CHAPTER 7

Differential graded Lie algebras

7.1. Differential graded vector spaces

Every vector space is considered over a fixed field $\mathbb{K}$; unless otherwise specified, by the symbol $\otimes$ we mean the tensor product $\otimes_{\mathbb{K}}$ over the field $\mathbb{K}$.

The category DG. By a graded vector space we mean a $\mathbb{K}$-vector spaces V endowed with a $\mathbb{Z}$-graded direct sum decomposition $V = \oplus_{i\in\mathbb{Z}} V^i$. The elements of V_i are called homogeneous of degree i.

If $V = \oplus_{n\in\mathbb{Z}} V^n \in \mathbf{G}$ we write $\deg(a; V) = i \in \mathbb{Z}$ if $a \in V_i$; if there is no possibility of confusion about V we simply denote $\overline{a} = \deg(a; V)$.

Definition 7.1.1. A DG-vector space is the data of a graded vector space $V = \oplus_{n\in\mathbb{Z}} V^n$ together a linear map $d\colon V \to V$, called **differential**, such that $d(V^n) \subset V^{n+1}$ for every n and $d^2 = d \circ d = 0$.

A morphism $f\colon (V, d_V) \to (W, d_W)$ of DG-vector spaces is a linear map $f\colon V \to W$ such that $f(V^n) \subset W^n$ for every n and $d_W f = f d_V$.

The category of DG-vector spaces will be denoted **DG**.

Thus, giving a morphism $f\colon (V, d_V) \to (W, d_W)$ of DG-vector spaces is the same of giving a sequence of linear maps $f_n\colon V^n \to W^n$ such that $d_W f_n = f_{n+1} d_V$ for every n.

Given a DG-vector space (V, d) we denote as usual by $Z(V) = \ker d$ the space of cycles, by $B(V) = d(V)$ the space of boundaries and by $H(V) = Z(V)/B(V)$ the cohomology of V.

A morphism in **DG** is called a **quasiisomorphism**, or a **weak equivalence**, if it induces an isomorphism in cohomology. A DG-vector space (V, d) is called **acyclic** if $H(V) = 0$, i.e. if it is weak equivalent to 0.

Remark 7.1.2. In a completely similar way we may define dg-vector spaces, in which differentials have degree -1, i.e. $d(V_i) \subset V_{i-1}$. A differential graded vector space is either a DG-vector space or a dg-vector space.

Example 7.1.3. Every complex of vector spaces

$$\cdots \to V^n \xrightarrow{d} V^{n+1} \xrightarrow{d} V^{n+2} \to \cdots$$

can be trivially considered as a DG-vector space.

Given a double complex $C^{i,j}$, $i, j \in \mathbb{Z}$, of vector spaces, with differentials

$$d_1\colon C^{i,j} \to C^{i+1,j}, \quad d_2\colon C^{i,j} \to C^{i,j+1}, \quad d_1^2 = d_2^2 = d_1 d_2 + d_2 d_1 = 0$$

we define the associated **total complex** as the DG-vector space

$$\operatorname{Tot}(C^{*,*}) = \bigoplus_{n\in\mathbb{Z}} \operatorname{Tot}(C^{*,*})^n, \quad \operatorname{Tot}(C^{*,*})^n = \bigoplus_{i+j=n} C^{i,j}, \quad d = d_1 + d_2.$$

The category **DG** contains products: more precisely if $\{(V_i, d_i)\}$ is a family of DG-vector spaces, we have

$$\prod_i V_i = \bigoplus_{n\in\mathbb{Z}} (\prod_i V_i)^n, \quad (\prod_i V_i)^n = \prod_i V_i^n, \qquad d(\{v_i\}) = \{d_i(v_i)\}, \qquad v_i \in V_i.$$

Künneth formulas. Given two DG-vector spaces V, W we may define their tensor produc $V \otimes W \in \mathbf{DG}$ and their **internal Hom** $\operatorname{Hom}^*_{\mathbb{K}}(V, W) \in \mathbf{DG}$ in the following way:

$$V \otimes W = \bigoplus_{n\in\mathbb{Z}} (V \otimes W)^n, \text{ where } (V \otimes W)^n = \bigoplus_{i+j=n} V^i \otimes W^j,$$

$$d(v \otimes w) = dv \otimes w + (-1)^{\overline{v}} v \otimes dw.$$

$$\operatorname{Hom}^*_{\mathbb{K}}(V, W) = \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}^n_{\mathbb{K}}(V, W), \qquad \operatorname{Hom}^n_{\mathbb{K}}(V, W) = \{f \colon V \to W \text{ linear} \mid f(V^i) \subset W^{i+n}\ \forall i\}.$$

$$d\colon \operatorname{Hom}^n_{\mathbb{K}}(V, W) \to \operatorname{Hom}^{n+1}_{\mathbb{K}}(V, W), \qquad df(v) = d_W(f(v)) - (-1)^n f(d_V(v)).$$

We point out that for every $V, W, Z \in \mathbf{DG}$ we have a natural isomorphism abelian groups

$$\operatorname{Hom}_{\mathbf{DG}}(V, W) = Z^0(\operatorname{Hom}^*_{\mathbb{K}}(V, W)),$$

$$\operatorname{Hom}_{\mathbf{DG}}(V \times W, Z) = \operatorname{Hom}_{\mathbf{DG}}(V, \operatorname{Hom}^*_{\mathbb{K}}(W, Z)).$$

Theorem 7.1.4 (Künneth formulas)**.** *Given a DG-vector space V, consider its cohomology $H^*(V) = \bigoplus_n H^n(V)$ as a DG-vector space with trivial differential. For every pair of DG-vector spaces there exists natural isomorphisms*

$$H^*(V \otimes W) = H^*(V) \otimes H^*(W), \qquad H^*(\operatorname{Hom}^*_{\mathbb{K}}(V, W)) = \operatorname{Hom}^*_{\mathbb{K}}(H^*(V), H^*(W)).$$

PROOF. See e.g. the book [120]. □

Koszul rule of signs.

Definition 7.1.5. Given $V, W \in \mathbf{DG}$, we define the **twisting involution**

$$\mathtt{tw} \in \operatorname{Hom}_{\mathbf{DG}}(V \otimes W, W \otimes V), \qquad \mathtt{tw}(v \otimes w) = (-1)^{\overline{v}\,\overline{w}} w \otimes v.$$

Using the **Koszul signs convention** means that we choose as *natural isomorphism* between $V \otimes W$ and $W \otimes V$ the twisting map `tw` and we make every commutation rule compatible with `tw`. More informally, to "get the signs right", whenever an "object of degree d passes on the other side of an object of degree h, a sign $(-1)^{dh}$ must be inserted".

Example 7.1.6. Assume that $f \in \operatorname{Hom}^*_{\mathbb{K}}(V, W)$ and $g \in \operatorname{Hom}^*_{\mathbb{K}}(H, K)$. Then the Koszul rule of signs implies that the correct definition of $f \otimes g \in \operatorname{Hom}^*_{\mathbb{K}}(V \otimes H, W \otimes K)$ is

$$(f \otimes g)(v \otimes h) = (-1)^{\overline{g}\,\overline{v}} f(v) \otimes g(h).$$

Notice that $\mathtt{tw} \circ (f \otimes g) \circ \mathtt{tw} = (-1)^{\overline{f}\,\overline{g}} g \otimes f$.

Shifting indices. Given a DG-vector space (V, d_V) and an integer p we can define the DG-vector space $(V[p], d_{V[p]})$ by setting

$$V[p]^n = V^{n+p}, \qquad d_{V[p]} = (-1)^p d_V.$$

Sometimes it is useful to use a different notation. Let s be a formal symbol of degree $+1$, so that s^p becomes a formal symbol of degree p, for every integer p. Then define

$$s^pV = \{s^pv \mid v \in V\}, \qquad \deg s^pv = p + \deg(v).$$

Setting $ds^p = 0$, according to Leibniz and Koszul rules we have

$$d(s^pv) = d(s^p)v + (-1)^p s^p d(v) = (-1)^p s^p d(v).$$

Clearly $(s^pV)^n = V^{n-p}$ and then $s^pV \simeq V[-p]$. Notice that the natural map

$$s^p \colon V \to s^pV, \qquad v \mapsto s^pv$$

belongs to $\operatorname{Hom}^p_{\mathbb{K}}(V, s^pV)$. Some authors call the sV the **suspension** of V, $s^{-1}V$ the **desuspension** of V and more generally s^pV the **p-fold suspension** of V.

