LINEAR SYMMETRIES FOR THE SL(3,C) TRIPLE MULTIPLICITIES.

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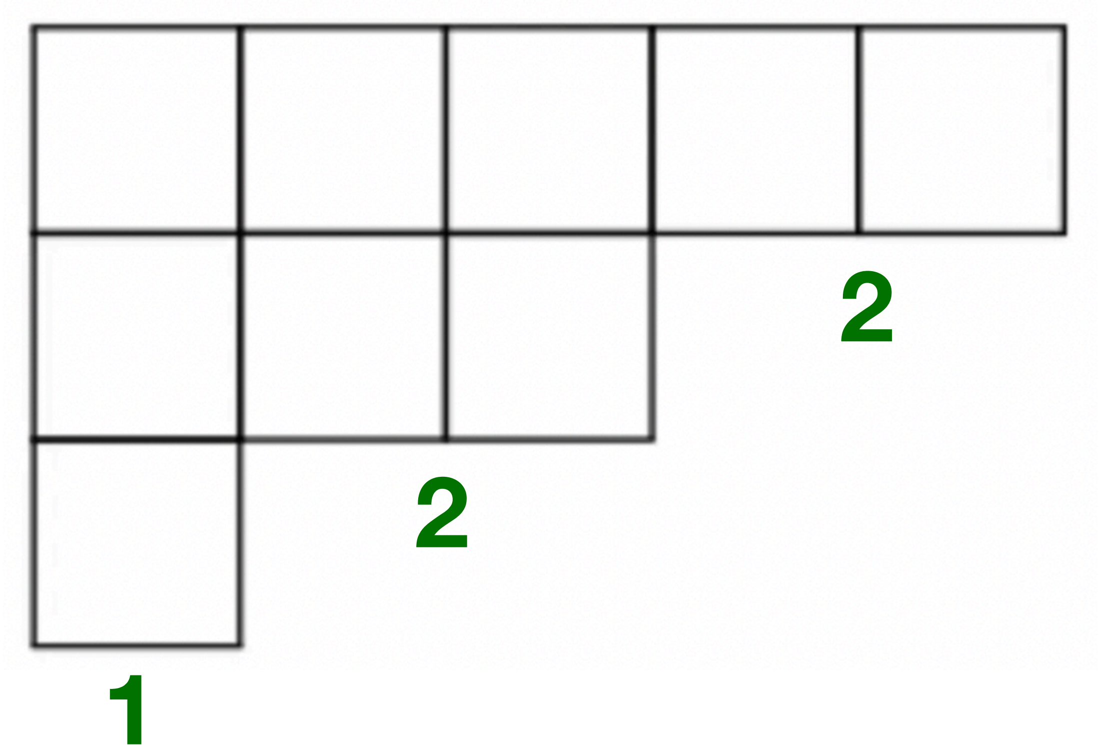
1. IRREDUCIBLE REPRESENTATIONS OF sl(3, C)

2. KOSTANT PARTITION FUNCTION AND VECTOR PARTITION FUNCTIONS.

3. THE TRIPLE MULTIPLICITIES OF SL(3,C)

YOUNG DIAGRAM



PARTITION

(5,3,1)

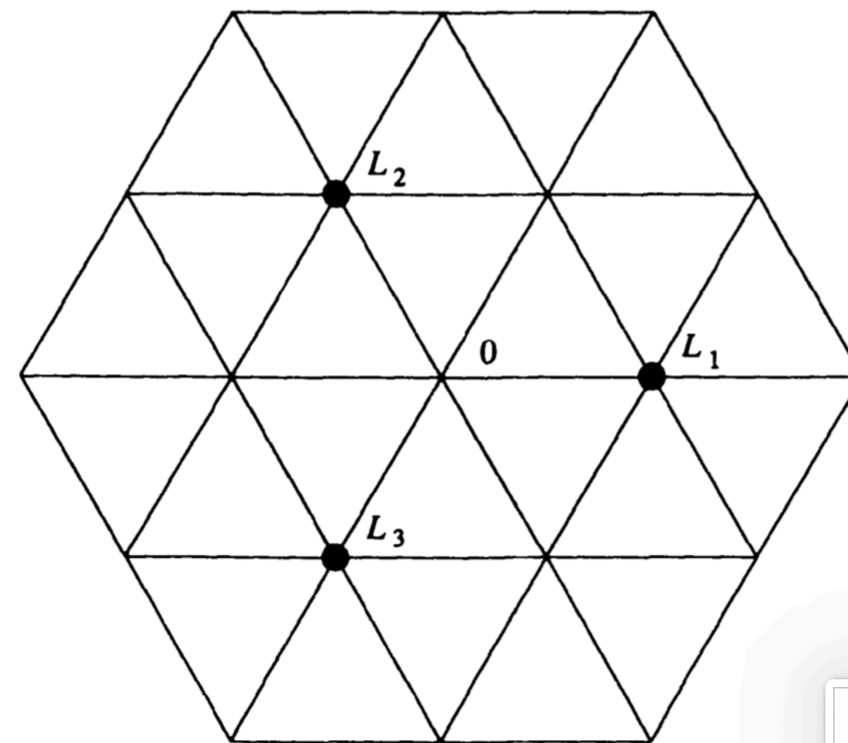
DYNKIN LABEL

(2,2,1)

The Standard Representation

$$\begin{pmatrix} \exp i\theta_1 & 0 & 0 \\ 0 & \exp i\theta_2 & 0 \\ 0 & 0 & \exp -i(\theta_1+\theta_2) \end{pmatrix}$$

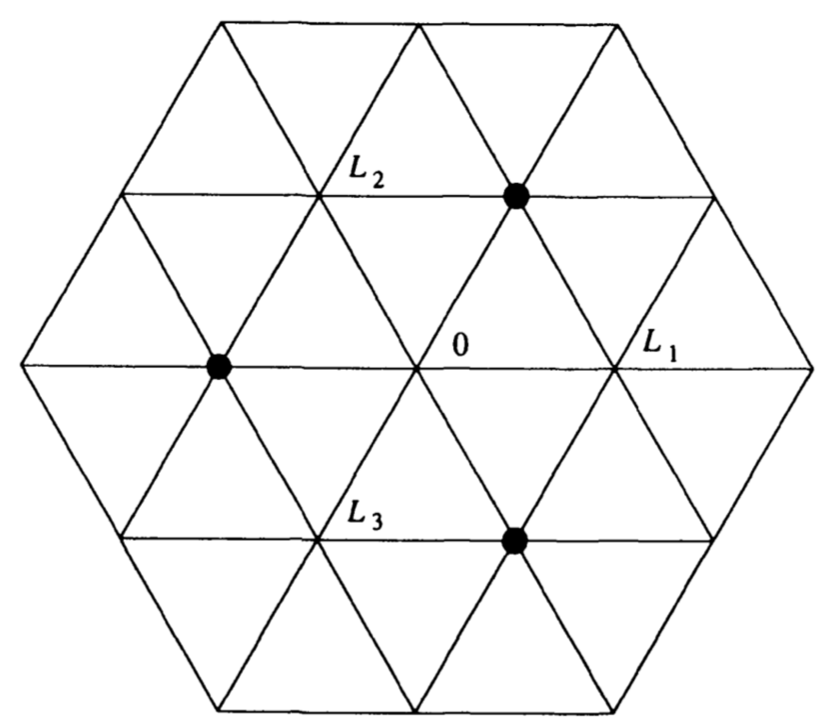
Dynkin label (1,0)
Partition (1,0)



