Identities involving Weyl groups from representation theory of quantum affine minimal W-algebras

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Setup: simple Lie algebras

$\mathfrak{g}$ complex simple finite dimensional Lie algebra
$\mathfrak{h} \subset \mathfrak{g}$ Cartan subalgebra
$\Delta \supset \Delta^+ \supset \Pi$, $(\mathfrak{g}, \mathfrak{h})$-root system, positive system, simple roots
These data induce

- a triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$
- a hyperplane arrangement in $\mathfrak{h}^*_{\mathbb{R}}$ associated to Δ and a reflection group acting on the complement, which turns out to be a Coxeter group with generators in bijection with Π

Example: $sl(n)$

$\mathfrak{g} = sl(n, \mathbb{C})$ complex $n \times n$ traceless matrices
$\mathfrak{h}$ complex $n \times n$ traceless diagonal matrices
Root space decomposition:

$$sl(n, \mathbb{C}) = \mathfrak{h} \oplus \sum_{i \neq j} \mathbb{C} e_{ij}$$

Roots:

$$\Delta = \{\epsilon_i - \epsilon_j \in \mathfrak{h}^* \mid i \neq j\}$$

We can choose $\Delta^+ = \{\epsilon_i - \epsilon_j \in \mathfrak{h}^* \mid i < j\}$, so that

$\mathfrak{n}_\pm$ strictly upper/lower triangular matrices

Weyl group:

$$W \cong S_n.$$

Representation Theory

Basic facts

- finite dimensional representations are completely reducible;
- finite dimensional irreducible representations are in bijection with the set $P^+ = \{\lambda \in \mathfrak{h}^* \mid \lambda(\alpha^\vee) \in \mathbb{Z}_+ \text{for all } \alpha \in \Pi\}$ of dominant integral weights;
- the irreducible representation $L(\lambda)$ attached to $\lambda \in P^+$ is a highest weight module: it is generated by a non zero vector v_λ such that

$$h(v_\lambda) = \lambda(h) \text{ for all } h \in \mathfrak{h} \text{ and } \mathfrak{n}_+(v_\lambda) = 0;$$

- more generally, the $L(\lambda)$, $\lambda \in \mathfrak{h}^*$ are the simple modules in the BGG category $\mathcal{O}$ consisting of $\mathfrak{h}$-diagonalizable modules with finite dimensional weight spaces and with a finite number of maximal weights.

Characters

For $M = \oplus M_\lambda \in \mathcal{O}$ one can define its character

$$ch\, M = \sum_\lambda \dim M_\lambda e^\lambda.$$

Set $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$.

Theorem

1. *Weyl character formula. If* $\lambda \in P^+$ *then*

$$chL(\lambda) = \frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})}$$

2. *Weyl denominator formula.*

$$e^\rho \prod_{\alpha\in\Delta^+}(1-e^{-\alpha}) = \sum_{w\in W}(-1)^{\ell(w)}e^{w(\rho)}.$$

Example: Vandermonde determinant

Specialize the denominator formula to the case of $sl(n)$. We have $\rho = \sum_{i=1}^{n}(n-i)\epsilon_i$, and setting $x_i = e^{\epsilon_i}$ we obtain

$$e^{\rho}\prod_{\alpha\in\Delta^+}(1-e^{-\alpha}) = x_1^{n-1}\cdots x_{n-1}\prod_{1\leqslant i<j\leqslant n}(1-x_i^{-1}x_j) = \prod_{1\leqslant i<j\leqslant n}(x_i-x_j)$$

and the denominator formula becomes

$$\prod_{1\leqslant i<j\leqslant n}(x_i-x_j) = \det(x_i^{j-1})_{i,j=1}^{n}.$$

Affine Lie algebras.

$\mathfrak{g}$ simple Lie algebra

$$\widehat{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}d$$

K is central, d acts as the Euler operator $t\,\frac{d}{dt}$. If we set $x_{(m)} = t^m \otimes x$, $x \in \mathfrak{g}$, the bracket is described by

$$[x_{(m)}, y_{(n)}] = [x, y]_{(m+n)} + m\delta_{m,-n}(x, y)K,$$

A Cartan subalgebra is $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$, and $\widehat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$, where

$$\Lambda_0(K) = 1, \quad \Lambda_0(d) = 0, \quad \delta(d) = 1, \quad \delta(K) = 0.$$

Basic structure theory

There is a root space decomposition

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{h}} \oplus \bigoplus_{\alpha \in \widehat{\Delta}} \mathfrak{g}_\alpha$$

where

$$\widehat{\Delta} = \{\alpha + \mathbb{Z}\delta \mid \alpha \in \Delta \cup \{0\}\} \setminus \{0\}.$$

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$\widehat{\mathfrak{g}}$ is a generated as a Lie algebra by $(e_\alpha)_{(0)}, \alpha \in \Pi$ and by $(e_{-\theta})_{(1)}$. Hence one has positive roots, simple roots and a triangular decomposition:

$$\widehat{\Pi} = \Pi \cup \{-\theta + \delta\}$$

$$\widehat{\Delta}^+ = \{\alpha + \mathbb{Z}_{\geqslant 0}\delta \mid \alpha \in \Delta^+\} \cup \{-\alpha + \mathbb{Z}_{>0}\delta \mid \alpha \in \Delta^+\} \cup \mathbb{Z}_{>0}\delta$$

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_+ \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_-$$

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$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_+ \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_-$$

The Weyl group $\widehat{W}$ is a Coxeter group with generators in bijection with $\widehat{\Pi}$.
Important fact: $\widehat{W}$ fixes δ.

