Catalan idempotents, from resummation theory to Rota-Baxter algebras and trees.

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Prehistory. (≈ 1996) Catalan derivations (J. Ecalle)

- act on an algebra of functions (resurgent functions)
- belong to an algebra of so-called "alien" operators

Middle age. (≈ 2006) The algebra of "alien" operators is isomorphic to Sym.

Up to the present days. Many properties related to algebraic and combinatorial structures, Rota-Baxter algebras, Quadrialgebras, Hopf algebras of plane forest, Tamari order.

- Lie idempotents and Noncommutative symmetric functions
- The family of Catalan idempotents
- Resummation theory and the Catalan idempotent as an alien operator
- The Catalan idempotents in the Lie Module
- The Catalan idempotents and Rota-Baxter algebras
- Catalan idempotents and trees

1 Lie idempotents and Noncommutative symmetric function.

1.1 Lie idempotents.

Let V a vector space over $\mathbb{K}$, $T(V)$ its tensor algebra and $L(V)$ its free Lie algebra

$$T(V)=\mathbb{K}1\oplus\left(\bigoplus_{n\geqslant 1} T_n(V)\right), \quad , \quad T_n(V)=V^{\otimes n}, \quad L_n(V)=L(V)\cap T_n(V).$$

There is a right action of $\mathbb{K}\mathfrak{S}_n$ on $T_n(V)$:

$$(v_1\otimes\cdots\otimes v_n)\cdot\sigma=v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)}.$$

and $\pi_n\in\mathbb{K}\mathfrak{S}_n$ is a Lie idempotent (resp. quasi-idempotent) if

$$\begin{array}{rccc}\Pi_n & : & T_n(V) & \rightarrow & T_n(V) \\ & & \boldsymbol{v} & \mapsto & \boldsymbol{v}\cdot\pi_n\end{array}$$

is an idempotent (resp. quasi-idempotent) such that $\operatorname{Im}\Pi_n=L_n(V)$.

$$\forall\boldsymbol{v}\in L_n(V), \qquad \Pi_n(\boldsymbol{v})=\boldsymbol{v} \quad (\text{resp.}\,\Pi_n(\boldsymbol{v})=\alpha\boldsymbol{v}).$$

Among idempotents, the Solomon idempotent and the Dynkin idempotent can be characterized as elements of the Hopf algebra Sym of noncommutative symmetric functions.

1.2 A short reminder on noncommutative symmetric functions.

Sym is the free associative algebra $\mathbb{K}\langle S_1, S_2, \ldots\rangle$ over an infinite sequence S_n, endowed with the grading $\deg S_n = n$ and the coproduct

$$\Delta\, S_n = \sum_{k=0}^{n} S_k \otimes S_{n-k} \quad (S_0 := 1)$$

If $\sigma_t = 1 + \sum_{n\geqslant 1} t^n S_n$, σ_t is group-like:

$$\Delta\sigma_t = \sigma_t \otimes \sigma_t.$$

Basises:

$$\mathrm{Sym}_n = \mathrm{Vect}\{S^I = S_{i_1} S_{i_2} \cdots S_{i_r}, I \vDash n\} = \mathrm{Vect}\{\Phi^I = \Phi_{i_1} \Phi_{i_2} \cdots \Phi_{i_r}, I \vDash n\}$$

With $I \vDash n$ compositions of n that is:

$$I = (i_1, \ldots, i_r) \text{ with } |I| = i_1 + \cdots + i_r = n \text{ and } i_k \in \mathbb{N}^{>0}$$

and

$$\sum_{n\geqslant 1} t^n \Phi_n = \log(\sigma_t) \qquad (\Delta\Phi_n = 1 \otimes \Phi_n + \Phi_n \otimes 1).$$

One can also consider the noncommutative ribbon Schur functions R_I as a basis:

$$S^I = \sum_{J \leqslant I} R_I, \text{ for instance } S^{113} = R_{113} + R_{23} + R_5.$$

or rather the signed ribbons basis. Let $\mathcal{E} = \{\emptyset\} \bigcup_{n \geqslant 1} \{\boldsymbol{\varepsilon} = \varepsilon_1 \ldots \varepsilon_n, \varepsilon_i = \pm\} = \bigcup_{n \geqslant 0} \mathcal{E}_n$. For $I = (i_1, \ldots, i_r) \vDash n$,

$$(-1)^{\ell(I)-1} R_I = R_{\varepsilon_1 \ldots \varepsilon_{n-1} \bullet} = R_{\boldsymbol{\varepsilon} \bullet} \qquad \text{with} \qquad \varepsilon_i = \begin{cases} - & \text{if } i \in \mathrm{Des}(I) = \{i_1, i_1 + i_2, i_1 + \cdots + i_{r-1}\} \\ + & \text{otherwise} \end{cases}$$

For instance,

$$R_{-++-+-+\bullet} = -R_{1322}$$

As for the product

$$R_{\boldsymbol{a}\bullet} R_{\boldsymbol{b}\bullet} = R_{\boldsymbol{a}+\boldsymbol{b}\bullet} - R_{\boldsymbol{a}-\boldsymbol{b}\bullet}$$

