Special bases for the swap algebras

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The Swap algebra

Schur–Weyl duality

Quantum information theory

Schur–Weyl duality

Let V be a vector space of finite dimension d over $\mathbb{C}$.
In the classical theory of Schur–Weyl a major role is played by the action of the symmetric group S_n on n elements on the n^{th} tensor power $V^{\otimes n}$ by exchanging the tensor factors.

The algebra of operators on $V^{\otimes n}$, generated by these permutations will be denoted by $\Sigma_n(d)$ and called *a d–swap algebra*.
It is the algebra formed by the elements which commute with the diagonal action of $GL(V)$.

The name comes from the use, in the physics literature, to call *swap* the exchange operator $(1,2) : u \otimes v \mapsto v \otimes u$ on $V^{\otimes 2}$.

In the literature on quantum information theory the states lying in $\Sigma_n(d)$ are called *Werner states* and widely used as source of examples, due to fundamental work of the physicist R. F. Werner.

1. A classical theorem states that the corresponding algebra homomorphism $\mathbb{C}[S_n] \to \Sigma_n(d) \subset End(V^{\otimes n})$ is injective if and only if $\dim V \geq n$.
2. When $d = \dim V < n$ the kernel of this map is the two sided ideal of $\mathbb{C}[S_n]$ generated by the antisymmetrizer

$$A_{d+1} := \sum_{\sigma \in S_{d+1}} \epsilon_\sigma \sigma, \quad \epsilon_\sigma \text{ the sign of the permutation.}$$

The algebra $\mathbb{C}[S_n]$ decomposes as direct sum of matrix algebras indexed by partitions, corresponding to the irreducible representations of S_n. As for $\Sigma_n(d)$ only the blocks relative to partitions of height $\leq d$ survive.

The problem

In the case $d = \dim V < n$ an interesting problem is to describe a basis of $\Sigma_n(d)$ formed by permutations.

In fact in the physics literature there are several examples of Hamiltonians lying in $\Sigma_n(d)$. Thus it may be convenient to express such Hamiltonian in a given special basis, This may be done as follows.

Definition

Let $0 < d$ be an integer and let $\sigma \in S_n$.
Then σ is called *d–bad* if σ has a descending subsequence of length d, namely, if there exists a sequence $1 \leq i_1 < i_2 < \cdots < i_d \leq n$ such that $\sigma(i_1) > \sigma(i_2) > \cdots > \sigma(i_d)$.
Otherwise σ is called *d–good*.

Remark

σ is d–good if any descending sub–sequence of σ is of length $\leq d-1$. If σ is d-good then σ is d'-good for any $d' \geq d$.
Every permutation is 1-bad.

Theorem

If $\dim(V) = d$ the $d+1$–good permutations form a basis of $\Sigma_k(V)$.

Step 1 The $d+1$ good permutations span.

Let us first prove that the $d+1$–good permutations span $\Sigma_k(d)$. So let σ be $d+1$–bad so that there exist $1 \leq i_1 < i_2 < \cdots < i_{d+1} \leq n$ such that $\sigma(i_1) > \sigma(i_2) > \cdots > \sigma(i_d + 1)$.

If A is the antisymmetrizer on the $d+1$ elements $\sigma(i_1), \sigma(i_2), \cdots, \sigma(i_d+1)$ we have that $A\sigma = 0$ in $\Sigma_k(V)$, that is, in $\Sigma_k(V)$, σ is a linear combination of permutations obtained from the permutation σ with some proper rearrangement of the indices $\sigma(i_1), \sigma(i_2), \cdots, \sigma(i_d+1)$.

These permutations are all lexicographically $< \sigma$.

One applies the same algorithm to any of these permutations which is still $d+1$–bad. This gives an explicit algorithm which stops when σ is expressed as a linear combination of $d+1$–good permutations □

Step 2 The Robinson, Schensted correspondence

In order to prove that the $d+1$–good permutations form a basis, it is enough to show that their number equals the dimension of $\Sigma_k(d)$.