Example: $\widehat{\mathfrak{g}}$ for $sl(n, \mathbb{C})$

Recall that $\mathfrak{h} =$ diagonal matrices in $sl(n, \mathbb{C})$.

$$\widehat{\mathfrak{g}} = sl(n, \mathbb{C}[t^{\pm 1}]) \oplus \mathbb{C}K \oplus \mathbb{C}d$$

$$\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$$

$$\widehat{\mathfrak{n}}_+ = \sum_{m>0} t^m \mathfrak{h} \oplus \sum_{i<j} \mathbb{C}e_{ij} \oplus \sum_{i \neq j} t\mathbb{C}[t]e_{ij}$$

The Weyl group $\widehat{W}$ can be realized as the set of permutations σ of integers such that

$$\sigma(x+n) = \sigma(x) + n, \quad \sum_{i=1}^{n} \sigma(i) = \binom{n}{2}.$$

Basic facts of representation theory

Basic facts of representation theory

- no nontrivial representation is finite dimensional;
- Category $\mathcal{O}$ makes sense, so we have Verma modules and highest weight modules
- The set $\widehat{P}^+ = \{\lambda \in \mathfrak{h}^* \mid \lambda(\alpha^\vee) \in \mathbb{Z}_+ \text{for all } \alpha \in \widehat{\Pi}\}$ of dominant integral weights indexes a special set of representations, called *integrable*;
- characters of modules in $\mathcal{O}$ can be defined; putting coordinates on $\widehat{\mathfrak{h}}$ one can discuss their convergence as complex analytic functions.

Characters

Set $\widehat{\rho} = h^\vee \Lambda_0 + \rho$.

Theorem

1. *Weyl-Kac character formula. If* $\lambda \in P^+$ *then*

$$chL(\lambda) = \frac{\sum_{w \in \widehat{W}} (-1)^{\ell(w)} e^{w(\lambda+\widehat{\rho})-\widehat{\rho}}}{\prod_{\alpha \in \widehat{\Delta}^+} (1 - e^{-\alpha})^{mult\,\alpha}}$$

2. *Weyl-Kac denominator formula.*

$$\prod_{\alpha \in \widehat{\Delta}^+} (1 - e^{-\alpha})^{mult\,\alpha} = \sum_{w \in \widehat{W}} (-1)^{\ell(w)} e^{w(\widehat{\rho})-\widehat{\rho}}.$$

Macdonald identities

Key Fact

$$\widehat{W} = W \ltimes T_M$$

where the elements of T_M are endomorphism of $\mathfrak{h}$ which can be viewed as translations, indexed by a certain lattice M, on a (quotient of a) suitable affine hyperplane isomorphic to $\mathfrak{h}^*_{\mathbb{R}}$.

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Estimating the sum $\sum_{w\in\widehat{W}}(-1)^{\ell(w)}e^{w(\widehat{\rho})-\widehat{\rho}}$ in Weyl-Kac denominator formula by setting $w = ut_\alpha$, we get the Macdonald identities:

$$\prod_{n>0}\Big((1-q^n)^{rk\mathfrak{g}}(1-q^n e^{-\alpha})\Big) = \sum_{\alpha\in M}\chi(h^\vee\alpha)q^{-\frac{c(h^\vee\alpha)}{2h^\vee}}$$

where $q = e^{-\delta}$, $c(\mu) = (\mu|\mu+2\rho)$ and $\chi(\mu)$ is given by the Weyl character formula (regardless that μ might be not dominant).

Jacobi triple product identity

$$\prod_{n>0}\left((1-q^n)^{rk\mathfrak{g}}(1-q^n e^{-\alpha})\right)=\sum_{\alpha\in M}\chi(h^\vee\alpha)q^{-\frac{c(h^\vee\alpha)}{2h^\vee}}$$

Consider the special case of $\widehat{sl(2)}$: we have

$$\Delta=\{\pm\alpha\},\quad h^\vee=2,\quad M=\mathbb{Z}\alpha,\quad \rho=\alpha/2,\quad W=\mathbb{Z}/2\mathbb{Z}$$

$$LHS:\quad \prod_{n>0}(1-q^n)(1-zq^n)(1-z^{-1}q^n)\quad (z=e^{-\alpha})$$

For the RHS: $\chi(n\alpha)=\frac{e^{n\alpha}-\epsilon^{-(n+1)\alpha}}{1-\epsilon^{-\alpha}}=\frac{z^{-n}-z^{n+1}}{1-z},\quad c(2n\alpha)=4n(2n+1):$ hence

$$\sum_{n\in\mathbb{Z}}\frac{z^{-2n}-z^{2n+1}}{1-z}q^{n(2n+1)}=\frac{1}{1-z}\sum_n(z^{-2n}q^{(2n+1)n}-z^{2n+1}q^{n(2n+1)})$$

$$=\frac{1}{1-z}\sum_m(-1)^m z^m q^{m(m-1)/2}$$

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Affine algebras and W-algebras

- W-algebras have been introduced in maximal generality by Kac-Roan-Wakimoto in 2005. They are obtained by *quantum hamiltonian reduction* from affine Lie algebras.
- Their main feature is to make available under a uniform construction a lot of examples arising in CFT (the *superconformal algebras*) and more....
- The starting data consists of a basic classical Lie superalgebra $\mathfrak{g}$ and an nilpotent element f in the even part of $\mathfrak{g}$.
- The above Lie superalgebras are simple Lie superalgebras $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ such that
  1. $\mathfrak{g}_{\bar{0}}$ is reductive
  2. $\mathfrak{g}$ has a nondegenerate supersymmetric invariant even bilinear form

  Looking at Kac's classification one gets the following list:
  $sl(m|n)$ $(m, n \geqslant 1, m \neq n)$, $psl(m|m)$ $(m \geqslant 2)$,
  $osp(m|n) = spo(n|m)$ $(m \geqslant 1, n \geqslant 2$ even$)$,
  $D(2,1;a)$ $(a \in \mathbb{C},\ a \neq 0, -1)$, $F(4)$, $G(3)$.