1.3 Descent algebra and Lie idempotents.

$$\forall \sigma \in \mathfrak{S}_n, \quad \mathrm{Sgn}(\sigma) = \varepsilon_1 \ldots \varepsilon_{n-1}, \qquad \varepsilon_i = \begin{cases} - & \text{if } \sigma(i) > \sigma(i+1) \\ + & \text{if } \sigma(i) < \sigma(i+1) \end{cases}$$

$$\mathrm{Sgn}((623145)) = -+-++$$

The descent algebra Σ_n of $\mathfrak{S}_n$ is spanned by

$$D_{\boldsymbol{\varepsilon}\bullet} = D_{\varepsilon_1 \ldots \varepsilon_{n-1}\bullet} = \sum_{\mathrm{Sgn}(\sigma) = \boldsymbol{\varepsilon}} \sigma$$

Let

$$\begin{array}{rcccl} \alpha & : & \Sigma_n & \to & \mathrm{Sym}_n \\ & & D_{\boldsymbol{\varepsilon}\bullet} & \mapsto & (-1)^{m(\boldsymbol{\varepsilon})} R_{\boldsymbol{\varepsilon}\bullet} \end{array} \qquad m(\boldsymbol{\varepsilon}) = \#\{i; \varepsilon_i = -\}$$

Theorem 1. *[Gelfand, Krob, Lascoux, Leclerc, Retakh, Thibon (1995)]. Let $F = \alpha(\pi)$ be an element of Sym_n, where $\pi \in \Sigma_n$. The following assertions are equivalent:*

1. *π is a Lie quasi-idempotent;*
2. *F is a primitive element for Δ;*
3. *F belongs to the Lie algebra $L(\Phi)$ generated by the Φ_n.*

Moreover, π is a Lie idempotent iff $F - \Phi_n$ is in the Lie ideal $[L(\Phi), L(\Phi)]$.

1.4 The exemple of the Solomon idempotent.

If $\sum_{n\geqslant 1} t^n\Phi_n = \log(\sigma_t) = \log(1+\sum_{n\geqslant 1} t^n S_n)$,

$$\Phi_n = \sum_{\boldsymbol{\varepsilon}=\varepsilon_1\ldots\varepsilon_{n-1}} \frac{m(\boldsymbol{\varepsilon})!p(\boldsymbol{\varepsilon})!}{n!} R_{\boldsymbol{\varepsilon}\bullet} \qquad m(\boldsymbol{\varepsilon}) = \#\{i;\varepsilon_i = -\}, p(\boldsymbol{\varepsilon}) = \#\{i;\varepsilon_i = +\}$$

$$\begin{aligned}
\Phi_3 &= \frac{0!2!}{3!}R_{++\bullet} + \frac{1!1!}{3!}R_{-+\bullet} + \frac{1!1!}{3!}R_{+-\bullet} + \frac{2!1!}{3!}R_{--\bullet} \\
&= \frac{1}{3}R_{++\bullet} + \frac{1}{6}R_{-+\bullet} + \frac{1}{6}R_{+-\bullet} + \frac{1}{6}R_{--\bullet} \\
&= \alpha\left(\frac{1}{3}(123) - \frac{1}{6}((213)+(312)) - \frac{1}{6}((132)+(231)) + \frac{1}{3}(321)\right) \\
&= \alpha\left(\frac{1}{6}([[1,2],3] + [[1,[2,3]])\right) \\
&= \alpha(\phi_3)
\end{aligned}$$

and

$$(v_1\otimes v_2\otimes v_3)\cdot\phi_3 = \frac{1}{6}([[v_1,v_2],v_3] + [[v_1,[v_2,v_3]]) \in L_3(V)$$

2 The family of Catalan idempotents.

Consider the Narayana polynomials :

$$\begin{aligned}
\mathrm{ca}_1^{a,b} &= 1\\
\mathrm{ca}_2^{a,b} &= a+b\\
\mathrm{ca}_3^{a,b} &= a^2+3\,a\,b+b^2\\
\mathrm{ca}_4^{a,b} &= a^3+6\,a^2\,b+6\,a\,b^2+b^3\\
\mathrm{ca}_5^{a,b} &= a^4+10\,a^3\,b+20\,a^2\,b^2+10\,a\,b^3+b^4
\end{aligned}$$

For any word $\boldsymbol{\varepsilon}=\varepsilon_1\ldots\varepsilon_r$, consider its "stack" decomposition:

$$\boldsymbol{\varepsilon}=(\eta_1)^{r_1}\ldots(\eta_s)^{r_s}=(\pm)^{r_1}(\mp)^{r_2}\ldots \qquad (---++-++++=(-)^3(+)^2(-)^1(+)^4).$$

