{m}(\mathfrak{f}))}} u.$$

Lemma

For any signature $\mathcal{S}$, the map $\mathsf{r}_{\mathbb{A}_{\mathrm{p}}(\mathcal{S})} : \mathbf{N}(\mathfrak{T}(\mathcal{S})) \to \mathbb{K}\langle \mathbb{A}_{\mathrm{p}}(\mathcal{S}) \rangle$ is injective.

Theorem [G., 2024+]

For any signature $\mathcal{S}$, the map r_{A} is a polynomial realization of $\mathbf{N}(\mathfrak{T}(\mathcal{S}))$.

Case of non-free operads

A congruence $\equiv$ of the free operad $\mathfrak{T}(\mathcal{S})$

- ☐ is compatible with the degree if $\mathfrak{t}_1 \equiv \mathfrak{t}_2$ implies $\mathrm{dg}(\mathfrak{t}_1) = \mathrm{dg}(\mathfrak{t}_2)$;
- ☐ is of finite type if the $\equiv$-equivalence class $[\mathfrak{t}]_\equiv$ of any $\mathcal{S}$-term $\mathfrak{t}$ is finite.

Theorem [G., 2024+]

Let $\mathcal{S}$ be a signature and $\equiv$ be a congruence of $\mathfrak{T}(\mathcal{S})$ which is compatible with the degree and of finite type. The associative algebra morphism

$$\phi : \mathbf{N}(\mathfrak{T}(\mathcal{S})/_\equiv) \to \mathbf{N}(\mathfrak{T}(\mathcal{S}))$$

satisfying

$$\phi(\mathsf{E}_{[\mathfrak{t}]_\equiv}) = \sum_{\mathfrak{t} \in [\mathfrak{t}]_\equiv} \mathsf{E}_\mathfrak{t}$$

for any $[\mathfrak{t}]_\equiv \in \mathfrak{T}(\mathcal{S})/_\equiv$ is an injective Hopf algebra morphism.

We have $\mathsf{As} \simeq \mathsf{Mag}/_{\equiv}$ where $\mathsf{Mag} := \mathfrak{T}(\mathcal{S})/_{\equiv}$, $\mathcal{S} := \mathcal{S}(2) = \{\mathsf{a}\}$, and $\equiv$ satisfies $\mathfrak{t}_1 \equiv \mathfrak{t}_2$ whenever $\mathrm{dg}(\mathfrak{t}_1) = \mathrm{dg}(\mathfrak{t}_2)$.

Each $\equiv$-equivalence class $[\mathfrak{t}]_{\equiv}$ is represented by the element $\alpha_{\mathrm{dg}(\mathfrak{t})}$ of As.

The map

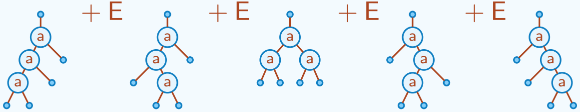
$$\phi : \mathbf{N}(\mathsf{Mag}/_{\equiv}) \simeq \mathbf{FdB} \to \mathbf{N}(\mathsf{Mag})$$

satisfies

$$\phi(\mathsf{E}_{\alpha_n}) = \sum_{\substack{\mathfrak{t} \in \mathfrak{T}(\mathcal{S}) \\ \mathrm{dg}(\mathfrak{t}) = n}} \mathsf{E}_{\mathfrak{t}}.$$

Example — An image by ϕ

$\phi \mathsf{E}_{\alpha_3} = \mathsf{E} \quad + \mathsf{E} \quad + \mathsf{E} \quad + \mathsf{E} \quad + \mathsf{E}$



By setting $\bar{\mathsf{r}}_A := \mathsf{r}_A \circ \phi$, we obtain a **polynomial realization of $\mathbf{FdB}$**.

Example — The $\mathbb{A}_{\mathrm{p}}(\mathcal{S})$-polynomial of an element of $\mathbf{FdB}$

$$\bar{\mathsf{r}}_{\mathbb{A}_{\mathrm{p}}(\mathcal{S})} \mathsf{E}_{\alpha_3} = \sum_{\ell_1,\ell_2,\ell_3 \in \mathbb{N}} \mathbf{a}^{\mathsf{a}}_{0^{\ell_1}} \; \mathbf{a}^{\mathsf{a}}_{0^{\ell_1} 1 0^{\ell_2}} \; \mathbf{a}^{\mathsf{a}}_{0^{\ell_1} 1 0^{\ell_2} 1 0^{\ell_3}} + \sum_{\ell_1,\ell_2,\ell_3 \in \mathbb{N}} \mathbf{a}^{\mathsf{a}}_{0^{\ell_1}} \; \mathbf{a}^{\mathsf{a}}_{0^{\ell_1} 1 0^{\ell_2}} \; \mathbf{a}^{\mathsf{a}}_{0^{\ell_1} 1 0^{\ell_2} 2 0^{\ell_3}}$$

$$+ \sum_{\ell_1,\ell_2,\ell_3 \in \mathbb{N}} \mathbf{a}^{\mathsf{a}}_{0^{\ell_1}} \; \mathbf{a}^{\mathsf{a}}_{0^{\ell_1} 1 0^{\ell_2}} \; \mathbf{a}^{\mathsf{a}}_{0^{\ell_1} 2 0^{\ell_3}} + \sum_{\ell_1,\ell_2,\ell_3 \in \mathbb{N}} \mathbf{a}^{\mathsf{a}}_{0^{\ell_1}} \; \mathbf{a}^{\mathsf{a}}_{0^{\ell_1} 2 0^{\ell_2}} \; \mathbf{a}^{\mathsf{a}}_{0^{\ell_1} 2 0^{\ell_2} 1 0^{\ell_3}}$$

$$+ \sum_{\ell_1,\ell_2,\ell_3 \in \mathbb{N}} \mathbf{a}^{\mathsf{a}}_{0^{\ell_1}} \; \mathbf{a}^{\mathsf{a}}_{0^{\ell_1} 2 0^{\ell_2}} \; \mathbf{a}^{\mathsf{a}}_{0^{\ell_1} 2 0^{\ell_2} 2 0^{\ell_3}}$$

Using the specialization $\pi : \mathbf{a}^{\mathsf{a}}_{v} \mapsto \mathbf{a}_{\ell(v)}$, we obtain

$$\pi \bar{\mathsf{r}}_{\mathbb{A}_{\mathrm{p}}(\mathcal{S})} \mathsf{E}_{\alpha_3} = 4 \sum_{\ell_1 < \ell_2 < \ell_3 \in \mathbb{N}} \mathbf{a}_{\ell_1} \mathbf{a}_{\ell_2} \mathbf{a}_{\ell_3} + \sum_{\substack{\ell_1,\ell_2,\ell_3 \in \mathbb{N} \\ \ell_1 < \ell_2, \; \ell_1 < \ell_3}} \mathbf{a}_{\ell_1} \mathbf{a}_{\ell_2} \mathbf{a}_{\ell_3}.$$

This map $\pi \circ \bar{\mathsf{r}}_{\mathbb{A}_{\mathrm{p}}(\mathcal{S})}$ is still injective and is hence another polynomial realization of $\mathbf{FdB}$.

By using similar methods, it is possible to build a **polynomial realization of the double tensor CHA**, constructed in [**Ebrahimi-Fard, Patras**, 2015].

Preprint *Polynomial realizations of Hopf algebras built from nonsymmetric operads* available at

$\ulcorner \mathbb{A}_{\mathrm{URL}} \urcorner$ = https://arxiv.org/abs/2406.12559

Grazie mille!