# FUNCTORS OF ARTIN RINGS, OBSTRUCTIONS AND FACTORIZATION

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In this chapter we collect some definitions and main properties of deformation functors.

In the first section, we introduce the notions of functor of Artin rings, of deformation functor and of the associated tangent and obstruction spaces. Moreover, we describe the deformations functor associated with a semicosimplicial Lie algebra.

The main references for this chapter are [2], [4], [7] and [8].

## 1. Basic definition

$\mathbb{K}$ is a field of characteristic 0. Denote by **Set** the category of sets in a fixed universe and by $\{*\}$ a fixed set of cardinality 1. Let $\mathbf{Art} = \mathbf{Art}_{\mathbb{K}}$ be the category of local Artinian $\mathbb{K}$-algebras with residue field $\mathbb{K}$ ($A/\mathfrak{m}_A = \mathbb{K}$); the morphisms in **Art** are local morphism.

We shall say that a morphism $\alpha\colon B \to A$ in **Art** is a *small surjection* if $\alpha$ is surjective and its kernel is annihilated by the maximal ideal $\mathfrak{m}_B$. The artinian property implies that every surjective morphism in **Art** can be decomposed in a finite sequence of small surjections and then a functor $F$ is smooth if and only if $F(B) \to F(A)$ is surjective for every small surjection $B \to A$.

A *small extension* is a small surjection together a framing of its kernel. More precisely a small extension $e$ in **Art** is an exact sequence of abelian groups

$$e\colon \quad 0 \longrightarrow M \longrightarrow B \xrightarrow{\alpha} A \longrightarrow 0,$$

such that $\alpha$ is a morphism in the category **Art** and $M$ is an ideal of $B$ annihilated by the maximal ideal $\mathfrak{m}_B$. In particular $M$ is a finite dimensional vector space over $B/\mathfrak{m}_B = \mathbb{K}$. A small extension as above is called *principal* if $M = \mathbb{K}$.

**Definition 1.1.** A *functor of Artin rings* is a covariant functor $F\colon \mathbf{Art} \to \mathbf{Set}$ such that $F(\mathbb{K}) = \{*\}$.

The functors of Artin rings are the objects of a category whose morphisms are the natural transformations of functors. For simplicity of notation, if $\phi\colon F \to G$ is a natural transformation, we denote by $\phi\colon F(A) \to G(A)$ the corresponding morphism of sets, for every $A \in \mathbf{Art}$.

**Example 1.2.** The *trivial* functor $*$ is the functor defined by $*(A) = \{*\}$, for every $A \in \mathbf{Art}$.

**Example 1.3.** Let $V$ be a $\mathbb{K}$-vector space. Then, $F, G\colon \mathbf{Art} \to \mathbf{Set}$, defined by

$$F(A) = V \otimes \mathfrak{m}_A, \qquad G(A) = \operatorname{Hom}_{\mathbb{K}}(V, V \otimes \mathfrak{m}_A)$$

are functors of Artin rings. Notice that $G(A)$ is the kernel of the morphism

$$\operatorname{Hom}_A(V \otimes A, V \otimes A) = \operatorname{Hom}_{\mathbb{K}}(V, V \otimes A) \to \operatorname{Hom}_{\mathbb{K}}(V, V \otimes \mathbb{K}) = \operatorname{Hom}_{\mathbb{K}}(V, V)$$

and then $G(A)$ is the set of $A$-linear endomorphism of $V \otimes A$ that are trivial modulus $\mathfrak{m}_A$.

**Example 1.4.** Let $R$ be a local complete $\mathbb{K}$-algebra with residue field $\mathbb{K}$. The functor

$$\mathrm{h}_R\colon \mathbf{Art} \to \mathbf{Set}, \qquad \mathrm{h}_R(A) = \mathrm{Hom}_{\mathbb{K}-\mathrm{alg}}(R, A),$$

is a functor of Artin rings.

1.1. **The exponential functor.** Let $L$ be a Lie algebra over $\mathbb{K}$, $V$ a $\mathbb{K}$-vector space and $\xi\colon L \to \mathrm{End}(V)$ a representation of $L$. For every $A \in \mathbf{Art}$, the morphism $\xi$ can be extended naturally to a morphism of Lie algebras $\xi\colon L \otimes A \to \mathrm{End}_A(V \otimes A)$. Taking the exponential we get a functorial map

$$\exp(\xi)\colon L \otimes \mathfrak{m}_A \to GL_A(V \otimes A), \qquad \exp(\xi)(x) = e^{\xi(x)} = \sum_{i=0}^{\infty} \frac{\xi(x)^n}{n!},$$

where $GL_A$ denotes the group of $A$-linear invertible morphisms.
Note that $\exp(\xi)(-x) = (\exp(\xi)(x))^{-1}$. If $\xi$ is injective then also $\exp(\xi)$ is injective (easy exercise).

**Theorem 1.5.** *In the notation above, the image of* $\exp(\xi)$ *is a subgroup. More precisely for every* $a, b \in L \otimes \mathfrak{m}_A$ *there exists* $c \in L \otimes \mathfrak{m}_A$ *such that* $e^{\xi(a)}e^{\xi(b)} = e^{\xi(c)}$ *and* $a + b - c$ *belong to the Lie ideal of* $L \otimes \mathfrak{m}_A$ *generated by* $[a, b]$.

*Proof.* This is an immediate consequence of the Campbell-Baker-Hausdorff formula. □

In the above notation denote $P = \mathrm{End}(V)$ and let $ad(\xi)\colon L \to \mathrm{End}(P)$ be the adjoint representation of $\xi$,

$$ad(\xi)(x)f = [\xi(x), f] = \xi(x)f - f\xi(x).$$

Then, for every $a \in L \otimes \mathfrak{m}_A$, $f \in \mathrm{End}_A(V \otimes A) = P \otimes A$ we have

$$e^{ad(\xi)(a)}f = e^{\xi(a)}fe^{-\xi(a)}.$$

1.2. **Automorphisms functor.** In this section every tensor product is intended over $\mathbb{K}$, i.e $\otimes = \otimes_{\mathbb{K}}$. Let $S \xrightarrow{\alpha} R$ be a morphism of commutative unitary $\mathbb{K}$-algebras, for every $A \in \mathbf{Art}$, we have natural morphisms $S \otimes A \xrightarrow{\alpha} R \otimes A$ and $R \otimes A \xrightarrow{p} R$, $p(x \otimes a) = x\overline{a}$, where $\overline{a} \in \mathbb{K}$ is the class of $a$ in the residue field of $A$.

**Lemma 1.6.** *Given* $A \in \mathbf{Art}$ *and a commutative diagram of morphisms of* $\mathbb{K}$*-algebras*

$$\begin{array}{ccc} S \otimes A & \xrightarrow{\alpha} & R \otimes A \\ \downarrow{\scriptstyle \alpha} & \overset{f}{\nearrow} & \downarrow{\scriptstyle p} \\ R \otimes A & \xrightarrow{p} & R \end{array}$$

*we have that* $f$ *is an isomorphism and* $f(R \otimes J) \subset R \otimes J$ *for every ideal* $J \subset A$.