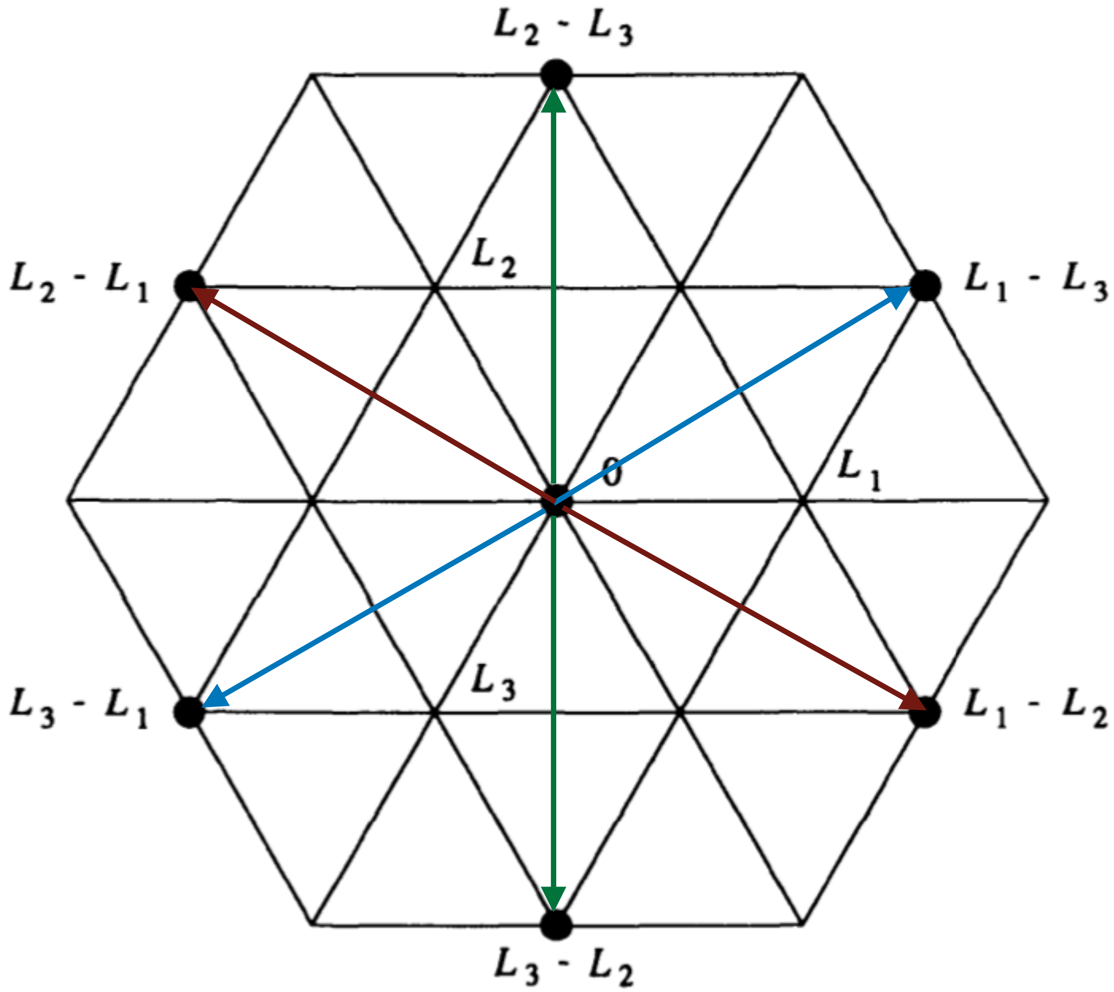
The dual of the standard representation

$$X \mapsto \pi(X)$$
$$X^* \mapsto -\pi(X)^t$$

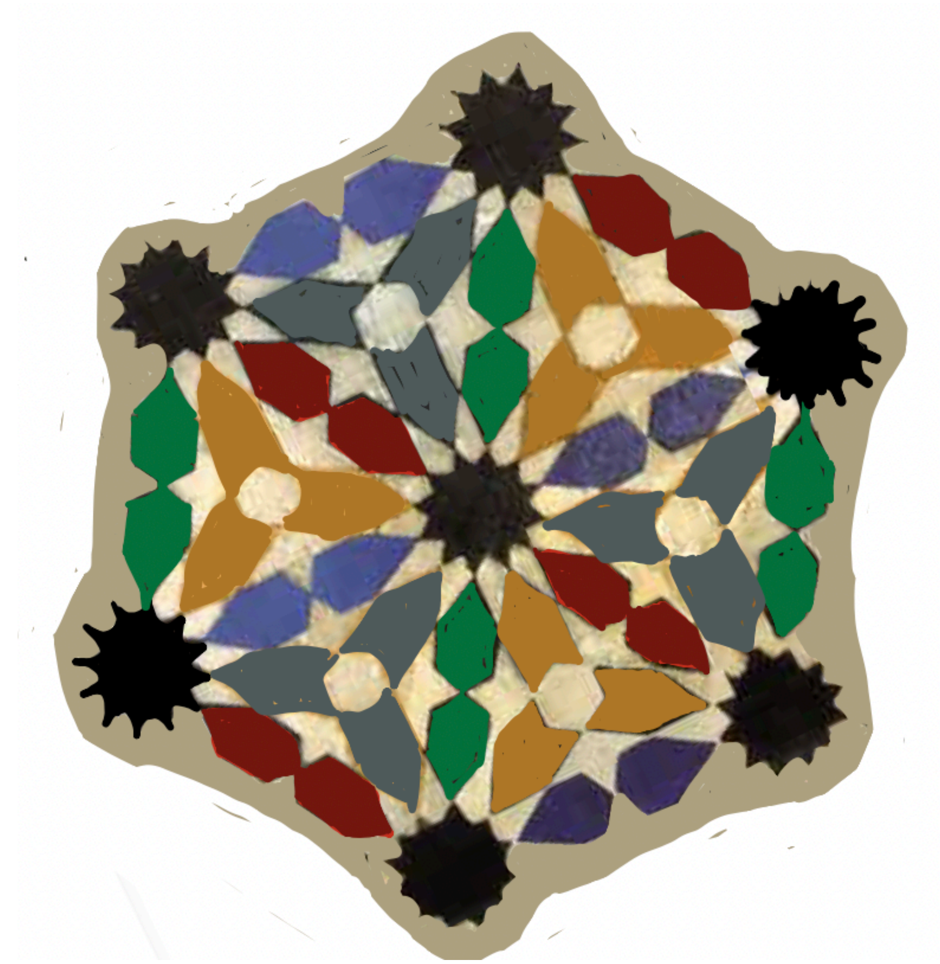
Dynkin label (0,1)
Partition (1,1)



THE ADJOINT REPRESENTATION OF $sl(3, \mathbb{C})$.



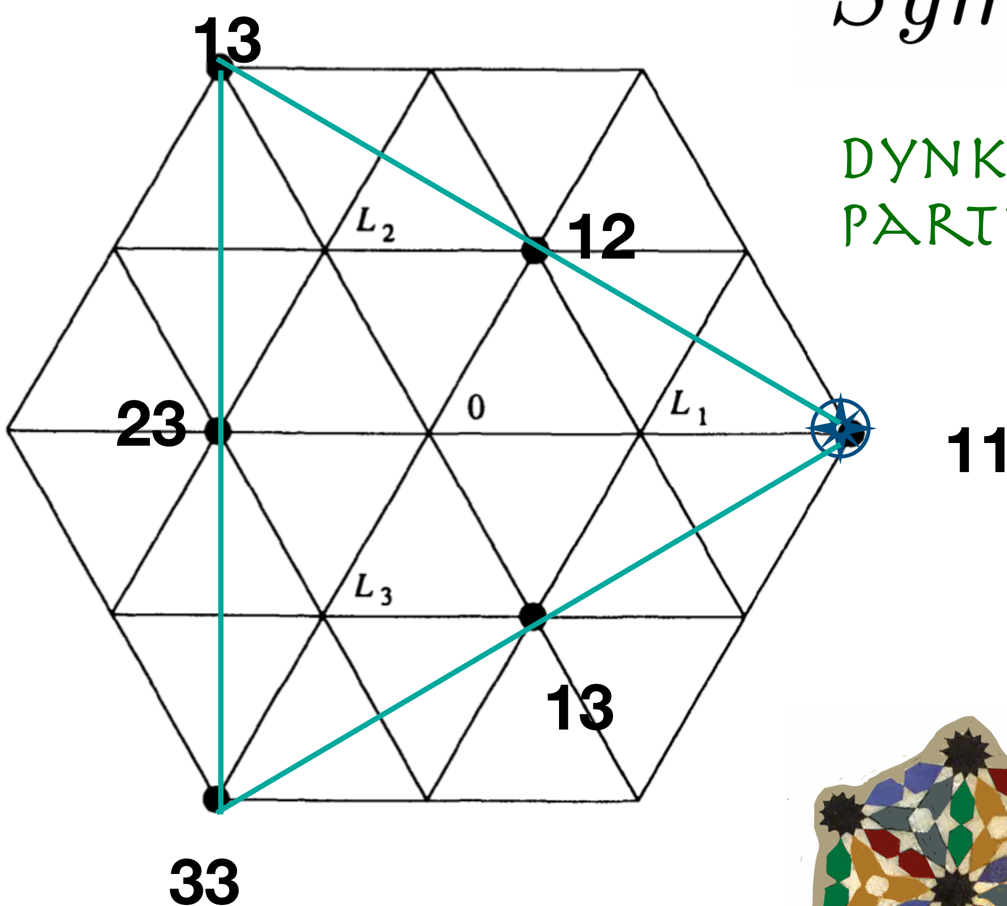
DYNKIN LABEL (1,1)
PARTITION (2,1)



ALCÁZAR DE SEVILLA

THE ROOTS ALLOWS US TO MOVE IN THE DIRECTIONS OF THE THREE LONG DIAGONALS OF THE RHOMBI

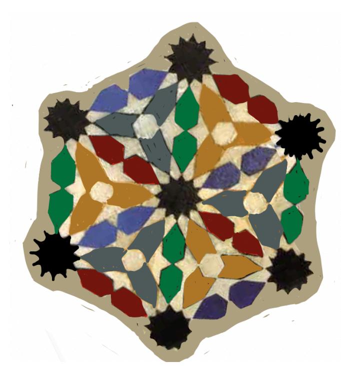
Symmetric powers of the standard representation

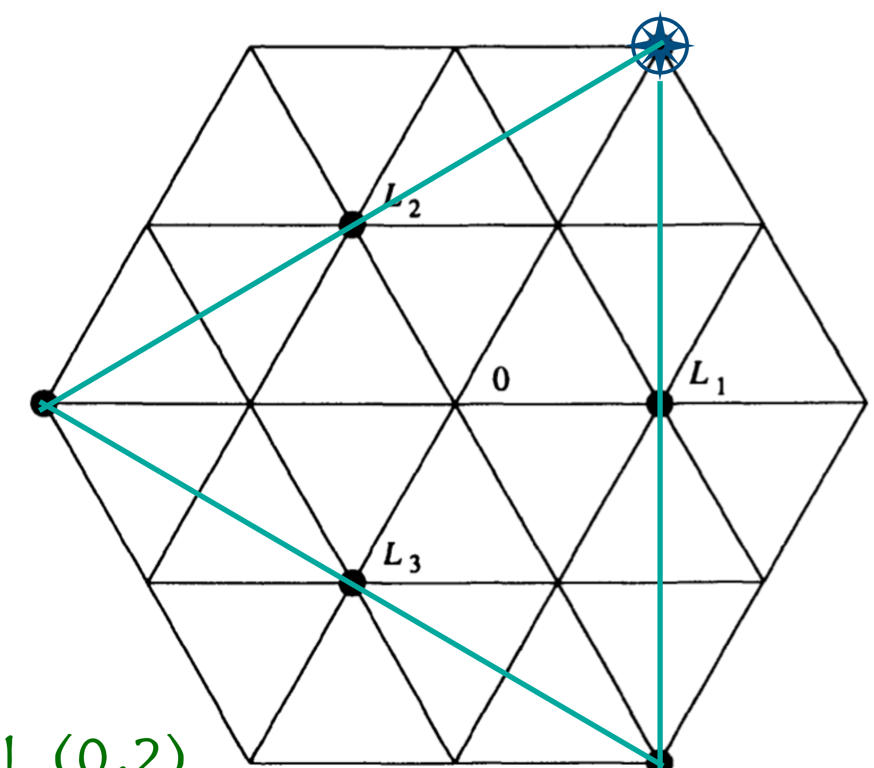


Sym^2V

Dynkin label (2,0)
Partition (2,0)

Maximal weights





Sym^2V^*

Dynkin label (0,2)
Partition (2,2)

Long diagonals of the rhombi

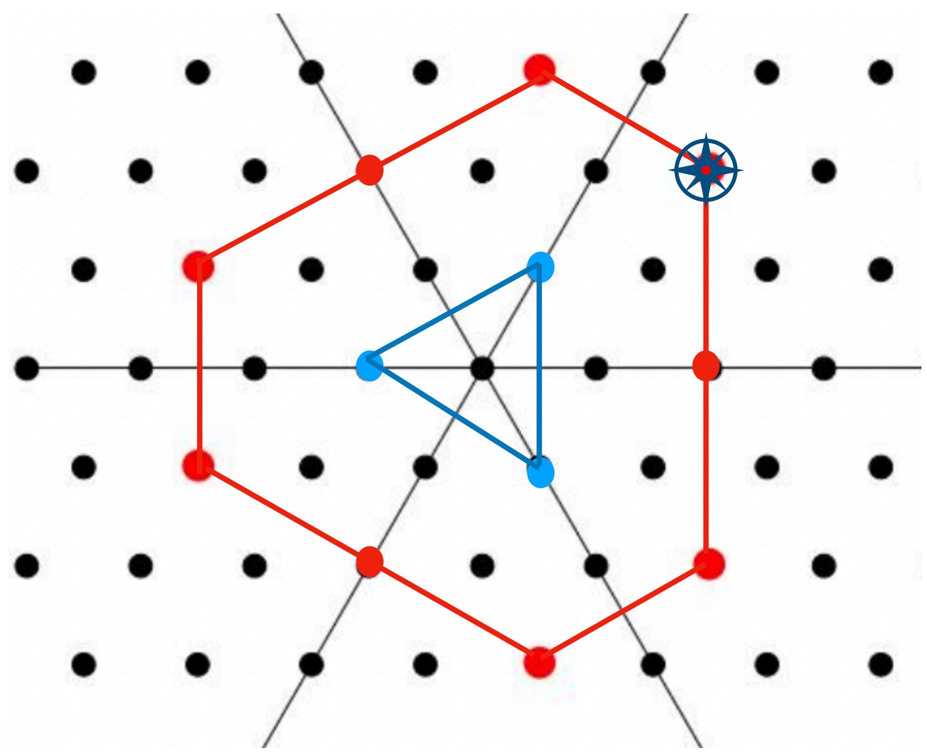
The irreducible representation of $\mathfrak{sl}(3, \mathbb{C})$.

$$\mathrm{Sym}^n V = \Gamma_{n,0} \quad \text{and} \quad \mathrm{Sym}^n V^* = \Gamma_{0,n}.$$

TRIANGLES

DYNKIN LABEL (n,0)
PARTITION (n,0)

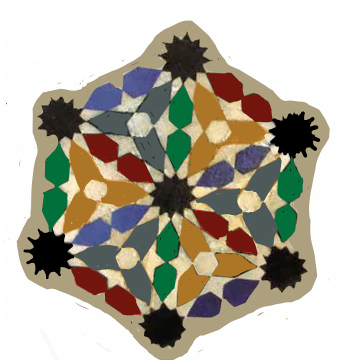
DYNKIN LABEL (0,n)
PARTITION (n,n)