This is insured by the RSK correspondence *Robinson, Schensted, Knuth*, a combinatorially defined bijection $\sigma \longleftrightarrow (P_\lambda, Q_\lambda)$ between permutations $\sigma \in S_n$ and pairs P_λ, Q_λ of standard tableaux of same shape λ, where $\lambda \vdash n$.

The $d + 1$ good permutations are linearly independent.

By a classical theorem of Schensted, if $\sigma \longleftrightarrow (P_\lambda, Q_\lambda)$ we have that $ht(\lambda)$ equals the length of a longest decreasing subsequence in the permutation σ. Hence σ is $d+1$-good if and only if $ht(\lambda) \leq d$.

Now the irreducible M_λ has a basis indexed by standard tableaux of shape λ.
Thus the algebra $\Sigma_k(V)$ has a basis indexed by pairs of standard tableaux (the matrix units) of shape λ with $ht(\lambda) \leq d$ and the claim follows □

Remark

This is just a counting argument not an explicit 1–1 correspondence between the two bases.

Dimension 2

Q–bits

Dimension of V is 2

Dimension 2

If $\dim V = 2$ there is a different possible choice which has also some specific merits

We call $\Sigma_n(2)$ the *n–swap algebra* and denote it simply Σ_n.

It is known that $\dim \Sigma_n = C_n$ the n^{th} Catalan number.

The list of the first 10 Catalan numbers is

$$1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796$$

Notice that we can consider these algebras as each included in the next

$$\Sigma_2(2) \subset \Sigma_3(2) \subset \cdots \subset \Sigma_n(2) \subset \cdots$$

Symmetric elements

The standard Hibert structure on $V = \mathbb{C}^2$ induces a Hilbert space structure on $V^{\otimes n}$ and the adjoint of a permutation σ is its inverse σ^{-1}, moreover one has a real (and also rational) structure and the permutations are real.

Real symmetric elements play a special role

A symmetric permutation is one equal to its inverse, usually called *involution*.

Main Theorem

We will see that the space of symmetric elements of $\Sigma_n(2)$ is linearly spanned by involutions.

Definition

The set $\mathcal{S}$ of *special permutations* is formed by the involutions and also by the permutations with cycles only of order 2,1 plus a single cycle of order 3.

The 3 cycle can be further normalised to be increasing.

A choice of special elements

Let us start with the basic antisymmetrizer which vanishes in $\Sigma_3(2)$.

The basic identity $A = 0$ in $\Sigma_3(2)$.

1. $$A = (1,2,3) + (1,3,2) - (1,2) - (1,3) - (2,3) + 1$$
2. $$(1,2,3) + (1,3,2) = (1,2) + (1,3) + (2,3) - 1. \qquad (1)$$
3. $$(1,3,2) = -(1,2,3) + (1,2) + (1,3) + (2,3) - 1$$
4. In S_3 all permutations are special, and a 3–cycle can be normalised

In $\Sigma_3(2)$ this is the only relation but when we pass to $\Sigma_n(2),\ n > 3$ we have the various relations

$$\sigma A \tau,\ \sigma, \tau \in S_n$$

which linearly span the ideal of relations.

Our main Theorem is the following

Theorem

1. *For each n the algebra $\Sigma_n(2)$ has a basis formed by special elements.*
2. *$\Sigma_n^+(2)$ has a basis over $\mathbb{C}$ formed by involutions.*
3. *The space of real and symmetric elements has a basis over $\mathbb{R}$ formed by involutions.*

Notice that items (2) and (3) are equivalent and follow from (1). In fact the involutions are symmetric.

Take a permutation of the form $g = ab$ with a a 3 cycle and b is a product of 2 or 1 cycles so $b = b^{-1}$ an involution.

Its symmetrization is $g + g^{-1} = (a + a^{-1})b$. .