*Proof.* $f$ is a morphism of $A$-algebras, in particular for every ideal $J \subset A$, $f(R \otimes J) \subset Jf(R \otimes A) \subset R \otimes J$. In particular, if $B = A/J$, then $f$ induces a morphism of $B$-algebras $\overline{f}\colon R \otimes B \to R \otimes B$. We claim that, if $\mathfrak{m}_A J = 0$, then $f$ is the identity on $R \otimes J$; in fact for every $x \in R$, $f(x \otimes 1) - x \otimes 1 \in \ker p = R \otimes \mathfrak{m}_A$ and then if $j \in J$, $x \in R$.

$$f(x \otimes j) = jf(x \otimes 1) = x \otimes j + j(f(x \otimes 1) - x \otimes 1) = x \otimes j.$$

Now we prove the lemma by induction on $n = \dim_{\mathbb{K}} A$, being $f$ the identity for $n = 1$. Let

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

be a small extension with $J \neq 0$. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R\otimes J & \longrightarrow & R\otimes A & \longrightarrow & R\otimes B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} & & \\ 0 & \longrightarrow & R\otimes J & \longrightarrow & R\otimes A & \longrightarrow & R\otimes B & \longrightarrow & 0 \end{array}$$

By induction $\overline{f}$ is an isomorphism and by snake lemma also $f$ is an isomorphism. □

**Definition 1.7.** For every $A \in \mathbf{Art}$ let $\mathrm{Aut}_{R/S}(A)$ be the set of commutative diagrams of graded $\mathbb{K}$-algebra morphisms

$$\begin{array}{ccc} S\otimes A & \longrightarrow & R\otimes A \\ \downarrow & \overset{f}{\nearrow} & \downarrow \\ R\otimes A & \longrightarrow & R \end{array}$$

According to Lemma 1.6, $\mathrm{Aut}_{R/S}$ is a functor from the category **Art** to the category of groups **Grp**. Here we consider $\mathrm{Aut}_{R/S}$ as a functor of Artin rings (just forgetting the group structure).

Let $\mathrm{Der}_S(R, R)$ be the space of $S$-derivations $R \to R$ of. If $A \in \mathbf{Art}$ and $J \subset \mathfrak{m}_A$ is an ideal then, since $\dim_{\mathbb{K}} J < \infty$ there exist natural isomorphisms

$$\mathrm{Der}_S(R, R) \otimes J = \mathrm{Der}_S(R, R \otimes J) = \mathrm{Der}_{S\otimes A}(R \otimes A, R \otimes J),$$

where a given derivation $d = \sum_i d_i \otimes j_i \in \mathrm{Der}_S(R, R) \otimes J$ corresponds to the $S \otimes A$-derivation

$$d\colon R \otimes A \to R \otimes J \subset R \otimes A, \qquad d(x \otimes a) = \sum_i d_i(x) \otimes j_i a.$$

For every $d \in \mathrm{Der}_{S\otimes A}(R \otimes A, R \otimes A)$, denote $d^n = d \circ \ldots \circ d$ the iterated composition of $d$ with itself $n$ times. The generalized Leibnitz rule gives

$$d^n(uv) = \sum_{i=0}^{n} \binom{n}{i} d^i(u) d^{n-1}(v), \qquad u, v \in R \otimes A.$$

In particular, note that, if $d \in \mathrm{Der}_S(R, R) \otimes \mathfrak{m}_A$, then $d$ is a nilpotent endomorphism of $R \otimes A$ and

$$e^d = \sum_{n\geq 0} \frac{d^n}{n!}$$

is a morphism of $\mathbb{K}$-algebras belonging to $\mathrm{Aut}_{R/S}(A)$.

**Proposition 1.8.** *For every $A \in \mathbf{Art}_{\mathbb{K}}$ the exponential*

$$\exp\colon \mathrm{Der}_S(R, R) \otimes \mathfrak{m}_A \to \mathrm{Aut}_{R/S}(A)$$

*is a bijection.*

*Proof.* This is obvious if $A = \mathbb{K}$; by induction on the dimension of $A$ we may assume that there exists a nontrivial small extension

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

such that $\exp\colon \mathrm{Der}_S(R,R) \otimes \mathfrak{m}_B \to \mathrm{Aut}_{R/S}(B)$ is bijective. We first note that if $d \in \mathrm{Der}_S(R,R) \otimes \mathfrak{m}_A$, $h \in \mathrm{Der}_S(R,R) \otimes J$ then $d^i h^j = h^j d^i = 0$ whenever $j > 0$, $j + i \geq 2$ and then $e^{d+h} = e^d + h$; this easily implies that exp is injective.
Conversely take a $f \in \mathrm{Aut}_{R/S}(A)$; by the inductive assumption there exists $d \in \mathrm{Der}_S(R,R) \otimes \mathfrak{m}_A$ such that $\overline{f} = \overline{e^d} \in \mathrm{Aut}_{R/S}(B)$; denote $h = f - e^d\colon R \otimes A \to R \otimes J$. Since $h(ab) = f(a)f(b) - e^d(a)e^d(b) = h(a)f(b) + e^d(a)h(b) = h(a)\overline{b} + \overline{a}h(b)$ we have that $h \in \mathrm{Der}_S(R,R) \otimes J$ and then $f = e^{d+h}$. □

The same argument works also if $S \to R$ is a morphism of sheaves of $\mathbb{K}$-algebras over a topological space and $\mathrm{Der}_S(R,R)$, $\mathrm{Aut}_{R/S}(A)$ are respectively the vector space of $S$-derivations of of $R$ and the $S \otimes A$-algebra automorphisms of $R \otimes A$ lifting the identity on $R$.

## 1.3. **Deformation functors.**

**Definition 1.9.** (cf. [7]) A functor $F\colon \mathbf{Art} \to \mathbf{Set}$ is *prorepresentable* if it is isomorphic to $\mathrm{h}_R$, for some local complete $\mathbb{K}$-algebra $R$ with residue field $\mathbb{K}$. $F$ is *representable* if it is isomorphic to $\mathrm{h}_R$, for some $R \in \mathbf{Art}$.

The category **Art** is closed under fiber products, i.e., every pair of morphisms $C \to A$, $B \to A$ may be extended to a commutative diagram

$$\begin{array}{ccc} B \times_A C & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & A, \end{array} \tag{1}$$

such that the natural map

$$\mathrm{h}_R(B \times_A C) \to \mathrm{h}_R(B) \times_{\mathrm{h}_R(A)} \mathrm{h}_R(C)$$

is bijective, for every $R$.

**Definition 1.10.** Let $F\colon \mathbf{Art} \to \mathbf{Set}$ be a functor of Artin rings; for every fiber product

$$\begin{array}{ccc} B \times_A C & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \xrightarrow{\beta} & A \end{array}$$

in **Art**, consider the induced map

$$\eta\colon F(B \times_A C) \to F(B) \times_{F(A)} F(C).$$

The functor $F$ is *homogeneous* if $\eta$ is bijective whenever $\beta$ is surjective [6, Definition 2.5].
The functor $F$ is a *deformation functor* if:

(1) $\eta$ is surjective, whenever $\beta$ is surjective;
(2) $\eta$ is bijective, whenever $A = \mathbb{K}$.

The name deformation functor comes from the fact that almost all functors arising in deformation theory satisfy the conditions of Definition 1.10. Every prorepresentable functor is a homogeneous deformation functor.

*Remark* 1.11. Our definition of deformation functors involves conditions that are slightly more restrictive than the classical Schlessinger conditions H1, H2 of [7] and the semi-homogeneity condition of [6]. The main motivations of this change are:

(1) Functors of Artin rings satisfying Schlessinger condition H1, H2 and H3 do not necessarily have a "good"obstruction theory (see [2, Example 6.8]).
(2) The definition of deformation functor extends naturally in the framework of derived deformation theory and extended moduli spaces [5].

The formal smoothness of $\operatorname{Spec}(R)$ is equivalent to the property that $A \to B$ surjective implies $\mathrm{h}_R(A) \to \mathrm{h}_R(B)$ surjective. This motivate the following definition.