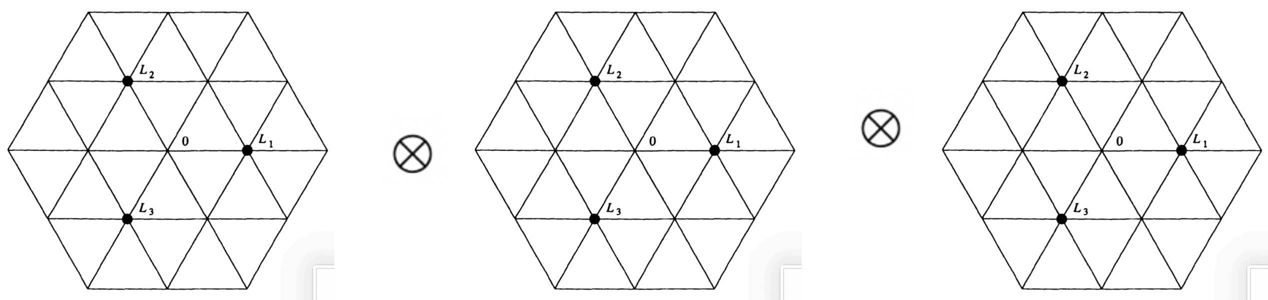
OUTER SHAPE
HEXAGON

$$\Gamma_{(1,2)}$$

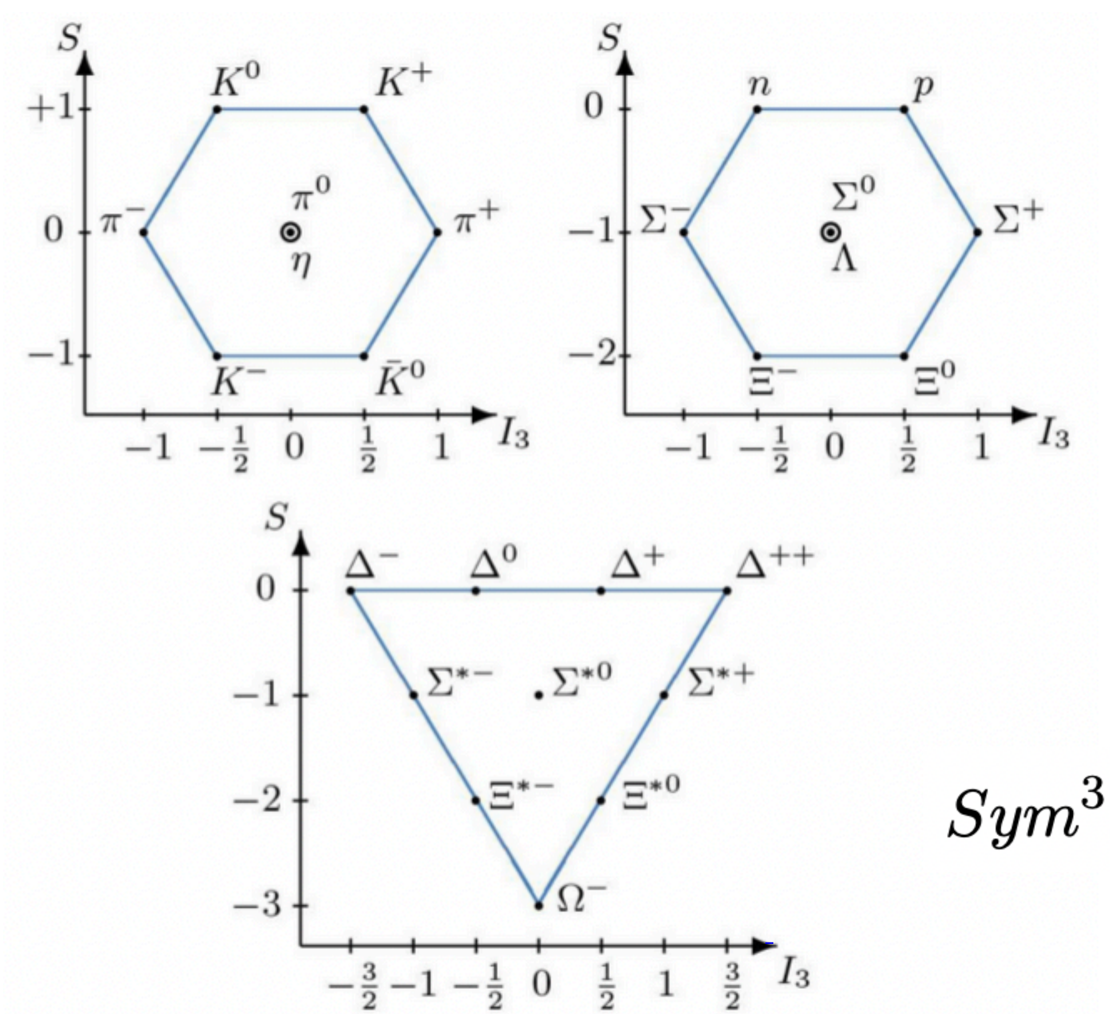
DYNKIN LABEL (1,2)
PARTITION (4,1)

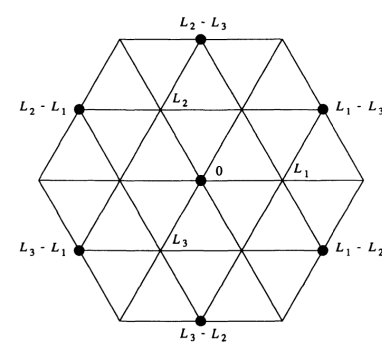


Gell-Mann and Ne'eman Eight-Fold Way

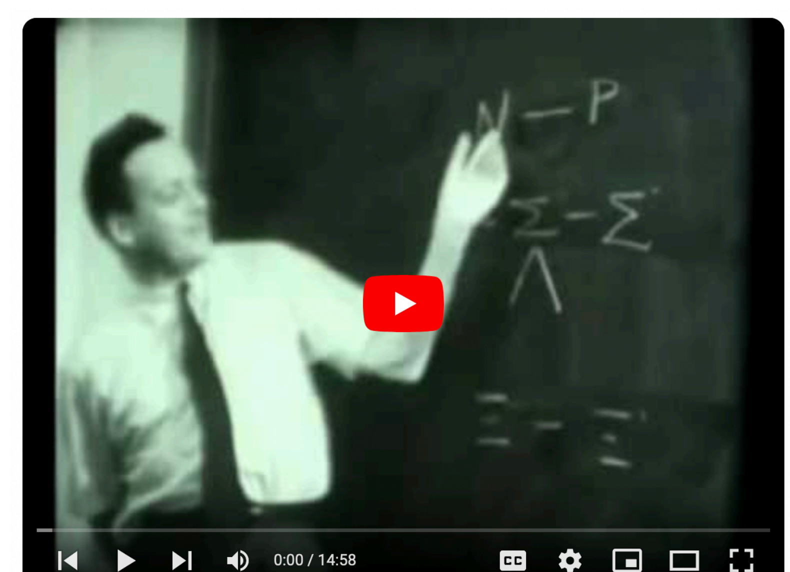


$$\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$$





$$Sym^3 V$$



Richard Feynman, Murray Gell-Mann, Juval Ne'eman: Strangeness Minus Three (BBC Horizon 1964) I

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2. KOSTANT PARTITION FUNCTION AND
VECTOR PARTITION FUNCTIONS.

3. THE TRIPLE MULTIPLICITIES OF $SL(3,C)$

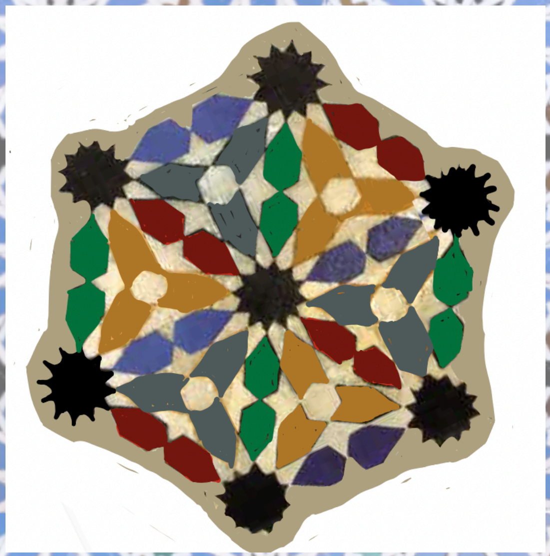
The Root Lattice

Vector space (real plane)