Since a is a 3 cycle, by relation (1) in the algebra Σ_3 we have that $a + a^{-1}$ is the sum of -1 and 3 transpositions. The claim follows.

Some numbers

The dimensions of the real symmetric elements are, $n = 1, \cdots, 10$

$$1, 2, 4, 10, 26, 76, 232, 750, 2494, 8524, \cdots$$

(see The On-Line Encyclopedia of Integer Sequences A007123 for many interesting informations on this sequence).

The number $I(n)$ of involutions in S_n $n = 1, \cdots, 10$

$$I(n) = 1, 2, 4, 10, 26, 76, 232, 764, 2620, 9496, \cdots$$

which is also equal (by the RSK correspondence) to the number of standard Young tableaux with n cells (O.E.I.S A000085).
So a curious fact is that these two sequences coincide up to $n = 7$.

We have thus that the involutions are a basis of the real symmetric elements for $n \leq 7$ and after that they have linear relations.

A combinatorial problem

For $n \geq 8$ give some combinatorial restrictions on involutions so that the ones satisfying these restrictions form a basis of $\Sigma_n^+(2)$.

We will prove the theorem by presenting an algorithm which given as input any permutation, writes it as a linear combination of special elements in Σ_n.

The algorithm

We start by writing in $\Sigma_4(2)$ a 4–cycle as sum of special elements, this is a simple computation using some relations deduced from A, which gives:

$$2(1,2,3,4) = \tag{2}$$

$$(1,4)(2,3)+(1,2)(3,4)-(1,3)(2,4)+(1,2,4)+(1,3,4)+(2,3,4)$$

$$+(1,2,3)-(1,2)-(1,4)-(3,4)-(2,3)+1.$$

Since all 4 cycles are conjugate we deduce that statement (1) is true for $\Sigma_4(2)$.

The algorithm

Now notice the following general fact: consider two cycles (a, A), (a, B) of lengths h, k respectively where A and B are strings of integers of lengths $h-1, k-1$ respectively and disjoint. Then their product is the cycle of length $h+k-1$:

$$(a, B)(a, A) = (a, A, B),$$

$$e.g.\ a = 1, \quad (1,2,3)(1,5,4,6) = (1,5,4,6,2,3). \tag{3}$$

The algorithm

Thus take a cycle of length $p > 4$ and, up to conjugacy we may take

$$c_p := (1, 2, 3, 4, 5, \dots, p) = (1, 5, \dots, p)(1, 2, 3, 4). \qquad (4)$$

In Σ_p we have thus that $2c_p$ equals $(1, 5, \dots, p)$ times the expression of Formula (2).
But then applying again Formula (3) we see that the resulting formula is a sum of permutations on p elements which are **not** full cycles.

The algorithm

By iterating then the operation on the cycles of length ℓ with $4 \leq \ell \leq p-1$ we have a preliminary.

Proposition

The cycle c_p *(formula* (4)*) is a linear combination in* $\Sigma_p(2)$ *of permutations which contain only cycles of length 1,2,3.*
Hence for all n *we have that* $\Sigma_n(2)$ *is spanned by permutations which contain only cycles of length 1,2,3.*

Example $p = 5$.

$$4(1,2,3,4,5) \overset{(4)}{=} 4(1,5)(1,2,3,4) = \tag{5}$$

$$(1,2)(3,5)+(1,2)(4,5)-(1,3)(2,5)+(1,3)(4,5)-(1,4)(2,5)-(1,4)(3,5)+(1,5)(2,4)$$

$$-(1,5)(3,4)-(2,3)(1,5)+(2,4,5)+(1,2,4)+(1,3,5)+(3,4,5)+(1,3,4)+(1,3,5)+(2,3,5)$$