**Definition 1.12.** A natural transformation $\phi\colon F \to G$ of functors of Artin rings is called *smooth* if, for every surjective morphism $A \to B$ in **Art**, the map $F(A) \to G(A) \times_{G(B)} F(B)$ is also surjective. A functor of Artin rings $F$ is called *smooth* if $F(A) \to F(B)$ is surjective, for every surjective morphism $A \to B$ in **Art**, i.e., the natural transformation $F \to *$ is smooth.

*Remark* 1.13. If $\phi\colon F \to G$ is a smooth natural transformation, then $\phi\colon F(A) \to G(A)$ is surjective for every $A$ (take $B = \mathbb{K}$).

**Exercise 1.14.**
(1) If $F \to G$ and $G \to H$ are smooth, then the composition $F \to H$ is smooth.
(2) If $u\colon F \to G$ and $v\colon G \to H$ are natural transformations of functors such that $u$ is surjective and $vu$ is smooth. Then, $v$ is smooth.
(3) If $F \to G$ and $H \to G$ are natural transformations of functor such that $F \to G$ is smooth, then $F \times_G H \to H$ is smooth.

**Lemma 1.15.** *Let $R$ be a local complete noetherian $\mathbb{K}$-algebra with residue field $\mathbb{K}$. The following conditions are equivalent:*

(1) *$R$ is isomorphic to a power series ring $\mathbb{K}[[x_1, \ldots, x_n]]$.*
(2) *The functor $\mathrm{h}_R$ is smooth.*
(3) *For every $s \geq 2$ the morphism*
$$\mathrm{h}_R\left(\frac{\mathbb{K}[t]}{(t^{s+1})}\right) \to \mathrm{h}_R\left(\frac{\mathbb{K}[t]}{(t^2)}\right)$$
*is surjective.*

*Proof.* The only nontrivial implication is [3 ⇒ 1]. Let $n$ be the embedding dimension of $R$, then we can write $R = \mathbb{K}[[x_1, \ldots, x_n]]/I$ for some ideal $I \subset (x_1, \ldots, x_n)^2$; we want to prove that $I = 0$. Assume therefore $I \neq 0$ and denote by $s \geq 2$ the greatest integer such that $I \subset (x_1, \ldots, x_n)^s$: we claim that
$$\mathrm{h}_R\left(\frac{\mathbb{K}[t]}{(t^{s+1})}\right) \to \mathrm{h}_R\left(\frac{\mathbb{K}[t]}{(t^2)}\right)$$
is not surjective. Choosing $f \in I - (x_1, \ldots, x_n)^{s+1}$, after a possible generic linear change of coordinates of the form $x_i \mapsto x_i + a_i x_1$, with $a_2, \ldots, a_k \in \mathbb{K}$, we may assume that

$f$ contains the monomial $x_1^s$ with a nonzero coefficient, say $f = cx_1^s + \ldots$; let $\alpha\colon R \to \mathbb{K}[t]/(t^2)$ be the morphism defined by $\alpha(x_1) = t$, $\alpha(x_i) = 0$ for $i > 1$. Assume that there exists $\tilde{\alpha}\colon R \to \mathbb{K}[t]/(t^{s+1})$ that lifts $\alpha$ and denote by $\beta\colon \mathbb{K}[[x_1, \ldots, x_n]] \to \mathbb{K}[t]/(t^{s+1})$ the composition of $\tilde{\alpha}$ with the projection $\mathbb{K}[[x_1, \ldots, x_n]] \to R$. Then $\beta(x_1) - t, \beta(x_2), \ldots, \beta(x_n) \in (t^2)$ and therefore $\beta(f) \equiv ct^s \not\equiv 0 \pmod{t^{s+1}}$. □

**Definition 1.16.** Given a functor of Artin rings $F\colon \mathbf{Art} \to \mathbf{Set}$ and a group functor of Artin rings $G\colon \mathbf{Art} \to \mathbf{Grp}$, by a $G$-action on $F$ we shall mean a natural transformation $G \times F \to F$ such that

$$G(A) \times F(A) \to F(A)$$

is a $G(A)$-action on $F(A)$ in the usual sense for every $A \in \mathbf{Art}$. Then one can clearly define in the obvious way the quotient functor $F/G$.

**Proposition 1.17.** *In the situation of Definition 1.16, if $F$ and $G$ are deformation functors and $G$ is smooth, then $F/G$ is a deformation functor and the natural projection $F \to F/G$ is smooth.*

*Proof.* Easy exercise. □

Later we will give lots of examples where $F$ and $G$ are homogeneous and $F/G$ is not homogeneous. Moreover it is possible to prove that over a field of characteristic 0 every group deformation functor is smooth.

1.4. **Examples of deformation functors.**

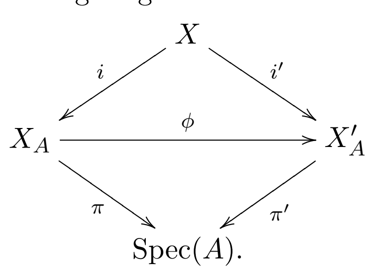
1.4.1. *Infinitesimal deformations of projective varieties.* Let $X$ be a projective variety over $\mathbb{K}$. An *infinitesimal deformation* of $X$ over $\mathrm{Spec}(A)$ is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & X_A \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \mathrm{Spec}(\mathbb{K}) & \xrightarrow{\ a\ } & \mathrm{Spec}(A), \end{array}$$

where $\pi$ is a proper and flat morphism, $a \in \mathrm{Spec}(A)$ is the closed point, $i$ is a closed embedding and $X \cong X_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\mathbb{K})$. If $A = \mathbb{K}[\epsilon]$ we call it a *first order* deformation of $X$.

*Remark* 1.18. Let $X_A$ be an infinitesimal deformation of $X$. By definition, it can be interpreted as a morphism of sheaves of algebras $\mathcal{O}_A \to \mathcal{O}_X$, such that $\mathcal{O}_A$ is flat over $A$ and $\mathcal{O}_A \otimes_A \mathbb{K} \to \mathcal{O}_X$ is an isomorphism.

Given another deformation $X'_A$ of $X$ over $\mathrm{Spec}(A)$, we say that $X_A$ and $X'_A$ are *isomorphic* if there exists an isomorphism $\phi : X_A \to X'_A$ over $\mathrm{Spec}(A)$, that induces the identity on $X$, that is, the following diagram is commutative



An infinitesimal deformation of $X$ over $\mathrm{Spec}(A)$ is called *trivial* if it is isomorphic to the infinitesimal product deformation, i.e., to the deformation

$$\begin{array}{ccc} X & \xrightarrow{i} & X \times \mathrm{Spec}(A) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{K}) & \xrightarrow{a} & \mathrm{Spec}(A). \end{array}$$

$X$ is called *rigid* if every infinitesimal deformation of $X$ over $\mathrm{Spec}(A)$ (for each $A \in$ **Art**) is trivial.