Lattice generated by the root vectors

Positive roots

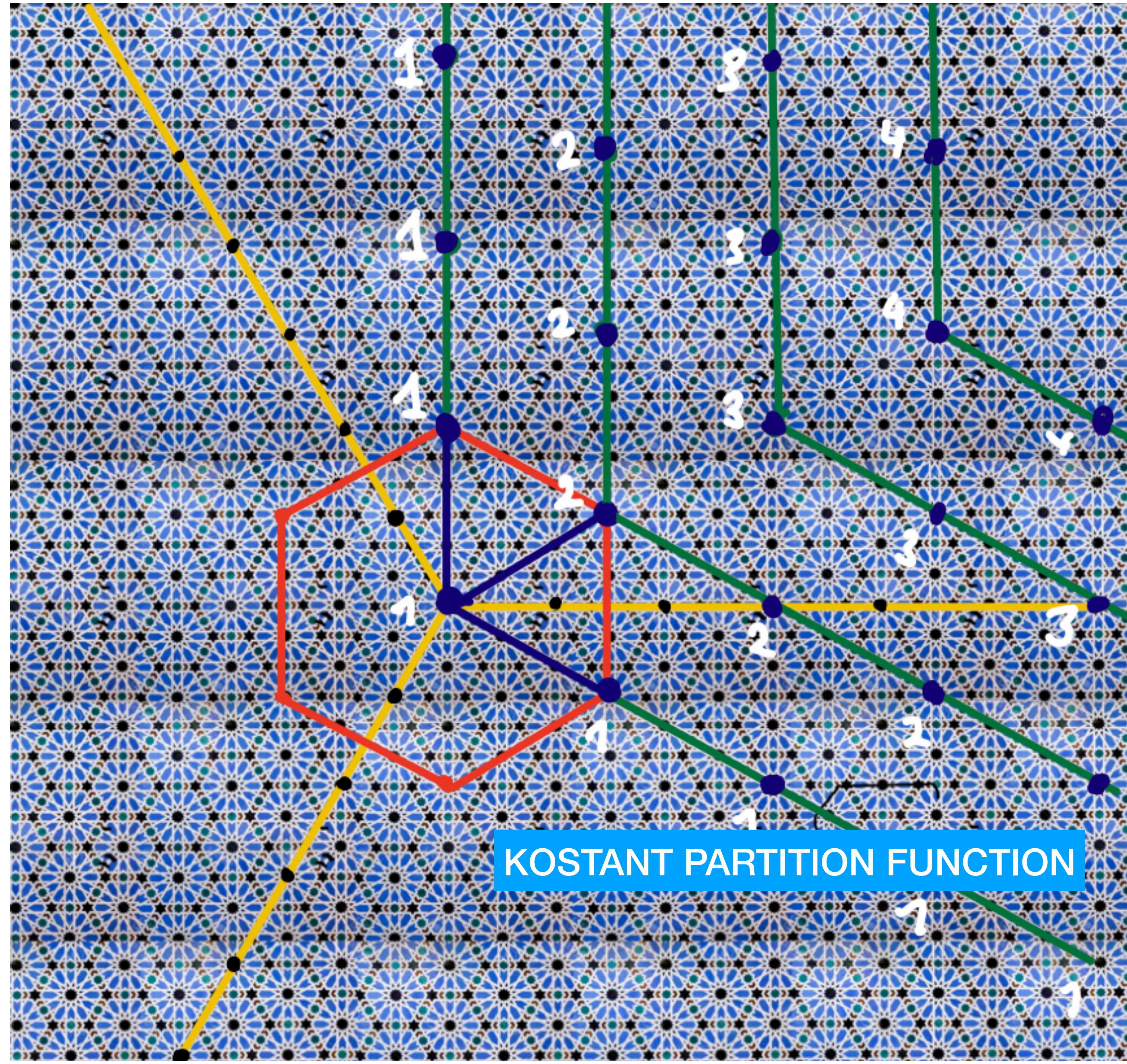
Positive simple roots



KONSTANT PARTITION FUNCTION

POSITIVE ROOTS

α_1, α_2 $\quad$ $\alpha_3 = \alpha_1 + \alpha_2$



P(μ) = THE NUMBER OF WAYS OF WRITING μ AS A SUM OF POSITIVE ROOT

$$\mu = n_1\alpha_1 + n_2\alpha_2$$

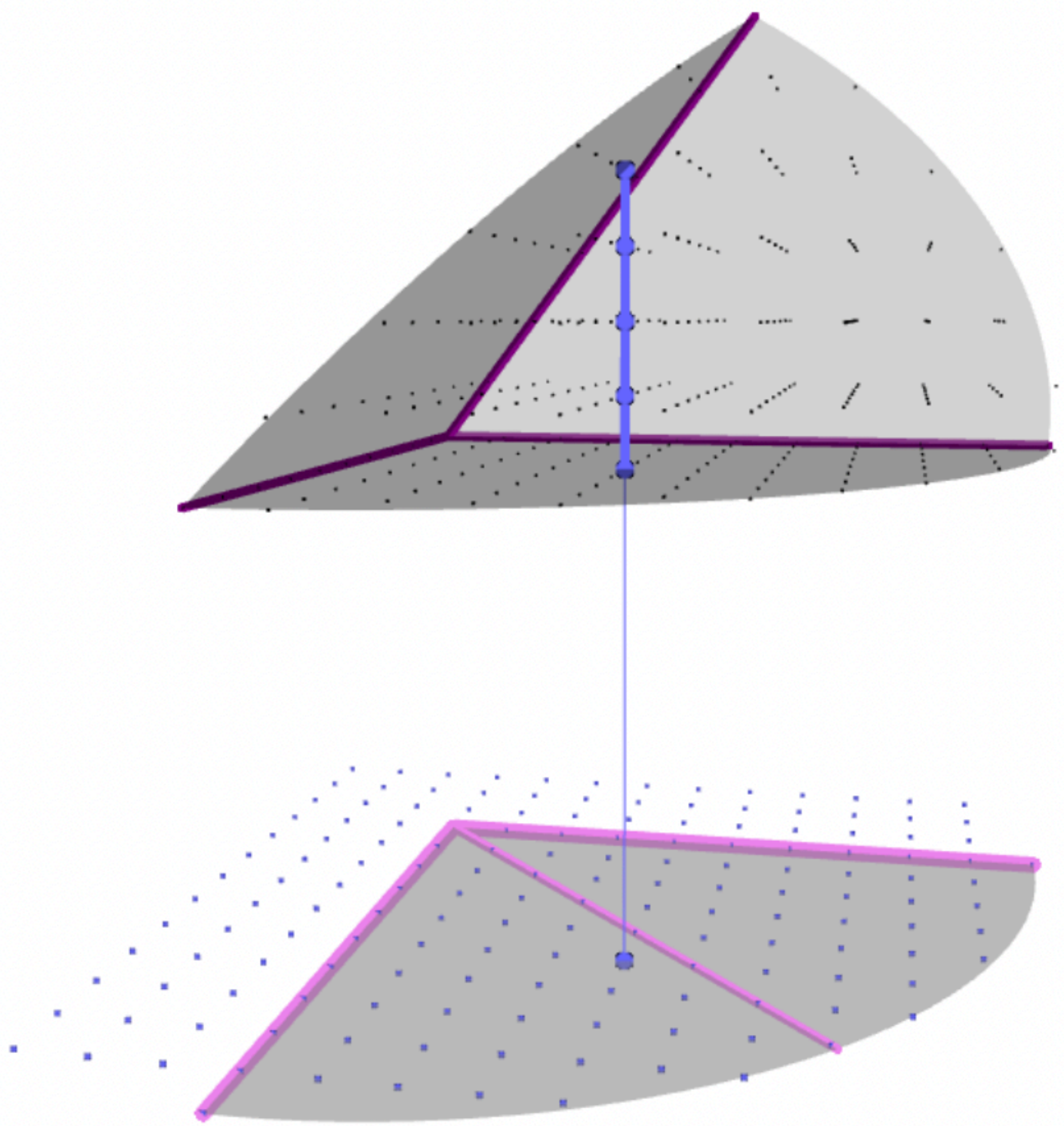
POINTED CONE :

NON-NEGATIVE REAL COEFFICIENTS.

LATTICE CONDITION

NON-NEGATIVE REAL COEFFICIENTS.

$$p(n_1\alpha_1 + n_2\alpha_2) = 1 + \min(n_1, n_2).$$



Konstant Partition Function

Fix a set of positive roots

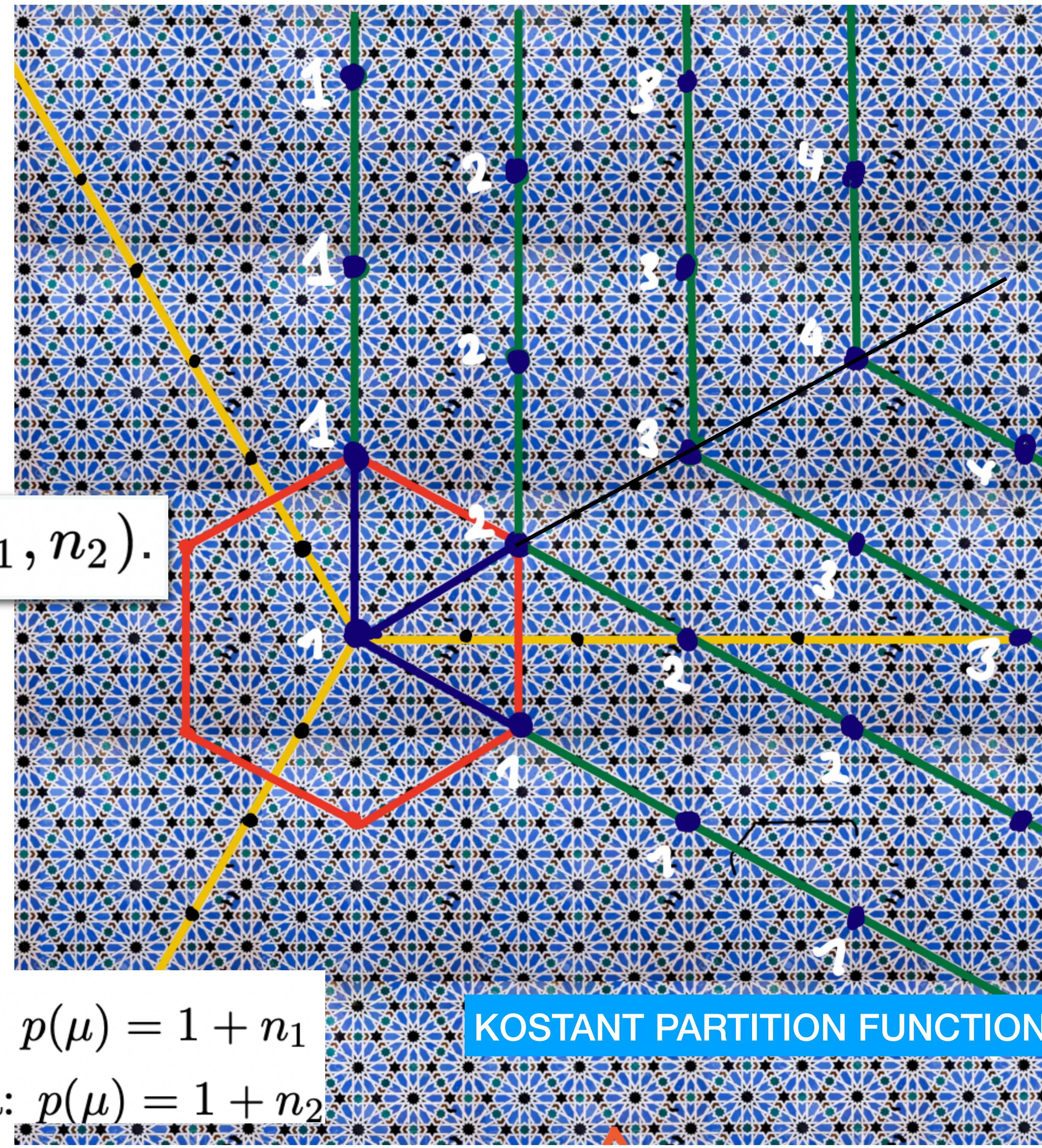
α_1, α_2 $\alpha_3 = \alpha_1 + \alpha_2$

Write

$$\mu = n_1\alpha_1 + n_2\alpha_2$$

Then

$$p(n_1\alpha_1 + n_2\alpha_2) = 1 + \min(n_1, n_2).$$



Pointed cone :

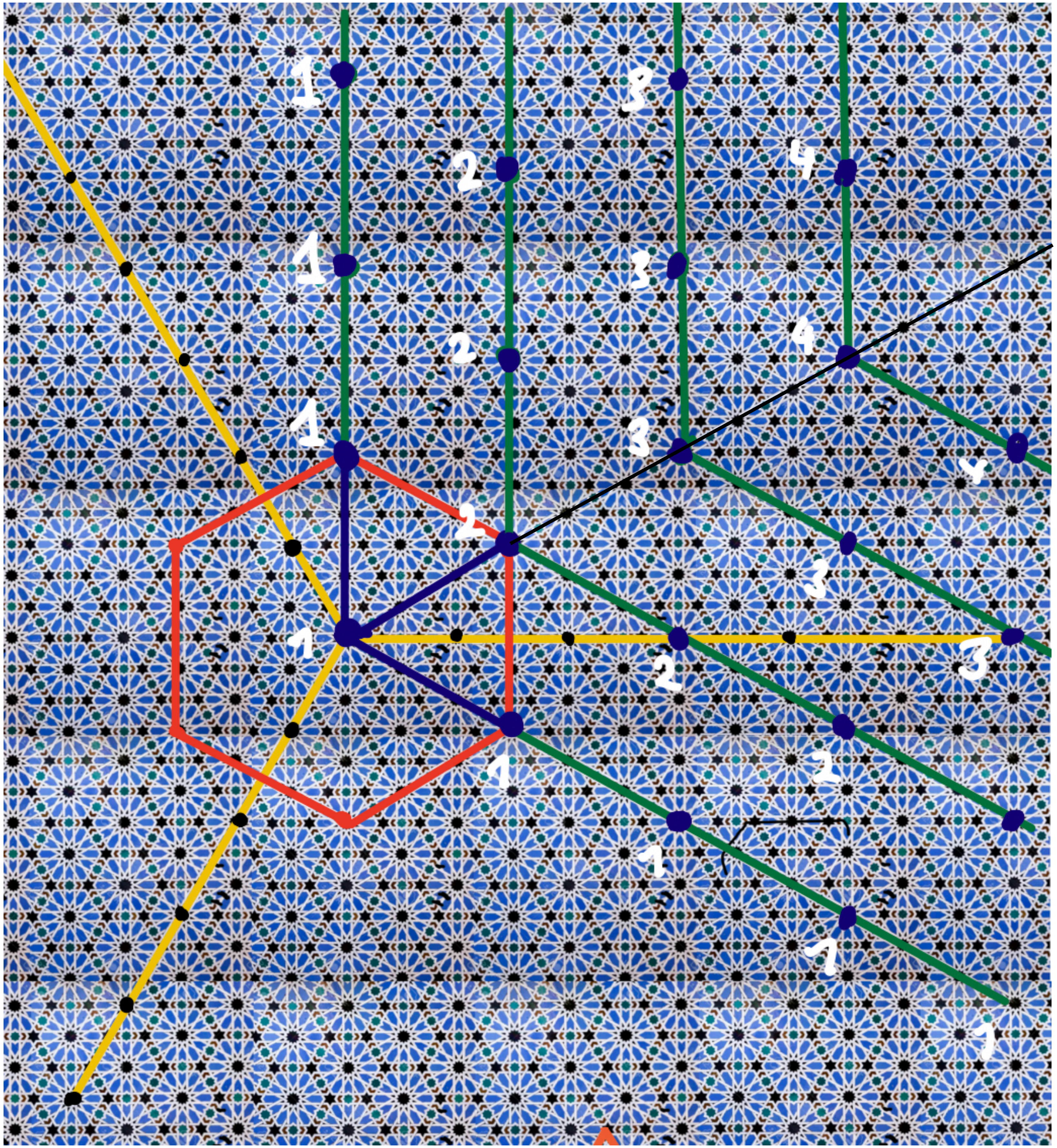
non-negative real coefficients.