$$+(1,2,3)-(2,3)-2(4,5)-(3,4)-2(1,2)-(2,4)-(1,3)-(3,5)-(1,5)+3$$

$$+2(1,4,5)(2,3)+2(1,2,5)(3,4)-2(1,3,5)(4,2)+2(2,3,4)(1,5)$$

Example

$$2(1,2,3,4,5,6) \stackrel{(4)}{=} 2(1,5,6)(1,2,3,4) = \tag{6}$$

$$(1,4,5,6)(2,3) + (1,2,5,6)(3,4) - (1,3,5,6)(4,2) - (1,4,2,5,6)$$

$$+(1,3,4,5,6) + (3,4,2)(1,5,6) + (1,2,3,5,6)$$

$$+(2,4)(1,5,6) - (3,4)(1,5,6) - (2,3)(1,5,6).$$

developing the 4 and the 5 cycles we have a sum of special elements plus the element $(3,4,2)(1,5,6)$ which is NOT special.

Writing the element $(3,4,2)(1,5,6)$

computer aided

and a bit of luck

Theorem 6 using Proposition 7

It is enough to prove that

In $\Sigma_6(2)$, a permutation of type $3, 3$ can be developed as linear combination of special elements

since then we apply recursively this to a product of k disjoint 3-cycles. If k is even we replace them all and if odd we remain with only one 3-cycle which can be normalized if necessary using Formula (1).

The computation in $\Sigma_6(2)$ in principle is similar to that in $\Sigma_4(2)$ but now we have to handle a priori many more relations and I had to be assisted by the software "Mathematica" in order to discover the needed relations.

What I have done is to ask the computer to analyse thousands of relation in $\Sigma_6(2)$ deduced from the antisymmetrizer $A = 0$.
After some messy and confusing results I got the following two relations, $A6$ is just A but thought of as in $\Sigma_6(2)$.

$$(5,6,1)(3,4)\mathrm{A6} = (1,2,4,3,5,6)+(1,4,3,2,5,6) - (2,5,6,1)(3,4) - (4,3,5,6,1) - (5,6,1)(4,3,2)+(5,6,1)(3,4)$$

$$(6,1)(4,5,3)\mathrm{A6} = (1,2,4,5,3,6) + (1,4,5,3,2,6) - (2,6,1)(4,5,3) - (4,5,3,6,1) - (6,1)(4,5,3,2) + (6,1)(4,5,3)$$

Each involves two 6 cycles and one permutation of type $3,3$

In Formula (6) the contribution to the expansion of $2(1,2,3,4,5,6)$ of an element of type $3,3$ is $+(3,4,2)(1,5,6)$.

Therefore the contributions of type $3,3$ of the 4 cycles of length 6 appearing in the previous Formulas are obtained by conjugating $(3,4,2)(1,5,6)$ with the permutation which has as string the same form of the cycle

Therefore the previous 2 relations multiplied by 2 are of the form

In the next Formulas by $\cdots$ I mean *sum of special elements*

$$-(3,4,5)(1,2,6)-(2,3,5)(1,4,6)+2(2,3,4)(1,5,6)+\cdots$$

$$(3,4,5)(1,2,6)-(2,3,5)(1,4,6)-2(3,4,5)(1,2,6)+\cdots \tag{7}$$

Subtracting the second from the first one has the desired Formula:

$$0=2(2,3,4)(1,5,6)+\cdots$$

a relation with a single permutation $2(2,3,4)(1,5,6)$ of type $(3,3)$ and the remaining elements special.
This gives the desired expression