Affine algebras and W-algebras

- The construction is homological: one builds up a complex out of the triple $(\mathfrak{g}, f, k)$, which turns out to be acyclic, and the corresponding H_0 is the (quantum affine) W-algebra $W^k(\mathfrak{g}, f)$.
- The most convenient language to deal with these objects is that of vertex operator algebras. I will not review the standard definitions but I will highlight an axiomatization which allows us to view the vertex algebras we are interested in as a one-parameter generalization of Lie algebras.
- Although the construction of $W^k(\mathfrak{g}, f)$ is rather technical, these algebras should be regarded as a simplification of $V^k(\mathfrak{g})$. It is not rare that one is able to lift information from $W^k(\mathfrak{g}, f)$ to $V^k(\mathfrak{g})$.

Interlude: Lie conformal vs vertex algebras

Definition

A *Lie conformal superalgebra* is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[T]$-module $R = R_{\bar{0}} \oplus R_{\bar{1}}$, endowed with a parity preserving $\mathbb{C}$-bilinear λ-bracket $R \otimes R \to \mathbb{C}[\lambda] \otimes R$, denoted by $[a_\lambda b]$, such that the following axioms hold:

1. $[Ta_\lambda b] = -\lambda[a_\lambda b], \quad T[a_\lambda b] = [Ta_\lambda b] + [a_\lambda Tb]$,
2. $[b_\lambda a] = -p(a,b)[a_{-\lambda-T} b]$,
3. $[a_\lambda[b_\mu c]] - p(a,b)[b_\lambda[a_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c]$.

One writes $[a_\lambda b] = \sum_{n\in\mathbb{Z}_+} \frac{\lambda^n}{n!}(a_{(n)}b)$, where the sum is finite, and the bilinear products $a_{(n)}b$ are called the n-th products of R.

Interlude: Lie conformal vs vertex algebras

Definition

A vertex algebra is a quintuple $(V, |0\rangle, T, [\,.\,_\lambda\,.\,], {:}\,{:})$, where

1. $(V, T, [\,.\,_\lambda\,.\,])$ is a Lie conformal superalgebra,
2. $(V, |0\rangle, T, {:}\,{:})$ is a unital differential (i.e. T is a derivation) superalgebra, satisfying suitable compatibility conditions,
3. the λ-bracket $[\,.\,_\lambda\,.\,]$ and the product ${:}\,{:}$ are related by the non-commutative Wick formula.

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If one puts

$$:ab := a_{(-1)}b,$$

it can be shown that setting of $a_{(n)}b = \frac{1}{(-n-1)!} : T^n(a)b :, n < -1$ one gets all the bilinear products $V \otimes V \to V$, $a \otimes b \mapsto a_{(n)}b$, $n \in \mathbb{Z}$, which give rise to the state field correspondence

$$a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} b z^{-n-1}$$

VOAs

Definition

A vertex operator algebra is a pair (V, L) where V is a vertex algebra and $L \in V$ is a vector such that

1. $[L_\lambda L] = TL + 2\lambda L + \frac{\lambda^3}{12}c$.
2. $L_{-1} = T$.
3. L_0 is semisimple with eigenvalues in $\frac{1}{2}\mathbb{Z}_+$.

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Remark

Condition (1) simply means that if $Y(L, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, then

$$[L_m, L_n] = L_{m+n} + \delta_{m,-n}\frac{m^3 - m}{12}c.$$

Affine vertex algebras

$\mathfrak{g}$ simple Lie algebra, $(\cdot,\cdot)$ normalized invariant form. Let $V^k(\mathfrak{g})$ be the universal enveloping vertex algebra of the Lie conformal algebra

$$Cur(\mathfrak{g}) = \mathbb{C}[\lambda] \otimes \mathfrak{g}[t^{\pm 1}] \oplus \mathbb{C}K$$

and λ-bracket

$$[a(t)_\lambda b(t)] = [a,b](t) + \lambda(a,b)K$$

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Reps of $V^k(\mathfrak{g})$ vs reps of $\widehat{\mathfrak{g}}$

With some care, one can establish a bijection between "good" representations of $V^k(\mathfrak{g})$ and "good" representations of $\widehat{\mathfrak{g}}$ (better, of $[\widehat{\mathfrak{g}},\widehat{\mathfrak{g}}]$) of level k (i.e, K acts as k).

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Sugawara construction

Let $h^\vee$ be the dual Coxeter number of $\mathfrak{g}$. If $k \neq -h^\vee$ then $V^k(\mathfrak{g})$ is a VOA with conformal vector $L^\mathfrak{g}$ and central charge $c_\mathfrak{g}$

$$L^\mathfrak{g} = \frac{1}{2(k+h^\vee)} \sum_{i=1}^{\dim \mathfrak{g}} :x_i x^i:, \quad c_\mathfrak{g} = \frac{k \dim \mathfrak{g}}{k+h^\vee}$$

where $\{x_i\}$, $\{x^i\}$ are dual bases of $\mathfrak{g}$ w.r.t. $(\cdot,\cdot)$.