Vector partition function:

chamber 1: $0 \leq n_1 \leq n_2$, formula: $p(\mu) = 1 + n_1$

chamber 2: $0 \leq n_2 \leq n_1$, formula: $p(\mu) = 1 + n_2$

Symmetry and Stability



Symmetry

Reflection around the line generated by

$$\alpha_3 = \alpha_1 + \alpha_2$$

Cyclic group of order two

Stability

Cyclic

Kostant multiplicity formula

$$\mathrm{mult}(\mu) = \sum_{w \in W} (-1)^{\ell(w)} p(w \cdot (\lambda + \rho) - (\mu + \rho)).$$

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THE LITTLEWOOD-RICHARDSON COEFFICIENTS

$$\ell = (\ell_1, \ell_2),\ m = (m_1, m_2) \text{ and } n = (n_1, n_2).$$

- THE MULTIPLICITY OF V_n IN THE TENSOR PRODUCT

  $V_\ell \otimes V_m$ IRREP OF $\mathfrak{sl}(3, \mathbb{C})$

- SCHUR FUNCTIONS $[s_\lambda]\ s_\mu s_\nu$

- THE TRIPLE MULTIPLICITIES

  DIMENSION OF $(V_\ell \otimes V_m \otimes V_n^*)^{SU(3)}$

THE TRIPLE MULTIPLICITIES

$$c(\ell; m; n) = \dim \left(V_\ell \otimes V_m \otimes V_n\right)^{SU(3)}$$

THUS $c^{\lambda}_{\mu.\nu}$ EQUALS $c(\ell; m{:}\, n^*)$

THE SUPPORT OF THE TRIPLE MULTIPLICITIES

SET OF DYNKIN LABELS WITH $c(\ell;m;n) \neq 0$

GENERATES A SUB LATTICE Λ_{TM} OF $\mathbb{Z}^6$

Λ_{TM} $\ell_1 + m_1 + n_1 \equiv \ell_2 + m_2 + n_2 \mod 3.$

$c(\ell;m;n^*)$

A LINEAR SYMMETRY FOR THE TRIPLE MULTIPLICITIES

LINEAR AUTOMORPHISM OF Λ_{TM}

$$c(\theta(\ell, m, n)) = c(\ell; m; n)$$

PERMUTATIONS OF THE DYNKIN LABELS $\mathfrak{S}_3$

$$c(\ell; m; n) = \dim\left(V_\ell \otimes V_m \otimes V_n\right)^{SU(3)}$$

DUALITY SYMMETRY $\mathfrak{S}_2$

$$(\ell, m, n) \leftrightarrow (\ell^*, m^*, n^*)$$

GROUP OF SYMMETRIES OF THE TRIPLE MULTIPLICITIES $SU(k)$

$\mathfrak{S}_2 \times \mathfrak{S}_3$ ORDER 12 $k \geq 3$

LITTLEWOOD-RICHARDSON COEFFICIENTS
O RATHER TRIPLE MULTIPLICITIES

Berenstein & Zelevinsky, 1991

Alcázar de Sevilla, 1090

Triple Multiplicities for $s\ell(r+1)$ and the Spectrum of the Exterior Algebra of the Adjoint Representation

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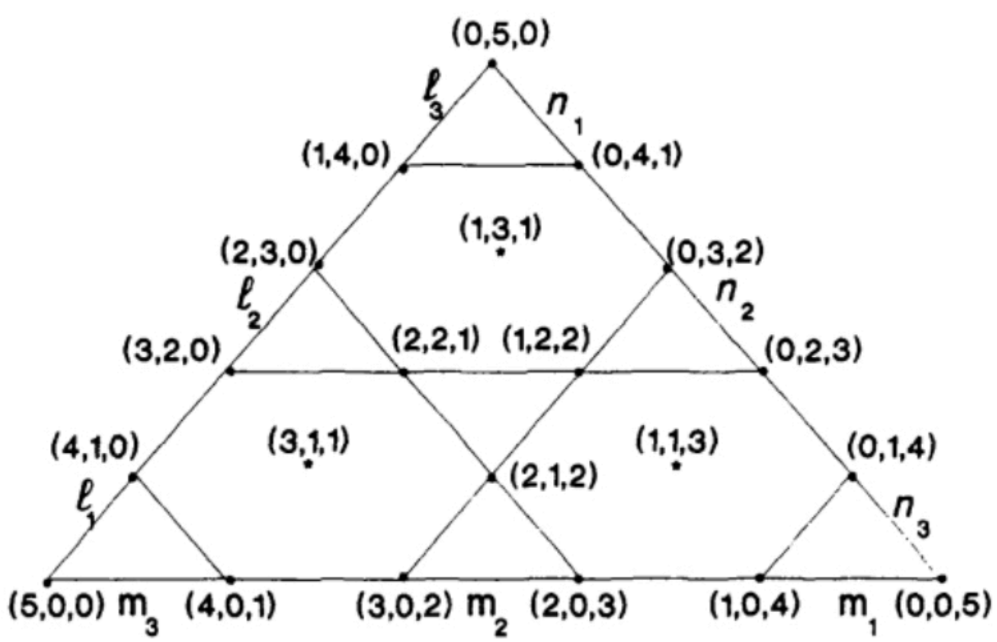
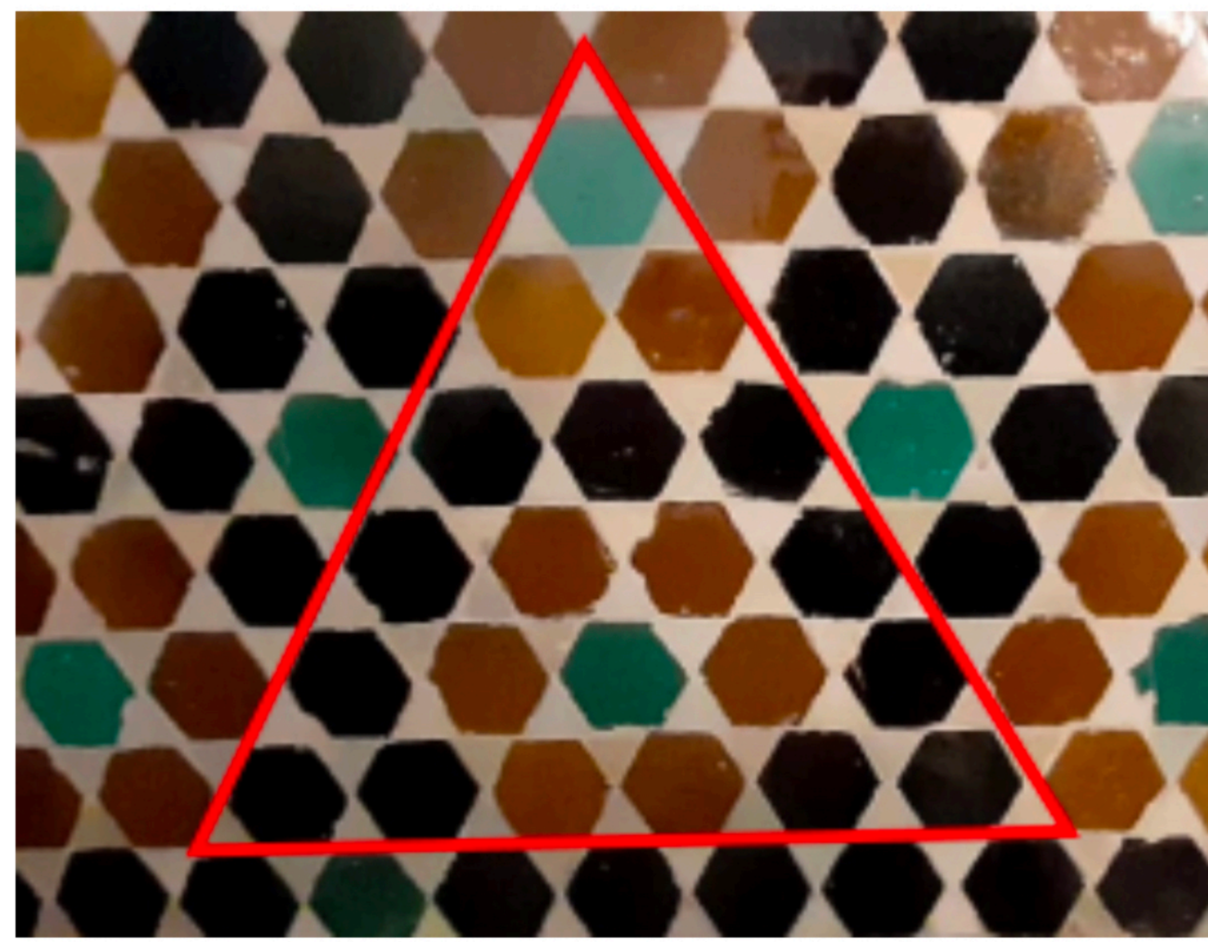
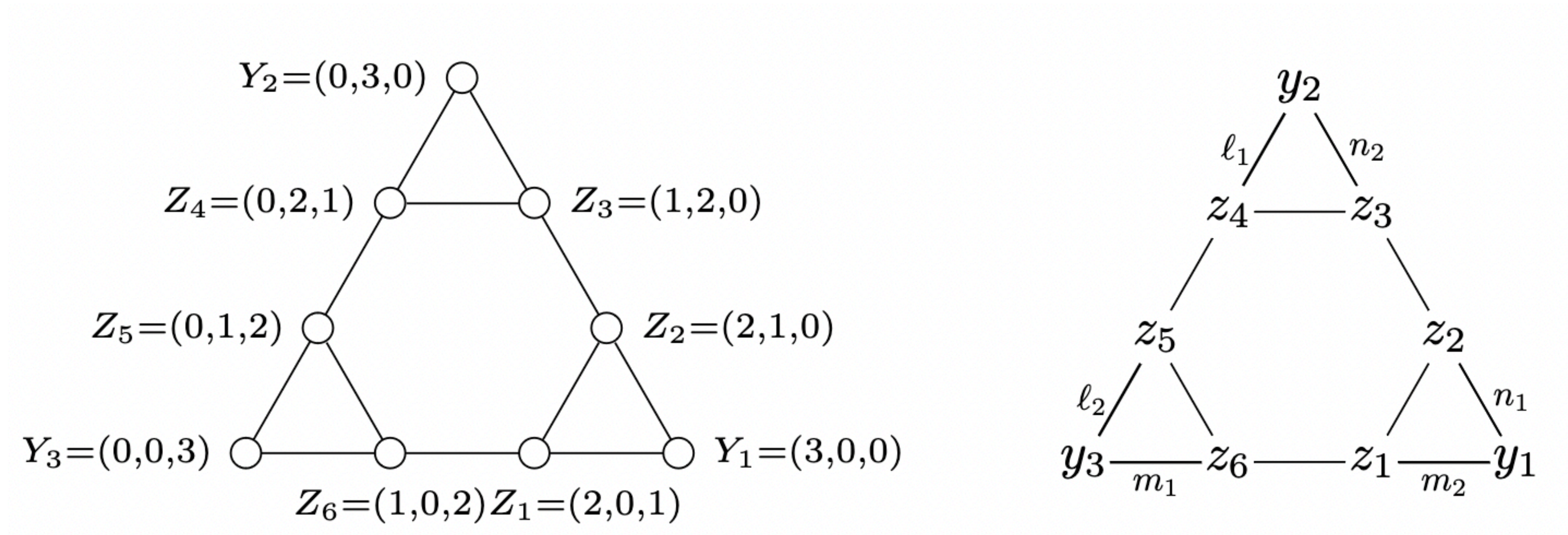
Received May 23, 1991, Revised October 10, 1991



We fix a natural number r and put $T = T_r = \{(i,j,k) \in \mathbf{Z}_+^3 : i+j+k = 2r-1\}$. Put also $H = H_r = \{(i,j,k) \in T_r$: all i,j,k are odd$\}$ and $G = G_r = T_r - H_r$. Thus T_r is the set of vertices of a regular triangular lattice filling the regular triangle with vertices $(2r-1,0,0)$, $(0,2r-1,0)$, and $(0,0,2r-1)$; this triangle is decomposed into the union of elementary triangles having all three vertices in G_r and of elementary hexagons centered at points of H_r (see Figure 1).