$$8(4,3,2)(5,6,1) = \tag{8}$$

$$(1,2) - 2(1,3) + (1,5) + 4(1,6) - 3(2,3) + 2(1,5)(2,3) - 2(1,6)(2,3) - 3(2,4) + 2(1,5)(2,4) - 2(1,6)(2,4) + 7(2,5) - 2(1,3)(2,5) - 4(1,4)(2,5) - 6(1,6)(2,5) - 2(2,6) - 2(1,3)(2,6) + 4(1,4)(2,6) + 2(1,5)(2,6) + (3,4) - 2(1,5)(3,4) - 2(1,6)(3,4) - 4(2,5)(3,4) - 4(3,5) + 4(1,4)(3,5) + 4(2,4)(3,5) + 4(2,6)(3,5) - 4(1,4)(2,6)(3,5) - 4(3,6) + 4(1,2)(3,6) - 4(1,4)(3,6) - 4(2,5)(3,6) + 4(1,4)(2,5)(3,6) - 2(4,5) + 2(1,2)(4,5) + 2(1,3)(4,5) + 4(2,3)(4,5) + 4(1,3)(2,6)(4,5) + 4(3,6)(4,5) - 4(1,2)(3,6)(4,5) - 2(1,2)(4,6) + 2(1,3)(4,6) - 4(1,3)(2,5)(4,6) + 4(1,2)(3,5)(4,6) + 6(5,6) - 4(1,2)(5,6) - 4(4,5)(1,2,3) + 4(5,6)(1,2,3) + (1,2,5) - 4(3,6)(1,2,5) - 2(1,2,6) + 4(4,5)(1,2,6) + 2(1,3,6) + 4(2,5)(1,3,6) - 4(4,5)(1,3,6) - 4(3,5)(1,4,2) + 4(5,6)(1,4,2) + 4(2,5)(1,4,3) - 4(5,6)(1,4,3) - 4(2,6)(1,4,5) + 4(3,6)(1,4,5) - (1,5,2) - 4(1,5,6) + 8(3,4)(1,5,6) + 2(1,6,3) - 2(1,6,5) + (2,3,5) + 2(2,3,6) - 4(4,5)(2,3,6) + 4(1,6)(2,4,3) - (2,4,5) + 4(1,6)(2,4,5) - (2,5,3) + 4(1,6)(2,5,3) + (2,5,4) - 2(2,5,6) + 2(1,4)(2,5,6) + 2(2,6,4) - 4(3,5)(2,6,4) - 2(1,4)(2,6,5) + (3,4,5) - 4(1,6)(3,4,5) + 2(3,4,6) - (3,5,4) + 2(1,2)(3,5,6) - 2(1,4)(3,5,6) + 4(2,5)(3,6,4) - 2(1,2)(3,6,5) + 2(1,4)(3,6,5) - 2(4,5,6) - 2(4,6,5)$$

General dimension

A classical problem

non commutative algebra

invariant theory

One may ask the same question for $\Sigma_n(d)$ and $d \geq 3$. The first problem is:
Determine the minimum $m = m(d)$ so that $\Sigma_{m+1}(d)$ is spanned by the permutations which are NOT $m+1$–cycles.

This number m has also other interesting interpretations

see

E. Aljadeff, A. Giambruno, C. Procesi, A. Regev.
Rings with polynomial identities and finite dimensional representations of algebras,
A.M.S. Colloquium Publications, vol. 66.2020; 630 pp

at page 331 also for the interesting history of this question

.

The known estimates for $m(d)$ are the lower bound $m(d) \geq \binom{d+1}{2}$ due to Kuzmin, and the upper bound $m(d) \leq d^2$ due to Razmyslov. Kuzmin conjectures that $m(d) = \binom{d+1}{2}$ which has been verified only for $d \leq 4$.

Other interpretations of $m(d)$

1. The same m is the maximum degree of the generators of invariants of $d \times d$ matrices.
2. It is also the minimum degree for which, given an associative algebra R over a field of characteristic 0, in which every element x satisfies $x^d = 0$ one has $R^{m(d)} = 0$.