Minimal quantum affine W-algebras

Choose a Cartan subalgebra $\mathfrak{h}$ for $\mathfrak{g}_{\bar{0}}$ and let Δ be the set of roots. Fix a minimal root $-\theta$ of $\mathfrak{g}$. We may choose root vectors e_θ and $e_{-\theta}$ such that

$$[e_\theta, e_{-\theta}] = x \in \mathfrak{h}, \qquad [x, e_{\pm\theta}] = \pm e_{\pm\theta}.$$

Due to the minimality of $-\theta$, the eigenspace decomposition of *ad* x defines a *minimal* $\frac{1}{2}\mathbb{Z}$-grading:

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1,$$

where $\mathfrak{g}_{\pm 1} = \mathbb{C}e_{\pm\theta}$.

Minimal quantum affine W-algebras

Furthermore, one has

$$\mathfrak{g}_0 = \mathfrak{g}^\natural \oplus \mathbb{C}x, \quad \mathfrak{g}^\natural = \{a \in \mathfrak{g}_0 \mid (a|x) = 0\}.$$

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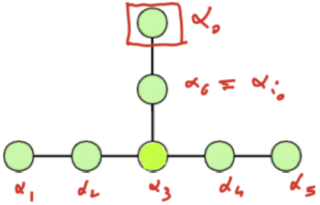
Example: If $\mathfrak{g} = E_6$, then $\mathfrak{g}^\natural$ is of type A_5 and $\mathfrak{g}_{1/2} = \Lambda^3\mathbb{C}^6$.



$$E_6 = \mathbb{C}e_\theta \oplus \Lambda^3\mathbb{C}^6 \oplus (sl(6) \oplus \mathbb{C}x) \oplus (\Lambda^3\mathbb{C}^6)^* \oplus \mathbb{C}e_{-\theta}$$

Minimal quantum affine W-algebras

Furthermore, one has

$$\mathfrak{g}_0 = \mathfrak{g}^\natural \oplus \mathbb{C}x, \quad \mathfrak{g}^\natural = \{a \in \mathfrak{g}_0 \mid (a|x) = 0\}.$$

Note that $\mathfrak{g}^\natural$ is the centralizer of the triple $\{e_{-\theta}, x, e_\theta\}$. We can choose $\mathfrak{h}^\natural = \{h \in \mathfrak{h} \mid (h|x) = 0\}$ as a Cartan subalgebra of the Lie superalgebra $\mathfrak{g}^\natural$, so that $\mathfrak{h} = \mathfrak{h}^\natural \oplus \mathbb{C}x$.
For a given choice of a minimal root $-\theta$, we normalize the invariant bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}$ by the condition

$$(\theta|\theta) = 2. \tag{3.1}$$

The dual Coxeter number $h^\vee$ of the pair $(\mathfrak{g}, \theta)$ is defined to be half the eigenvalue of the Casimir operator of $\mathfrak{g}$ corresponding to $(\cdot|\cdot)$, normalized by (3.1).

Minimal quantum affine W-algebras

Kac, Wakimoto and Roan associated a vertex algebra $W^k(\mathfrak{g}, f)$, called a *universal W-algebra*, to each triple $(\mathfrak{g}, f, k)$, where $\mathfrak{g}$ is a basic Lie superalgebra, f is a nilpotent element of $\mathfrak{g}_{\bar{0}}$, and $k \in \mathbb{C}$, by applying the quantum Hamiltonian reduction functor H_f to the affine vertex algebra $V^k(\mathfrak{g})$.

In particular, it was shown that, for k non-critical, $W^k(\mathfrak{g}, f)$ has a Virasoro vector L, making it a conformal vertex algebra, and a set of free generators was constructed. For k non-critical the vertex algebra $W^k(\mathfrak{g}, f)$ has a unique simple quotient, denoted by $W_k(\mathfrak{g}, f)$.

We will consider the case $f = e_{-\theta}$ where $-\theta$ is a minimal root. The Virasoro vector L of $W^k_{\min}(\mathfrak{g})$ has central charge

$$c(\mathfrak{g}, k) = \frac{k\, sdim \mathfrak{g}}{k + h^\vee} - 6k + h^\vee - 4.$$

Presentation of $W^k_{\min}(\mathfrak{g})$

Recall that $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1, \mathfrak{g}_0 = \mathfrak{g}^\natural \oplus \mathbb{C}x$.

Theorem (KW)

(a) The vertex algebra $W^k_{\min}(\mathfrak{g})$ is strongly and freely generated by elements $J^{\{a\}}$, where a runs over a basis of $\mathfrak{g}^\natural$, $G^{\{v\}}$, where v runs over a basis of $\mathfrak{g}_{-1/2}$, and the Virasoro vector L.

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Rough explanation: a vector space basis $W^k_{\min}(\mathfrak{g})$ is given by monomials in (iterated) normal orders of the above generators and their derivatives.

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(b) The elements $J^{\{a\}}$, $G^{\{v\}}$ are primary of conformal weight 1 and $3/2$, respectively, with respect to L.