Theorem 2. *[Ecalle, M., 1996], [M., Novelli, Thibon, 2008]*

For $n\geqslant 2$,

$$C_n^{a,b}=\sum_{\boldsymbol{\varepsilon}=(\varepsilon_1,\ldots,\varepsilon_n)=(\eta_1)^{r_1}\ldots(\eta_s)^{r_s}}\left(\prod_{\eta_i=+,i<s}a\right)\left(\prod_{\eta_i=-,i<s}b\right)\mathrm{ca}_{r_1}^{a,b}\ldots\mathrm{ca}_{r_s}^{a,b}\,R_{\boldsymbol{\varepsilon}\bullet}$$

is primitive and thus corresponds to a Lie (quasi)-idempotent.

$$\begin{aligned}
C_2^{a,b} &= R_{+\bullet}+R_{-\bullet}\\
C_3^{a,b} &= (a+b)\,R_{++\bullet}+a\,R_{+-\bullet}+b\,R_{-+\bullet}+(a+b)\,R_{--\bullet}\\
C_4^{a,b} &= (a^2+3\,a\,b+b^2)\,R_{+++\bullet}+a\,(a+b)\,R_{++-\bullet}+a\,b\,R_{+-+\bullet}+(a+b)\,b\,R_{-++\bullet}\\
&\quad +a\,(a+b)\,R_{+--\bullet}+a\,b\,R_{--\bullet}+(a+b)\,b\,R_{--+\bullet}+(a^2+3\,a\,b+b^2)\,R_{---\bullet}
\end{aligned}$$

3 Resummation theory and the Catalan idempotent as an alien operator.

3.1 Algebras of resurgent functions and alien operators.

In resummation theory :

$$\tilde{\varphi}(t) = \sum_{n \geqslant 0} a_n t^{n+1} \in \mathbb{C}[[t]] \xrightarrow{\mathcal{B}} \hat{\varphi}(\zeta) = \sum_{n \geqslant 0} \frac{a_n}{n!} \zeta^n \in \mathbb{C}\{\zeta\} \xrightarrow{\mathcal{L}_\theta} \varphi(t) = \int_0^{e^{i\theta}\infty} \hat{\varphi}(\zeta) e^{-\zeta/t} d\zeta$$

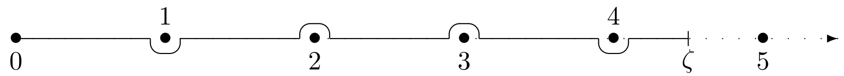
$\mathrm{Res}_{\mathbb{N}}$ is a an algebra of functions $\hat{\varphi}$ holomorphic near the origin, analytically continuable along any path that follows $\mathbb{R}^+$ and dodges each point of $\mathbb{N}^*$.

Moreover, $\mathrm{Res}_{\mathbb{N}}$ is stable by convolution

$$(\hat{\varphi} * \hat{\psi})(\zeta) = \int_0^{\zeta} \hat{\varphi}(\zeta_1) \hat{\psi}(\zeta - \zeta_1) d\zeta_1.$$

and stable under the action of **alien operators**.

Let $\mathcal{E}=\{\emptyset\}\bigcup_{n\geqslant 1}\{\boldsymbol{\varepsilon}=\varepsilon_1\ldots\varepsilon_n,\varepsilon_i=\pm\}=\bigcup_{n\geqslant 0}\mathcal{E}_n$, one can label the analytic continuations of a function $\hat{\varphi}$ by such sequences: For $\zeta\in]4,5[$, $\hat{\varphi}^{+---+}(\zeta)$ is the continuation along the path



Let $A_\emptyset=\mathrm{Id}$ $(\mathrm{ALIEN}_0=\mathbb{C}D_\emptyset)$ and for $\boldsymbol{\varepsilon}\in\mathcal{E}_{n-1}$ let $A_{\boldsymbol{\varepsilon}\bullet}\in\mathrm{ALIEN}_n$ the endomorphism of $\mathrm{Res}_{\mathbb{N}}$ such that:

$$\forall\hat{\varphi}\in\mathrm{Res}_{\mathbb{N}},\forall\zeta\approx 0,\quad (A_{\boldsymbol{\varepsilon}\bullet}\hat{\varphi})(\zeta)=\hat{\varphi}^{\boldsymbol{\varepsilon}+}(\zeta+l(\boldsymbol{\varepsilon}\bullet))-\hat{\varphi}^{\boldsymbol{\varepsilon}-}(\zeta+l(\boldsymbol{\varepsilon}\bullet))$$

$$(A_{+--+\bullet}\hat{\varphi})(\zeta)=\hat{\varphi}^{+--++}(\zeta+5)-\hat{\varphi}^{+--+-}(\zeta+5)$$

Proposition 3. *For* $(\varepsilon,\boldsymbol{\eta})\in\mathcal{E}$,

$$A_{\boldsymbol{\varepsilon}\bullet}A_{\boldsymbol{\eta}\bullet}=A_{\boldsymbol{\eta}+\boldsymbol{\varepsilon}\bullet}-A_{\boldsymbol{\eta}-\boldsymbol{\varepsilon}\bullet}$$

thus ALIEN *is a graded algebra with unit* $\mathbf{1}=A_\emptyset$.