For every deformation $X_A$ of $X$ over $\mathrm{Spec}(A)$ and every morphism $A \to B$ in **Art** ($\mathrm{Spec}(B) \to \mathrm{Spec}(A)$), there exists an associated deformation of $X$ over $\mathrm{Spec}(B)$, called *pull-back deformation*, induced by a base change:

$$\begin{array}{ccc} X & \longrightarrow & X_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{K}) & \longrightarrow & \mathrm{Spec}(B). \end{array}$$

**Definition 1.19.** The *infinitesimal deformation functor* $\mathrm{Def}_X$ of $X$ is defined as follows:

$$\mathrm{Def}_X : \mathbf{Art} \to \mathbf{Set},$$

$$A \longmapsto \mathrm{Def}_X(A) = \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{infinitesimal deformations} \\ \text{of } X \text{ over } \mathrm{Spec}(A) \end{array} \right\}.$$

**Proposition 1.20.** $\mathrm{Def}_X$ *is a deformation functor, i.e., it satisfies the conditions of Definition 1.10*

*Proof.* See [7, Section 3]. □

1.4.2. *Infinitesimal deformations of locally free sheaves.* Let $X$ be a projective scheme and $\mathcal{E}$ a locally free sheaf of $\mathcal{O}_X$-modules on $X$. An *infinitesimal deformation of* $\mathcal{E}$ over $\mathrm{Spec}(A)$ is a locally free sheaves of $\mathcal{O}_X \otimes A$-modules $\mathcal{E}_A$ on $X \times \mathrm{Spec}(A)$, together with a morphism $\pi_A : \mathcal{E}_A \to \mathcal{E}$, such that $\pi_A : \mathcal{E}_A \otimes_A \mathbb{K} \to \mathcal{E}$ is an isomorphism.

Given another deformation $\mathcal{E}'_A$ of $\mathcal{E}$ over $\mathrm{Spec}(A)$, we say that $\mathcal{E}_A$ and $\mathcal{E}'_A$ are *isomorphic* if there exists an isomorphism of shaeves of $\mathcal{O}_X \otimes A$-modules $\phi : \mathcal{E}_A \to \mathcal{E}'_A$ over $\mathrm{Spec}(A)$, that commutes with the morphisms $\pi_A : \mathcal{E}_A \otimes_A \mathbb{K} \to \mathcal{E}$ and $\pi'_A : \mathcal{E}'_A \otimes_A \mathbb{K} \to \mathcal{E}$, i.e., $\pi_A \circ \phi = \pi_A$.

For every deformation $\mathcal{E}_A$ of $\mathcal{E}$ over $\mathrm{Spec}(A)$ and every morphism $A \to B$ in **Art** ($\mathrm{Spec}(B) \to \mathrm{Spec}(A)$), there exists an associated deformation of $\mathcal{E}$ over $\mathrm{Spec}(B)$, called *pull-back deformation*, induced by a base change:

$$\begin{array}{ccc} p^*\mathcal{E}_A & \longrightarrow & \mathcal{E}_A \\ \downarrow & & \downarrow \\ X \times \mathrm{Spec}(B) & \xrightarrow{p} & X \times \mathrm{Spec}(A). \end{array}$$

**Definition 1.21.** The *infinitesimal deformation functor* $\mathrm{Def}_{\mathcal{E}}$ of $\mathcal{E}_A$ is defined as follows:

$$\mathrm{Def}_{\mathcal{E}} : \mathbf{Art} \to \mathbf{Set},$$
$$A \longmapsto \mathrm{Def}_{\mathcal{E}}(A) = \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{infinitesimal deformations} \\ \text{of } \mathcal{E} \text{ over } \mathrm{Spec}(A) \end{array} \right\}.$$

**Proposition 1.22.** $\mathrm{Def}_{\mathcal{E}}$ *is a deformation functor, i.e., it satisfies the conditions of Definition 1.10.*

*Proof.* See [7, Section 3]. □

*Remark* 1.23. Given a projective scheme $X$, we have defined a deformation fo a locally free sheaf $\mathcal{E}$ over $\mathrm{Spec}(A)$, as a sheaf $\mathcal{E}_A$ on $X \times \mathrm{Spec}(A)$, i.e., we are considering the trivial deformations of $X$. More generally, we can define infinitesimal deformations of the pair $(X, \mathcal{E})$ whenever we allow deformations of $X$ too.

1.4.3. *Infinitesimal deformations of maps.*

**Definition 1.24.** Let $f : X \to Y$ be a holomorphic map and $A \in \mathbf{Art}$. An *infinitesimal deformation of f over* $\mathrm{Spec}(A)$ is a commutative diagram of complex spaces

$$\begin{array}{ccc} X_A & \xrightarrow{\mathcal{F}} & Y_A \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\mu} & \\ & S, & \end{array}$$

where $S = \mathrm{Spec}(A)$, $(X_A, \pi, S)$ and $(Y_A, \mu, S)$ are infinitesimal deformations of $X$ and $Y$, respectively (Definition 1.19), $\mathcal{F}$ is a holomorphic map that restricted to the fibers over the closed point of $S$ coincides with $f$.

If $A = \mathbb{K}[\epsilon]$ we have a *first order deformation* of $f$.

**Definition 1.25.** Let

$$\begin{array}{ccc} X_A & \xrightarrow{\mathcal{F}} & Y_A \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\mu} & \\ & S & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X'_A & \xrightarrow{\mathcal{F}'} & Y'_A \\ & {\scriptstyle\pi'}\searrow \quad \swarrow{\scriptstyle\mu'} & \\ & S & \end{array}$$

be two infinitesimal deformations of $f$. They are *equivalent* if there exist bi-holomorphic maps $\phi : X_A \to X'_A$ and $\psi : Y_A \to Y'_A$ (that are equivalence of infinitesimal deformations of $X$ and $Y$, respectively) such that the following diagram is commutative:

$$\begin{array}{ccc} X_A & \xrightarrow{\mathcal{F}} & Y_A \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ X'_A & \xrightarrow{\mathcal{F}'} & Y'_A. \end{array}$$

**Definition 1.26.** The *functor of infinitesimal deformations* of a holomorphic map $f : X \to Y$ is

$$\mathrm{Def}(f) : \mathbf{Art} \to \mathbf{Set},$$

$$A \longmapsto \mathrm{Def}(f)(A) = \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{infinitesimal deformations of} \\ f \text{ over } \mathrm{Spec}(A) \end{array} \right\}.$$

**Proposition 1.27.** $\mathrm{Def}(f)$ *is a deformation functor, since it satisfies the conditions of Definition 1.10.*

*Proof.* It follows from the fact that the functors $\mathrm{Def}_X$ and $\mathrm{Def}_Y$ of infinitesimal deformations of $X$ and $Y$ are deformation functors. □

## 2. Tangent space

**Definition 2.1.** Let $F\colon \mathbf{Art} \to \mathbf{Set}$ be a deformation functor. The set

$$T^1F = F\left(\frac{\mathbb{K}[t]}{(t^2)}\right)$$

is called the *tangent space* of $F$.

**Proposition 2.2.** *The tangent space of a deformation functor has a natural structure of vector space over $\mathbb{K}$. For every natural transformation of deformation functors $F \to G$, the induced map $T^1F \to T^1G$ is linear.*

*Proof.* (See [7, Lemma 2.10]) Since $F(\mathbb{K})$ is just one point, by Condition 2. of Definition 1.10, there exists a bijection $F\left(\frac{\mathbb{K}[t]}{(t^2)} \times_{\mathbb{K}} \frac{\mathbb{K}[t]}{(t^2)}\right) \cong F\left(\frac{\mathbb{K}[t]}{(t^2)}\right) \times F\left(\frac{\mathbb{K}[t]}{(t^2)}\right)$.