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(b) The elements $J^{\{a\}}$, $G^{\{v\}}$ are primary of conformal weight 1 and 3/2, respectively, with respect to L.
Rough explanation: being primary means having a certain λ-bracket with L. In this case

$$[L_\lambda J^{\{a\}}] = (L_{-1} + \lambda)J^{\{a\}}, \quad [L_\lambda G^{\{v\}}] = (L_{-1} + \tfrac{3}{2}\lambda)G^{\{v\}}$$

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(b) The elements $J^{\{a\}}$, $G^{\{v\}}$ are primary of conformal weight 1 and 3/2, respectively, with respect to L.
(c) If κ_0 is the Killing form of $\mathfrak{g}_0$, then

$$[J^{\{a\}}{}_\lambda J^{\{b\}}] = J^{\{[a,b]\}} + \lambda \underbrace{\left((k + h^\vee/2)(a|b) - \tfrac{1}{4}\kappa_0(a,b)\right)}_{\beta_k(a,b)},\ \ a, b \in \mathfrak{g}^\natural,$$

$$[J^{\{a\}}{}_\lambda G^{\{u\}}] = G^{\{[a,u]\}},\ \ a \in \mathfrak{g}^\natural,\ u \in \mathfrak{g}_{-1/2}.$$

(d) There are explicit formulas yielding $[G^{\{u\}}{}_\lambda G^{\{v\}}]$ for $u, v \in \mathfrak{g}_{-1/2}$.

Presentation of $W^k_{\min}(\mathfrak{g})$

Proposition, AKMPP, J-alg

Let $u, v \in \mathfrak{g}_{-1/2}$. There is a monic quadratic polynomial $p_{\mathfrak{g}}$ such that

$$G^{\{u\}}{}_{(2)}G^{\{v\}} = 4(e_\theta|[u,v])p_{\mathfrak{g}}(k)|0\rangle.$$

Moreover, the linear polynomial $k_i(k)$, $i \in I$, defined by $k_i(k) = k + \frac{1}{2}(h^\vee - h^\vee_{0,i})$, divides $p_{\mathfrak{g}}(k)$ and

$$G^{\{u\}}{}_{(1)}G^{\{v\}} = 4\sum_{i\in I}\frac{p_{\mathfrak{g}}(k)}{k_i(k)}J^{\{([[e_\theta,u],v])^\natural_i\}}$$

where $\mathfrak{g}^\natural = \bigoplus_i \mathfrak{g}^\natural_i$ and $(a)^\natural_i$ denotes the orthogonal projection of $a \in \mathfrak{g}_0$ onto $\mathfrak{g}^\natural_i$.

Collapsing levels

Definition

We say that a level k is *collapsing* if $W_k^{\min}(\mathfrak{g}) = V_{\beta_k}(\mathfrak{g}^\natural)$.

Theorem (AKMPP)

Let $\mathfrak{g}^\natural = \bigoplus_{i\in I} \mathfrak{g}_i^\natural$. *Then* k *is a collapsing level, if and only if* $p_{\mathfrak{g}}(k) = 0$

$$W_k^{\min}(\mathfrak{g}) = \bigotimes_{i\in I : k_i \neq 0} V_{k_i}(\mathfrak{g}_i^\natural). \tag{3.2}$$

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Proposition (Arakawa-Moreau for $\mathfrak{g}$ even J.I.M. Jussieu; AKMPP, J-alg)

$W_k^{\min}(\mathfrak{g}) = \mathbb{C}|0\rangle$ if and only if either $k = -\frac{h^\vee}{6} - 1$ and either $\mathfrak{g}$ is a Lie algebra belonging to the Deligne's series $(A_2, G_2, D_4, F_4, E_6, E_7, E_8)$ or $\mathfrak{g} = psl(m|m)$ $(m \geqslant 2)$, $\mathfrak{g} = osp(n+8|n)$ $(n \geqslant 2)$, $\mathfrak{g}$ is of type $F(4)$, $G(3)$ (for both choices of θ), or $\mathfrak{g} = spo(2|1)$; or $k = -\frac{1}{2}$ and $\mathfrak{g} = spo(n|m)$ $(n \geqslant 1, m \neq n+1)$.

Identities from representation theory of W-algebras

Key idea

The character formulas for representations of W-algebras are obtained from character formulas for representations of affine algebras by using the quantum hamiltonian reduction functor. This is known to map a Verma module for $\widehat{\mathfrak{g}}$ of highest weight λ to a Verma module for $W^k_{\min}(\mathfrak{g})$ of highest weight $\tilde{\lambda}$ explicitly expressible in terms of λ.
Replace the use of the trivial representation in the Weyl or Weyl-Kac denominator formulas by that of a collapsing level k_0 such that $W_{k_0}(\mathfrak{g}) = \mathbb{C}$. If one knows a formula for the character of the $\widehat{\mathfrak{g}}$-module $L(\lambda)$, using the reduction functor one gets usually interesting identities.

Verma and irreducible modules

Fix a basis $\{v_i \mid i \in I\}$ of $\mathfrak{g}_{-1/2}$ and a basis $\{u_i \mid i \in J\}$ of $\mathfrak{g}^\natural$. Set $A^{\{i\}} = J^{\{u_i\}}$ if $i \in J$, $A^{\{i\}} = G^{\{v_i\}}$ if $i \in I$, and $A^{\{0\}} = L$.