The Koszul rule of signs gives immediately a canonical isomorphism

$$s^pV \otimes s^qW \to s^{p+q}(V \otimes W), \qquad s^pv \otimes s^qw \mapsto (-1)^{q\overline{v}} s^{p+q}(v \otimes w),$$

Similarly we have $\operatorname{Hom}^*_{\mathbb{K}}(s^pV, s^qW) \simeq s^{q-p}\operatorname{Hom}^*_{\mathbb{K}}(V, W)$.

Definition 7.1.7. For a morphism of DG-vector spaces $f\colon V \to W$ we will denote by C_f the suspension of the mapping cone of f. More explicitely $C_f = V \oplus sW$ and the differential is

$$\delta\colon C_f^n = V^n \oplus W^{n-1} \to C_f^{n+1} = V^{n+1} \oplus W^n, \qquad \delta(v,w) = (dv, f(v) - dw).$$

The projection $p\colon C_f \to V$ and the inclusion $i\colon sW \to C_f$ are morphisms of DG-vector spaces and we have a long exact cohomology sequence

$$\cdots \to H^i(V) \xrightarrow{f} H^i(W) \xrightarrow{i} H^{i+1}(C_f) \xrightarrow{p} H^{i+1}(V) \xrightarrow{f} H^{i+1}(W) \to \cdots$$

In particular, given a commutative square

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & E \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{\beta} & F \end{array}$$

if both α and β are quasiisomorphisms, then also the induced map $C_f \to C_g$ is a quasiisomorphism.

7.2. DG-algebras

Definition 7.2.1. A DG-algebra (short for Differential graded commutative algebra) is the data of a DG-vector space A and a morphism of DG-vector spaces

$$A \otimes A \to A, \qquad a \otimes b \mapsto ab,$$

called *product*, which is associative and invariant under the twisting involution.

More concretely, this means that for $a, b, c \in A$ we have:

(1) (associativity) $(ab)c = a(bc)$,
(2) (graded commutativity) $ab = (-1)^{\overline{a}\,\overline{b}} ba$,
(3) (graded Leibniz) $d(ab) = d(a)b + (-1)^{\overline{a}} a d(b)$.

A morphism of DG-algebras is simply a morphism of DG-vector spaces commuting with products. The category fo DG-algebras will be denoted by **DGA**. A DG-algebra A is called **unitary** if there exists a unit $1 \in A^0$.

Example 7.2.2. Every commutative $\mathbb{K}$-algebra can be considered as a DG-algebra concentrated in degree 0.

Example 7.2.3. The de Rham complex of a smooth manifold, endowed with wedge product is a DG-algebra.

Example 7.2.4 (Koszul algebras)**.** Let V be a vector space and consider the graded algebra

$$A = \bigoplus_{n \le 0} A^n, \qquad A^{-n} = \bigwedge^n V\,,$$

with the wedge product as a multiplication map. Given a linear map $f\colon V \to \mathbb{K}$, we may define a differential $d\colon A^{-i} \to A^{-i+1}$, $i \ge 0$:

$$d = f\lrcorner\colon \bigwedge^i V \to \bigwedge^{i-1} V\,,$$

where the contraction operator $\lrcorner$ is defined by the formula

$$f\lrcorner(v_1 \wedge \cdots \wedge v_h) = \sum_{j=1}^{h} (-1)^{j-1} f(v_j) v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_h.$$

Example 7.2.5. The de Rham complex of algebraic differential forms on the affine line will be denoted by $\mathbb{K}[t, dt]$. We may write

$$\mathbb{K}[t, dt] = \mathbb{K}[t] \oplus \mathbb{K}[t]dt$$

where t, dt are indeterminates of degrees $\overline{t} = 0$, $\overline{dt} = 1$ and the differential d is determined by the "obvious" equality $d(t) = dt$ and therefore $d(p(t) + q(t)dt) = p(t)'dt$. The inclusion $\mathbb{K} \to \mathbb{K}[t, dt]$ and the evaluation maps

$$e_s\colon \mathbb{K}[t, dt] \to \mathbb{K}\,, \qquad p(t) + q(t)dt \mapsto p(s), \qquad s \in \mathbb{K}\,,$$

are morphisms of DG-algebras.

Lemma 7.2.6. *In characteristic 0, every evaluation morphism* $e_s\colon \mathbb{K}[t,dt]\to\mathbb{K}$ *is a quasiisomorphism.*

Proof. If $i\colon \mathbb{K}\to\mathbb{K}[t,dt]$ is the natural inclusion, we have $e_s\circ i = Id$ and then it is sufficient to prove that i is a quasiisomorphism. This is obvious since every cocycle of $\mathbb{K}[t,dt]$ is of type $a+q(t)dt$ with $a\in\mathbb{K}$ and $q(t)dt$ is exact, being the differential of $\int_0^t q(s)ds$. □

The tensor product of two DG-algebras is still a DG-algebra; clearly we need to take attention to Koszul sign convention. If A, B are DG-algebras, then the product on $A\otimes B$ is defined as the linear extension of

$$(a_1\otimes b_1)(a_2\otimes b_2) = (-1)^{\overline{b_1}\,\overline{a_2}}a_1a_2\otimes b_1b_2.$$

7.3. Differential graded Lie algebras

In this section $\mathbb{K}$ will be a field of characteristic 0.

Definition 7.3.1. A **differential graded Lie algebra** (DGLA for short) is the data of a DG-vector space (L,d) with a bilinear bracket $[,]\colon L\times L\to L$ satisfying the following condition:

1. $[\ ,\ ]$ is homogeneous skewsymmetric: this means $[L^i,L^j]\subset L^{i+j}$ and $[a,b]+(-1)^{\overline{a}\overline{b}}[b,a]=0$ for every a,b homogeneous.
2. Every triple of homogeneous elements a,b,c satisfies the (graded) Jacobi identity
$$[a,[b,c]] = [[a,b],c]+(-1)^{\overline{a}\,\overline{b}}[b,[a,c]].$$
3. (graded Leibniz) $d[a,b]=[da,b]+(-1)^{\overline{a}}[a,db]$.

We should take attention that skewsymmety implies that $[a,a]=0$ only if a is of even degree, while for odd degrees we have the following result.

Lemma 7.3.2. *Let L be a DGLA and $a\in L$ homogeneous of odd degree. Then* $[a,[a,a]]=0$.

Proof. By graded Jacobi and graded skesymmetry we have

$$[a,[a,a]] = [[a,a],a]-[a,[a,a]] = -2[a,[a,a]].$$

□

Example 7.3.3. If $L=\oplus L^i$ is a DGLA then L^0 is a Lie algebra in the usual sense. Conversely, every Lie algebra can be considered as a DGLA concentrated in degree 0.

Example 7.3.4. Let A be a DG-algebra and L a DGLA. Then the DG-vector space $L\otimes A$ has a natural structure of DGLA with bracket

$$[x\otimes a, y\otimes b] = (-1)^{\overline{a}\,\overline{y}}[x,y]\otimes ab.$$

Example 7.3.5. Let V be a DG-vector space. Then the total Hom complex $\operatorname{Hom}^*_{\mathbb{K}}(V,V)$ has a natural structure of DGLA with bracket

$$[f,g] = fg-(-1)^{\deg(f)\deg(g)}gf.$$

Notice that the differential on $\operatorname{Hom}^*_{\mathbb{K}}(V,V)$ is equal to the adjoint operator $[d,-]$, where d is the differential of V.

Example 7.3.6. Let E be a holomorphic vector bundle on a complex manifold M. We may define a DGLA $L=\oplus L^p$, $L^p=\Gamma(M,\mathcal{A}^{0,p}(\mathcal{E}nd(E)))$ with the Dolbeault differential and the natural bracket. More precisely if e,g are local holomorphic sections of $\mathcal{E}nd(E)$ and ϕ,ψ differential forms we define $d(\phi e)=(\overline{\partial}\phi)e$, $[\phi e,\psi g]=\phi\wedge\psi[e,g]$.

Example 7.3.7. Let T_M be the holomorphic tangent bundle of a complex manifold M. The **Kodaira-Spencer DGLA** is defined as $KS(M)=\oplus\Gamma(M,\mathcal{A}^{0,p}(T_M))[-p]$ with the Dolbeault differential; if $z_1,\dots,z_n$ are local holomorphic coordinates we have $[\phi d\overline{z}_I,\psi d\overline{z}_J]=[\phi,\psi]d\overline{z}_I\wedge d\overline{z}_J$ for $\phi,\psi\in\mathcal{A}^{0,0}(T_M)$, $I,J\in\bigwedge^*\{1,\dots,n\}$.

There is an obvious notion of morphism of differential graded Lie algebras: it is a morphism of DG-vector spaces commutaing with brackets. The category of differential graded Lie algebras will be denoted **DGLA**.

