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Topics in algebra–deformation theory

Lecture 1

"Hard to construct" finite dimensional compact manifold.

Sets arise usually as sets of equivalence classes

e.g. $N =$ equivalence classes of finite sets, with equivalence existence of a bijection.

e.g. finite simple groups/isomorphism.

e.g. M_g moduli space of curves of genus g, "essentially smooth".

Need tools to provide compactness and smoothness of these spaces. Tools come from algebraic geometry (geomtric invariant theory) and analysis (compatness theorems, Fredholm properties) for compactness. For smoothness, one has resolution of singularities (which changes the space), Lie group and homgeoneous space mthods, general position arguments, Sard lemma, and deformation theory.

GOAL OF COURSE: to develop techniques which produce an enormous class of examples of "quasismooth" moduli spaces, which are nice enough to have characteristic classes.

For compactness, geometric invariant theory is not good enough (only one succesful example—space of curves in algebraic varieties or almost complex manifolds). There is no good notion of "stable surface" to give a good moduli space.

PHILOSPHY OF DEFORMATION THEORY

Infinitesimal study of moduli spaces. Intuitive picture (Arnol'd):

Begin with infinite dimenisonal vector space V, containing a closed subspace S of structures given by some equations.

e.g. $X =$ closed smooth manifold. $V =$ almost complex structures. (Locally a vector space). $S =$ integrable complex structures.

Next, one has an infinite dimensional Lie group acting on V and preserving S.

The moduli space is S/G (e.g. equivalence classes of complex structures, in the previous examples).

Fix m in the moduli space M. Pick a representative $\tilde{m}$ in S. Consdier the orbit $G\tilde{m}$, which is a smooth manifold, and pick a transversal manifold ("slice") T, and intersect with S to get a space whose germ at $\tilde{m}$ is called a miniversal, or transveral deformation.

PRE-LEMMA. Any family of structures containing $\tilde{m}$ is induced from the miniversal deformation. Any two miniversal deformations are isomorphic.

Good situation: stabilizer of $\tilde{m}$ is discrete. In this case, the miniversal deformation is the universal deformation – it is completely unique (the equivalence between any two realizations is canonical).

EXAMPLE. 1st order deformations of associative algebras. Let A be a vector space over k (e.g. C). If you choose a basis e_i for A as a vectorspace, we get structure constants c_{ij}^k in C. Here, our space V is of dimension n^3 if A has dimension n; S is the space of associative products, given by a system of quadratic equations. the group G is $\mathrm{Aut}(A)$.

For 1st order deformation, suppose that $C_{ij}^k(h) = c_{ij}^k + \tilde{c}_{ij}^k h + O(h^2)$. Impose associativity and divide by transfomrations $e_i \mapsto g_{ij}e_j$, where $g_{ij} = \delta_{ij} + \tilde{g}_{ij}h + \ldots$.

It's convenient to consider algebras over the dual numbers $C[h]/(h^2)$. In particular, one consider algebra structures on $A_h = A[h]/h^2$. Consider products on A_h which reduce to the old product on A.

We get $a * b = ab + hf(a,b)$... Get a condition on f:

$$f(ab,c) + f(a,b)c = f(a,bc) + af(b,c).$$

Now if we consider automorphisms which are the identity when $h = 0$, we consider linear maps $g : A \to A$ giviing $T(a) = a + hg(a)$. the inverse is given by $-g$.

The new product $a *' b$ pulled back from $*$ by T is given by replacing $f(a,b)$ by

$$f(a,b) + g(a)b + ag(b) - g(ab).$$

RESULT.

$$\mathrm{Hom}(A,A) \xrightarrow{d_1} \mathrm{Hom}(A \otimes A, A) \xrightarrow{d_2} \mathrm{Hom}(A \otimes A \otimes A, A)$$

with d_1 and d_2 given by formulas based on above.

$$\{\text{equiv classes of 1st order deformations}\} = \ker d_2 / \mathrm{im} d_1.$$

One can also map $d_0 : A \to \mathrm{Hom}(A,A)$ by $d_0(a)(x) = ax - xa$.

So $\ker d_1/\mathrm{im} d_0$ is derivations/inner derivations.

All this is called (lower) Hochschild cohomology of A with coefficients in A. it is denoted $H^*(A,A)$.

We can name several of the Hochschild cohomology spaces:

$$H^0(A,A) = \text{center of } A,$$

$$H^1(A,A) = \text{exterior derivations of } A,$$

$$H^2(A,A) = \text{1st order deformations of } A,$$

$$H^3(A,A) = \text{obstructions to deformations of } A$$

(when it vanishes, every first order deformation can be prolonged to a formal deformation).

What is the meaning of the higher cohomology? Analogy from Gelfand. We know the geometric meaning of the first derivative (slope) and of the second derivative (curvature), and of the vanishing of the second derivative (inflection). The higher derivatives don't have individual meaning, but they are coefficients of the Taylor series. In the same way, one should try to think of all the cohomology as the "Taylor coefficients" of a single object.

EXERCISE. "Formal deformation theory is not very realistic."

Let A_λ by $C[x_1, x_2, x_3, x_4]$ with the relations

$$x_2 x_1 = 1,$$

$$x_3(x_1 - 1) = 1,$$

$$x_4(x_1 - \lambda) = 1.$$

1. Construct a basis $e_i(\lambda)$ of A_λ ($\lambda \in C^{0,1}$) such that the structure constants are rational functions in λ.

2. Prove that the formal 1st order derivation is trivial for each value of λ.

3. Prove that A_λ and A_μ are isomorphic iff μ is in $\{\lambda, 1/\lambda, 1-\lambda, 1/(1-\lambda), \lambda/(1-\lambda), (\lambda-1)/\lambda\}$.

In fact, $H^2(A_\lambda, A_\lambda) = 0$.

The moral of this exercise is that formal deformation theory is not realistic for infinite dimensional algebras.

A SCIENTIFIC APPROACH TO FIRST ORDER DEFORMATIONS (Grothendieck-SGA I, Schlessinger)

1st order: moduli space is not just a set but a groupoid (category in which all morphisms are invertible). Such a category gives rise to a set of equivalence classes (orbits) and a "Galois group" (isotropy group) in each class, which is a group defined modulo (inner) isomorphism.

Groupoids arise in equivalence problems because there are usually many equivalences between two objects.

SECOND BASIC IDEA: Introduce a category of parameter spaces. For such space P, there is associated a groupoid of objects parametrized by P.

1st order defomration theory. Consider the parameter space whose function algebra is the dual numbers.

Kontsevich, Lecture 2

8/25/94

Associative algebras were Example 1 of a deformation theory. The groupoid which replaces the tangent space to moduli space is the action groupoid for the action of the 1-cochains on the 2-cocycles by addition of the coboundary. (The tangent space to moduli space is the orbit space of this groupoid.) This groupoid will be discussed further in a subsequent lecture.