A highest weight module M of highest weight (ν, ℓ_0), $\nu \in (\mathfrak{h}^\natural)^*$, $\ell_0 \in \mathbb{C}$ is called a Verma module if

$$\mathcal{B} = \left\{ \left(A^{\{1\}}_{-m_1}\right)^{b_1} \cdots \left(A^{\{s\}}_{-m_s}\right)^{b_s} v_{\nu,\ell_0} \right\}$$

where $b_i \in \mathbb{Z}_+$, $b_i \leqslant 1$ if $i \in I$, $m_i \geqslant 0$ with $m_i > 0$ if $u_i \in \mathfrak{n}^\natural_+ \oplus \mathfrak{h}^\natural$, is a basis of M. Note that

$$J^{\{h\}}_0 v_{\nu,\ell_0} = \nu(h) v_{\nu,\ell_0}, \quad L_0 v_{\nu,\ell_0} = \ell_0 v_{\nu,\ell_0}.$$

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$$J_0^{\{h\}} v_{\nu,\ell_0} = \nu(h) v_{\nu,\ell_0}, \quad L_0 v_{\nu,\ell_0} = \ell_0 v_{\nu,\ell_0}.$$

The Verma module of highest weight (ν, ℓ_0) exists, is unique up to isomorphism, and is denoted by $M^W(\nu, \ell_0)$.

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If M is a highest weight module of highest weight (ν, ℓ_0) then M is a quotient of $M^W(\nu, \ell_0)$. The unique irreducible quotient is denoted by $L^W(\nu, \ell_0)$.

Properties of the reduction functor

These results are due to Arakawa and Kac-Wakimoto: there is a functor

$$\mathcal{H} : V^k(\mathfrak{g})\text{-modules} \longrightarrow W^k_{\min}(\mathfrak{g})\text{-modules}$$

such that

- If $M \in \mathcal{O}_k$ then
$$\mathcal{H}(M) = H_0(M)$$
In particular, the functor $\mathcal{H}$ is exact on $\mathcal{O}_k$.
- If M is a highest weight module over $\widehat{\mathfrak{g}}$ of highest weight $\Lambda \in \widehat{\mathfrak{h}}^*_k$, $\mathcal{H}(M)$ is either zero or a highest weight module over $W^k_{\min}(\mathfrak{g})$ of highest weight (ν, ℓ) with
$$\nu = \Lambda_{|\mathfrak{h}^\natural}, \quad \ell = \frac{(\Lambda|\Lambda + 2\widehat{\rho})}{2(k + h^\vee)} - \Lambda(x + D).$$
- The functor $\mathcal{H}$ maps Verma modules to Verma modules.
- Let $\lambda \in \widehat{\mathfrak{h}}^*_k$. If $\lambda((\delta - \theta)^\vee) \in \{0, 1, 2, \dots\}$, then $\mathcal{H}(L(\lambda)) = \{0\}$. Otherwise, $\mathcal{H}(L(\lambda))$ is an irreducible highest weight $W^k_{\min}(\mathfrak{g})$-module.

We shall coordinatize $\widehat{\mathfrak{h}}$ by letting

$$(\tau, z, u) = 2\pi i(z - \tau D + uK) \ ,$$

where $z \in \mathfrak{h}$ We shall assume that $\mathrm{Im}\ \tau > 0$ in order to guarantee the convergence of characters, and set $q = e^{2\pi i\tau}$. Define the character of a $\widehat{\mathfrak{g}}$-module M by

$$ch(M) = \mathrm{tr}_M \ e^{2\pi i(z-\tau D+uK)}.$$

For any highest weight $\widehat{\mathfrak{g}}$-module M of level $k \neq -h^\vee$ the series $ch(M)$ converges to an analytic function in the interior of the domain

$$Y_> := \{h \in \widehat{\mathfrak{h}}|\ (\alpha|h) > 0 \text{ for all } \ \alpha \in \widehat{\Delta}^+\}$$

Characters

The character $ch\,M$ of a non-twisted highest weight $W^k_{\min}(\mathfrak{g})$-module M is defined as the trace of $q^{L_0} J_0^{\{h\}}$, $h \in \mathfrak{h}^\natural$, where $q = exp(-\tau(x + D))$. Similarly, the character $ch\,M$ of a Ramond twisted highest weight $W^k_{\min}(\mathfrak{g})$-module M is defined as the trace of $q^{L_0^{\rm tw}} J_0^{\{h\},\rm tw}$, $h \in \mathfrak{h}^\natural$,

Characters

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Similarly, the character $ch\,M$ of a Ramond twisted highest weight $W^k_{\min}(\mathfrak{g})$-module M is defined as the trace of $q^{L_0^{\mathrm{tw}}}J_0^{\{h\},\mathrm{tw}}$, $h \in \mathfrak{h}^\natural$,
Let's go back to $\widehat{\mathfrak{g}}$. Introduce the Weyl denominator

$$\hat{R} = \prod_{\alpha \in \widehat{\Delta}^+} (1 - s(\alpha)e^{-\alpha})^{s(\alpha) mult \alpha}$$

where $s(\alpha) = (-1)^{p(\alpha)}$.
Let M be the highest weight $\widehat{\mathfrak{g}}$-module with the highest weight Λ of level $k \neq -h^\vee$, and suppose that ch_M extends to a meromorphic function on Y with at most simple poles at the hyperplanes T_α, where $\alpha \in (\widehat{\Delta}^+_{re})_{\bar{0}}$.