Example 7.3.8. The fiber product $L \times_H M$ of two morphisms $f\colon L \to H$, $g\colon M \to H$ of DGLA is a DGLA with bracket

$$[(a,x),(b,y)] = ([a,b],[x,y]).$$

Definition 7.3.9. A **quasiisomorphism** of DGLAs is a morphism of DGLA which is a quasi-isomorphism of DG-vector spaces. Two DGLA's are said to be **quasiisomorphic** if they are equivalent under the equivalence relation generated by quasiisomorphisms.

Example 7.3.10. Denote by $\mathbb{K}[t,dt]$ the differential graded algebra of polynomial differential forms over the affine line and, for every differential graded Lie algebra L denote $L[t,dt] = L \otimes \mathbb{K}[t,dt]$. As a graded vector space $L[t,dt]$ is generated by elements of the form $aq(t)+bp(t)dt$, for $p,q \in \mathbb{K}[t]$ and $a,b \in L$. The differential and the bracket on $L[t,dt]$ are

$$d(aq(t)+bp(t)dt) = (da)q(t) + (-1)^{\deg(a)}aq(t)'dt + (db)p(t)dt,$$

$$[aq(t),ch(t)] = [a,c]q(t)h(t), \quad [aq(t),ch(t)dt] = [a,c]q(t)h(t)dt.$$

For every $s \in \mathbb{K}$, the evaluation morphism

$$e_s\colon L[t,dt] \to L, \quad e_s(aq(t)+bp(t)dt) = q(s)a$$

is a morphism of differential graded Lie algebras. According to Lemma 7.2.6 and Künneth formulas, it is also a quasiisomorphism of DGLA.

Example 7.3.11. Let $f\colon L \to H$, $g\colon M \to H$ be two morphisms of differential graded Lie algebras. Their **homotopy fiber product** is defined as

$$L \times_H^h M := \{(l,m,h(t)) \in L \times M \times H[t,dt] \mid h(0) = f(l),\ h(1) = g(m)\},$$

where for every $s \in \mathbb{K}$ we denote for simplicity $h(s) = e_s(h(t))$. It is immediate to verify that it is a differential graded Lie algebras and that the natural projections

$$L \times_H^h M \to L, \qquad L \times_H^h M \to M,$$

are surjective morphisms of DGLAs.

Remark 7.3.12. In the notation of Example 7.3.11, it is an easy exercise to prove that, if $f\colon L \to H$ is a quasiisomorphism, then the projection $L \times_H^h M \to M$ is a quasiisomorphism. This is a consequence of a more general results that we will prove in **??**.

Using this fact it is immediate to observe that two differential graded Lie algebas L, M are quasiisomorphic if and only if there exists a DGLA K and two quasiisomorphisms $K \to L$, $K \to M$.

The cohomology of a DGLA is itself a differential graded Lie algebra with the induced bracket and zero differential:

Definition 7.3.13. A DGLA L is called formal if it is quasiisomorphic to its cohomology DGLA $H^*(L)$.

We will see later on, that there exists differential graded Lie algebras that are not formal.

Lemma 7.3.14. *For every DG-vector space V, the differential graded Lie algebra* $\operatorname{Hom}^*(V,V)$ *is formal.*

Proof. For every index i we choose a vector subspace $H^i \subset Z^i(V)$ such that the projection $H^i \to H^i(V)$ is bijective. The graded vector space $H = \oplus H^i$ is a quasiisomorphic subcomplex of V. The subspace $K = \{f \in \operatorname{Hom}^*(V,V) \mid f(H) \subset H\}$ is a differential graded Lie subalgebra and there exists a commutative diagram of complexes with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{\alpha} & \operatorname{Hom}^*(V,V) & \longrightarrow & \operatorname{Hom}^*(H,V/H) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle Id} & & \\
0 & \longrightarrow & \operatorname{Hom}^*(H,H) & \longrightarrow & \operatorname{Hom}^*(H,V) & \longrightarrow & \operatorname{Hom}^*(H,V/H) & \longrightarrow & 0
\end{array}$$

The maps α and β are morphisms of differential graded Lie algebras. The complex $\operatorname{Hom}^*(H,V/H)$ is acyclic and γ is a quasiisomorphism, therefore also α and β are quasiisomorphisms. □

7.4. Further examples of differential graded Lie algebras

Given a graded vector space V and a bilinear map $\bullet\colon V\times V\to V$ such that $V^i\bullet V^j\subset V^{i+j}$, the vector

$$A(x,y,z)=(x\bullet y)\bullet z-x\bullet(y\bullet z)$$

is called the **associator** of the triple x,y,z: the product $\bullet$ is associative if and only if $A(x,y,z)=0$ for every x,y,z.

Lemma 7.4.1. *Assume that the associator is graded symmetric in the last two variables, i.e.* $A(x,y,z)=(-1)^{\overline{y}\,\overline{z}}A(x,z,y)$. *Then the graded commutator*

$$[x,y]=x\bullet y-(-1)^{\overline{x}\,\overline{y}}y\bullet x$$

satisfies the graded Jacobi identity.

Proof. Straightforward. □

Example 7.4.2 (The Gerstenhaber bracket)**.** Let A be a vector space and, for every integer $n\ge 0$ let $V^n=\operatorname{Hom}_{\mathbb{K}}(\bigotimes^{n+1}A,A)$ be the space of multilinear maps

$$f\colon \underbrace{A\times\cdots\times A}_{n+1}\to A.$$

The **Gerstenhaber product** is defined as

$$\bullet\colon V^n\times V^m\to V^{n+m},$$

$$(f\bullet g)(a_0,\ldots,a_{n+m})=\sum_{i=0}^{n}(-1)^{im}f(a_0,\ldots,a_{i-1},g(a_i,\ldots,a_{i+m}),a_{i+m+1},\ldots,a_{n+m}).$$

It is easy to verify that the associator is graded symmetric in the last two variables and then the graded commutator

$$[x,y]=x\bullet y-(-1)^{\overline{x}\,\overline{y}}y\bullet x,$$

called **Gerstenhaber bracket** satisfies graded Jacobi identity. Notice that for an element $m\in V^1$ we have $[m,m]=2m\bullet m$ and

$$m\bullet m(a,b,c)=m(m(a,b),c)-m(a,m(b,c)).$$

Therefore $[m,m]=0$ if and only if $m\colon A\times A\to A$ is an associative product.

Example 7.4.3 (The Hochschild DGLA)**.** Let A be an associative $\mathbb{K}$-algebra and denote by $m\colon A\times A\to A$, $m(a,b)=ab$, the multiplication map. We have seen that the graded vector space

$$Hoch^*(A)=\bigoplus_{n\ge 0}Hoch^n(A),\qquad Hoch^n(A)=\operatorname{Hom}_{\mathbb{K}}(\bigotimes^{n+1}A,A),$$

endowed with Gerstenhaber bracket is a graded Lie algebra. The **Hochschild differential** is defined as the linear map

$$d\colon Hoch^n(A)\to Hoch^{n+1}(A),\qquad d(f)=-[f,m].$$

In a more explicit form, for $f\in Hoch^n(A)$ we have

$$df(a_0,\ldots,a_{n+1})=a_0f(a_1,\ldots,a_{n+1})+(-1)^nf(a_0,\ldots,a_n)a_{n+1}\\ -\sum_{i=0}^{n}(-1)^if(a_0,\ldots,a_{i-1},a_ia_{i+1},a_{i+2},\ldots,a_{n+1}).$$

Setting

$$\delta\colon Hoch^n(A)\to Hoch^{n+1}(A),\qquad \delta(f)=[m,f]=(-1)^nd(f),$$

Jacobi identity gives:

(1) $\delta^2(f)=[m,[m,f]]=\frac{1}{2}[[m,m],f]=0$ since m is associative and then $[m,m]=0$,
(2) $\delta[f,g]=[\delta f,g]+(-1)^{\overline{f}}[f,\delta g]$.

Therefore the triple $(Hoch^*(A),\delta,[,])$ is a differential graded Lie algebra.

Example 7.4.4 (Derivations)**.** Let A be a DG-algebra over the field $\mathbb{K}$.

Definition 7.4.5. The DGLA of derivations of a DG-algebra A is $\operatorname{Der}^*_{\mathbb{K}}(A,A)=\bigoplus_n \operatorname{Der}^n_{\mathbb{K}}(A,A)$, where $\operatorname{Der}^n_{\mathbb{K}}(A,A)$ is the space of derivations of degree n defined as

$$\operatorname{Der}^n_{\mathbb{K}}(A,A)=\{\phi\in\operatorname{Hom}^n_{\mathbb{K}}(A,A)\mid \phi(ab)=\phi(a)b+(-1)^{n\overline{a}}a\phi(b)\}.$$

In particular the differential of A is a derivation of degree $+1$. It is easy to prove that derivations are closed under graded commutator and then $\operatorname{Der}^*_{\mathbb{K}}(A,A)$ is a DG-Lie subalgebra of $\operatorname{Hom}^*_{\mathbb{K}}(A,A)$.