EXAMPLE 2. Deformations of Lie algebras.

Start with a Lie algebra g over k, a field of characteristic zero.

First order deformations $= H^2(g, g)$ (Eilenberg-MacLane)

RULE OF SIGNS: Draw a permutation by arrows. The number of intersection points among these arrows is (mod 2) the sign of the permutation. Now apply this to the permutation of variables occurring in the terms of the coboundary formula.

The cohomologies have the same interpretation as in the associative case.

Example 3 (for completeness). Commutative associative algebras, not necessarily with unit. Here we have the Harrison complex which controls the deformation theory. Here, the cocycles are in degree>0, and H^2 is again the 1st order deformations.

FACT (generalization of exercise from the first lecture). Let

$$R = C[x_1, ..., x_n]/(f_1, ..., f_m).$$

Suppose that the algebraic variety given by setting the f_i to zero is smooth (maximal rank of the derivative matrix). "Closed points" of this (smooth affine algebraic) variety are homomorphisms from R to C.

For such varieties, the Harrison cohomology of the function algebra is zero in degrees$>$ 1. But the varieties are in general deformable. This means that the Harrison cohomology sees only the singularities.

For all three of the standard algebraic structures, we have: 1st order deformations $= H^2$(standard complex).

Now we will go on to some geometric examples.

Example 4. Local systems.

$X =$ topological space (CW complex), $G =$ Lie group, $G^\delta = G$ with discrete topology. We will refer to G^δ bundles as "local systems".

There are three different descriptions of local systems.

A. Sheaf theoretic: local system is given by a covering U_i of X by open sets, transition functions $\gamma_{ij} : U_i \cup U_j \to G$ which are locally constant and satisfy the cocycle condition for compatibility. Equivalence is given by a common refinement of two coverings and a system of maps to G which conjugate one system of transition functions to the other.

B. Group theoretic: Suppose that X is connected. Then equivalence classes of local systems are naturally isomorphic to equivalence classes of homomorphisms from $\pi_1(X)$ to G. (If X is not connected, one can use the fundamental groupoid instead of the fundamental group.)

C. Differential geometric: If X is a smooth manifold, we can look at the space of flat connections on G bundles modulo gauge transformations.

WHAT IS THE DEFORMATION THEORY IN THIS SITUATION.

Since G is a Lie group, we have a good notion of local system depending smoothly on parameters, and so we have a good notion of first-order deformation.

In terms of description A, the first order deformations of a local system E are equivalence classes of pairs $(\tilde{E}, i)$, where $\tilde{E}$ is a TG-local system and i is an isomorphism from E to the G-local system induced from $\tilde{E}$.

Algebraic view: points of G are continuous homomorphisms from $C^\infty(G)$ to R. Points of TG are continuous homomorphisms to the dual numbers.

EXERCISE. Let A be any commutative associative R-algebra of finite dimension containing a nilpotent ideal of codimension 1. (Artin algebra). Then continuous functions from $C^\infty(G)$ to A naturally form a Lie group. "Higher order tangent bundle".

The description A gives the first order deformations as Cech cohomology $H^1(X, \mathrm{ad}E)$, where $\mathrm{ad}E$ is the sheaf of Lie algebras associated to E.

Description B gives a picture of the first order deformations of a homomorphism ρ as the first cohomology of $\pi = \pi_1(X, x)$ with coefficients in $\mathrm{ad}\rho$.

Description C gives the first order deformations as the first de Rham cohomology of X with coefficients in the flat bundle $\mathrm{ad}E$.

The three cohomologies are thus the same, but the "RIGHT" one is the cohomology of the local system. (Its higher cohomology is "correct".) The second description is "wrong". The third one gives an explicit complex.

EXAMPLE 5. Holomorphic vector bundles. Here X is a complex manifold. We have two descriptions.

Description A: have cover, with holomorphic transition functions to $GL(N, C)$....

Description B: flat connections in $\bar{\partial}$ directions. Suppose that we have a C^∞ vector bundle E over X. The complexified tangent bundle of X splits canonically into T^{10} and T^{01} (holomorphic and antiholomorphic), where T^{01} is a formally integrable distribution.

Now we also have a decomposition of 1 forms into Ω^{10} and Ω^{01}. A connection in the $\bar{\partial}$ direction is a C-linear map from sections of E to sections of $E \otimes \Omega^{01}$ satisfying the Leibniz rule

$$\bar{\nabla}(f\xi) = f\bar{\nabla}\xi + \xi \otimes \bar{\partial} f.$$

Now we can prolong $\bar{\nabla}$ to a differential on all $E \otimes \Omega^{0k}$, and flatness is the condition that the square of this differential is zero.

THEOREM (corollary of Newlander-Nirenberg theorem). Holomorphic structures $\Longleftrightarrow$ flat $\bar{\partial}$ connections.

So we find that deformations in picture B are given by Dolbeault cohomology of X in End$E$. (Maybe this should be called picture C.)

So we have one basic formula with Cech cohomology and one formula with an explicit complex.

EXAMPLE 6. Deformation of complex structures.

Description A. Charts and transition functions. $f_i : U_i \to C^n$ embeddings, with transitions given by holomorphisms.

Description B. Smooth manifold X with integrable almost complex structure.

Deformation theory in description B. Think of almost complex structure as a subbundle of the complexified tangent bundle. Deformation is given by a map to the tangent space of the appropriate grassmannian. So first order deformations are sections gamma of Hom(T^{01}, T^{10}), or "Beltrami differentials": i.e. type 0,1 forms with values in holomorphic tangent bundle. The infinitesimalized integrability condition becomes $\bar{\partial}\gamma = 0$, while one divides by the image of $\bar{\partial}$ to get the equivalence classes. Thus one gets the deformation space to be the first Dolbeault cohomology of X with values in TX, while Description A gives the Cech cohomology with values in the corresponding sheaf.

So here we find that the deformation space is $H^1(X$, sheaf of Lie algebras), with an explicit complex computing this cohomology.

In the algebraic setting, we have a complex but no spaces.

In all situations (algebraic and geometric), the explicit complex which computes the cohomology is a Z-graded differential Lie superalgebra. It is an "art" to discover these objects for a general deformation theory.

A Z-graded differential Lie algebra is a:

graded vector space

brackets from $g^k \times g^l$ to g^{k+l}

differential from g^k to g^{k+1} satisfying $d^2 = 0$

graded antisymmetry and graded Jacobi identity

graded derivation rule

Structures (near a given one) are the same as elements $\gamma \in g^1$ satisfying the equation $d\gamma + [\gamma, \gamma] = 0$. (Maurer-Cartan equation);

equivalences arise from the action of g^0.