The algorithm

Final remarks

In the algorithm one thus uses the following *substitutional rules*.

$$2(a, b, c, d) = \tag{9}$$

$$(a, d)(b, c)+(a, b)(c, d)-(a, c)(d, b)-(a, d, b)+(a, c, d)+(b, c, d)+(a, b, c)+(b, d)-(c, d)-(b, c).$$

and:

$$(a, B)(a, A) = (a, A, B), \quad e.g.\ (1, 2, 3)(1, 5, 4, 6) = (1, 5, 4, 6, 2, 3) \tag{10}$$

for a cycle $\mathtt{C} := (a, b, c, d, A)$ of length $n + 4$, where A is of length $n > 0$, we have

$$\mathtt{C} := (a, b, c, d, A) = (a, A)(a, b, c, d) = \tag{11}$$

$$(a, d, A)(b, c) + (a, b, A)(c, d) - (a, c, A)(d, b) - (a, d, b, A)$$
$$+(a, c, d, A) + (b, c, d) + (a, b, c, A) + (b, d) - (c, d) - (b, c).$$

This formula now contains only cycles of length $< n + 4$.

Summarizing the algorithm

1. Take a permutation decomposed into cycles
2. Apply the reduction Formula to all 4–cycles
3. Split cycles of length $p > 4$ as product of a 4–cycle and a $p - 4 + 1$ cycle.
4. Apply the reduction Formula to the resulting 4–cycle.
5. Continue until all cycles are of length ≤ 3.
6. Apply the final reduction formula to $3, 3$ cycles.

$$8(d,c,b)(e,f,a) = \tag{12}$$

$$(a,b) - 2(a,c) + (a,e) + 4(a,f) - 3(b,c) + 2(a,e)(b,c) - 2(a,f)(b,c) - 3(b,d) + 2(a,e)(b,d) - 2(a,f)(b,d) + 7(b,e) - 2(a,c)(b,e) - 4(a,d)(b,e) - 6(a,f)(b,e) - 2(b,f) - 2(a,c)(b,f) + 4(a,d)(b,f) + 2(a,e)(b,f) + (c,d) - 2(a,e)(c,d) - 2(a,f)(c,d) - 4(b,e)(c,d) - 4(c,e) + 4(a,d)(c,e) + 4(b,d)(c,e) + 4(b,f)(c,e) - 4(a,d)(b,f)(c,e) - 4(c,f) + 4(a,b)(c,f) - 4(a,d)(c,f) - 4(b,e)(c,f) + 4(a,d)(b,e)(c,f) - 2(d,e) + 2(a,b)(d,e) + 2(a,c)(d,e) + 4(b,c)(d,e) + 4(a,c)(b,f)(d,e) + 4(c,f)(d,e) - 4(a,b)(c,f)(d,e) - 2(a,b)(d,f) + 2(a,c)(d,f) - 4(a,c)(b,e)(d,f) + 4(a,b)(c,e)(d,f) + 6(e,f) - 4(a,b)(e,f) - 4(d,e)(a,b,c) + 4(e,f)(a,b,c) + (a,b,e) - 4(c,f)(a,b,e) - 2(a,b,f) + 4(d,e)(a,b,f) + 2(a,c,f) + 4(b,e)(a,c,f) - 4(d,e)(a,c,f) - 4(c,e)(a,d,b) + 4(e,f)(a,d,b) + 4(b,e)(a,d,c) - 4(e,f)(a,d,c) - 4(b,f)(a,d,e) + 4(c,f)(a,d,e) - (a,e,b) - 4(a,e,f) + 8(c,d)(a,e,f) + 2(a,f,c) - 2(a,f,e) + (b,c,e) + 2(b,c,f) - 4(d,e)(b,c,f) + 4(a,f)(b,d,c) - (b,d,e) + 4(a,f)(b,d,e) - (b,e,c) + 4(a,f)(b,e,c) + (b,e,d) - 2(b,e,f) + 2(a,d)(b,e,f) + 2(b,f,d) - 4(c,e)(b,f,d) - 2(a,d)(b,f,e) + (c,d,e) - 4(a,f)(c,d,e) + 2(c,d,f) - (c,e,d) + 2(a,b)(c,e,f) - 2(a,d)(c,e,f) + 4(b,e)(c,f,d) - 2(a,b)(c,f,e) + 2(a,d)(c,f,e) - 2(d,e,f) - 2(d,f,e)$$

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