For $n\geqslant 1$, let

$$\Delta_n^+=A_{+^{n-1}\bullet},\qquad \Delta_n=\sum_{\boldsymbol{\varepsilon}=(\varepsilon_1,\ldots,\varepsilon_{n-1})}\frac{(\#\{\varepsilon_i=+\})!(\#\{\varepsilon_i=-\})!}{n!}A_{\boldsymbol{\varepsilon}\bullet}$$

then $\sum\Delta_n=\log(1+\sum\Delta_n^+)$

and ALIEN is the polynomial algebra generated by the family $\{\Delta_n^+\}$ (or $\{\Delta_n\}$).

3.2 Derivations, automorphisms and the Catalan derivation.

The action of these operators induces a coproduct δ on ALIEN: $\forall \hat{\varphi}, \hat{\psi} \in \mathrm{Res}_{\mathbb{N}}$,

$$\begin{aligned} \Delta_n^+(\hat{\varphi} * \hat{\psi}) &= \sum_{k=0}^{n} \Delta_k^+(\hat{\varphi}) * \Delta_{n-k}^+(\hat{\psi}) \longrightarrow \delta(\Delta_n^+) = \sum_{k=0}^{n} \Delta_k^+ \otimes \Delta_{n-k}^+ \quad (\Delta_0^+ = 1) \\ \Delta_n(\hat{\varphi} * \hat{\psi}) &= \Delta_n(\hat{\varphi}) * \hat{\psi} + \hat{\varphi} * \Delta_n(\hat{\psi}) \longrightarrow \delta(\Delta_n) = 1 \otimes \Delta_n + \Delta_n \otimes 1 \end{aligned}$$

It turns ALIEN into a graded Hopf algebra where:

$\rightarrow$ The derivations Δ_n are primitives elements.

$\rightarrow$ The automorphisms $\Delta^+ = 1 + \sum \Delta_n^+$ is group-like elements.

Theorem 4. ALIEN *is isomorphic to the Hopf algebra* Sym

A natural isomorphism send Δ_n^+ to S_n, Δ_n to Φ_n and $A_{\varepsilon_1 \ldots \varepsilon_n \bullet}$ to $R_{\varepsilon_n \ldots \varepsilon_1 \bullet}$.

Compare

$$\begin{aligned} A_{\boldsymbol{\varepsilon}\bullet} A_{\boldsymbol{\eta}\bullet} &= A_{\boldsymbol{\eta}+\boldsymbol{\varepsilon}\bullet} - A_{\boldsymbol{\eta}-\boldsymbol{\varepsilon}\bullet} \\ R_{\boldsymbol{\varepsilon}\bullet} R_{\boldsymbol{\eta}\bullet} &= A_{\boldsymbol{\varepsilon}+\boldsymbol{\eta}\bullet} - A_{\boldsymbol{\varepsilon}-\boldsymbol{\eta}\bullet} \end{aligned} .$$

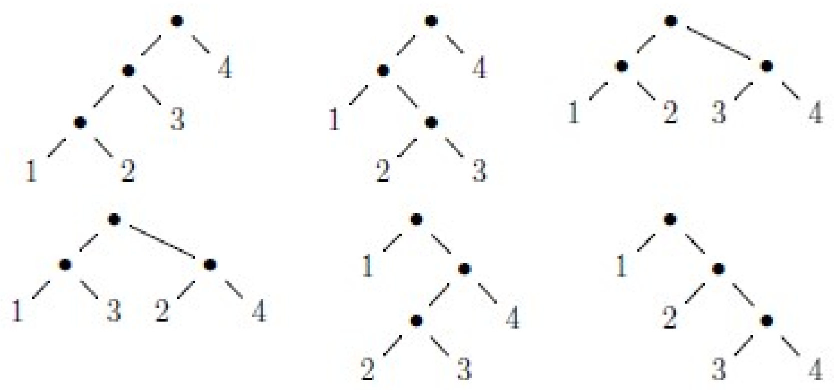
4 The Catalan idempotents in the Lie Module.

For the Solomon idempotent:

$$\begin{aligned}\phi_3 &= \frac{1}{3}(123) - \frac{1}{6}((213)+(312)) - \frac{1}{6}((132)+(231)) + \frac{1}{3}(321) &\in \mathbb{K}\mathfrak{S}_3 \\ &= \frac{1}{6}([[1,2],3]+[[1,[2,3]]) &\in \mathrm{Lie}(3)\end{aligned}$$

The Poincaré-Birkhoff-Witt basis of $\mathrm{Lie}(n)$ (of dimension $(n-1)!$) is indexed by complete binary trees with n leaves labelled by $[n]$ such that, for each internal node, the smallest (resp. greatest) label is in the left (resp. right) subtree.