Kontsevich, Lecture 3

August 30, 1994

(Notes by Alan Weinstein)

A GENERAL SCHEME FOR FORMAL DEFORMATION THEORY IN CHARACTERISTIC ZERO

Start with a D(Z)GLA Γ over a field k of characteristic zero.

Let V be the vector space Γ^1. S is the subset consisting of those γ satisfying the quadratic equation $d\gamma + \frac{1}{2}[\gamma,\gamma] = 0$.

Instead of a group G acting on S, we have the Lie algebra $g = \Gamma^0$ acting on Γ^1 by affine vector fields: $\alpha \in \Gamma^0$ maps to the affine vector field on Γ^1, $\dot{\gamma} = [\alpha,\gamma] - d\alpha$.

Exercise: this is a Lie algebra homomorphism preserving the equation for S.

We will check the latter: let $K(\gamma) = d\gamma + \frac{1}{2}[\gamma,\gamma] = 0$. Then we show that $\dot{K}(\gamma) = 0$ for every α.

We use the chain rule: $\dot{K}(\gamma) = d\gamma + [\dot{\gamma},\gamma] = d([\alpha,\gamma] - d\alpha) + [[\alpha,\gamma] - d\alpha,\gamma] == [d\alpha,\gamma] + [\alpha,d\gamma] - dd\alpha + ... = [\alpha,d\gamma] + [[\alpha,\gamma],\gamma] = -\frac{1}{2}[\alpha,[\gamma,\gamma]] + \frac{1}{2}[[\alpha,\gamma],\gamma] + \frac{1}{2}[[\alpha,\gamma],\gamma] = 0$. (we used curvature zero for γ, plus Jacobi).

Now the notion of orbit space for Lie algebra actions in infinite dimensions is not very useful. So we go on to...

ARTIN RINGS

Definition (useless): A commutative associative ring A with unit is an Artin ring if it has no infinite descending chain of ideals. ("dual" to notion of Noetherian ring).

Structure theorem: an Artin ring A is a finite direct sum of local Artin rings A_α. Each of these A_α has a maximal ideal m_α which is nilpotent. In addition, each quotient of A_α by a power of m_α is finite dimensional.

Fix a field k. Consider those A over k for which $A/m \sim k$. As a vector space $A = k \oplus m$, where m is a nilpotent finite dimensional algebra over k.

EXAMPLES: $k[h]/(h^n)$, versions with several variables.

NOW to a DZGLA γ, we associate a function from local Artin k-algebras to groupoids.

The objects of the groupoid attached to A (with maximal ideal m) will be elements γ of $\Gamma \otimes m$ satisfying the Maurer-Cartan equation $d\gamma + \frac{1}{2}[\gamma,\gamma] = 0$.

To describe the morphisms, our first step is to introduce the nilpotent Lie algebra $\Gamma^0 \otimes m$.

Second step: to every nilpotent Lie algebra g over k we can associate the group of formal symbols $\exp(x), x \in g$, with multiplication given by the Campbell-Baker-Hausdorff formula.

CLAIM: Group($\Gamma^0 \otimes m$) acts on the set of objects in our category. The action is: for ϕ in the group, we get the map

$$\gamma \mapsto \phi\gamma\phi^{-1} - d\phi\phi^{-1}$$

(action of gauge transformations on connections).

Here, the notation

$$\exp(\alpha)\gamma\exp(-\alpha) = \sum_n (\mathrm{ad}\alpha)^n(\gamma)/n!$$

Also,

$$d\phi\phi^{-1} = d\exp\alpha\,\exp(-\alpha)$$

is defined by

$$d\exp\alpha = \left(\int_0^1 \exp(t\alpha)d\alpha\right)\exp((1-t)\alpha)dt,$$

$$d\phi\phi^{-1} = \sum_n (1/(n+1)!)(\mathrm{ad}\alpha)^n(d\alpha).$$

(Some discussion here about using divided powers to handle the case of finite characteristic.)

NOW WE DEFINE THE GROUPOID to be the action groupoid of this action. In other words, $\mathrm{Mor}(\Gamma_1, \Gamma_2)$ = those ϕ which map Γ_1 to Γ_2, with composition the group product.

THE IDEA IS THAT although we cannot integrate the elements of the original Lie algebra, we can integrate them when we make the Lie algebra nilpotent by tensoring with an nilpotent algebra. Also, we "code" the orbit space of the Lie algebra action by using its action groupoid instead.

DZGLA STRUCTURES ON STANDARD COMPLEXES

Example 4. G-local systems on manifold X, G a Lie group.

if we fix a choice of copnnection ∇_0 giving rise to the flat bundle ξ, then Γ^k = $k-$forms on X with values in $\mathrm{ad}\xi$.

Locally, we can choose trivializations, so that we have differential forms with values in g. The forms themselves form a Z-graded commutative differential associative super algebra under wedge product. The tensor product of this object with the Lie algebra g automatically gets the structure of DZGLA.

Here. the Maurer-Cartan equation for γ says that $\nabla_0 + \gamma$ is flat. The action of γ_0 is the action of the infinitesimal gauge transformations.

The story is essentially the same in...

Example 5. Holomorphic vector bundles.

On the other hand, something nontrivial and "funny" happens in...

Example 6. Deformation of complex structures.

X a C^∞ manifold, J a complex structure on X. Let γ^k be the tensor product of holomorphic vector fields with forms of type $0, k$. Typical element is $f_I, jd\bar{z}_I\partial/\partial z_j$. ($I$ is a multi-index)

Brackets and differential are given by local formulas. We look instead at the formal completion of the "f" part. it is formal power series in $\mathrm{Re}z$ and $\mathrm{Im}z$, or equivalently in z and $\bar{z}$.

Thus the formal completion of γ is

$$(C[[\bar{z}]] \otimes \wedge^*(d\bar{z}))\hat{\otimes}(C[[z]] \otimes \langle\partial/\partial z\rangle).$$

This is the tensor product of a differential graded commutative algebra with a lie algebra, as before.

PREVIOUS PICTURE	NEW PICTURE

almost complex structures	Γ_1 = Beltrami differentials
contains	contains
integrable AC structures	solutions of $\bar\partial$ Maurer-Cartan
acted on by	acted on by
diff$X$	smooth sections of
	holomorphic tangent bundle

It is strange for both of these Lie algebras (vector fields and smooth sections of holomorphic tangent bundle) to have the same orbits.

An open domain in the AC structures can be identified with an open domain in the Beltrami differentials.

The graph of a Beltrami differential is a subbundle of the holomorphic plus antiholomorphic tangent bundle. When the differential γ is small enough, the graph is transverse to its conjugate.

CLAIM. Integrability of the almost structure graph(γ) is equivalent to the zero-curvature equation for γ.