Consider the map

$$\begin{gathered} + : \frac{\mathbb{K}[t]}{(t^2)} \times_{\mathbb{K}} \frac{\mathbb{K}[t]}{(t^2)} \to \frac{\mathbb{K}[t]}{(t^2)}, \\ (a + bt, a + b't) \longmapsto a + (b + b')t. \end{gathered}$$

Then using the previous bijection, the map $+$ induces the addition on $F\left(\frac{\mathbb{K}[t]}{(t^2)}\right)$:

$$F\left(\frac{\mathbb{K}[t]}{(t^2)}\right) \times F\left(\frac{\mathbb{K}[t]}{(t^2)}\right) \stackrel{\cong}{\to} F\left(\frac{\mathbb{K}[t]}{(t^2)} \times_{\mathbb{K}} \frac{\mathbb{K}[t]}{(t^2)}\right) \stackrel{F(+)}{\to} F\left(\frac{\mathbb{K}[t]}{(t^2)}\right).$$

Analogously, for the multiplication by a scalar $k \in \mathbb{K}$ we consider the map:

$$\begin{gathered} k : \frac{\mathbb{K}[t]}{(t^2)} \to \frac{\mathbb{K}[t]}{(t^2)}, \\ a + bt \longmapsto a + (kb)t. \end{gathered}$$

It is an easy exercise to prove that the axioms of vector space are satisfied. The linearity of the map $T^1F \to T^1G$ induced by a natural transformation of deformation functors $F \to G$ follows by the definition of the $\mathbb{K}$-vector space structure on $T^1F$ and $T^1G$. □

It is notationally convenient to reserve the letter $\epsilon$ to denote elements of $A \in \mathbf{Art}$ annihilated by the maximal ideal $\mathfrak{m}_A$, and in particular of square zero.

**Example 2.3.** The tangent space of the functor $\mathrm{h}_R$, defined in Example 1.4, is

$$T^1\,\mathrm{h}_R = \mathrm{Hom}_{\mathbb{K}-\mathrm{alg}}(R, \mathbb{K}[\epsilon]) = \mathrm{Hom}_{\mathbb{K}}\left(\frac{\mathfrak{m}_R}{\mathfrak{m}_R^2}, \mathbb{K}\right).$$

Therefore $T^1\,\mathrm{h}_R$ is isomorphic to the Zariski tangent space of $\mathrm{Spec}(R)$ at its closed point.

**Definition 2.4.** Given a functor $F$ and $R$ a local complete $\mathbb{K}$-algebra, $R$ is said to be an *hull* for $F$ if we are given a morphism $h_R \to F$ which is smooth and bijective on tangent spaces.

*Remark* 2.5. (Exercise) An hull, if it exists, is unique up to non-canonical isomorphism.

The notion of hull is a weaker version of prorepresentability and it is related to the notion of semiuniversal deformation. The majority of deformation functors arising in concrete cases are not proprepresentable but they admit an hull as it shown in the following theorem.

**Theorem 2.6** (Schlessinger, [7]). *Let $F$ be a deformation functor with finite dimensional vector space. Then, there exists a local complete noetherian $\mathbb{K}$-algebra $R$ with residue field $\mathbb{K}$ and a smooth natural transformation $\mathrm{h}_R \to F$ inducing an isomorphism on tangent spaces $T^1 \mathrm{h}_R = T^1 F$. Moreover $R$ is unique up to non-canonical isomorphism.*

*Proof.* We will prove later as a consequence of a more general statement (the factorization theorem). ☐

**Lemma 2.7.** *Let $\eta\colon F \to G$ be a natural tranformation of deformation functors.*

1. *If $G$ is homogeneous and $\eta\colon T^1F \to T^1G$ is injective, then $\eta\colon F(A) \to G(A)$ is injective for every $A$ and $F$ is homogeneous.*
2. *If $F$ is smooth and $\eta\colon T^1F \to T^1G$ is surjective, then $G$ is a smooth functor and $\eta$ is a smooth morphism.*

*Proof.* Every small principal extension

$$0 \to \mathbb{K} \xrightarrow{\alpha} B \xrightarrow{\beta} A \to 0,$$

there exists an isomorphism

$$B \times_{\mathbb{K}} \mathbb{K}[\varepsilon] \to B \times_A B, \qquad (b, \overline{b} + k\varepsilon) \mapsto (b, b + k\alpha(\varepsilon))$$

and then, for every deformation functor $G$ a surjective map

$$\theta\colon G(B) \times T^1G = G(B \times_{\mathbb{K}} \mathbb{K}[\epsilon]) \to G(B) \times_{G(A)} G(B)$$

commuting with the projection on the first factor and such that $\theta(x, 0) = (x, x)$. If $G$ is homogeneous, then $\theta$ is bijective.

Assume now $G$ homogeneous and $\eta\colon T^1F \to T^1G$ injective. We will prove by induction on the length of $B \in \mathbf{Art}$ that $\eta\colon F(B) \to G(B)$ is injective. Let $x, y \in F(B)$ such that $\eta(x) = \eta(y) \in G(B)$ and let

$$0 \to \mathbb{K} \xrightarrow{\alpha} B \xrightarrow{\beta} A \to 0$$

be a principal small extension. By induction $\beta(x) = \beta(y) \in F(A)$ and then there exists $v \in T^1F$ such that $\theta(x, v) = (x, y)$. Thus $\theta(\eta(x), \eta(v)) = (\eta(x), \eta(y))$ and, since $G$ is homogeneous this implies $\eta(v) = 0$ and then $v = 0, x = y$. This proves that $\eta$ is always injective the homogeneity of $F$ is trivial.

Assume now $F$ smooth and $\eta\colon T^1F \to T^1G$ surjective. We need to prove that for every principal small extension as above, the map

$$(\beta, \eta)\colon F(B) \to F(A) \times_{G(A)} G(B)$$

is surjective. Let $(x, y) \in F(A) \times_{G(A)} G(B)$, since $F$ is smooth there exists $z \in F(B)$ such that $\beta(z) = x$; denoting $w = \eta(z)$ we have $(w, y) \in G(B) \times_{G(A)} G(B)$ and then there exists $v \in T^1G$ such that $\theta(w,v) = (w,y)$. Now $\eta\colon T^1F \to T^1G$ is surjective and then $v = \eta(u)$ and $\theta(z,u) = (z,r)$ with $\beta(r) = \beta(z) = x$ and $\eta(r) = y$. □

## 3. Obstructions

In the set-up of functors of Artin rings, with the term obstructions we intend *obstructions for a deformation functor to be smooth.*

**Definition 3.1.** Let $F$ be a functor of Artin rings. An *obstruction theory* $(V, v_e)$ for $F$ is the data of a $\mathbb{K}$-vector space $V$ and for every small extension in **Art**

$$e\colon \quad 0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

of an *obstruction map* $v_e\colon F(A) \to V \otimes M$ satisfying the following properties:

(1) If $A = \mathbb{K}$ then $v_e(F(\mathbb{K})) = 0$.
(2) (base change) For every commutative diagram

$$\begin{array}{ccccccccccc} e_1\colon & 0 & \longrightarrow & M_1 & \longrightarrow & B_1 & \longrightarrow & A_1 & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle \alpha_M} & & \downarrow{\scriptstyle \alpha_B} & & \downarrow{\scriptstyle \alpha_A} & & \\ e_2\colon & 0 & \longrightarrow & M_2 & \longrightarrow & B_2 & \longrightarrow & A_2 & \longrightarrow & 0. \end{array}$$

with $e_1, e_2$ small extensions and $\alpha_A, \alpha_B$ morphisms in **Art**, we have

$$v_{e_2}(\alpha_A(a)) = (Id_V \otimes \alpha_M)(v_{e_1}(a)) \qquad \text{for every} \quad a \in F(A_1).$$

*Remark* 3.2. It has to be observed that, to give a morphism of sets $v_e\colon F(A) \to V \otimes M$ is the same that to give a map $v_e\colon F(A) \times M^\vee \to V$ such that $v_e(a,-)\colon M^\vee \to V$ is linear for every $a \in F(A)$.