For $n=4,$ the basis PBW_4 is indexed by

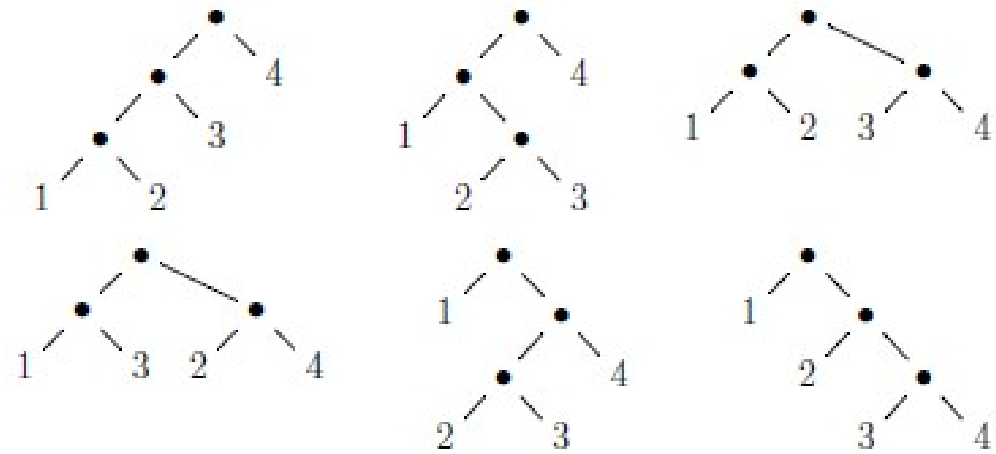


corresponding to

$$\begin{array}{ccccc} [[[1,2],3],4] & , & [[1,[2,3]],4] & , & [[1,2],[3,4]] \\ [[1,3],[2,4]] & , & [1,[[2,3],4]] & , & [1,[2,[3,4]]] \end{array}$$

Theorem 5. *[Foissy, M., Novelli, Thibon, 2022]. For $n \geqslant 2$ the Lie quasi-idempotent $abC_n^{a,b}$ corresponds in $\mathrm{Lie}(n)$ to*

$$abC_n^{a,b} = \sum_{T \in \mathrm{PBW}_n} a^{\#\{left\ leaves\}} b^{\#\{right\ leaves\}} T$$



$$\begin{aligned} abC_4^{a,b} \;=\; & ab^3[[[1,2],3],4] + a^2b^2[[1,[2,3]],4] + a^2b^2[[1,2],[3,4]] \\ + \; & a^2b^2[[1,3],[2,4]] + a^2b^2[1,[[2,3],4]] + a^3b[1,[2,[3,4]]] \end{aligned}$$

Proof based on a fixed point equation in the Malvenuto-Reutenauer Hopf algebra involving its quadrialgebra structure.

5 The Catalan idempotents and Rota-Baxter algebras.

5.1 Rota-Baxter algebras and Birkhoff decomposition.

A commutative algebra $\mathcal{A}$ over a field $\mathbb{K}$ is a Rota-Baxter algebra if $\mathcal{A} = \mathcal{A}^- \oplus \mathcal{A}^+$ where $\mathcal{A}^+$ and $\mathcal{A}^-$ are subalgebras. If P_+ (resp. P_-) is the projection A^+ (resp. $\mathcal{A}^-$) parallel to $\mathcal{A}^-$ (resp. $\mathcal{A}^+$) then,

$$\begin{aligned} P_+(xy) &= P_+(xP_+(y) + P_+(x)y) - P_+(x)P_+(y) \\ P_+(xP_+(y)) &= P_+(x)P_+(y) + P_+(P_-(x)y) \end{aligned}$$

In the $\mathcal{A}$-module $\mathrm{Sym}_{\mathcal{A}}$, there exists a unique Birkhoff decomposition $(\sigma_x^+, \sigma_x^-) \in \mathrm{Sym}_{\mathcal{A}^+} \times \mathrm{Sym}_{\mathcal{A}^-}$ of $\sigma_x = 1 + \sum x^n S_n$ ($x \in \mathcal{A}$):

$$\sigma_x^+ = \sigma_x^- \sigma_x.$$

Moreover, if $V: \mathcal{A}^+ \to \mathbb{K}$ is a linear map such that:

$$\forall (x, y) \in \mathcal{A}^+ \times \mathcal{A}^+, \qquad V(xy) = 0$$

then $D_x = V(\sigma_x^+ - 1)$ is primitive in Sym.

One has

$$\sigma_x^+ = 1 + \sum_{\boldsymbol{\varepsilon}} c^+(\boldsymbol{\varepsilon}\bullet)R_{\boldsymbol{\varepsilon}\bullet}, \quad V(\sigma_x^+ - 1) = \sum_{\boldsymbol{\varepsilon}} c(\boldsymbol{\varepsilon}\bullet)R_{\boldsymbol{\varepsilon}\bullet}.$$

Let

$$P_{\varepsilon_1\ldots\varepsilon_n}(x) = P_{\varepsilon_n}(xP_{\varepsilon_1\ldots\varepsilon_{n-1}}(x)),$$

for instance,

$$P_{+--}(x) = P_-(xP_-(xP_+(x)))$$

then

$$c^+(\boldsymbol{\varepsilon}\bullet) = P_{\boldsymbol{\varepsilon}+}(x), \quad c(\boldsymbol{\varepsilon}\bullet) = V(P_{\boldsymbol{\varepsilon}+}(x)).$$

5.2 The case of the Catalan idempotents.

Consider $\mathcal{A} = \frac{1}{z}\mathbb{K}[\frac{1}{z}] \oplus \mathbb{K}[[z]] = \mathcal{A}^+ \oplus \mathcal{A}^-$, with $V = \mathrm{Residue}$.