PROOF. Fix γ. We get a new almost complex structure with its $T^{0,1}_{\rm new}$ generated by $\xi_i = \partial/\partial\bar z_i + \sum \gamma_{ij}\partial/\partial z_j$. Now compute the commutators of these complex vector fields. their vanishing is equivalent to the equation of zero curvature.

We have two Lie algebras acting on the space of almost complex structures. They have different orbits, BUT when restricted to the integrable structures they have the same orbits.

WHAT IS THE EXPLANATION? Both algebras (smooth vector fields, smooth sections of the holomorphic tangent bundle) lie in the larger Lie algebra of smooth sections of the complexified tangent bundle.

On integrable complex structures. this action has a big kernel

so the image of this map ?.?.?.??? (I'm lost here) is the same for any complement of the kernel. This is the case for our two Lie algebras.

One can even look at the Artin algebra picture and see that, not only are the orbits for the two algebras the same, but the groupoids obtained by the Artin algebra approach are equivalent.

REMARKS

1. The set of almost complex structures is complex (open set of a complex grassmannian).

2. The identification of open domain in complex structures with an open domain in Γ^1 is holomorphic. The Maurer-Cartan equation is a complex quadratic equation. then we get a functor from local Artin C-algebras to groupoids.

Kontsevich, Lecture 4

September 1, 1994

Notes by Alan Weinstein

SUPERMATHEMATICS

This is a way to resolve all questions of $\pm$ signs with just one rule.

A super vector space is a Z_2-graded vector space.

(A large part of mathematics can be formulated in terms of vector spaces, rather than sets. Fix a field, preferably of characteristic zero. An associative algebra is a vector space $V \in OB(\mathrm{Vect}_k)$, plus a morphism $m : V \otimes V \to V$ satisfying an associativity condition which can be expressed in terms of an equation $m(m \otimes 1) = m(1 \otimes m)$. Similarly, commutativity can be expressed in a similar way.

This leads to the notion of Tensor Category (Saavedra LNM 265, Deligne-Milne in LNM 900). This is "representation theory without a group".

DATA: C an abelian k-linear category. (All morphism spaces are k-vector spaces, have direct sums, kernels and images of morphisms.) An example is the category of modules over an associative algebra.

Next, have a functor $\otimes : C \times C \to C$ which is biadditive, bilinear over k. Also have identity object ONE. Hom(ONE,ONE)$= k$.

Also have two isomorphisms of functors:

commutativity: $\otimes P_{12} \to \otimes$ (P_{12} is the flip),

associativity: $\otimes(\otimes \times \mathrm{Id}_C) \to \otimes(\mathrm{Id}_C \times \otimes)$,

identity: $U \otimes 1 \to U$.

(The formulas look like the formulas in the definition of an associative algebra!)

These objects satisfy a lot of axioms: for instance:

The square of the commutativity transformation is the identity.

Pentagon diagram: 4 objects, lots of associativity transformations. Allows one to remove parentheses in tensor products.

Hexagon diagram: (permute $U \otimes V$ with W either all at once or in two steps).

Identity axioms. etc.

REASON FOR OMITTING BRACKETS IS A TOPOLOGICAL THEOREM

Introduce a CW complex in which the 0-cells are configurations of brackets in a product (of a given length). 1-cells are associativity isomorphisms. 2-cells are pentagons coming from the pentagon axiom. 4-gons coming from functoriality of the tensor product.

THEOREM (Stasheff). This CW complex is 1-connected. This implies that the isomorphisms corresponding to all closed loops are the identity. (Not so trivial to prove!)

Meaning of the hexagon axiom. The symmetric group acts on the n-fold tensor product. This breaks up into a direct sum of representations parametrized by Young diagrams.

EXAMPLES of TENSOR CATEGORIES

(0) vector spaces over k.

(1) representations of a group Γ.

(2) modules over a cocommutative Hopf k-algebra A.

(3) (exotic example) supervector spaces Super_k. Objects are Z_2 graded vector spaces V. Homomorphisms are gradation preserving homomorphisms. So, as a category, this is isomorphic to $\mathrm{Vect}_k \oplus \mathrm{Vect}_k =$ modules over $k[p]/(p^2 - p) =$ representations of Z_2.

The tensor product is $(U_0, U_1) \otimes (V_0, V_1) = (U_0 \otimes V_0 \oplus U_1 \otimes V_1, U_1 \otimes V_0 \oplus U_0 \otimes V1)$ the commutativity functor is –flip on the factor $U_1 \otimes V_1$, usual flip elsewhere.

FACT: all axioms of a tensor category hold. This explains why the "rule of signs always works".

This tensor category is almost the representations of Z_2.

SEMISIMPLE TENSOR CATEGORIES: each object is a finite sum of simple objects.

EXERCISE (topic for reflection). Define tensor product of two semisimple tensor categories in such a way that the tensor product of the representation categories of two finite groups becomes the representation category of their product.

Then we can show that

$\mathrm{Super}_k \otimes \mathrm{Repr}_k(Z_2) = \mathrm{Repr}_k(Z_2) \otimes \mathrm{Repr}_k(Z_2)$. In some sense, Super_k is the representations of a "twisted form of Z_2."

ANALOG OF FINITE-DIMENSIONAL VECTOR SPACES

A rigid tensor category is a tensor category C together with a duality functor $*: C_{\mathrm{op}} \to C$ together with functorial isomorphism $V^{**} \to V$ plus a "really boring list of axioms". These give rise to a map rank:$Ob\, C \to k = \mathrm{Hom(ONE,ONE)}$ by the composition ONE$\to V \otimes V^* \to$ ONE.

In the rigid tensor category of supervector spaces, the rank of (V_0, V_1) is $\dim V_0 - \dim V_1$.

THEOREM (Deligne, Grothendieck festschrift). Let k be an algebraically closed field of characteristic zero, C a rigid tensor category. if all ranks like in 0,1,2,3,..., then there is a fibre functor:

$C \to \mathrm{Vect}_k$ faithful and commuting with all structures

and a commutative Hopf algebra A such that C is the category of comodules over A.

STRUCTURE THEOREM FOR COMMUTATIVE HOPF ALGEBRAS

$A =$ projective limit A_α, where A_α is finitely generated, i.e. functions on an affine scheme of finite type which is in fact an algebraic group.

Thus C is the category of representations of an affine proalgebraic group.

Milne-Deligne gave examples of rigid tensor categories in which the rank function takes noninteger values. (Base field is rational functions in a variable t.)

CONJECTURE: Rigid tensor categories with ranks in Z should be of two types: comodules over commutative Hopf algebras or comodules over supercommutative Hopf algebras.

APPLICATION OF SUPERMATHEMATICS

Can identify symplectic and orthogonal geometry.