Similarly, if L is a DGLA, then $\operatorname{Der}^*_{\mathbb{K}}(L,L)=\bigoplus_n \operatorname{Der}^n_{\mathbb{K}}(L,L)$, where

$$\operatorname{Der}^n_{\mathbb{K}}(L,L)=\{\phi\in\operatorname{Hom}^n_{\mathbb{K}}(L,L)\mid \phi[a,b]=[\phi(a),b]+(-1)^{n\overline{a}}[a,\phi(b)]\}$$

is a DG-Lie subalgebra of $\operatorname{Hom}^*_{\mathbb{K}}(L,L)$.

Example 7.4.6 (Differential operators)**.** Let A be a DG-algebra over the field $\mathbb{K}$ with unit $1\in A^0$. We may consider A as an abelian DG-Lie subalgebra of $\operatorname{Hom}^*_{\mathbb{K}}(A,A)$, where every $a\in A$ is identified with the operator

$$a\colon A\to A,\qquad a(b)=ab.$$

For every integer k we will denote by

$$\operatorname{Diff}_k(A)=\bigoplus_{n\in\mathbb{Z}}\operatorname{Diff}^n_k(A)\subset\operatorname{Hom}^*_{\mathbb{K}}(A,A)$$

the graded subspace of differential operators of order $\le k$: it is defined recursively by setting $\operatorname{Diff}_k(A)=0$ for $k<0$ and

$$\operatorname{Diff}_k(A)=\{f\in\operatorname{Hom}_{\mathbb{K}}(A,A)\mid [f,a]\in\operatorname{Diff}_{k-1}(A)\ \forall a\in A\}$$

for $k\ge 0$. Notice that $f\in\operatorname{Diff}_0(A)$ if and only if $f(a)=f(1)a$ and every derivation belongs to $\operatorname{Diff}_1(A)$.

A very simple induction on $h+k$ gives that

$$\operatorname{Diff}_k(A)\operatorname{Diff}_h(A)\subset\operatorname{Diff}_{h+k}(A),\qquad [\operatorname{Diff}_k(A),\operatorname{Diff}_h(A)]\subset\operatorname{Diff}_{h+k-1}(A).$$

In particular, the spaces $\operatorname{Diff}_1(A)$ and $\operatorname{Diff}(A)=\bigcup_k\operatorname{Diff}_k(A)$ are DG-Lie subalgebras of $\operatorname{Hom}^*_{\mathbb{K}}(A,A)$.

7.5. Maurer-Cartan equation and gauge action

Definition 7.5.1. The **Maurer-Cartan equation** (also called the deformation equation) of a DGLA L is

$$da+\frac{1}{2}[a,a]=0,\qquad a\in L^1.$$

The solutions of the Maurer-Cartan equation are called the Maurer-Cartan elements of the DGLA L. The set of such solutions will be denoted $MC(L)\subset L^1$.

It is plain that Maurer-Cartan equation commutes with morphisms of differential graded Lie algebras.

The notion of nilpotent Lie algebra extends naturally to the differential graded case; in particular for every DGLA L and every proper ideal I of a local artinian $\mathbb{K}$-algebra the DGLA $L\otimes I$ is nilpotent.

Assume now that L is a nilpotent DGLA, in particular L^0 is a nilpotent Lie algebras and we can consider its exponential group $\exp(L^0)$. By Jacobi identity, for every $a\in L^0$ the corresponding adjoint operator

$$\operatorname{ad}a\colon L\to L,\qquad (\operatorname{ad}a)b=[a,b],$$

is a nilpotent derivation of degree 0 and then its exponential

$$e^{\operatorname{ad}a}\colon L\to L,\qquad e^{\operatorname{ad}a}(b)=\sum_{n\ge 0}\frac{(\operatorname{ad}a)^n}{n!}(b),$$

is an isomorphism of graded Lie algebras, i.e. for every $b,c\in L$ we have

$$e^{\operatorname{ad}a}([b,c])=[e^{\operatorname{ad}a}(b),e^{\operatorname{ad}a}(c)].$$

In particular the quadratic cone $\{b\in L^1\mid [b,b]=0\}$ is stable under the adjoint action of $\exp(L^0)$.

The **gauge action** is a derived from the adjoint action via the next construction. Given a DGLA $(L,[,],d)$ we can construct a new DGLA $(L',[,]',d')$ by setting $(L')^i=L^i$ for every $i\neq 1$, $(L')^1=L^1\oplus\mathbb{K}\,d$ (here d is considered as a formal symbol of degree 1) with the bracket and the differential

$$[a+vd,b+wd]'=[a,b]+vd(b)+(-1)^{\overline{a}}wd(a),\qquad d'(a+vd)=[d,a+vd]'=d(a).$$

Since $(L')^{[n]}\subset L^{[n-1]}+dL^{[n-2]}$ for every $n\geq 3$, if L is nilpotent, then also L' is nilpotent.

The natural inclusion $L\subset L'$ is a morphism of DGLA; denote by ϕ the affine embedding $\phi\colon L^1\to(L')^1$, $\phi(x)=x+d$. The image of ϕ is stable under the adjoint action and then it makes sense the following definition.

Definition 7.5.2. Let L be a nilpotent DGLA. The **gauge action** of $\exp(L^0)$ on L^1 is defined as

$$e^a*x=\phi^{-1}(e^{\operatorname{ad}a}(\phi(x)))=e^{\operatorname{ad}a}(x+d)-d.$$

Explicitely

$$\begin{aligned}e^a*x&=\sum_{n\geq 0}\frac{1}{n!}(\operatorname{ad}a)^n(x)+\sum_{n\geq 1}\frac{1}{n!}(\operatorname{ad}a)^n(d)\\&=\sum_{n\geq 0}\frac{1}{n!}(\operatorname{ad}a)^n(x)-\sum_{n\geq 1}\frac{1}{n!}(\operatorname{ad}a)^{n-1}(da)\\&=w+\sum_{n\geq 0}\frac{(\operatorname{ad}a)^n}{(n+1)!}([a,x]-da).\end{aligned}$$

Lemma 7.5.3. *Let L be a nilpotent DGLA, then:*

(1) *the set of Maurer-Cartan solutions is stable under the gauge action;*
(2) *$e^a*x=x$ if and only if $[x,a]+da=0$;*
(3) *for every $x\in\operatorname{MC}(L)$ and every $u\in L^{-1}$ we have $e^{[x,u]+du}*x=x$.*

PROOF. [1] For an element $a\in L^1$ we have

$$d(a)+\frac{1}{2}[a,a]=0\qquad\iff\qquad[\phi(a),\phi(a)]'=0$$

and the quadratic cone $\{b+d\in(L^1)'\mid[b+d,b+d]'=0\}$ is stable under the adjoint action of $\exp(L^0)$.

[2] Since $\operatorname{ad}a$ is nilpotent, the operator $\sum_{n\geq 0}\dfrac{(\operatorname{ad}a)^n}{(n+1)!}=\dfrac{e^{\operatorname{ad}a}-1}{\operatorname{ad}a}$ is invertible.

[3] Setting $a=[x,u]+du$ we have

$$[x,a]+da=[x,[x,u]]+[x,du]+d[x,u]=\frac{1}{2}[[x,x],u]+[dx,u]=0.$$

□

Remark 7.5.4. For every $a\in L^0$, $x\in L^1$, the polynomial $\gamma(t)=e^{ta}*x\in L^1\otimes\mathbb{K}[t]$ is the solution of the "Cauchy problem"

$$\begin{cases}\dfrac{d\gamma(t)}{dt}=[a,\gamma(t)]-da\\ \gamma(0)=x\end{cases}$$

7.6. Deformation functors associated to a DGLA

In order to introduce the basic ideas of the use of DGLA in deformation theory we begin with an example where technical difficulties are reduced at minimum. Consider a finite complex of vector spaces

$$(V,\overline{\partial}):\qquad 0\longrightarrow V^0\xrightarrow{\overline{\partial}}V^1\xrightarrow{\overline{\partial}}\cdots\xrightarrow{\overline{\partial}}V^n\longrightarrow 0.$$

Given a local artinian $\mathbb{K}$-algebra A with maximal ideal $\mathfrak{m}_A$ and residue field $\mathbb{K}$, we define a deformation of $(V,\overline{\partial})$ over A as a complex of A-modules of the form

$$0\longrightarrow V^0\otimes A\xrightarrow{\overline{\partial}_A}V^1\otimes A\xrightarrow{\overline{\partial}_A}\cdots\xrightarrow{\overline{\partial}_A}V^n\otimes A\longrightarrow 0$$

such that its residue modulo $\mathfrak{m}_A$ gives the complex $(V, \overline{\partial})$. Since, as a $\mathbb{K}$ vector space, $A = \mathbb{K} \oplus \mathfrak{m}_A$, this last condition is equivalent to say that

$$\overline{\partial}_A = \overline{\partial} + \xi, \qquad \text{where} \quad \xi \in \operatorname{Hom}^1(V, V) \otimes \mathfrak{m}_A.$$

The "integrability" condition $\overline{\partial}_A^2 = 0$ becomes

$$0 = (\overline{\partial} + \xi)^2 = \overline{\partial}\xi + \xi\overline{\partial} + \xi^2 = d\xi + \frac{1}{2}[\xi, \xi],$$

where d and $[\ ,\ ]$ are the differential and the bracket on the differential graded Lie algebra $\operatorname{Hom}^*(V, V) \otimes \mathfrak{m}_A$.