If $x = \frac{a}{z} + \frac{b}{1-z} = \frac{a}{z} + b + b\,z + b\,z^2 + \cdots$ then, for $n \geqslant 2$,

$$D_{x,n} = a\,b\,C_n^{a,b}$$

$$a\,b\,C_3^{a,b} \;=\; a\,b(a+b)\,R_{++\bullet} + a^2 b\,R_{+-\bullet} + a\,b^2\,R_{-+\bullet} + a\,b(a+b)\,R_{--\bullet}$$

and

$$\begin{aligned} c^+(++\bullet) &= P_+\bigg(\Big(\frac{a}{z} + \frac{b}{1-z}\Big)P_+\Big(\Big(\frac{a}{z} + \frac{b}{1-z}\Big)P_+\Big(\frac{a}{z} + \frac{b}{1-z}\Big)\Big)\bigg) = P_+\bigg(\Big(\frac{a}{z} + \frac{b}{1-z}\Big)P_+\Big(\Big(\frac{a}{z} + \frac{b}{1-z}\Big)\frac{a}{z}\Big) \\ &= P_+\bigg(\Big(\frac{a}{z} + \frac{b}{1-z}\Big)\Big(\frac{a^2}{z^2} + \frac{a\,b}{z}\Big)\bigg) = \frac{a^3}{z^3} + \frac{2a^2 b}{z^2} + \frac{a^2 b + a\,b^2}{z} \end{aligned}$$

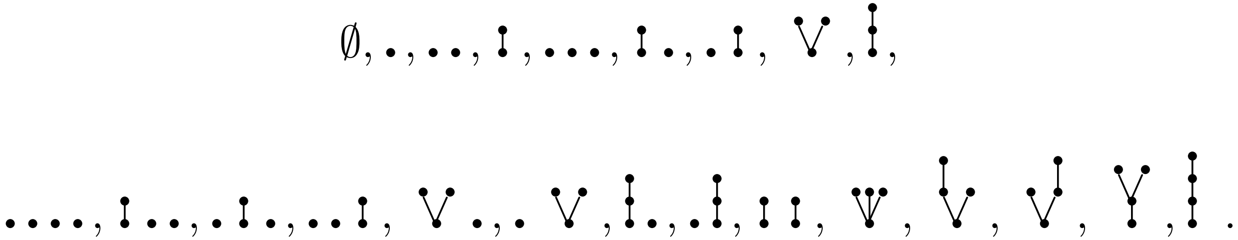
Smaller idempotents : replace $x = \frac{a}{z} + \frac{b}{1-z}$ by $x = \sum_{n \geqslant 0} a_n z^{-n-1}$ and isolate monomials in $a_0, a_1, \ldots$

$$D_{x,3} = (a_0^2\,a_2 + a_0\,a_1^2)\,R_{++\bullet} + (a_0^2\,a_2)\,R_{+-\bullet} + (a_0\,a_1^2)\,R_{-+\bullet} + (a_0^2\,a_2 + a_0\,b_1^2)\,R_{--\bullet}$$

6 Catalan idempotents and trees

6.1 $\mathcal{H}^*_{\mathrm{NCK}}$ and the morphism $\mathrm{Sym} \longrightarrow \mathcal{H}^*_{\mathrm{NCK}}$

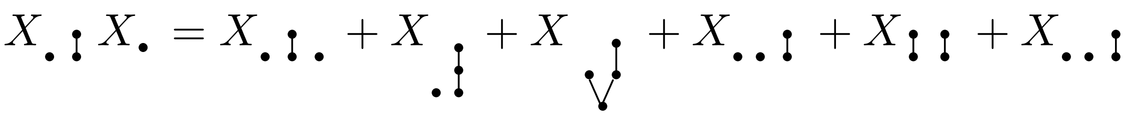
Consider planar rooted forest $\boldsymbol{F} = \bigcup_{n \geqslant 0} \boldsymbol{F}_n$:



$\mathcal{H}^*_{\mathrm{NCK}}$ is the graded Hopf algebra with linear basis $\{X_F, F \in \boldsymbol{F}\}$ and deconcatenation coproduct:

$$\Delta X_F = \sum_{F = F_1 F_2} X_{F_1} \otimes X_{F_2}.$$

The product $X_{F_1} X_{F_2}$ is the sum of X_F over all the graftings F of F_1 into F_2:



The algebra map Θ such that:

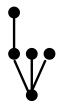
$$\Theta(S_n) = \sum_{F \in \boldsymbol{F}_n} X_F$$

is a Hopf morphism.