$V =$ (super) vector space, B bilinear form on V with values in ONE. Can construct $\Pi V = V \otimes k^{0|1}$ (odd version of V) and a new form $\tilde{B}$ on ΠV by $\tilde{B} = B \otimes \sim$, where $\sim$: $k^{0|1} \otimes k^{0|1} \to$ ONE is the bilinear form with coefficient one.

COROLLARY: $Sp(2n) = O(-2n)$.

Interpretation: (forget supermath for a moment)

Let g be a Lie subalgebra of $gl(V)$, V finite dimensional. Suppose that the bilinear form $\mathrm{tr}(XY)$ is nondegenerate on g. This leads to many numerical invariants of g, as follows. Choose an orthonormal base X_i of g. Look at the structure constants c_{ijk} in this base, which are totally skew symmetric.

Now fix a word divided into three letter subwords, in some alphabet. Suppose that each letter appears twice in the word. For instance: $ijk\ jik$. Then we can construct the sum

$$\sum_{i,j,k} c_{ijk} c_{jik}.$$

This number is independent of the choice of orthonormal basis.

Now all such words are labeled by trivalent graphs. (vertex = subword, edge = letter).

Now look at the algebras

–4	–3	–2	–1	0	1	2	3	4	5	6
Sp(4)		Sp(2)		0	0	O(2)	O(3)	...		

Exercise: any of the invariants above is given by the values of a polynomial in n.

e.g. dimension of $O(n) = n(n-1)/2$, of $Sp(m)$ is $m(m+1)/2$.

NICEST PROOF OF THIS EXERCISE uses the Π object in supermathematics.

COULD ALSO LOOK AT $osp(n|2m)$–defined by a nondegenerate even bilinear form.

WHERE DOES THE DE RHAM COMPLEX COME FROM?

$A^{0|1}$ is the superscheme whose function ring is the symmetric algebra $S^*(k^{0|1}) = k^{1|1} = k[\epsilon]$ where ϵ is an odd variable.

Aut$A^{0|1}$ is the function algebra of a super group scheme. Its comodules are Z-graded complexes.

On a manifold X, we have the scheme of maps from $A^{0|1}$ to X. On it acts the automorphisms of $A^{0|1}$.....

Kontsevich, Lecture 5

September 6, 1994

Plan for today: explain more about supermathematics (differential and algebraic geometry). Next time: definitions of DGLA structures on standard complexes, in these terms.

Quillen notation: write $\pm$ for

$$(-1)^{\text{sign of permutation of odd symbols}},$$

and $\mp$ for $-\pm$.

For example, in a super Lie algebra, $[x,y] = \mp[x,y]$, $[x,[y,z]] = [[x,y],z] \pm [y,[x,z]]$.

DIFFERENTIAL GEOMETRY

A supermanifold is a topological space with a sheaf O of topological supercommutative associative algebras with unit which is locally like the standard model $R^{n|m}$ – its underlying space is R^n, and the "functions" on an open subset are $C^\infty(U) \otimes S(R^{0|M\,*})$. (We write it this way rather than as a wedge product.)

Simple theorem (exercise). Every $n|m$ dimensional supermanifold Y is isomorphic to one coming from an m-dimensional vector bundle V on an ordinary n-dimensional manifold X. (Functions are sections of the wedge powers of V^*.)

Exercise (on composition of maps): Consider $R^{1|2k}$, mapped to R by the formula

$$y = x + x_{i1}\eta_1 + ... + x_{ik}\eta_k.$$

Now let $z = \sin(y)$. What is $z(x,\xi,\eta)$?

SUPER VECTOR BUNDLE OVER supermanifold Y is a sheaf of O_Y modules which is locally free and finitely generated (i.e.locally $O_Y \otimes R^{k|l}$).

If V is a super vector bundle, tot$V$ is its total space considered as a supermanifold.

OPERATIONS ON VECTOR BUNDLES

direct sum, tensor product, dual, CHANGE OF PARITY operator Π (tensor with $R^{0|1}$).

Associated with a supermanifold Y are 4 bundles

$$TY, \Pi TY, T^*Y, \Pi T^*Y.$$

BIG EXERCISE

1. Define a structure of Lie superalgebra on the sections of TY.

2. Define an odd vector field D on the total space of ΠTY such that $[D, D] = 0$. Note that the functions on totΠTY are called differential forms on Y.

There are 3 versions of differential forms. Let x_i, ξ_j be coordinates on Y.

(a) all C^∞ functions in $D\xi_j$;

(b) all polynomials in $D\xi_j$;

(c) all distributions in $D\xi_j$.

We will use only the choice (b) (If Y is an ordinary manifold, this problem does not arise.)

3. Define a closed (even) 2-form ω on totT^*Y, non-degenerate. Its inverse is a bivector field on totT^*Y, which gives a Poisson bracket on functions on totT^*Y, making them a Lie superalgebra.

4. Define an ODD closed 2 form on totΠT^*Y to get an ODD Poisson structure, and get again a Lie superalgebra structure which in the case where Y is even is the Schouten bracket on the multivector fields.

PROBLEM: In the presence of odd coordinates, one can't integrate differential forms. One can see this by looking at changes of coordinates. SOLUTION: Berezin integral. Requires introduction of "integral forms" which can be integrated–but not multiplied.

A QUASI INTRODUCTION TO ALGEBRAIC GEOMETRY

(most of what we say should work in arbitrary tensor category)

Affine schemes over k = commutative associative algebras with unit, but with arrows reversed. $O(S)$ is the algebra of functions on S, Spec(A) is the scheme of A.

k-points of Spec(A) are algebra homomorphisms $A \to k$,

Can superize the above in the obvious way.

EXAMPLES. A. V super vector space. Consider $S*(V)$,the direct sum of symmetric powers of V, defined as the coinvariants of the (super) action of the symmetric groups on the tensor powers of V.

Notation: when dim$V = n|m$, finite, Spec$S(V) = A^{n|m}$.

A general (not free) finitely generated affine scheme corresponds to the quotient of such an algebra by a Z_2-graded ideal.

B. Scheme of homomorphisms. A, B comm assoc with 1 algebras, B finite dimensional. Then there is an affine scheme Map(Spec$B$, Spec$A$) whose k-points are homomorphisms from A to B.

Define $C = O$(Map) by the finite functorial property.

For any scheme spec$R$, there should be a functorial isomorphism

OrdinaryMap(Spec$R$,Map(Spec$B$, spec$A$)) = OrdinaryMap(Spec$R\times$ Spec$B$, Spec$A$), which equals OrdinaryMap(Spec$R \otimes B$, A);

which implies that

$$\mathrm{Hom}(C, R) = \mathrm{Hom}(A, R \otimes B).$$

let b_i be a homogeneous base of B with $b_0 = 1$. Then a homomorphism form A to $R \otimes B$ is of the form $a \mapsto \sum f_i(a) \otimes b$.