Two deformations $\overline{\partial}_A, \overline{\partial}'_A$ are isomorphic if there exists a commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & V^0 \otimes A & \xrightarrow{\overline{\partial}_A} & V^1 \otimes A & \xrightarrow{\overline{\partial}_A} & \cdots & \xrightarrow{\overline{\partial}_A} & V^n \otimes A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi_1} & & & & \downarrow{\scriptstyle \phi_n} & & \\
0 & \longrightarrow & V^0 \otimes A & \xrightarrow{\overline{\partial}'_A} & V^1 \otimes A & \xrightarrow{\overline{\partial}'_A} & \cdots & \xrightarrow{\overline{\partial}'_A} & V^n \otimes A & \longrightarrow & 0
\end{array}$$

such that every ϕ_i is an isomorphism of A-modules whose specialization to the residue field is the identity. Therefore we can write $\phi := \sum_i \phi_i = Id + \eta$, where $\eta \in \operatorname{Hom}^0(V, V) \otimes \mathfrak{m}_A$ and, since $\mathbb{K}$ is assumed of characteristic 0 we can take the logarithm and write $\phi = e^a$ for some $a \in \operatorname{Hom}^0(V, V) \otimes \mathfrak{m}_A$. The commutativity of the diagram is therefore given by the equation $\overline{\partial}'_A = e^a \circ \overline{\partial}_A \circ e^{-a}$. Writing $\overline{\partial}_A = \overline{\partial} + \xi$, $\overline{\partial}'_A = \overline{\partial} + \xi'$ and using the relation $e^a \circ b \circ e^{-a} = e^{\operatorname{ad} a}(b)$ we get

$$\xi' = e^{\operatorname{ad} a}(\overline{\partial} + \xi) - \overline{\partial} = \xi + \frac{e^{\operatorname{ad} a} - 1}{\operatorname{ad} a}([a, \xi] + [a, \overline{\partial}]) = \xi + \sum_{n=0}^{\infty} \frac{\operatorname{ad} a^n}{(n+1)!}([a, \xi] - da).$$

In particular both the integrability condition and isomorphism are entirely written in terms of the DGLA structure of $\operatorname{Hom}^*(V, V) \otimes \mathfrak{m}_A$. This leads to the following general construction.

Denote by **Art** the category of local artinian $\mathbb{K}$-algebras with residue field $\mathbb{K}$ and by **Set** the category of sets (we ignore all the set-theoretic problems, for example by restricting to some universe). Unless otherwise specified, for every objects $A \in$ **Art** we denote by $\mathfrak{m}_A$ its maximal ideal.

Let $L = \oplus L^i$ be a DGLA over $\mathbb{K}$, we can define the following three functors:

(1) The exponential functor $\exp_L \colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Grp}$,

$$\exp_L(A) = \exp(L^0 \otimes \mathfrak{m}_A).$$

It is immediate to see that $\exp_L$ is smooth.

(2) The Maurer-Cartan functor $MC_L \colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ defined by

$$MC_L(A) = MC(L \otimes \mathfrak{m}_A) = \left\{ x \in L^1 \otimes \mathfrak{m}_A \;\middle|\; dx + \frac{1}{2}[x, x] = 0 \right\}.$$

(3) The gauge action of the group $\exp(L^0 \otimes \mathfrak{m}_A)$ on the set $MC(L \otimes \mathfrak{m}_A)$ is functorial in A and gives an action of the group functor $\exp_L$ on the Maurer-Cartan functor MC_L. The quotient functor $\operatorname{Def}_L = MC_L/G_L$ is called the **deformation functor** associated to the DGLA L. For every $A \in \mathbf{Art}_{\mathbb{K}}$ we have

$$\operatorname{Def}_L(A) = \frac{MC(L \otimes \mathfrak{m}_A)}{\exp(L^0 \otimes \mathfrak{m}_A)} = \frac{\left\{ x \in L^1 \otimes \mathfrak{m}_A \mid dx + \frac{1}{2}[x, x] = 0 \right\}}{\text{gauge action}}.$$

Both MC_L and Def_L are deformation functors in the sense of Definition 4.1.5; in fact MC_L commutes with fiber products, while, according to Proposition 4.2.7 Def_L is a deformation functor and the projection $\mathrm{MC}_L \to \operatorname{Def}_L$ is smooth.

The reader should make attention to the difference between the deformation functor Def_L *associated to a DGLA* L *and the functor of deformations of a DGLA* L.

Lemma 7.6.1. *If* $L \otimes \mathfrak{m}_A$ *is abelian then* $\operatorname{Def}_L(A) = H^1(L) \otimes \mathfrak{m}_A$. *In particular*

$$T^1\operatorname{Def}_L = \operatorname{Def}_L(\mathbb{K}[\epsilon]) = H^1(L) \otimes \mathbb{K}\epsilon, \qquad \epsilon^2 = 0.$$

Proof. The Maurer-Cartan equation reduces to $dx = 0$ and then $\mathrm{MC}_L(A) = Z^1(L) \otimes \mathfrak{m}_A$. If $a \in L^0 \otimes \mathfrak{m}_A$ and $x \in L^1 \otimes \mathfrak{m}_A$ we have

$$e^a * x = x + \sum_{n \geq 0} \frac{ad(a)^n}{(n+1)!}([a,x] - da) = x - da$$

and then $\mathrm{Def}_L(A) = \dfrac{Z^1(L) \otimes \mathfrak{m}_A}{d(L^0 \otimes \mathfrak{m}_A)} = H^1(L) \otimes \mathfrak{m}_A$. □

It is clear that every morphism $\alpha \colon L \to N$ of DGLA induces morphisms of functors $G_L \to G_N$, $MC_L \to MC_N$. These morphisms are compatible with the gauge actions and therefore induce a morphism between the deformation functors $\mathrm{Def}_\alpha \colon \mathrm{Def}_L \to \mathrm{Def}_N$.

Obstructions for MC_L and Def_L. . Let L be a differential graded Lie algebra. We want to show that MC_L has a "natural" obstruction theory $(H^2(L), v_e)$.

Let's consider a small extension in **Art**

$$e\colon \qquad 0 \longrightarrow M \longrightarrow A \longrightarrow B \longrightarrow 0$$

and let $x \in \mathrm{MC}_L(B) = \{x \in L^1 \otimes \mathfrak{m}_B \mid dx + \frac{1}{2}[x,x] = 0\}$; we define an obstruction $v_e(x) \in H^2(L \otimes M) = H^2(L) \otimes M$ in the following way: first take a lifting $\tilde{x} \in L^1 \otimes \mathfrak{m}_A$ of x and consider $h = d\tilde{x} + \frac{1}{2}[\tilde{x}, \tilde{x}] \in L^2 \otimes M$; we have

$$dh = d^2\tilde{x} + [d\tilde{x}, \tilde{x}] = [h, \tilde{x}] - \frac{1}{2}[[\tilde{x}, \tilde{x}], \tilde{x}].$$

Since $[L^2 \otimes M, L^1 \otimes \mathfrak{m}_A] = 0$ we have $[h, \tilde{x}] = 0$, by Jacobi identity $[[\tilde{x}, \tilde{x}], \tilde{x}] = 0$ and then $dh = 0$. Define $v_e(x)$ as the class of h in $H^2(L \otimes M) = H^2(L) \otimes M$; the first thing to prove is that $v_e(x)$ is independent from the choice of the lifting $\tilde{x}$; every other lifting is of the form $y = \tilde{x} + z$, $z \in L^1 \otimes M$ and then

$$d\tilde{y} + \frac{1}{2}[y, y] = h + dz.$$

It is evident from the above computation that $(H^2(L), v_e)$ is a complete obstruction theory for the functor MC_L.