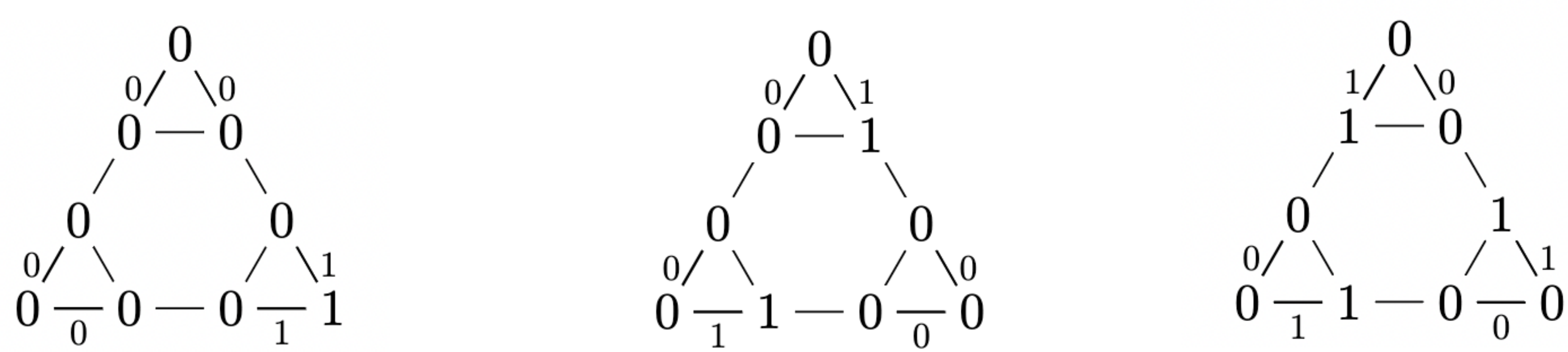


Berenstein-Zelevinski triangles



Sides of the hexagon sum as much as the opposite sides

$$z_1 - z_4 = z_5 - z_2 = z_3 - z_6.$$

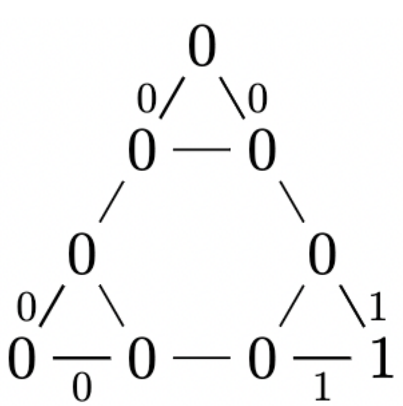


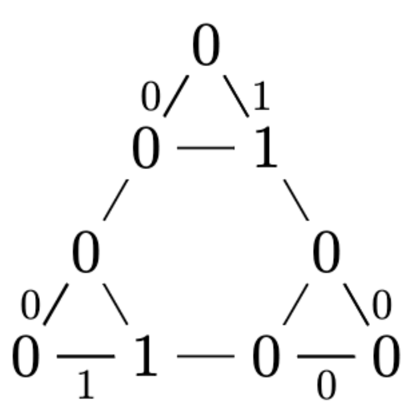
Join work with Emmanuel Briand and Stefan Trandafir

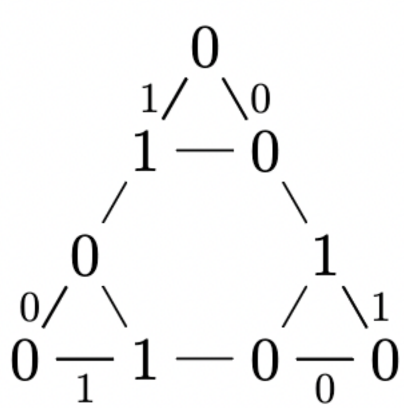
Vector space $\mathcal{L}_{\mathrm{BZ}}$ Real labelling of BZ triangles

BZ cone: The cone of all points with non-negative labelings

BZ triangle: An element of lattice of integral points in the BZ cone





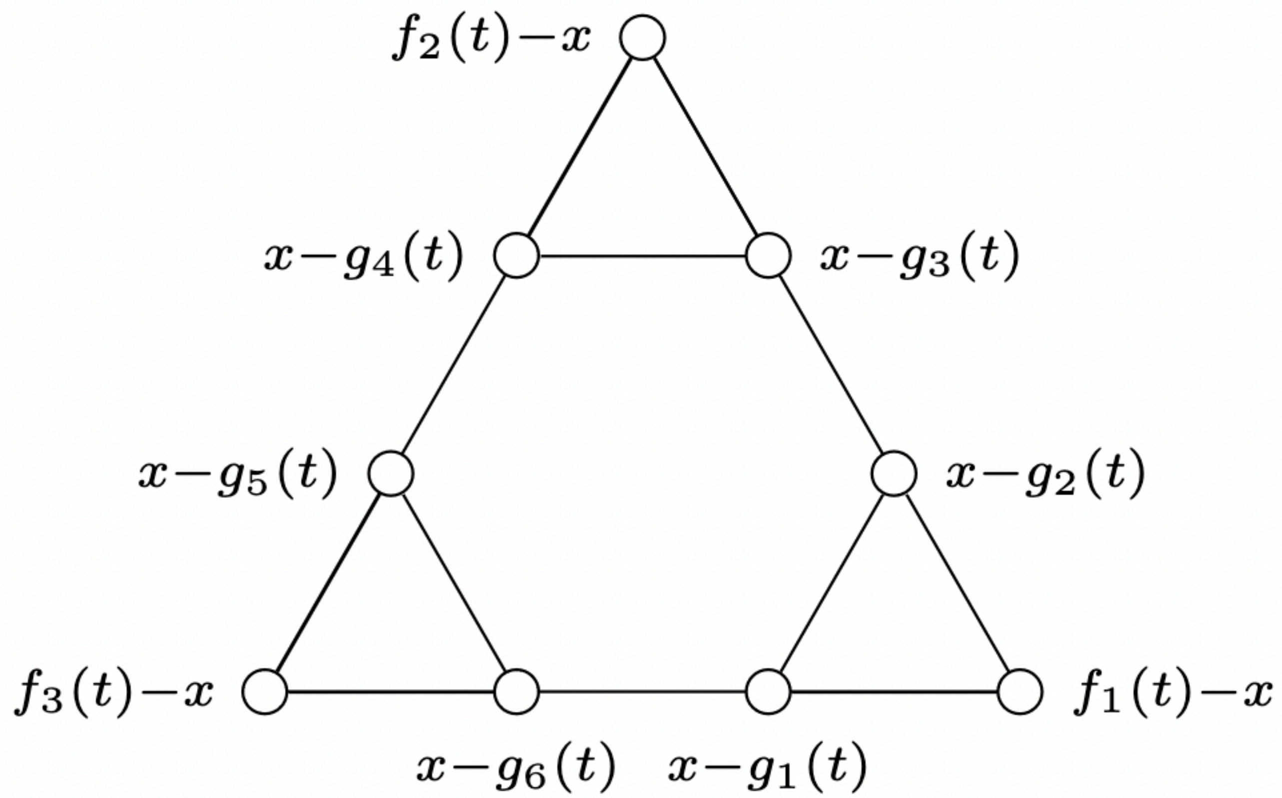


$$
\begin{array}{lll}
f_1(t) = 0, & f_2(t) = \ell_1 - m_2 - \omega(t), & f_3(t) = \ell_2 - n_1 + \omega(t), \\
g_1(t) = -m_2, & g_3(t) = \ell_1 - m_2 - n_2 - \omega(t), & g_5(t) = -n_1 + \omega(t), \\
g_2(t) = -n_1, & g_4(t) = -m_2 - \omega(t), & g_6(t) = \ell_2 - m_1 - n_1 + \omega(t)
\end{array}
$$

$$
\text{with } \omega(t) = \tfrac{1}{3}\left(\ell_1 + m_1 + n_1 - \ell_2 - m_2 - n_2\right).
$$

A parametrization of the space of BZ-triangles

$$
\begin{cases} \forall i, x \le f_i(t), \\ \forall j, x \ge g_j(t). \end{cases}
$$



The rays of the BZ cone

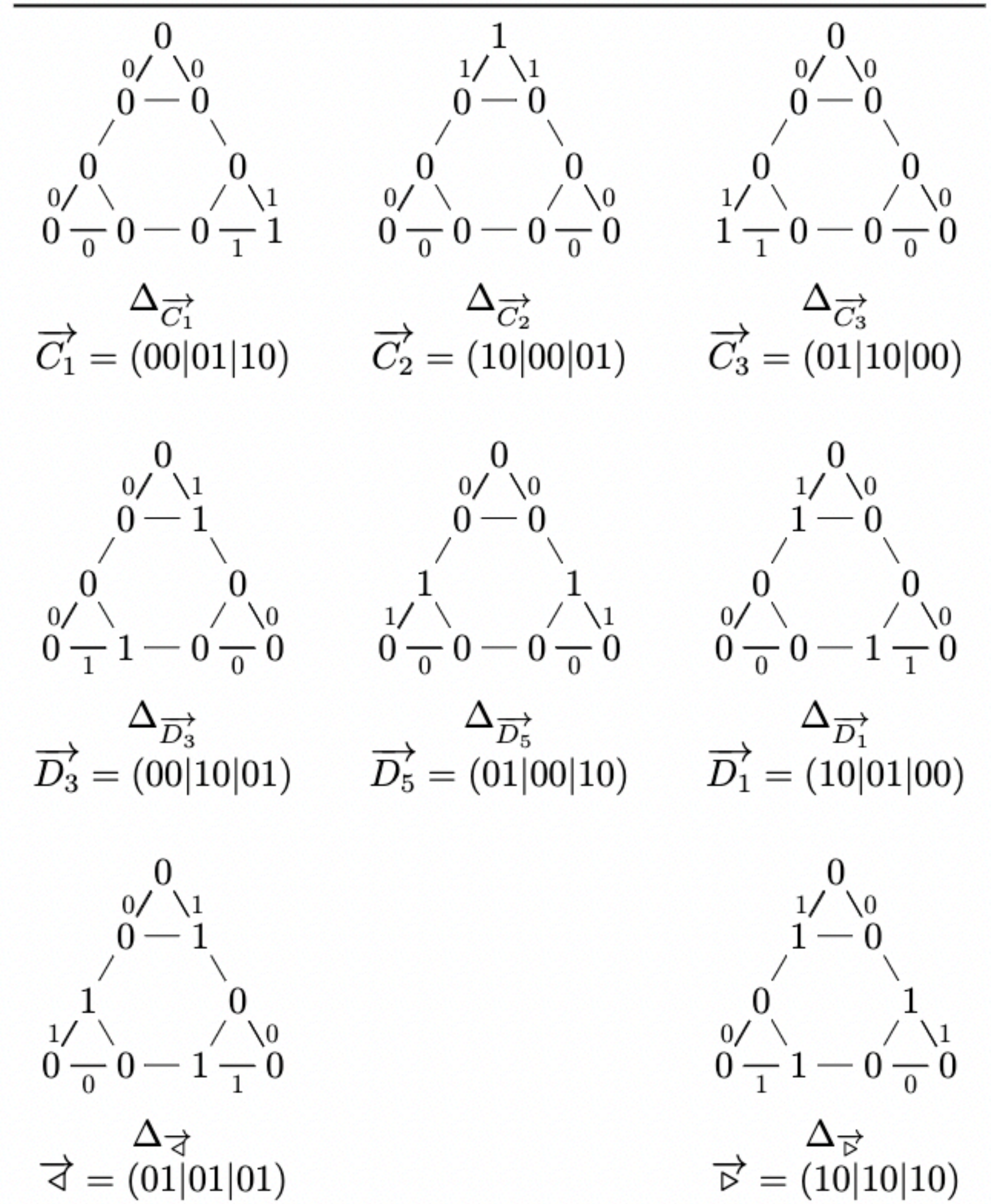
Relation

$$\Delta_{\vec{D_1}} + \Delta_{\vec{D_3}} + \Delta_{\vec{D_5}} = \Delta_{\vec{\triangleleft}} + \Delta_{\vec{\triangleright}}$$