The name *obstruction theory* is motivated by the following result.

**Lemma 3.3.** *Let $(V, v_e)$ be an obstruction theory for a functor of Artin rings $F$, let*

$$e\colon \quad 0 \longrightarrow M \xrightarrow{\alpha} B \xrightarrow{\beta} A \longrightarrow 0$$

*be a small extension and $x \in F(A)$. If $x$ lifts to $F(B)$, i.e. if $x \in \beta(F(B))$, then $v_e(x) = 0$.*

*Proof.* Assume $x = \beta(y)$ for some $y \in F(B)$ and consider the morphism of small extension

$$\begin{array}{ccccccccccc} e'\colon & 0 & \longrightarrow & M & \xrightarrow{(0,\alpha)} & B \times_A B & \xrightarrow{p_1} & B & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \beta} & & \\ e\colon & 0 & \longrightarrow & M & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & A & \longrightarrow & 0. \end{array}$$

where $p_1$ and $p_2$ are the projections. By base change property $v_e(x) = v_{e'}(y)$. Now consider the morphism of small extensions

$$\begin{array}{ccccccccccc} e'\colon & 0 & \longrightarrow & M & \xrightarrow{(0,\alpha)} & B \times_A B & \xrightarrow{p_1} & B & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \pi} & & \\ e''\colon & 0 & \longrightarrow & M & \xrightarrow{(0,\alpha)} & \mathbb{K} \oplus \alpha(M) & \xrightarrow{\beta} & \mathbb{K} & \longrightarrow & 0. \end{array}$$

where $\pi\colon B \to \mathbb{K}$ is the projection and $\gamma(a,b) = (\pi(a), a-b)$. Again by base change property $v_{e'}(y) = v_{e''}(\pi(y)) = 0$. □

**Definition 3.4.** An obstruction theory $(V, v_e)$ for $F$ is called *complete* if the converse of Lemma 3.3 holds; i.e., the lifting exists if and only if the obstruction vanishes.

Clearly, if $F$ admits a complete obstruction theory then it admits infinitely ones; it is in fact sufficient to embed $V$ in a bigger vector space. One of the main interest (and problem) is to look for the "smallest" complete obstruction theory.

*Remark* 3.5. Let $e\colon\ 0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$ be a small extension and $a \in F(A)$; the obstruction $v_e(a) \in V \otimes M$ is uniquely determined by the values $(Id_V \otimes f)v_e(a) \in V$, where $f$ varies along a basis of $\mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K})$. On the other hand, by base change we have $(Id_V \otimes f)v_e(a) = v_\epsilon(a)$, where $\epsilon$ is the small extension

$$\epsilon: \quad 0 \longrightarrow \mathbb{K} \longrightarrow \frac{B \oplus \mathbb{K}}{\{(m, -f(m)) \mid m \in M\}} \longrightarrow A \longrightarrow 0.$$

This implies that every obstruction theory is uniquely determined by its behavior on principal small extensions.

**Definition 3.6.** A morphism of obstruction theories $(V, v_e) \to (W, w_e)$ is a linear map $\theta\colon V \to W$ such that $w_e = (\theta \otimes Id)v_e$, for every small extension $e$.
An obstruction theory $(O_F, ob_e)$ for $F$ is called *universal* if, for every obstruction theory $(V, v_e)$, there exists a unique morphism $(O_F, ob_e) \to (V, v_e)$.

**Theorem 3.7** ([2])**.** *Let $F$ be a deformation functor, then:*

(1) *There exists the universal obstruction theory $(O_F, ob_e)$ for $F$, and such obstruction theory is complete.*
(2) *Every element of the universal obstruction target $O_F$ is of the form $ob_e(a)$, for some principal extension*

$$e: \quad 0 \longrightarrow \mathbb{K} \longrightarrow B \longrightarrow A \longrightarrow 0$$

*and some $a \in F(A)$.*

*Proof.* The proof is quite long and it is postponed to Section 4 □

It is clear that the universal obstruction theory $(O_F, ob_e)$ is unique up to isomorphism and depends only by $F$ and not by any additional data.

**Definition 3.8.** The *obstruction space* of a deformation functor $F$ is the universal obstruction target $O_F$.

**Corollary 3.9.** *Let $(V, v_e)$ be a complete obstruction theory for a deformation functor $F$. Then, the obstruction space $O_F$ is isomorphic to the vector subspace of $V$ generated by all the obstructions arising from principal extensions.*

*Proof.* Denote by $\theta\colon O_F \to V$ the morphism of obstruction theories. Every principal obstruction is contained in the image of $\theta$ and, since $V$ is complete, the morphism $\theta$ is injective. □

*Remark* 3.10. The majority of authors use Corollary 3.9 as a definition of obstruction space.

**Example 3.11.** Let $R$ be a local complete $\mathbb{K}$-algebra with residue field $\mathbb{K}$ and $n = \dim T^1\,\mathrm{h}_R = \dim \mathfrak{m}_R/\mathfrak{m}_R^2$ its embedding dimension. Then, we can write $R = P/I$, where $P = \mathbb{K}[[x_1, \ldots, x_n]]$ and $I \subset \mathfrak{m}_P^2$. We claim that

$$T^2\,\mathrm{h}_R := \mathrm{Hom}_P(I, \mathbb{K}) = \mathrm{Hom}_{\mathbb{K}}(I/\mathfrak{m}_P I, \mathbb{K})$$

is the obstruction space of $\mathrm{h}_R$. In fact, for every small extension

$$e\colon \quad 0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

and every $\alpha \in \mathrm{h}_R(A)$, we can lift $\alpha$ to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & P & \longrightarrow & R & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle ob_a(\alpha)} & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & M & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

with $\beta$ a morphism of $\mathbb{K}$-algebras. It is easy to verify that

$$ob_e(\alpha) = \beta_{|I} \in \mathrm{Hom}_{\mathbb{K}}(I/\mathfrak{m}_P I, M) = T^2\,\mathrm{h}_R \otimes M$$

is well defined, it is a complete obstruction and that $(T^2\,\mathrm{h}_R, ob_e)$ is the universal obstruction theory for the functor $\mathrm{h}_R$ (see [2, Prop. 5.3]).

Let $\phi\colon F \to G$ be a natural transformation of deformation functors. Then, $(O_G, ob_e \circ \phi)$ is an obstruction theory for $F$; therefore, there exists an unique linear map $ob_\phi\colon O_F \to O_G$ which is compatible with $\phi$ in the obvious sense.

**Theorem 3.12** (Standard smoothness criterion)**.** *Let $\phi\colon F \to G$ be a morphism of deformation functors. The following conditions are equivalent:*

1. *$\phi$ is smooth.*
2. *$T^1\phi\colon T^1F \to T^1G$ is surjective and $ob_\phi\colon O_F \to O_G$ is bijective.*
3. *$T^1\phi\colon T^1F \to T^1G$ is surjective and $ob_\phi\colon O_F \to O_G$ is injective.*

*Proof.* In order to avoid confusion we denote by $ob_e^F$ and $ob_e^G$ the obstruction maps for $F$ and $G$ respectively.