Since $1 \mapsto 1$, we have $f_0(1) = 1, f_i(1) = 0$ for $i \neq 0$.

The multiplicativity of the homomorphisms gives:

$$\sum f_i(a_1, a_2) \otimes b_i = \sum_{jk} \pm f_j(a_1) f_k(a_2) \otimes b_j b_k$$

If the structure constants of B are given by $b_i b_j = \sum c_{ijk} b_i$ we find the relations

$$f_i(a_1, a_2) = \sum_{jk} \pm f_j(a_1) f_k(a_2) c_{ijk}.$$

These are relations on abstract symbols $f_j(a)$ which, together with the relations $f(\lambda a_1 + \mu a_2) = \lambda f(a_1) + \mu f(a_2)$, define the structure of the algebra whose Spectrum is C.

DIFFEOMORPHISMS of $0|1$ dimensional space

Consider $S = \mathrm{Map}(A^{0|1}, A^{0|1}). A = B = O(A^{0|1}) = k^{1|1}$. Let ξ be the odd coordinate on $A^{0|1}$. for such a map we have $f(\xi) = a + b\xi$. The generators are a (odd) and b (even).

The function ring is $k[b, a]$.

Composition of functions gives a coproduct on this algebra given by

$\Delta(b) = b \otimes b$,

$\Delta(a) = a \otimes 1 + b \otimes a$.

Let S^* be the automorphisms of $A^{0|1}$.

This is a closed subscheme of $S \times S$ (pairs of automorphisms with their inverses). S^* is a group object in superschemes, so $O(S^*)$ is a Hopf algebra.

We write $S^* = G_m \times G_a$, where G_m is $\mathrm{Spec}[b, b^{-1}]$ and G_a is $A^{0|1}$.

REPRESENTATIONS OF THE GROUP SCHEME S^*

A representation of S^* is a super vector space V with a comodule structure $\rho : V \to O(S^*) \otimes V = V \otimes k[b, b^{-1}, a]$.

$v \mapsto \sum P_n(v) \otimes b^n \pm Q_n(v) \otimes ab^n$, where almost all $P_n(v)$ and $Q_n(v)$ are zero for any given v.

Now we need commutativity of some diagrams to specify that we have a coalgebra action (compatibility with coproduct and counit). These translate into identities for the P_n and Q_n. (I haven't copied all the calculations from the blackboard.)

We get:

$P_k \circ P_l = 0,\ k \neq l$,

$P_n \circ P_n = P_n$,

$\sum P_n = \mathrm{Id}_V$.

in other words, we have commuting projections which give a direct sum decomposition of V making it into a Z-graded vector space.

We also conclude that Q_k maps V^k to V^{k+1}, with its square zero.

So we get exactly COMPLEXES!

THE "CORRECT OBJECT" which arises in practice is not the full tensor category of complexes of super vector spaces, but rather those for which V^{even} is even and V^{odd} is odd.

ON THE ORIGIN OF THE DE RHAM COMPLEX

Let X be an affine superscheme. Then $\text{tot}\Pi TX = \text{Map}(A^{0|1}, X)$. Then $O(\text{tot}\Pi TX)$ is the algebra generated by a and da, for $a \in O(X)$, with relations given by those in the ordinary algebra of functions, together with $d(ab) = adb \pm adb$.

By general nonsense, the scheme $S^* = \text{Aut}(A^{0|1})$ acts on Map, making it into a differential graded algebra.

Kontsevich, Lecture 6

September 8, 1994

Notes by K.

LIE BRACKETS ON STANDARD COMPLEXES IN ALGEBRA

Recall: moduli problem in geometry (flat/holomorphic bundles, complex structures) $\Longrightarrow$ D(Z)GLA $\Longrightarrow$ functor on Artin algebras.

We will construct today Lie brackets on complexes from algebraic Examples 1,2,3 (Lect. 1,2).

For simplicity we will describe some general constructions in terms of ordinary vector spaces. Everything generalizes to the case of tensor categories, e.g. superspaces.

FREE ALGEBRAS

Notation: for V - vector space$/k$

$\text{Assoc}(V) :=$ free associative algebra (without 1) generated by V.

As a vector space, $\text{Assoc}(V) = V \oplus V \otimes V \oplus V \otimes V \otimes V + ...$ Variant with unit: $\text{Assoc}_1(V) = 1 \oplus V \oplus V \otimes V + ...$.

Analogously, $\text{CoAssoc}(V) :=$ co-free co-associative co-algebra co-generated by V. (Also, $\text{CoAssoc}_1(V) = ...$).

Again, as a space, $\text{CoAssoc}(V) = V \oplus V \otimes V + ...$

Co-product on $A :=\text{CoAssoc}(V)$

$\Delta : A \to A \otimes A$

$\Delta(v_1 \otimes ... \otimes v_n) == \sum_{k:0<k<n}(v_1 \otimes ... \otimes v_k) \otimes (v_{k+1} \otimes ..v_n)$.

If we use CoAssoc_1 then the summation is over $\{0 \leq k \leq n\}$.

DERIVATIONS

For any algebraic structure $A \Longrightarrow$ Lie algebra $\text{Der}(A)$. As a vector space $\text{Der}(A) = \{\text{Automorphisms } T \text{ of } (A \otimes k[h]/h^2) \text{ as an algebra over } k[h]/h^2, T = \text{Id}_A \bmod h\} = \{\text{Automorphisms } 1 + hD, \text{ where } D : A \to A \text{ is a linear map obeying Leibniz rule}\}$.

In tensor categories: Ordinary $\text{Der}(A)$ – ordinary Lie algebra, also there is $\text{Der}(A)$ – Lie algebra in the category.

DERIVATIONS OF FREE ALGEBRAS

As a vector space $\text{Der}(\text{Assoc}(V))=\text{Hom}(V, \text{Assoc}(V))$.

Reason: homomorphism $1 + hD : A \otimes k[h]/h^2 \to A \otimes k[h]/h^2$ is determined by its restriction to the space of generators V.

Analogously, $\text{Der}(\text{CoAssoc}(V)) = \text{Hom}(\text{CoAssoc(V)}, V) = \text{Product}_{n\geq 1} \text{Hom}(V^{\otimes n}, V)$ contains as a Lie subalgebra $\sum_{n>=1} \text{Hom}(V^{\otimes n}, V)$ (we will use the last one).