Fundamental BZ triangles

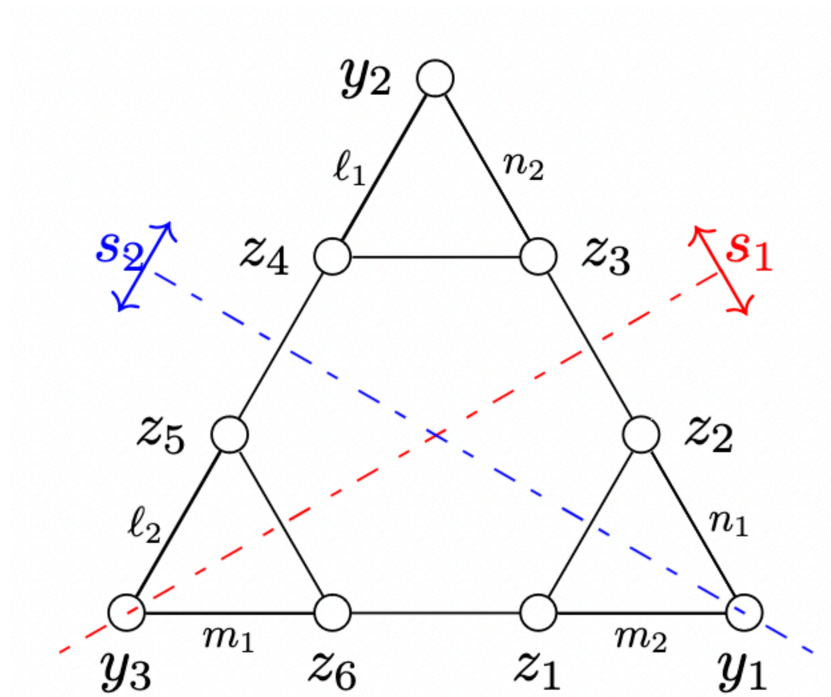
Generate as vector space
and as lattice

Fundamental BZ triangles



Berenstein-Zelevinsky

Symmetries of the BZ triangles



Generated by

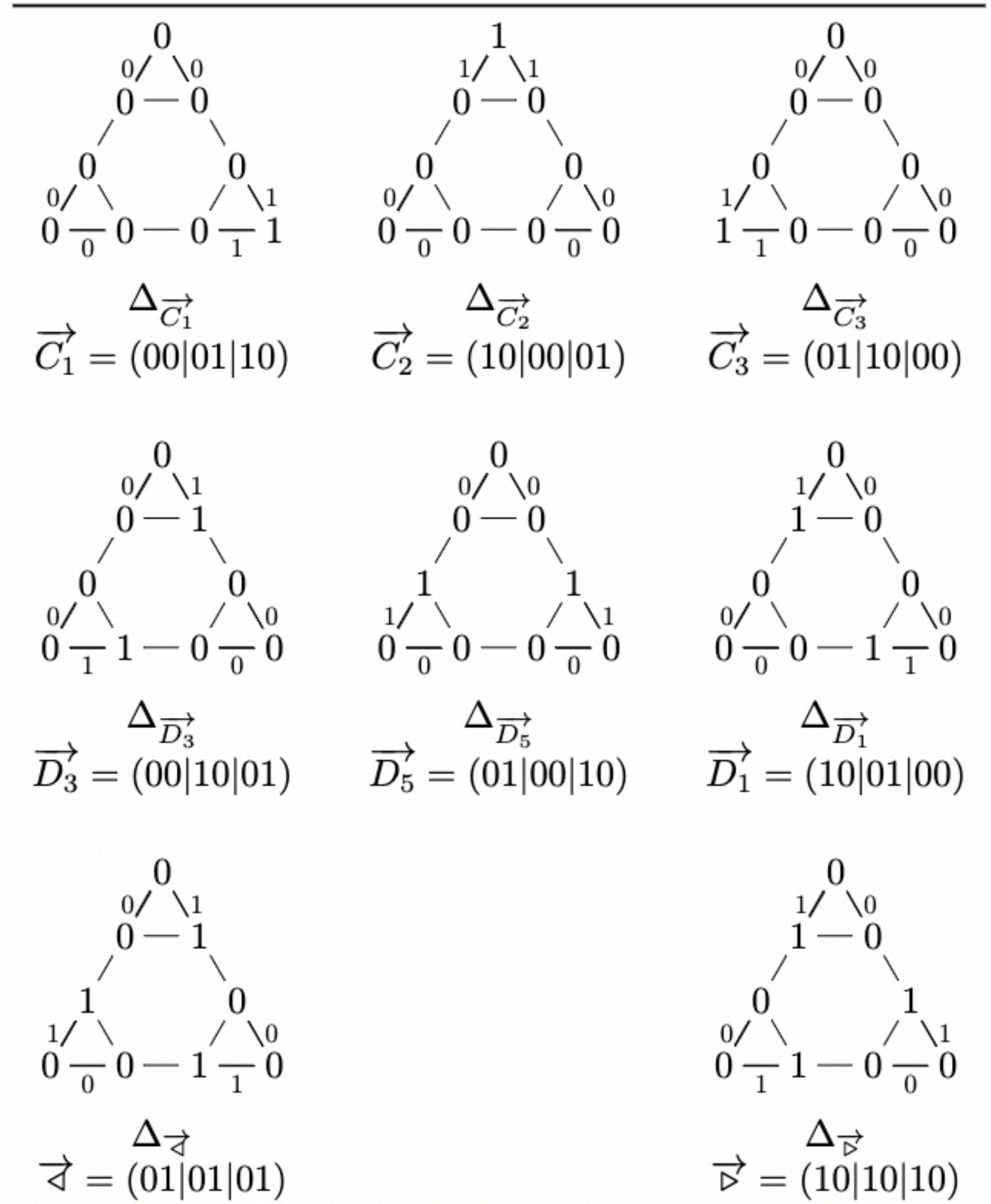
$$(\ell; m; n) \leftrightarrow (m^*; \ell^*; n^*)$$

$$(\ell; m; n) \leftrightarrow (\ell^*; n^*; m^*).$$

A not the group of permutations of the Dynkin labels
(it is isomorphic to it)

Does not contain the duality symmetry

FUNDAMENTAL BZ TRIANGLES



A LINEAR SYMMETRY OF THE SPACE OF BZ TRIANGLES PERMUTES

MINIMAL RAY GENERATORS FOR THE CONE BZ

ANY LINEAR SYMMETRY SHOULD STABILIZE

$$\{\Delta_{\vec{D_3}}, \Delta_{\vec{D_5}}, \Delta_{\vec{D_1}}\}$$

AND $\{\Delta_{\vec{\triangleleft}}, \Delta_{\vec{\triangleright}}\}$

$$\Delta_{\vec{D_1}} + \Delta_{\vec{D_3}} + \Delta_{\vec{D_5}} = \Delta_{\vec{\triangleleft}} + \Delta_{\vec{\triangleright}}$$

THUS, IT SHOULD ALSO STABILIZE

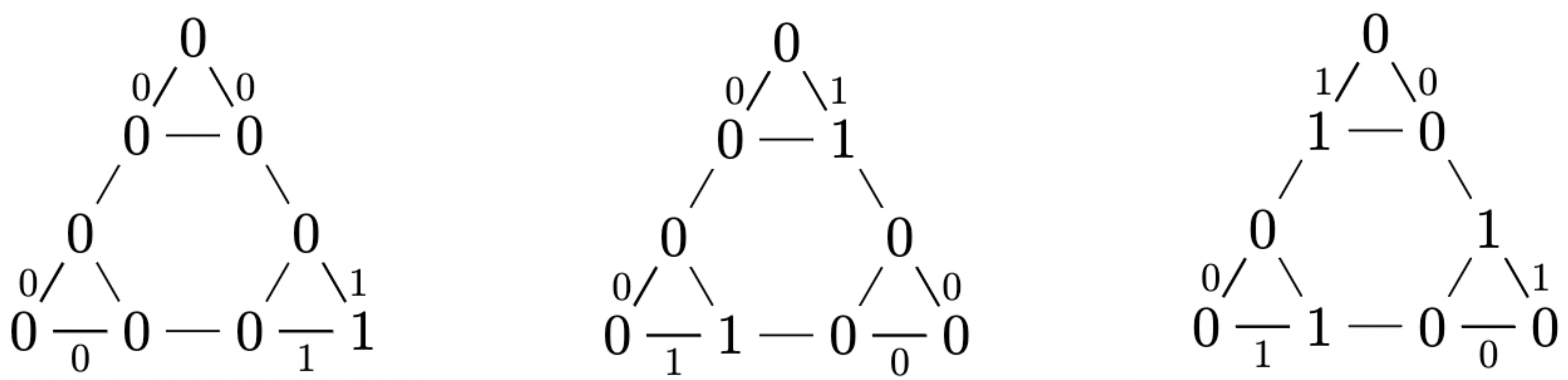
$$\{\Delta_{\vec{C_1}}, \Delta_{\vec{C_2}}, \Delta_{\vec{C_3}}\}$$

GROUP OF LINEAR SYMMETRIES OF ORDER 72

$$\mathfrak{S}_{\{\Delta_{\vec{C_1}},\Delta_{\vec{C_2}},\Delta_{\vec{C_3}}\}} \times \mathfrak{S}_{\{\Delta_{\vec{D_3}},\Delta_{\vec{D_5}},\Delta_{\vec{D_1}}\}} \times \mathfrak{S}_{\{\Delta_{\vec{\triangleleft}},\Delta_{\vec{\triangleright}}\}}.$$

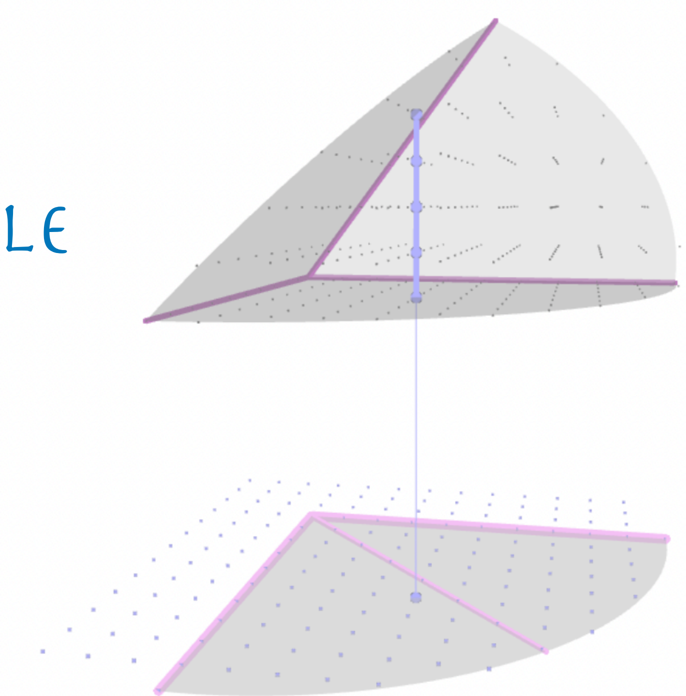
A LINEAR MAP $\quad pr : \mathcal{L}_{\mathrm{BZ}} \to \mathbb{R}^6$

$$\ell_1 = y_2 + z_4, \quad m_1 = y_3 + z_6, \quad n_1 = y_1 + z_2$$
$$\ell_2 = y_3 + z_5, \quad m_2 = y_1 + z_1, \quad n_2 = y_2 + z_3$$