[$1 \Rightarrow 2$] Every smooth morphism is in particular surjective; therefore, if $\phi$ is smooth then the induced morphisms $T^1F \to T^1G$, $O_F \to O_G$ are both surjective.
Assume that $ob_\phi(\xi) = 0$ and write $\xi = ob_e^F(x)$, for some $x \in F(A)$ and some small extension $e\colon 0 \longrightarrow \mathbb{K} \longrightarrow B \longrightarrow A \longrightarrow 0$. Since $ob_e^G(\phi(x)) = 0$, the element $x$ lifts to a pair $(x, y') \in F(A) \times_{G(A)} G(B)$ and then the smoothness of $\phi$ implies that $x$ lifts to $F(B)$.
[$3 \Rightarrow 1$] We need to prove that for every small extension $e\colon 0 \longrightarrow \mathbb{K} \longrightarrow B \longrightarrow A \longrightarrow 0$ the map

$$F(B) \to F(A) \times_{G(A)} G(B)$$

is surjective. Fix $(x, y') \in F(A) \times_{G(A)} G(B)$ and let $y \in G(A)$ be the common image of $x$ and $y'$. Then $ob_e^G(y) = 0$ because $y$ lifts to $G(B)$, hence $ob_e^F(x) = 0$ by injectivity of $ob_\phi$. Therefore $x$ lifts to some $x'' \in F(B)$. In general $y'' = \phi(x'')$ is not equal to $y'$. However, $(y'', y') \in G(B) \times_{G(A)} G(B)$ and therefore there exists $v \in T^1G$ such that $\theta(y'', v) = (y'', y')$ where

$$\theta\colon G(B) \times T^1G = G(B \times_{\mathbb{K}} \mathbb{K}[\epsilon]) \to G(B) \times_{G(A)} G(B)$$

is induced by the isomorphism

$$B \times_{\mathbb{K}} \mathbb{K}[\epsilon] \to B \times_A B, \qquad (b, \overline{b} + \alpha\epsilon) \mapsto (b, b + \alpha\epsilon).$$

By assumption $T^1F \to T^1G$ is surjective, $v$ lifts to a $w \in T^1F$ and setting $\theta(x'', w) = (x'', x')$ we have that $x'$ is a lifting of $x$ which maps to $y'$, as required. □

*Remark* 3.13. In most concrete cases, given a natural transformation $F \to G$ it is very difficult to calculate the map $O_F \to O_G$, while it is generally easy to describe complete obstruction theories for $F$ and $G$ and a compatible morphism between them. In this situation, only the implication $[3 \Rightarrow 1]$ of the standard smoothness criterion holds.

**Corollary 3.14.** *Let $F$ be a deformation functor and $\mathrm{h}_R \to F$ a smooth natural transformation. Then, the dimension of $O_F$ is equal to the minimum number of generators of an ideal $I$ defining $R$ as a quotient of a power series ring, i.e., $R = \mathbb{K}[[x_1, \dots, x_n]]/I$.*

*Proof.* Apply Nakayama's lemma to the $\mathbb{K}[[x_1, \dots, x_n]]$-module $I$ and use Example 3.11. □

## 4. Proof of Theorem 3.7

We need some care to avoid set theoretic difficulties. First of all, we work on a fixed universe. For every $n \geq 0$, choose a $\mathbb{K}$-algebra $\mathcal{O}_n$ isomorphic to the power series ring $\mathbb{K}[[x_1, \dots, x_n]]$ and consider the category **Art** whose objects are Artinian quotients of the $\mathcal{O}_n$'s and morphisms are local morphisms of $\mathbb{K}$-algebras. Let **Fvsp** the category whose objects are $\{0, \mathbb{K}, \mathbb{K}^2, \dots\}$ and morphisms are linear maps.

For a $V \in$ **Fvsp**, we denote by $V^\vee$ its $\mathbb{K}$-dual. If $A \in$ **Art**, we will denote by $\mathfrak{m}_A$ its maximal ideal.

By $\varepsilon$ and $\varepsilon_i$ we will always mean indeterminates annihilated by the maximal ideal, and in particular of square zero (e.g., the algebra $\mathbb{K}[\varepsilon]$ has dimension 2 and $\mathbb{K}[\varepsilon_1, \varepsilon_2]$ has dimension 3 as a $\mathbb{K}$-vector space).

**Definition 4.1.** A *small extension* $e$ in **Art** is a short exact sequence

$$e: \quad 0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

where $B \to A$ is a morphism in **Art**, $M \in$ **Fvsp** and the image of $M \to B$ is annihilated by the maximal ideal of $B$. In the sequel of the paper, for every small extension $e$ as above, we shall let $K(e) = M$, $S(e) = B$, $T(e) = A$ (the letters should be a reminder of kernel, source, target).

**Definition 4.2.** We denote by **Smex** the category whose objects are small extensions in **Art**. A morphism of small extensions $\alpha\colon e_1 \to e_2$ is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & B_1 & \longrightarrow & A_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_M} & & \downarrow{\scriptstyle \alpha_B} & & \downarrow{\scriptstyle \alpha_A} & & \\ 0 & \longrightarrow & M_2 & \longrightarrow & B_2 & \longrightarrow & A_2 & \longrightarrow & 0. \end{array}$$

The category **Smex** is small, in the sense that the class of its objects is a set.

For $A \in$ **Art** and $M \in$ **Fvsp** let $\mathrm{Ex}(A, M)$ be the set isomorphism classes of small extensions of $A$ with kernel $M$. Denote by $0 \in \mathrm{Ex}(A, M)$ the trivial extension

$$0: \quad 0 \longrightarrow M \longrightarrow A \oplus M \longrightarrow A \longrightarrow 0$$

where the product in $A \oplus M$ is $(a, m)(a', m') = (aa', a_0m' + a_0'm)$, and $a \to a_0$ is the quotient map $A \to \mathbb{K}$. A small extension is trivial if and only if it splits.

If $f : M \to N$ is a morphism in **Fvsp** and $\pi : C \to A$ is a morphism in **Art**, we shall denote by

$$f_*\colon \operatorname{Ex}(A, M) \to \operatorname{Ex}(A, N), \qquad \pi^*\colon \operatorname{Ex}(A, M) \to \operatorname{Ex}(C, M)$$

the induced maps, defined as follows:
Given an extension $e:\ 0 \to M \to B \to A \to 0$ in $\operatorname{Ex}(A, M)$, define $f_*e$ as the extension

$$0 \to N \to \frac{B \oplus N}{\{(m, f(m)) \mid m \in M\}} \to A \to 0.$$

Define $\pi^*e$ as the extension

$$0 \to M \to B \times_A C \to C \to 0.$$

**Exercise 4.3.** In the above set-up prove that $f_*\pi^* = \pi^* f_*\colon \operatorname{Ex}(A, M) \to \operatorname{Ex}(C, N)$.

**Exercise 4.4.** In the notation of Definition 4.2, prove that $\alpha_{M*}(e_1) = \alpha^*_A(e_2) \in \operatorname{Ex}(M_2, A_1)$.

Given two small extension

$$e_1:\quad 0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0,$$

$$e_2:\quad 0 \longrightarrow N \longrightarrow C \longrightarrow A \longrightarrow 0,$$

with the same target, we define $e_1 \oplus e_2 \in \operatorname{Ex}(A, M \times N)$ as

$$e_1 \oplus e_2:\quad 0 \to M \times N \to B \times_A C \to A \to 0.$$

We have a natural structure of vector space on $\operatorname{Ex}(M, A)$ where the sum is defined as

$$e_1 + e_2 = +_*(e_1 \oplus e_2), \qquad \text{where} \quad +\colon M \times M \to M$$

and the scalar multiplication is induced by the corrsponding operation on $M$.