Lemma 7.6.2. *The complete obstruction theory described above for the functor MC_L is invariant under the gauge action and then it is also a complete obstruction theory for Def_L.*

Proof. Since the projection $\mathrm{MC}_L \to \mathrm{Def}_L$ is smooth, it is sufficient to apply the general properties of universal obstruction theories. It is instructive to give also a direct and elementary proof of this lemma. Let x, y be two gauge equivalent solutions of the Maurer-Cartan equation in $L \otimes \mathfrak{m}_B$ and let $\tilde{x} \in L^1 \otimes \mathfrak{m}_A$ be a lifting of x. It is sufficient to prove that there exists a lifting $\tilde{y}$ of y such that

$$h := d\tilde{x} + \frac{1}{2}[\tilde{x}, \tilde{x}] = d\tilde{y} + \frac{1}{2}[\tilde{y}, \tilde{y}].$$

Let $a \in L^0 \otimes \mathfrak{m}_B$ be such that $e^a * x = y$, choose a lifting $\tilde{a} \in L^0 \otimes \mathfrak{m}_A$ and define $e^{\tilde{a}} * \tilde{x} = \tilde{y}$. We have

$$d\tilde{y} + \frac{1}{2}[\tilde{y}, \tilde{y}] = [\tilde{y} + d, \tilde{y} + d]' = [e^{\mathrm{ad}\,\tilde{a}}(\tilde{x} + d), e^{\mathrm{ad}\,\tilde{a}}(\tilde{x} + d)]' = e^{\mathrm{ad}\,\tilde{a}}[\tilde{x} + d, \tilde{x} + d]' = e^{\mathrm{ad}\,\tilde{a}}(h) = h.$$

□

Finally, it is clear that every morphism of differential graded Lie algebras $f \colon L \to M$ induces natural transformations of functors

$$f \colon \mathrm{MC}_L \to \mathrm{MC}_M, \qquad f \colon \mathrm{Def}_L \to \mathrm{Def}_M.$$

7.7. Semicosimplicial groupoids

For reader convenience we recall some basic notion of category theory. For simplicity of notation, if $\mathbf{C}$ is a category we shall write $x \in \mathbf{C}$ if x is an object and $f \in \mathrm{Mor}_{\mathbf{C}}$ if f is a morphism.

Definition 7.7.1. A **small category** is a category whose class of objects is a set.

Example 7.7.2. The category of finite ordinals $\mathbf{\Delta}$ is a small category.

Definition 7.7.3. A **groupoid** is a small category such that every morphism is an isomorphism. We will denote by **Grpd** the category of groupoids.

Notice that, for a groupoid G and every object $g \in G$ the set $\mathrm{Mor}_G(g,g)$ is a group.

Example 7.7.4. The fundamental groupoid $\pi_{\leq 1}(X)$ of a topological space X is defined in the following way: the set of objects is X, while the morphisms are the continuos path up to homotopy of paths.

Definition 7.7.5. For a groupoid G we well denote by $\pi_0(G)$ the set of isomorphism classes of objects of G and, for every $g \in G$ we will denote $\pi_1(G,g) = \mathrm{Mor}_G(g,g)$.

It is clear that every equivalence of groupoids $f\colon G \to E$ induce a bijection $f\colon \pi_0(G) \to \pi_0(E)$.

Definition 7.7.6. Let

$$G^{\Delta}: \qquad G_0 \rightrightarrows G_1 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; G_2 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \cdots$$

be a semicosimplicial groupoid with face maps $\partial_i\colon G_n \to G_{n+1}\colon \partial_l\partial_k = \partial_{k+1}\partial_l$, for any $l \leq k$.

The **total groupoid** $\mathrm{Tot}(G^{\Delta})$ is defined in the following way (cf. [**25**, **46**, **55**]):

(1) The objects of $\mathrm{Tot}(G^{\Delta})$ are the pairs (l,m) with $l \in G_0$ and m is a morphism in G_1 between $\partial_0 l$ and $\partial_1 l$ such that the diagram

$$\begin{array}{ccccc} \partial_0\partial_0 l & \xrightarrow{\partial_0 m} & \partial_0\partial_1 l & = & \partial_2\partial_0 l \\ \Vert & & & & \downarrow{\scriptstyle \partial_2 m} \\ \partial_1\partial_0 l & \xrightarrow{\partial_1 m} & \partial_1\partial_1 l & = & \partial_2\partial_1 l \end{array}$$

is commutative in G_2.

(2) The morphisms between (l_0,m_0) and (l_1,m_1) are morphisms a in G_0 between l_0 and l_1 making the diagram

$$\begin{array}{ccc} \partial_0 l_0 & \xrightarrow{m_0} & \partial_1 l_0 \\ {\scriptstyle \partial_0 a}\downarrow & & \downarrow{\scriptstyle \partial_1 a} \\ \partial_0 l_1 & \xrightarrow{m_1} & \partial_1 l_1 \end{array}$$

commutative in G_1.

Example 7.7.7. caso delle SCLA. Da scrivere

7.8. Deligne groupoids

Definition 7.8.1. Let L be a nilpotent differential graded Lie algebra. The **Deligne groupoid** of L is the groupoid $\mathrm{Del}(L)$ defined as follows:

(1) the set of objects is $\mathrm{MC}(L)$,
(2) the morphisms are

$$\mathrm{Mor}_{\mathrm{Del}(L)}(x,y) = \{e^a \in \exp(L^0) \mid e^a * x = y\}, \qquad x,y \in \mathrm{MC}(L).$$

Definition 7.8.2. Let L be a nilpotent differential graded Lie algebra. The **irrelevant stabilizer** of a Maurer-Cartan element $x \in \mathrm{MC}(L)$ is defined as the subgroup (see Lemma 7.5.3):

$$I(x) = \{e^{du+[x,u]} \mid u \in L^{-1}\} \subset \mathrm{Mor}_{\mathrm{Del}(L)}(x,x).$$

Lemma 7.8.3. *Let L be a nilpotent differential graded Lie algebra, $a \in L^0$ and $x \in \mathrm{MC}(L)$. Then*
$$e^a I(x) e^{-a} = I(y), \qquad \text{where} \qquad y = e^a * x.$$
In particular $I(x)$ is a normal subgroup of $\mathrm{Mor}_{\mathrm{Del}(L)}(x,x)$ and there exists a natural isomorphism
$$\frac{\mathrm{Mor}_{\mathrm{Del}(L)}(x,y)}{I(x)} = \frac{\mathrm{Mor}_{\mathrm{Del}(L)}(x,y)}{I(y)},$$
with $I(x)$ and $I(y)$ acting in the obvious way.

Proof. Recall that for every $a, b \in L^0$ we have
$$e^a e^b e^{-a} = e^{\operatorname{ad} a}(e^b) = e^c, \qquad \text{where} \quad c = e^{\operatorname{ad} a}(b).$$
and then $e^a e^{[x,u]+du} e^{-a} = e^c$, where, setting $v = e^{\operatorname{ad} a}(u)$, we have
$$c = e^{\operatorname{ad} a}([x,u] + du) = e^{\operatorname{ad} a}([x+d,u]') = [e^{\operatorname{ad} a}(x+d), v]' = [y,v] + dv.$$
□

Definition 7.8.4 ([55, 68, 125])**.** The **reduced Deligne groupoid** of a nilpotent differential graded Lie algebra L is the groupoid $\overline{\mathrm{Del}}(L)$ having as objects the Maurer-Cartan elements of L and as morphisms
$$\mathrm{Mor}_{\overline{\mathrm{Del}}(L)}(x,y) := \frac{\mathrm{Mor}_{\mathrm{Del}(L)}(x,y)}{I(x)} = \frac{\mathrm{Mor}_{\mathrm{Del}(L)}(x,y)}{I(y)}, \qquad x, y \in \mathrm{MC}(L).$$

In order to verify that $\overline{\mathrm{Del}}(L)$ is a groupoid we only need to verify that the (associative) multiplication map
$$\mathrm{Mor}_{\mathrm{Del}(L)}(y,z) \times \mathrm{Mor}_{\mathrm{Del}(L)}(x,y) \to \mathrm{Mor}_{\mathrm{Del}(L)}(x,z), \qquad (e^a, e^b) \mapsto e^a e^b,$$
factors to a morphism
$$\frac{\mathrm{Mor}_{\mathrm{Del}(L)}(y,z)}{I(y)} \times \frac{\mathrm{Mor}_{\mathrm{Del}(L)}(x,y)}{I(x)} \to \frac{\mathrm{Mor}_{\mathrm{Del}(L)}(x,z)}{I(x)}\ ,$$
and this follows immediately from Lemma 7.8.3

Notice that $\pi_0(\mathrm{Del}(L)) = \pi_0(\overline{\mathrm{Del}}(L)) = \mathrm{Def}(L)$.