Characters

Then

$$ch_{H(M)}(h) = \frac{q^{\frac{(\Lambda|\Lambda+2\widehat{\rho}))}{2(k+h^\vee)}}}{\prod_{j=1}^\infty (1-q^j)^{\dim \mathfrak{h}}} \, (\hat{R}\, ch_M)(H)$$

$$\times \prod_{n=1}^\infty \prod_{\alpha\in\widehat{\Delta}^+, (\alpha|x)=0} ((1-s(\alpha)e^{-(n-1)K-\alpha})^{-s(\alpha)}(1-s(\alpha)e^{-nK+\alpha})^{-s(\alpha)})(H)$$

$$\times \prod_{n=1}^\infty \prod_{\alpha\in\Delta_+, (\alpha|x)=\frac{1}{2}} (1-s(\alpha)e^{-nK+\alpha})^{-s(\alpha)}(H)$$

where, as before, $s(\alpha) = (-1)^{p(\alpha)}$ and

$$H := (\tau, -\tau x + h, 0) = 2\pi i(-\tau D - \tau x + h), \;\; h \in \mathfrak{h}^\natural.$$

Characters

Note that the denominators of the characters of non-twisted (resp. σ_R-twisted) Verma modules over $W^k_{\min}(\mathfrak{g})$ are given by

$$\widehat{F}^{NS} = \prod_{n=1}^{\infty} \frac{(1-q^n)^{\dim \mathfrak{h}} \prod_{\alpha \in \Delta_+^{\natural}} (1-q^{n-1}e^{-\alpha})(1-q^n e^{\alpha})}{\prod_{\alpha \in \Delta_{-1/2}} (1+q^{n-\frac{1}{2}} e^{\alpha_{|\mathfrak{h}^\natural}})},$$

$$\widehat{F}^{R} = \prod_{n=1}^{\infty} \frac{(1-q^n)^{\dim \mathfrak{h}} \prod_{\alpha \in \Delta_+^{\natural}} (1-q^{n-1}e^{-\alpha})(1-q^n e^{\alpha})}{\prod_{\eta \in (\overline{\Delta}^+_{1/2})'} (1+q^{n-1}e^{-\eta}) \prod_{\eta \in \overline{\Delta}^+_{1/2}} (1+q^n e^{\eta})},$$

Character formulas

We are able to compute the characters of untwisted and Ramond twisted irreducible highest weight modules $W_k^{\min}(\mathfrak{g})$, when k is in the unitary range. There are two cases to consider. In the first case, called *massive*, the $W_k^{\min}(\mathfrak{g})$-module is obtained by quantum Hamiltonian reduction of typical modules over the corresponding affine Lie algebra, and in the second case, called *massless*, from the maximally atypical ones. The corresponding character formulas are obtained by quantum Hamiltonian reduction (using the Conjecture TQHR in the Ramond twisted case). The basic starting point are formulas for characters of representations of affine superalgebras due to Gorelik- Kac.

Denominator Identities

Idea: Take k_0 such that $W^{\min}_{k_0}(\mathfrak{g}) = \mathbb{C}$ and use character formulas and the decomposition $\widehat{W}^\natural = W^\natural \ltimes M^\natural$.

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Theorem

For explicit constants c, b we have

$$\widehat{F}^{NS} = e^{-\rho^\natural} \sum_{\bar{w} \in W^\natural} \sum_{\alpha \in M^\natural} \det(\bar{w}) \frac{e^{\bar{w}(\rho^\natural + b\alpha)}}{\prod_{\beta \in \Pi_{\bar{1}}} (1 + e^{-\bar{w}(\beta_{|\mathfrak{h}^\natural})} q^{-\frac{2}{u}(\alpha|\beta) + \frac{1}{2}})} q^{\frac{b}{u}(\alpha|\alpha) + \frac{2}{u}(\rho^\natural|\alpha)}.$$

If Conjecture TQHR holds, then

$$\widehat{F}^{R} = \frac{e^{\rho_R - \rho^\natural}}{1 + \epsilon(\sigma_R)} \sum_{\bar{w} \in W^\natural} \sum_{\alpha \in M^\natural} \det(\bar{w}) \frac{e^{\bar{w}(\rho^\natural - \rho_R + b\alpha)}}{\prod_{\beta \in \Pi_{\bar{1}}^{\nu}} (1 + e^{-\bar{w}(\beta_{|\mathfrak{h}^\natural})} q^{-\frac{2}{u}(\alpha|\beta)})} q^{\frac{b}{u}(\alpha|\alpha) + \frac{2}{u}(\rho^\natural - \rho_R|\alpha)}.$$

Example: $spo(2|1)$

In the NS sector we have

$$\mathbb{C}|0\rangle = W^{\min}_{-1/2}(spo(2|1)) = H(L(-\frac{1}{2}\Lambda_0)).$$

It follows from KW that $-\frac{1}{2}\Lambda_0$ is an admissible weight for $spo(2|1)^\wedge$, hence

$$\widehat{R}\ ch\, L(-\frac{1}{2}\Lambda_0) = \sum_{w\in \widehat{W}_{int}} det(w) e^{w(-\frac{1}{2}\Lambda_0+\widehat{\rho})-\widehat{\rho}},$$

with $\widehat{W}_{int}$ the Weyl group of the root subsystem generated by the set of simple roots $\Pi_{int} = \{\theta/2, \delta - \theta/2\}$.