THE LATTICE OF INTEGRAL POINTS OF THE BZ CONE IS SENT ONTO Λ_{TM}

IT SENDS THE BZ CONE TO THE CONE OF THE TRIPLE MULTIPLICITIES



$$c(\ell; m; n) = \# \left(pr^{-1}(\ell; m; n) \cap \mathrm{lat}(\mathsf{BZ})\right)$$

BERENSTEIN-ZELEVINSKI

A linear symmetry of the triple multiplicities

$$\overrightarrow{C_1} + \overrightarrow{C_2} + \overrightarrow{C_3} = \overrightarrow{D_1} + \overrightarrow{D_3} + \overrightarrow{D_5} = \overrightarrow{\triangleleft} + \overrightarrow{\triangleright}.$$

↑ new

There are no other relations with all coefficients positive

Any linear symmetry of the triple multiplicities stabilizes the TM cone

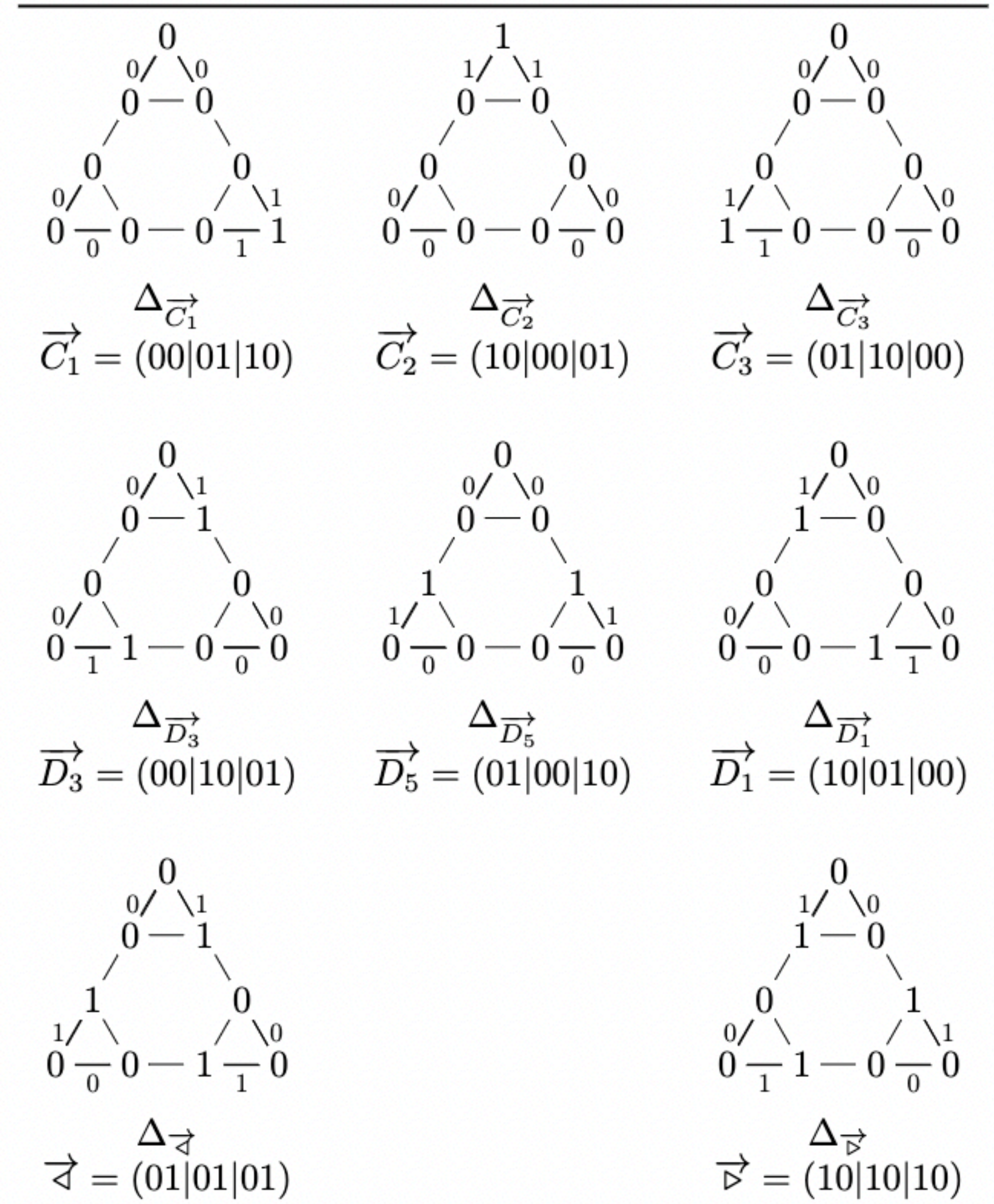
Thus, permutes its rays.

$$\{\overrightarrow{C_1}, \overrightarrow{C_2}, \overrightarrow{C_3}, \overrightarrow{D_1}, \overrightarrow{D_3}, \overrightarrow{D_5}, \overrightarrow{\triangleleft}, \overrightarrow{\triangleright}\}.$$

Minimal ray generators

Triple multiplicities

projections



A linear symmetry of the triple multiplicities

$$\overrightarrow{C_1}+\overrightarrow{C_2}+\overrightarrow{C_3}=\overrightarrow{D_1}+\overrightarrow{D_3}+\overrightarrow{D_5}=\overrightarrow{\triangleleft}+\overrightarrow{\triangleright}.$$

stabilizes $\{\overrightarrow{\triangleleft},\overrightarrow{\triangleright}\}$

stabilizes —-or swaps them!

$$\{\overrightarrow{C_1},\overrightarrow{C_2},\overrightarrow{C_3}\}\text{ and }\{\overrightarrow{D_1},\overrightarrow{D_3},\overrightarrow{D_5}\}$$

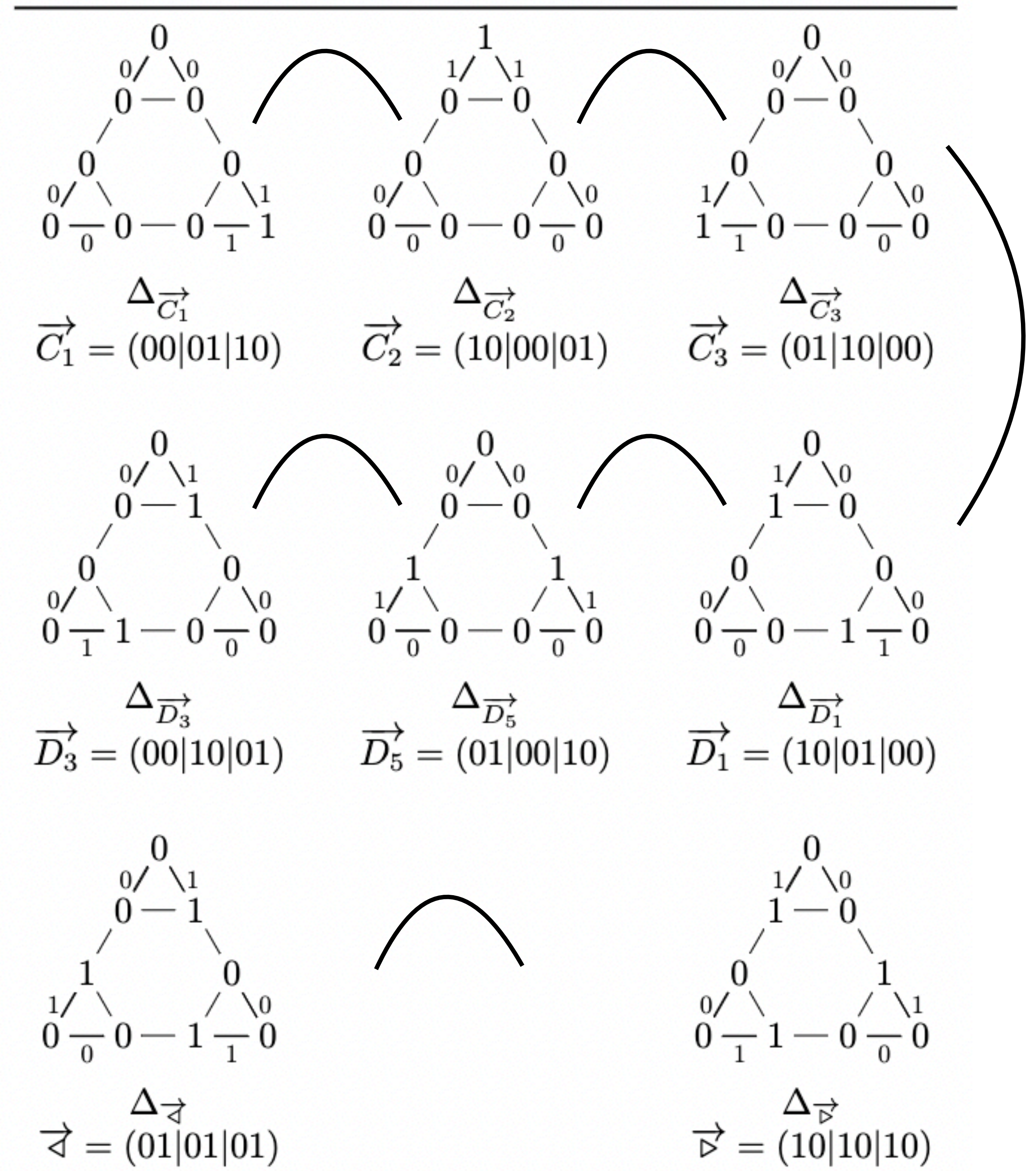
thus stabilizes

$$\{\overrightarrow{\triangleleft},\overrightarrow{\triangleright}\}\text{ and }\{\{\overrightarrow{C_1},\overrightarrow{C_2},\overrightarrow{C_3}\},\{\overrightarrow{D_1},\overrightarrow{D_3},\overrightarrow{D_5}\}\}$$

isomorphic $\mathfrak{S}_2\times(\mathfrak{S}_3\wr\mathfrak{S}_2)$

order $2\times(2\times(3!)^2)$

Triple multiplicities



symmetries of the outer triangle

symmetries of the inner hexagon

The support of the triple multiplicities is a cone.

$$\begin{cases} \forall i \in \{1,2,3\}, & x \leq f_i(t); \\ \forall j \in \{1,2,3,4,5,6\}, & x \geq g_j(t) \end{cases}$$

$$\max_q g_q(t) \leq x \leq \min_p f_p(t)$$

System of 18 inequalities:

$$\forall i \in \{1,2,3\},\ \forall j \in \{1,2,3,4,5,6\},\ g_j(t) \leq f_i(t).$$

The quasi polynomial:

$$c(t) = 1 + \max(0, \min_p f_p(t) - \max_q g_q(t)).$$

THE 18 CHAMBERS ARE FULL DIMENSIONAL

THE GROUP OF SYMMETRIES OF THE BZ TRIANGLES ACTS TRANSITIVELY ON THE CHAMBER COMPLEX

CHAMBERS ARE SIMPLICIAL,

HAVE 6 RAYS, 5 EXTERNAL $t_1, \ldots, t_5$

INTERNAL RAY $(11|11|11)$

$$c(t) = 1 + \mathrm{Vol}_{\Lambda_{\mathrm{TM}}}\left(\Pi(t_1, \ldots, t_5, t)\right)$$

fundamental domains of the lattice have volume 1

RANK GENERATING FUNCTION

$$(1+3q+3q^2)^2(1+2q)(1+q)^3.$$

AD UTRUMQUE