Every cosimplicial nilpotent DGLA gives a semicosimplicial reduced Deligne groupoid and then a total groupoid. Here we are interested to a particular case of this construction.

Definition 7.8.5. Given a pair of morphisms $h, g \colon L \to M$ of nilpotent differential graded Lie algebras define
$$\overline{\mathrm{Del}}(h,g) = \mathrm{Tot}\left(\overline{\mathrm{Del}}(L) \underset{g}{\overset{h}{\rightrightarrows}} \overline{\mathrm{Del}}(M) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; 0 \cdots \right), \qquad \mathrm{Def}(h,g) = \pi_0(\overline{\mathrm{Del}}(h,g)).$$

Lemma 7.8.6. *For any pair of morphisms $h, g \colon L \to M$ of nilpotent differential graded Lie algebras we have*
$$\mathrm{Def}(h,g) = \frac{\left\{(x, e^a) \in \mathrm{MC}(L) \times \exp(M^0) \;\middle|\; e^a * h(x) = g(x)\right\}}{\sim},$$
where $(x, e^a) \sim (y, e^b)$ if there exists $\alpha \in L^0$ such that $e^\alpha * x = y$ and the diagram
$$\begin{array}{ccc} h(x) & \xrightarrow{e^a} & g(x) \\ \Big\downarrow{\scriptstyle e^{h(\alpha)}} & & \Big\downarrow{\scriptstyle e^{g(\alpha)}} \\ h(y) & \xrightarrow{e^b} & g(y) \end{array}$$
is commutative in the reduced Deligne groupoid of M.

Proof. Immediate from definition. □

Notice that for $M = 0$ we reobtain the usual set $\mathrm{Def}(L)$.

Given a pair of morphisms $h, g \colon L \to M$ of differential graded Lie algebras, for every $A \in \mathbf{Art}$ we get a pair of morphisms of nilpotent DGLA $h, g \colon L \otimes \mathfrak{m}_A \to M \otimes \mathfrak{m}_A$ and therefore we are in the position to define in the obvius way the functor
$$\mathrm{Def}_{(h,g)} \colon \mathbf{Art} \to \mathbf{Set}.$$

Proposition 7.8.7. *In the notation above,* $\mathrm{Def}_{(h,g)}$ *is a deformation functor with tangent and obstruction spaces equal to* $H^1(C_{h-g})$ *and* $H^2(C_{h-g})$ *respectively, being* C_{h-g} *the suspended mapping cone of the morphism of DG-vector spaces* $h-g\colon L\to M$*.*

Proof. The fact that $\mathrm{Def}_{(h,g)}$ is a deformation functor follows from Proposition 4.2.7, while it is straighforward to prove the equality $T^1\mathrm{Def}_{(h,g)}=H^1(C_{h-g})$. We now compute obstructions using the description given in Lemma 7.8.6. Let

$$0\to I\to A\to B\to 0$$

be a small extension and $(\hat{x},e^{\hat{a}})\in \mathrm{MC}(L\otimes\mathfrak{m}_B)\times\exp(M^0\otimes\mathfrak{m}_B)$ be such that $e^{\hat{a}}*h(\hat{x})=g(\hat{x})$. Choose a lifting $(x,e^a)\in L^1\otimes\mathfrak{m}_A\times\exp(M^0\otimes\mathfrak{m}_A)$ and consider the elements

$$r=dx+\frac{1}{2}[x,x]\in L^2\otimes I,\qquad s=e^a*h(x)-g(x)\in M^1\otimes I,\qquad t=(r,s)\in C^2_{h-g}\otimes I.$$

We first prove that $dt=0$; we already know that $dr=0$; since

$$\begin{aligned}g(r)&=dg(x)+\frac{1}{2}[g(x),g(x)]=d(e^a*h(x))-ds+\frac{1}{2}[e^a*h(x),e^a*h(x)]\\&=\frac{1}{2}[e^a*h(x)+d,e^a*h(x)+d]'-ds=\frac{1}{2}[e^{\mathrm{ad}\,a}(h(x)+d),e^{\mathrm{ad}\,a}(h(x)+d)]'-ds\\&=\frac{1}{2}e^{\mathrm{ad}\,a}[h(x)+d,h(x)+d]'-ds=e^{\mathrm{ad}\,a}h(r)+ds=h(r)-ds.\end{aligned}$$

we have $(h-g)r-ds=0$ and then t is a cocycle in C_{h-g}.

If x is replaced with $x+u$, $u\in L^1\otimes I$ and a is replaced with $a+v$, $v\in M^0\otimes I$, the element (r,s) will be replaced with $(r+du,s+(h-g)u-dv)$. This implies that the cohomology class of t in $H^2(C_{h-g})\otimes I$ is well defined and is a complete obstruction. □

7.9. Homotopy invariance of deformation functors

We shall say that a functor $\mathrm{F}\colon \mathbf{DGLA}\to\mathbf{C}$ is homotopy invariant if for every quasiisomorphism f of DGLA the morphism $\mathrm{F}(f)$ is an isomorphism in the category $\mathbf{C}$. The main theme of this chapter is to prove that the functor

$$\mathrm{Def}\colon \mathbf{DGLA}\to\{\text{Deformation functors}\}$$

is homotopy invariant.

We have already pointed out that every morphism $f\colon L\to N$ of DGLA induces a morphism of associated deformation functors $f\colon \mathrm{Def}_L\to\mathrm{Def}_N$.

Theorem 7.9.1. *Let* $f\colon L\to N$ *be a morphism of differential graded Lie algebras. Assume that the morphism* $f\colon H^i(L)\to H^i(N)$ *is:*

(1) *surjective for* $i=0$*,*
(2) *bijective for* $i=1$*,*
(3) *injective for* $i=2$*.*

Then $f\colon \mathrm{Def}_L\to\mathrm{Def}_N$ *is an isomorphism of functors.*

Corollary 7.9.2. *Let* $L\to N$ *be a quasiisomorphism of DGLA. Then the induced morphism* $\mathrm{Def}_L\to\mathrm{Def}_N$ *is an isomorphism.*

In this chapter we give a proof of the above results that uses obstruction theory and the standard smoothness criterion for deformation functors (Theorem 4.5.12). Before doing this we need some preliminary results of independent interest.

One of the most frequent wrong interpretations of Corollary 7.9.2 asserts that if $L\to N$ is a quasiisomorphism of nilpotent DGLA then $\mathrm{MC}(L)/\exp(L^0)\to\mathrm{MC}(N)/\exp(N^0)$ is a bijection. This is false in general: consider for instance $L=0$ and $N=\oplus N^i$ with $N^i=\mathbb{C}$ for $i=1,2$, $N^i=0$ for $i\neq 1,2$, $d\colon N^1\to N^2$ the identity and $[a,b]=ab$ for $a,b\in N^1=\mathbb{C}$.

Lemma 7.9.3. *Let $f\colon L \to N$ be a morphism of differential graded Lie algebras. If $f\colon H^1(L) \to H^1(N)$ is surjective and $f\colon H^2(L) \to H^2(N)$ is injective, then the morphism $f\colon \mathrm{Def}_L \to \mathrm{Def}_N$ is smooth.*

PROOF. Since $H^1(L)$ is the tangent space of Def_L and $H^2(L)$ is a complete obstruction space, it is sufficient to apply the standard smoothness criterion. □

Example 7.9.4. Let $L = \oplus L^n$ be a DGLA such that $[L^1, L^1] \cap Z^2(L) \subset B^2(L)$. Then Def_L is smooth. In fact, consider the differential graded Lie subalgebra $N = \oplus N^i \subset L$ where:

(1) $N^i = 0$ for every $i \le 0$,
(2) $N^1 = L^1$,
(3) $N^2 = [L^1, L^1] + B^2(L)$,
(4) $N^i = L^i$ for every $i > 2$.

By assumption $H^2(N) = 0$ and then Def_N is smooth. Since $H^1(N) \to H^1(L)$ is surjective, the morphism $\mathrm{Def}_N \to \mathrm{Def}_L$ is smooth.

Example 7.9.5. Let $L = \oplus L^i$ be a DGLA and choose a vector space decomposition $N^1 \oplus B^1(L) = L^1$.