Example: $spo(2|1)$

Applying the functor H and using Arakawa theorem, we obtain

$$\prod_{n\geqslant 1}\frac{1-q^n}{1+q^{n-\frac{1}{2}}}=\sum_{w\in\widehat{W}_{int}} det(w)q^{-(w(-\frac{1}{2}\Lambda_0+\widehat{\rho})-\widehat{\rho})(x+D)}. \tag{4.1}$$

In our special case we have

$$t_{n\theta}(-\frac{1}{2}\Lambda_0+\widehat{\rho})-\widehat{\rho}=-\frac{1}{2}\Lambda_0-2n\theta-n(1+4n)\delta,$$

while

$$s_\theta t_{n\theta}(-\frac{1}{2}\Lambda_0+\widehat{\rho})-\widehat{\rho}=-\frac{1}{2}\Lambda_0-(2n+\frac{1}{2})\theta-n(1+4n)\delta,$$

Example: $spo(2|1)$

Hence

$$\prod_{n\geqslant 1}\frac{1-q^n}{1+q^{n-\frac{1}{2}}}=\sum_{n\in\mathbb{Z}}(q^{4n^2-n}-q^{4n^2+3n+\frac{1}{2}})$$

$$=\sum_{n=0}^{\infty}(q^{4n^2-n}-q^{4n^2+3n+\frac{1}{2}})+\sum_{n=1}^{\infty}(q^{4n^2+n}-q^{4n^2-3n+\frac{1}{2}})$$

$$=\sum_{m\in 4\mathbb{Z}_+-1}q^{\frac{1}{4}m(m+1)}-\sum_{m\in 4\mathbb{Z}_++1}q^{\frac{1}{4}m(m+1)}+\sum_{m\in 4\mathbb{N}}q^{\frac{1}{4}m(m+1)}-\sum_{m\in 4\mathbb{N}-2}q^{\frac{1}{4}m(m+1)}$$

$$=\sum_{m=0}^{\infty}(-q^{\frac{1}{2}})^{m(m+1)/2}.$$

Replacing q by q^2 and then changing the sign of q, we obtain the Gauss identity for the generating series of triangular numbers:

$$\prod_{n\geqslant 1}\frac{1-q^{2n}}{1-q^{2n+1}}=\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}}.$$

Example: $spo(2|1)$

In the Ramond sector, taking $k_0 = -\frac{1}{2}$ and using Conjecture TQHR, we have

$$H(L(-\frac{1}{2}\Lambda_0)) = L^W(0,0) \oplus L^W(0,0) = \mathbb{C}^2.$$

The character of the $\widehat{\mathfrak{g}}^{\mathrm{tw}}$-module $L(-\frac{1}{2}\Lambda_0)$ is given by

$$ch\, L(-\frac{1}{2}\Lambda_0) = \sum_{w \in \widehat{W}_{int}} det(w) \frac{e^{w(-\frac{1}{2}\Lambda_0 + \widehat{\rho}^{\mathrm{tw}}) - \widehat{\rho}^{\mathrm{tw}}}}{\widehat{R}^{\mathrm{tw}}}$$

so, applying the twisted quantum Hamiltonian reduction functor, the identity becomes

$$2 \prod_{n \geqslant 1} \frac{1 - q^n}{1 + q^{n-1}} = \prod_{n \geqslant 1} \frac{1 - q^n}{1 + q^n} = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2},$$

which is Gauss identity for the generating series of square numbers.

Example: $F(4)$

$(W^\natural = \{\pm 1\}^3 \rtimes \mathfrak{S}_3, \quad \vartheta_0(x) = \prod_{j=1}^{\infty}(1 - xq^{j-1})(1 - x^{-1}q^j))$

NS sector
$$\frac{\varphi(q)^4 \prod\limits_{1\leqslant i\leqslant 3} \vartheta_0(y_i^{-2}) \prod\limits_{1\leqslant i\lessgtr j\leqslant 3} \vartheta_0(y_i^{-2}y_j^{-2}) \prod\limits_{1\leqslant i\lessgtr j\leqslant 3} \vartheta_0(y_i^{-2}y_j^{2})}{\vartheta_1(y_1y_2y_3)\vartheta_1(y_1y_2y_3^{-1})\vartheta_1(y_1y_2^{-1}y_3)\vartheta_1(y_1^{-1}y_2y_3)}$$

$$= y_1^{-5}y_2^{-3}y_3^{-1} \sum_{\substack{m,r,t\in\mathbb{Z} \\ m+t+r\equiv 0 \mod 2}} q^{2m^2+2r^2+2t^2+\frac{5m+3r+t}{2}} \left(\sum_{\bar{w}\in W^\natural} \det(\bar{w}) \frac{\bar{w}(y_1^{8m+5}y_2^{8r+3}y_3^{8t+1})}{1+q^{b_{m,r,t}}\bar{w}(y_1y_2y_3)} \right),$$

Ramond sector
$$\frac{\varphi(q)^4 \prod\limits_{1\leqslant i\leqslant 3} \vartheta_0(y_i^{-2}) \prod\limits_{1\leqslant i\lessgtr j\leqslant 3} \vartheta_0(y_i^{-2}y_j^{-2}) \prod\limits_{1\leqslant i\lessgtr j\leqslant 3} \vartheta_0(y_i^{-2}y_j^{2})}{\vartheta_0(-y_1^{-1}y_2^{-1}y_3^{-1})\vartheta_0(-y_1^{-1}y_2^{-1}y_3)\vartheta_0(-y_1^{-1}y_2y_3^{-1})\vartheta_0(-y_1y_2^{-1}y_3^{-1})}$$

$$= y_1^{-4}y_2^{-2} \sum_{\substack{m,r,t\in\mathbb{Z} \\ m+t+r\equiv 0 \mod 2}} q^{2m^2+2r^2+2t^2+2m+r} \left(\sum_{\bar{w}\in W^\natural} \det(\bar{w}) \frac{\bar{w}(y_1^{8m+4}y_2^{8r+2}y_3^{8t})}{1+q^{\bar{b}_{m,r,t}}\bar{w}(y_1y_2^{-1}y_3^{-1})} \right).$$