Let $F$ be a deformation functor; for $A \in$ **Art**, $M \in$ **Fvsp** and $a \in F(A)$ define

$$F(A, M, a) = \{e \in \operatorname{Ex}(A, M) \mid a \text{ lifts to } F(S(e))\}.$$

**Lemma 4.5.** *Let $F$ be a deformation functor, then:*

(1) *For $A \in$ **Art** and $a \in F(A)$ we have*

$$F(A, M, a) \oplus F(A, N, a) \subset F(A, M \oplus N, a)$$

*for every $M, N \in$ **Fvsp**.*

(2) *For $A \in$ **Art**, $a \in F(A)$ and $f\colon M \to N$*

$$f_*F(A, M, a) \subset F(A, N, a).$$

(3) *For $A \in$ **Art**, $a \in F(A)$ and $\pi\colon A \to B$*

$$\pi^*F(B, M, \pi(a)) = F(A, M, a).$$

*In particular, $F(A, M, a)$ is a vector subspace of* $\operatorname{Ex}(A, M)$.

*Proof.* Immediate from the definition of deformation functors. □

**Lemma 4.6.** *Let $F$ be a deformation functor, $A \in$ **Art**, $M \in$ **Fvsp**, $e \in \operatorname{Ex}(A, M)$ and $a \in F(A)$. Then*

$$e \in F(A, M, a) \textit{ if and only if } \quad f_*e \in F(A, \mathbb{K}, a) \textit{ for every } f \in M^\vee.$$

*Proof.* Let $e$ be the small extension

$$0\longrightarrow M\longrightarrow B\xrightarrow{\pi}A\longrightarrow 0$$

and assume $f_*e\in F(A,\mathbb{K},a)$ for every $f\in M^\vee$. We prove that $a$ lifts to $F(B)$ by induction on $\dim_{\mathbb{K}} M$; If $\dim M=1$ there is nothing to prove.
Assume $\dim M>1$ and let $f\in M^\vee$ with proper kernel $N\subset M$. Consider the following small extensions and morphisms:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & B & \xrightarrow{\pi'} & A' & \longrightarrow & 0\\
 & & \big\downarrow{\scriptstyle i} & & \| & & \big\downarrow{\scriptstyle \delta} & & \\
0 & \longrightarrow & M & \longrightarrow & B & \xrightarrow{\pi} & A & \longrightarrow & 0\\
 & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \pi'} & & \| & & \\
0 & \longrightarrow & \mathbb{K} & \longrightarrow & A' & \xrightarrow{\delta} & A & \longrightarrow & 0
\end{array}$$

where the bottom row is $f_*(e)$; call $e'$ the top row. We have $i_*e'=\delta^*e$ and then, for every $h\in M^\vee$ we have $\delta^*h_*e=h_*\delta^*e=h_*i_*e'$.
By assumption $a$ lifts to some $a'\in F(A')$ and, since $M^\vee\to N^\vee$ is surjective, we may apply the inductive assumption to the small extension $e'$ and then $a'$ lifts to $F(B)$. $\square$

**Lemma 4.7.** *Let $F$ be a deformation functor, and let $f,g:B\to A$ be morphisms in* **Art***. Assume that $b\in F(B)$ and $f(b)=g(b)=a\in F(A)$. Then*

$$f^*=g^*:\frac{\operatorname{Ex}(A,M)}{F(A,M,a)}\hookrightarrow\frac{\operatorname{Ex}(B,M)}{F(B,M,b)}.$$

*Proof.* The injectivity is clear since $f^*F(A,M,a)=g^*F(A,M,a)=F(B,M,b)$. Let

$$e:\qquad 0\xrightarrow{i}M\to C\xrightarrow{p}A\to 0$$

be a small extension. We want to prove that $f^*e-g^*e\in F(B,M,b)$.
Consider the small extension

$$\nabla:\quad 0\longrightarrow M\xrightarrow{(i,0)=(0,-i)}D=\frac{C\times_{\mathbb{K}}C}{\{(m,m)\mid m\in M\}}\xrightarrow{(p,p)}A\times_{\mathbb{K}}A\longrightarrow 0$$

and the morphism $\phi\colon B\to A\times_{\mathbb{K}}A$, $\phi(x)=(f(x),g(x))$. Then $f^*e-g^*e=\phi^*\nabla$ and then it is sufficient to prove that $\nabla\in F(A\times_{\mathbb{K}}A,M,\phi(b))$, i.e. that $\phi(b)$ lifts to $D$. Since $F(A\times_{\mathbb{K}}A)\to F(A)\times F(A)$ is bijective, we must have $\phi(b)=\delta(a)$, where $\delta\colon A\to A\times_{\mathbb{K}}A$ is the diagonal. It is now sufficient to observe that $\delta$ lifts to a morphism $A\to D$. $\square$

**Definition 4.8.** Let $F$ be a deformation functor. For every $A\in$ **Art** and $a\in F(A)$ denote by

$$H(A,a)=\frac{\operatorname{Ex}(A,\mathbb{K})}{F(A,\mathbb{K},a)}.$$

Denote also by $\mathbf{O}_F$ the subcategory of $\mathbf{Vect}_{\mathbb{K}}$ with objects the $H(A,a)$'s, for $A\in$ **Art** and $a\in F(A)$, and morphisms the injective linear maps $f^*\colon H(A,a)\to H(B,b)$, where $f\colon B\to A$ is a morphism in **Art** such that $f(b)=a$.

The category $\mathbf{O}_F$ is filtrant. This means that [3, Def. 1.11.2]:

(1) Given morphisms $H(A,a)\to H(B,b)$ and $H(A,a)\to H(C,c)$, there exist morphisms $H(B,b)\to H(S,s)$ and $H(C,c)\to H(S,s)$ such that the resulting diagram is commutative.

(2) Given two morphisms $f^*, g^* \colon H(A,a) \to H(B,b)$ there exist a morphism $H(B,b) \to H(C,c)$ such that the composed morphisms $H(A,a) \to H(C,c)$ coincide.

Moreover, it is required that $I$ is nonempty and connected.
The Lemma 4.7 says that 2) holds in the stronger sense that, given two objects, there is at most one morphism between them. In view of this, 1) is equivalent to saying that, given any two objects, there is a third to which they both map (the commutativity of the diagram is ensured by 4.7). Given $B, C$ in **Art** and elements $b \in F(B)$, $c \in F(C)$, take $S = B \times_{\mathbb{K}} C$; since $F$ is a deformation functor there exists $s \in F(S)$ mapping to $b \in F(B)$ and to $c \in F(C)$.

Since $\mathbf{O}_F$ is filtrant, the colimit construction interchanges with the forgetful functor $\mathbf{Vect}_{\mathbb{K}} \to \mathbf{Set}$, i.e., the set

$$O_F := \operatorname{colim} \mathbf{O}_F = \bigcup_{\mathbf{O}_F} H(A,a)/\sim,$$

where $\sim$ is the equivalence relation generated by $v \sim f^*v$, is a vector space over $\mathbb{K}$ and the natural maps $H(A,a) \to O_F$ are injective morphisms of vector spaces.
The space $O_F$ is the obstruction space of an obstruction theory $(O_F, ob_e)$, where for every small extension

$$e: \quad 0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

we define $ob_e \colon F(A) \times M^\vee \to O_F$ by $ob_e(a,f) = \theta(e)$, where $\theta$ is the composition map

$$\theta\colon \operatorname{Ex}(A,M) \xrightarrow{f_*} \operatorname{Ex}(A,\mathbb{K}) \to H(A,a) \hookrightarrow O_F.$$

It is straightforward to verify that $(O_F, ob_e)$ is an obstruction theory, while Lemma 4.6 tell us that it is a complete obstruction.
Finally, the universal property of colimits gives the universality of $(O_F, ob_e)$.

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