Consider the DGLA $N = \oplus N^i$ where $N^i = 0$ if $i < 1$ and $N^i = L^i$ if $i > 1$ with the differential and bracket induced by L. The natural inclusion $N \to L$ gives isomorphisms $H^i(N) \to H^i(L)$ for every $i \ge 1$. In particular the morphism $\mathrm{Def}_N \to \mathrm{Def}_L$ is smooth and induce an isomorphism on tangent spaces $T^1\mathrm{Def}_N = T^1\mathrm{Def}_L$.

Let now $f\colon L \to M$ be a fixed morphism of differential graded Lie algebras and denote by $p_0, p_1\colon L \times L \to L$ the projections.

The commutative diagram of differential graded Lie algebras

$$\begin{array}{ccccc} L & \xleftarrow{p_0} & L \times L & \xrightarrow{p_1} & L \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle f} \\ M & \xleftarrow[f p_0]{} & L \times L & \xrightarrow[f p_1]{} & M \end{array}$$

induce a natural transformation of functors

$$\eta\colon \mathrm{Def}(p_0, p_1) \to \mathrm{Def}(fp_0, fp_1).$$

Lemma 7.9.6. *In the above set-up, if $f\colon H^0(L) \to H^0(M)$ is surjective and $f\colon H^1(L) \to H^1(M)$ is injective, then the morphism η is smooth.*

PROOF. According to Proposition 7.8.7 and standard smoothness criterion it is sufficient to prove that $f\colon H^1(C_{p_0-p_1}) \to H^1(C_{fp_0-fp_1})$ is surjective and $f\colon H^2(C_{p_0-p_1}) \to H^2(C_{fp_0-fp_1})$ is injective. This follows by a straighforward diagram chasing on the morphism of exact sequences

$$\begin{array}{ccccccccccc} H^0(L) & \to & H^1(C_{p_0-p_1}) & \to & H^1(L\times L) & \to & H^1(L) & \to & H^2(C_{p_0-p_1}) & \to & H^2(L\times L) \\ \downarrow{\scriptstyle f} & & \downarrow & & \| & & \downarrow{\scriptstyle f} & & \downarrow & & \| \\ H^0(M) & \to & H^1(C_{fp_0-fp_1}) & \to & H^1(L\times L) & \to & H^1(M) & \to & H^2(C_{fp_0-fp_1}) & \to & H^2(L\times L) \end{array}$$

□

PROOF OF THEOREM 7.9.1. Using the notation introduced above, we have already proved that the morphisms

$$f\colon \mathrm{Def}_L \to \mathrm{Def}_M, \qquad \eta\colon \mathrm{Def}(p_0, p_1) \to \mathrm{Def}(fp_0, fp_1)$$

are smooth. Given $A \in \mathbf{Art}$ we need to prove that if $x, y \in \mathrm{MC}_L(A)$ and there exists $b \in M^0 \otimes \mathfrak{m}_A$ such that $e^b * f(x) = f(y)$, then x is gauge equivalent to y. Using the notation of Lemma 7.8.6, since $(x, y, e^b) \in \mathrm{Def}(fp_0, fp_1)(A)$ and η is smooth, there exists $(u, v, e^a) \in \mathrm{Def}(p_0, p_1)(A)$ such that $\eta(u, v, e^a) = (x, y, e^b)$, i.e.

$$(u, v, e^{f(a)}) \sim (x, y, e^b)$$

and this implies in particular that there exists $\alpha \in (L^0 \times L^0) \otimes \mathfrak{m}_A$ such that

$$e^a * u = v, \quad e^\alpha * (u, v) = (x, y)\ ,$$

and then x, y are gauge equivalent. □

Definition 7.9.7. Let L be a DGLA and $x, y \in \mathrm{MC}(L)$. We shall say that x and y are **homotopy equivalent** if there is some $\xi \in \mathrm{MC}(L[t, dt])$ such that $e_0(\xi) = x$ and $e_1(\xi) = y$. Here $L[t, dt] = L \otimes \mathbb{K}[t, dt]$ and $e_0, e_1 \colon L[t, dt] \to L$ are the evaluation maps at $t = 0$ and $t = 1$ respectively.

We will denote by $\pi_0(\mathrm{MC}_\bullet(L))$ the quotient of $\mathrm{MC}(L)$ under the equivalence relation generated by homotopy.[1]

The construction of $\pi_0(\mathrm{MC}_\bullet)$ is functorial and then we may define a functor

$$\pi_0(\mathrm{MC}_\bullet)_L \colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}, \qquad \pi_0(\mathrm{MC}_\bullet)_L(A) = \pi_0(\mathrm{MC}(L \otimes \mathfrak{m}_A)).$$

Corollary 7.9.8. *For every differential graded Lie algebra L, the projection $\mathrm{MC}_L \to \pi_0(\mathrm{MC}_\bullet)_L$ factors to an isomorphism of functors $\mathrm{Def}_L \to \pi_0(\mathrm{MC}_\bullet)_L$.*

Proof. Let L be be DGLA, since the inclusion $L \to L[t, dt]$ is a quasiisomorphism the natural transformation $\mathrm{Def}_L \to \mathrm{Def}_{L[t,dt]}$ is an isomorphism. Given $A \in \mathbf{Art}$ and $x, y \in \mathrm{MC}(L \otimes \mathfrak{m}_A)$, it is sufficient to prove that x, y are gauge equivalent if and only if they are homotopy equivalent. Recall that x is gauge equivalent to y if there is some $a \in L^0 \otimes \mathfrak{m}_A$ such that $e^a * x = y$, whereas x is homotopy equivalent to y if there is some $z(t) \in \mathrm{MC}(L[t, dt] \otimes \mathfrak{m}_A)$ such that $z(0) = x$ and $z(1) = y$.

So first, assume $e^a * x = y$; then we can consider $x \in \mathrm{MC}(L \otimes \mathfrak{m}_A) \subset \mathrm{MC}(L[t, dt] \otimes \mathfrak{m}_A)$. Since $\exp(L^0[t] \otimes \mathfrak{m}_A)$ acts by gauge on $\mathrm{MC}(L[t, dt] \otimes \mathfrak{m}_A)$, for every t we can set $z(t) = e^{ta} * x$. Then $z(0) = x$ and $z(1) = y$.

On the other hand, notice that $\mathrm{MC}_L = \mathrm{MC}_{L^{\geq 1}}$ (in fact Maurer-Cartan only depends on L^1 and L^2), so $\mathrm{Def}_L = \mathrm{Def}_{L^{\geq 0}}$ and $\mathrm{MC}_{L[t,dt]} = \mathrm{MC}_{L^{\geq 0}[t,dt]}$. So it is not restrictive to assume that $L = \oplus_{n \geq 0} L^n$. In this case, $L[t, dt]^0 = L^0[t]$. Now let $z(t) \in \mathrm{MC}(L[t, dt] \otimes \mathfrak{m}_A)$. Then, as we have a smooth morphism $i \colon \mathrm{Def}_L(A) \to \mathrm{Def}_{L[t,dt]}(A)$, we must have some $x \in \mathrm{MC}_L(A)$ which is gauge equivalent to $z(t)$ in $L[t, dt] \otimes \mathfrak{m}_A$. So we have $a(t) \in L[t, dt]^0 \otimes \mathfrak{m}_A = L^0[t] \otimes \mathfrak{m}_A$ such that $e^{a(t)} * x = z(t)$. Now $z(0) = e^{a(0)} * x$ and $z(1) = e^{a(1)} * x$, and this imply that $z(0)$ is gauge equivalent to $z(1)$. □

Remark 7.9.9. The first consequences of Corollary 7.9.8 is that the bifunctor

$$\pi_0(\mathrm{MC}_\bullet) = \mathrm{Def} \colon \mathbf{DGLA} \times \mathbf{Art} \to \mathbf{Set}$$

is completely determined by the Maurer-Cartan bifunctor

$$\mathrm{MC} \colon \mathbf{DGLA} \times \mathbf{Art} \to \mathbf{Set}.$$

7.10. Exercises

Exercise 7.10.1. Let

$$G^\Delta : \qquad G_0 \rightrightarrows G_1 \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} G_2 \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots$$

be a semicosimplicial groupoid. Assume that for every i the natural map $G_i \to \pi_0(G_i)$ is an equivalence, i.e., every G_i is equivalent to a set. Then also $\mathrm{tot}(G^\Delta)$ is equivalent to a set and, more precisely, to the equalizer of the diagram of sets

$$\pi_0(G_0) \rightrightarrows \pi_0(G_1).$$

[1]Using this notation we have implicitely assumed that there exists a groupoid $\mathrm{MC}_\bullet(L)$ having $\mathrm{MC}(L)$ as objects and $\mathrm{MC}(L[t, dt])$ as morphisms. This is almost true, in the sense that there exists a natural structure of ∞-groupoid on $\mathrm{MC}_\bullet$: we will give the precise definition later on.