Brackets: $f : V^{\otimes n} \to V, g : V^{\otimes m} \to V$, $[f,g] : V^{\otimes m+n-1} \to V$

$$\begin{aligned}
&[f,g](v_1 \otimes ... \otimes v_{n+m-1}) = \\
&\qquad \sum_{k=1,n} f(v_1 ... \otimes v_{k-1} \otimes g(v_k \otimes v_{k+1} ... \otimes v_{k+m-1}) \otimes ... \otimes v_{n+m-1}) \\
&\quad - \sum_{l=1,m} g(v_1 \otimes ... \otimes f(v_l \otimes ...) ... \otimes v_{n+m-1})
\end{aligned}$$

Non-commutative analog of Lie algebra of polynomial vector fields.

TENSOR CATEGORY OF COMPLEXES (AND Z-GRADED SPACES)

Complexes of vector spaces + morphisms of complexes of degree 0.

Tensor product: $[(V,d) \otimes (U,d)]^n := \sum_k (V^k \otimes U^{n-k})$.

Differential$_n := \sum (d_k \otimes 1) + (-1)^k (1 \otimes d_{n-k})$.

Commutativity map: $(-1)^{kl} : V^k \otimes U^l \to U^l \otimes V^k$.

Z-graded spaces:=complexes with zero differential.

Notation: for complex $C, C[1] := (k$ in degree $-1) \otimes C$.

$C[1]^k = C^{k+1}, d_k$ of $C[1] = -d_{k+1}$ of C.

DGLA ASSOCIATED WITH VECTOR SPACE

A – vector space $\Longrightarrow \Gamma :=$ Der(CoAssoc $A[1]$) Lie algebra in the tensor category of complexes. Picture of Γ:

-2	-1	0	1	2	$\dots$
0	0 (or A if use CoAssoc$_1$)	Hom(A, A)	Hom$(A \otimes A, A)$	Hom$(A^{\otimes 3}, A)$	$\dots$

Lemma: Associative product $m : A \otimes A \to A$ is equivalent to $m \in \Gamma^1, [m,m] = 0$.

Proof: compute $[m,m](v_1 \otimes v_2 \otimes v_3)$, use formula for [,].

Fix such $m \implies$ differential on $\Gamma, dx = [m, x]$.

Exercise: Check that $d =$ Hochschild differential shifted by 1.

Brackets on $C(A,A)[1]$ called Gerstenhaber brackets.

Trivial Theorem: 2 functors: Artin algebras $\to$ Groupoids coincide:

1) Artin algebra R with the ideal $M \mapsto$

Objects: R-linear products on $R \otimes A =$ initial product mod M,

Morphisms: R-linear isomorphisms equal to 1 mod M,

2) Functor constructed in Lecture 3 from (Γ, d).

Proof: Maurer-Cartan equation ($d\gamma + \frac{1}{2}[\gamma,\gamma] = 0$) is equivalent to $[m+\gamma, m+\gamma] = 0$. Gauge Action of Lie algebra Γ^0 became adjoint action after the shift of Γ by m.

Remark: if one wants to consider isomorphisms of associative algebras MODULO interior automorphisms : change a little bit construction of morphisms in groupoids (in the functor associated with DGLA) using γ^{-1}.

OTHER ALGEBRAIC STRUCTURES (commutative and Lie algebras)

Naive idea: imitate construction for associative algebras - works, but with changing of roles of commutative and Lie algebras !

Functors Lie, Comm: vector spaces $\to$ free algebras, Also functors CoLie, CoComm, Comm$_1$, CoComm$_1$.

As vector spaces: $\mathrm{Comm}(V) = V + S^2(V) + S^3(V) + \ldots = \mathrm{CoComm}(V)$. $\mathrm{Lie}(V) = V + \wedge^2(V) +$ more complicated terms $= \mathrm{CoLie}(V)$.

Usual definition of $\mathrm{Lie}(V)$: on $A := \mathrm{Assoc}_1(V)$ define a coproduct $\Delta : A \to A \otimes A$. (Homomorphism of algebras). On generators $\Delta(v) = v \otimes 1 + 1 \otimes v$.

DEF: $LI(V) := \{a \in A | \Delta(a) = a \otimes 1 + 1 \otimes a\}$.

Exercise:1) $a, b \in \mathrm{Lie}(A)$ then $(ab - ba) \in \mathrm{Lie}(A)$;

2) give a definition of CoLie analogous to the def of Lie.

Let ? be Comm or Lie. As for associative algebras we have

$\mathrm{Der}(\mathrm{Co?}(V)) = \mathrm{Hom}(\mathrm{Co?}(V), V) = \prod \mathrm{Hom}(\text{homogeneous components of } \mathrm{Co?}(V), V)$ contains $\sum \ldots$. Last Lie algebra for $V = A[1]$ is denoted by $\Gamma_?(A)$.

LEMMA: 1) structure of Lie algebra on $A \iff \gamma \in \Gamma_{\mathrm{Comm}}(A)^1, [\gamma, \gamma] = 0$;

2) structure of commutative associative algebra on $A \iff \gamma \in \Gamma_{\mathrm{Lie}}(A)^1, [\gamma, \gamma] = 0$.

Explanation of 1) (leave 2) as an exercise):

Picture of $\mathrm{CoComm}(A[1])$:

-3	-2	-1	0	1	2	...
$\wedge^3(A)$	$\wedge^2(A)$	A	0 (or 1)	0	0	...

because $(\wedge^k(A))[k] = S^k(A[1])$.

Picture of $\gamma_{\mathrm{Comm}}(A)$:

-2	-1	0	1	2
0	0 (or A)	$\mathrm{Hom}(A, A)$	$\mathrm{Hom}(\wedge^2(A), A)$	$\mathrm{Hom}(\wedge^3(A), A)$

$\gamma \in \Gamma_{\mathrm{Comm}}(A)^1$: skew-symmetric bilinear operation on A, $[\gamma, \gamma] = 0 \iff$ Jacobi identity.

We can repeat all the story as for associative algebras, [,] on Eilenberg-MacLane complex $C(A, A)[1]$ was introduced by Nijenhuis-Richardson (1967) (here A is a Lie algebra).

For commutative algebra A $(\Gamma_{\mathrm{Lie}}(A), d)$ is called Harrison complex, it is a subcomplex of the Hochschild complex (in fact, sub DGLA).

Thus, we accomplished our task and constructed DGLA structures on all standard complexes from examples 1–6.

SITUATION IS NOT COMPLETELY SATISFACTORY

in Geometry: we used analytic methods (de Rham, $\bar{d}$ complexes). There are closely related Questions which are more algebraic: moduli of flat bundles over finite simplicial complexes, moduli of algebraic vector bundles, moduli of algebraic varieties. DGLA should be constructed over arbitrary field. Also, analytic complex are not useful for direct computations.

in Algebra: What to do with other algebraic structures? How to explain (or avoid) strange duality in the definition of standard complexes?

We need to develop a better understanding of DGLA.