For x in a Rota-Baxter algebra, the Birkhoff decomposition $\sigma_x^+ = \sigma_x^- \sigma_x$ gives

$$\Theta(\sigma_x^+) = \sum_{F \in \boldsymbol{F}} c^+(F) X_F$$

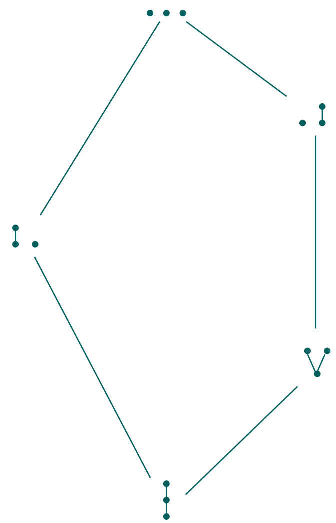
with $c^+(F_1 F_2) = c^+(F_1) c^+(F_2)$ and, for instance,

$c^+($  $)$

$$= P_+ \left(\underbrace{P_+(P_+(x)x)} \underbrace{P_+(x)}_{c^+(\bullet)} \underbrace{P_+(x)}_{c^+(\bullet)} x \right).$$

$c^+($ $)$

6.2 The Catalan idempotents in $\mathcal{H}^*_{\mathrm{NCK}}$



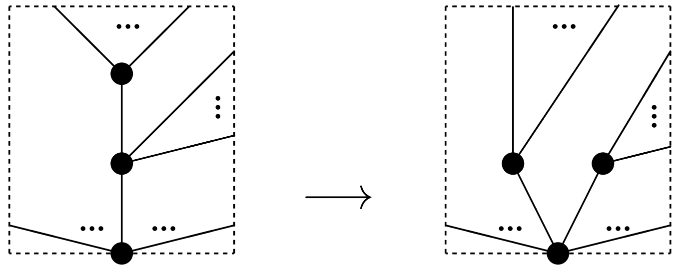
$$
\begin{array}{rclrcl}
c^+(\bullet\bullet\bullet) & = & \frac{a_0^3}{z^3} & d(\bullet\bullet\bullet) & = & \frac{a_0^3}{z^3}\\
c^+(\bullet\,\mathrm{I}) & = & \frac{a_0^3}{z^3}+\frac{a_0^2a_1}{z^2} & d(\bullet\,\mathrm{I}) & = & \frac{a_0^2a_1}{z^2}\\
c^+(\mathrm{I}\,\bullet) & = & \frac{a_0^3}{z^3}+\frac{a_0^2a_1}{z^2} & d(\mathrm{I}\,\bullet) & = & \frac{a_0^2a_1}{z^2}\\
c^+(\mathrm{V}) & = & \frac{a_0^3}{z^3}+\frac{a_0^2a_1}{z^2}+\frac{a_0^2a_2}{z} & d(\mathrm{V}) & = & \frac{a_0^2a_2}{z}\\
c^+(\vdots) & = & \frac{a_0^3}{z^3}+\frac{2a_0^2a_1}{z^2}+\frac{a_0^2a_2}{z}+\frac{a_0a_1^2}{z} & d(\vdots) & = & \frac{a_0a_1^2}{z}
\end{array}
$$

Theorem 6. *[M., Novelli, Thibon, 2024] For any forest F,*

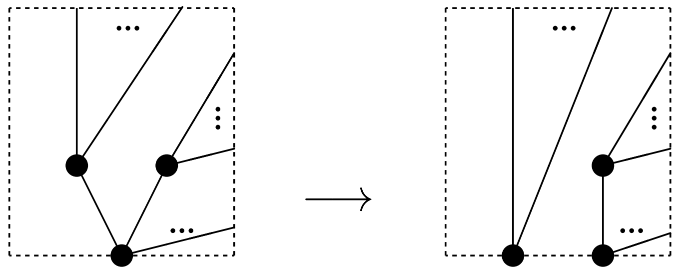
$$c^+(F)=\sum_{G\geqslant F} d(G)$$

Consider for a forest F and on any vertex that is not a leaf, define the transformations:

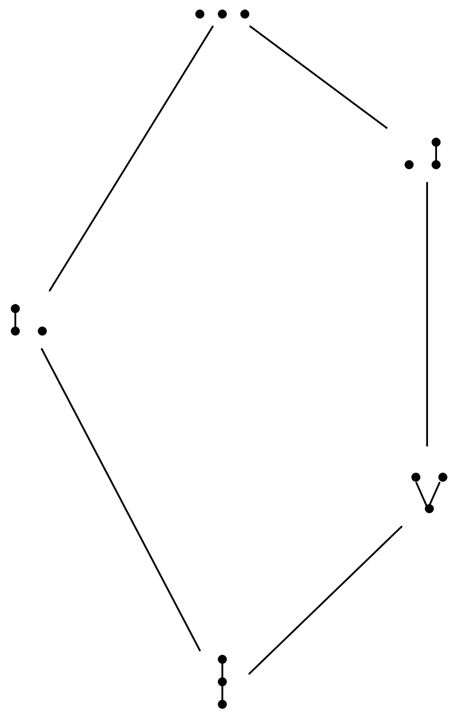
First kind:

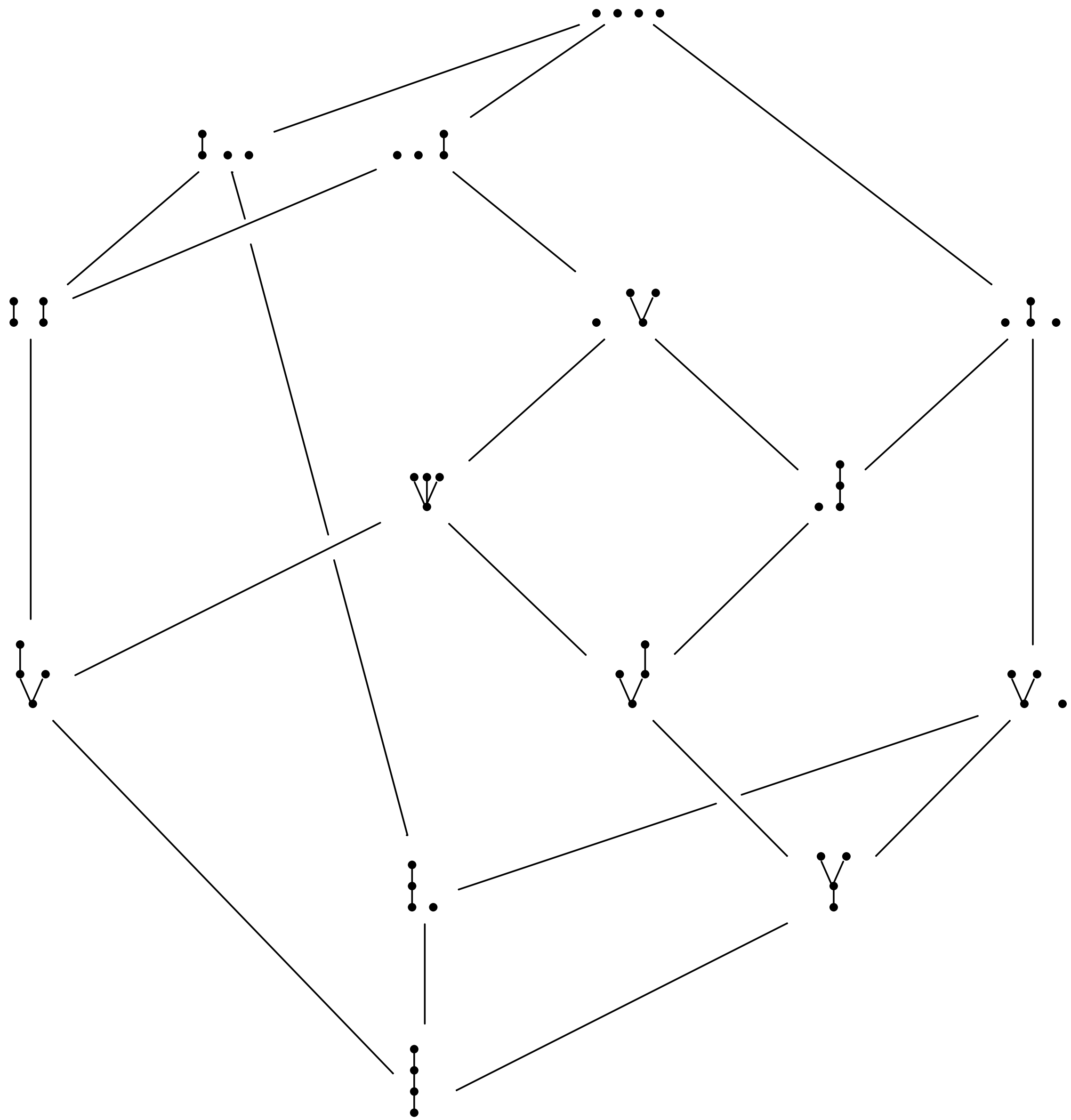


Second kind:



Let F, G be two plane forests. We shall say that $G \geqslant F$ if G can be obtained from F by a finite number of the preceding transformations.

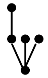




6.3 Rota-Baxter algebras and the Tamari order.

For any Rota-Baxter algebra, the coefficients $c^+(F)$ are sums over Tamari intervals of coefficients $d(G)$ obtained by the same way, but alternating the signs on each level [Foissy,M., in progress]:

Recall that:

$c^+($  $) = P_+ \left(\underbrace{P_+(P_+(x)x)}_{c^+(\,{\scriptstyle\vdots}\,)} \underbrace{P_+(x)}_{c^+(\bullet)} \underbrace{P_+(x)}_{c^+(\bullet)} x \right).$

then:

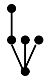
$d($  $) = P_+ \left(P_-(P_+(x)x)P_-(x)P_-(x)x\right)$

Table of contents