GENERALITIES ON DGLA

In practice there are two examples of DGLA:

1) From deformation theory: usually sits in degrees 0,1,2,... Sometimes we have Γ^{-1}.

2) From rational homotopy theory (Quillen, Sullivan):

$$\ldots \to \Gamma^{-3} \to \Gamma^{-2} \to \Gamma^{-1} \to 0 \to 0 \to \ldots$$

I'll explain 2) later. Good to have in mind topological analogies.

The first basic construction in DGLA is

(CO)-HOMOLOGY

Start to explain in the case of ordinary Lie algebras:

g/field $k \implies$ two complexes $C_*(g,1), C^*(g,1)$ (chains, co-chains) – DG commutative algebra, DG co-commutative co-algebra...

I consider chains as more fundamental object because one can get cochains by passing to the dual complex.

Simplest definition of cochains: imagine that $k = R, \dim g < \infty,\ g =$ Lie algebra of a Lie group G.

$C^*(g,R) := (\Omega^*(G))^G$ (use left action of G on itself) $= (\wedge^*(g))^*$.

We already have the definition of the chain co-algebra: $\mathrm{CoComm}_1(g[1])$ with the differential associated with [,].

Theorem: 1) $g/R =$ Lie algebra of compact connected Lie group $G \implies H^*(g,R) = H^*(G,R)$ (as of a topological space).

2)g/Q is nilpotent, $G :=$ abstract group associated with g (see Lecture 3) $\implies H^*(g,Q) = H^*(K(G,1),Q)$.

Kontsevich, Lecture 7

Notes by Alan Weinstein

9/13/94

Homological meaning of $H^*(g,1)$: it is Ext of g-modules (1,1),where 1 is the trivial 1 dimensional representation.

Meaning of Ext: g-modules are the same as $U(g)$-modules.

Choose a free resolution of 1:

$$\begin{array}{ccccccccccc}
\ldots & U(g)\otimes\wedge^2 g & \longrightarrow & U(g)\otimes g & \longrightarrow & U(g) & \longrightarrow & 0 & \longrightarrow & 0 & \ldots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\ldots & 0 & \longrightarrow & 0 & \longrightarrow & 1 & \longrightarrow & 0 & \longrightarrow & 0 & \ldots
\end{array}$$

vertical arrows give a quasiisomorphism of these complexes.

Assuming g finite dimensional, the dual spaces to the spaces in this complex are the differential forms with formal coefficients around the identity in the Lie group. Since the formal neighborhood is contractible, there is a Poincare lemma, whose proof uses the Euler vector field transported from the Lie algebra via the exponential map.

Now consider $\mathrm{Hom}_g(U(g)\otimes\wedge^k g, 1)$, dual to $\wedge^* g$.

Analogously, one can define Ext^* of g-modules $(1,V)$ as the cohomology of the complex whose cochains are multilinear alternating maps from g to V.

EXERCISE: write explicitly the differential in this complex.

One can also define Homology and chains.

We also have $C^*(g,g)$ with coefficients in the adjoint representation. (Chevalley-Eilenberg complex), essential to deformation theory. It's a bit surprising that these constructions arising from abelian category theory have application to deformation theory.

STANDARD (QUILLEN) CHAIN COMPLEX FOR DGLA

$\Gamma \to C(\Gamma,1)$, a Z-graded space which is the sum of the symmetric powers of $\Gamma[1]$.

Differential $d = d_1 + d_2$.

Consider Γ just as a Z-graded algebra, and let d_1 be the differential in its chain complex (from S^k to S^{k+1}).

For d_2, forget the bracket and let d_2 be the differential, from S^k to S^k. These two differentials anticommute, so their sum is a differential.

ANOTHER WAY

Γ is a Lie algebra in the tensor category of complexes.

$C(\Gamma, 1)$ will be a complex in this category, i.e. a bicomplex, with spaces C^{ij} and differentials d_1 and d_2 raising the first and second degrees respectively. Here, C^{ij} is zero for positive i and for negative i is the j-th tensor power (in the tensor category of complexes) of the $(-i)$-th exterior power of Γ. (Big bi-diagram here which I can't reproduce. AW)

Now we take the total complex of this bicomplex.

A bicomplex is a module over $\mathrm{Aut}(A^{0|1})$ times $\mathrm{Aut}(A^{0|1})$, which becomes a module over $\mathrm{Aut}(A^{0|1})$ (i.e. a complex) via the diagonal embedding of these Lie superalgebras.

The construction above "is purely formal and has nothing to do with derived functors."

$C(\Gamma, 1)$ is a differential graded coalgebra (cocommutative, with counit).

CENTRAL FACT

Theorem (proof next time). Assume that Γ is a DGLA with nonnegative degrees, with $H^0(\Gamma) = 0$ (i.e. $d_0 : \Gamma^0 \to \Gamma^1$ is injective).

Then $(H_0(\Gamma, 1))^*$ is a complete pro-(local Artin) algebra.

The functor from local Artin algebras R to the set $\mathrm{Hom}_{\mathrm{continuous}}((H_0(\Gamma, 1))^*, R)$, considered as a groupoid with only identity morphisms is equivalent to the deformation functor associated with Γ.

This theorem was proposed by Drinfeld (letter 1988), Deligne (letter 1989), Feigin, ...

A FEW MORE WORDS ABOUT (cocommutative, coassociative, counital) COALGEBRAS

Any such coalgebra A is a union of finite dimensional subcoalgebras.

Proof: $\Delta a =$ finite sum of $x_1 \otimes y_1$. The linear span A_a of the x_i (which equals that of the y_i, by cocommutativity) is finite dimensional. A computation (too fast to type! AW) shows that A_a is a sub- coalgebra.

Also, The sum of two finite-dimensional subcoalgebras is another one.

QED

The dual space A_a^* in the finite dimensional case is an Artin algebra. In general, it is a limit of finite dimensional Artin algebras.

COCONNECTED COALGEBRAS

$A = k \oplus \tilde{A}$, $\tilde{A}$ coalgebra without unit.

$\tilde{A}$ should be conilpotent in the sense that higher products disappear. (equivalently, all finite dimensional subcoalgebras are duals of local Artin algebras).

A gives rise to a functor on local Artin algebras

$$R \to \mathrm{Hom}_{\mathrm{cont}}(A^*, R) \to \mathrm{Hom}_{\mathrm{coal}}(R^*, A).$$

Continuous homomorphisms from $A*$ to k are the same as elements of $A^* \otimes k$ satisfying certain identities.

We will always have co-connected coalgebras. Start from cofree algebras and pass to some homology.

QUASI-ISOMORPHISMS OF DGLA'S

A homomorphism $f : \Gamma_1 \to \Gamma_2$ is a quasiisomorphism if it induces an isomorphism of cohomology spaces.

THEOREM. A quasiisomorphism f induces a quasiisomorphism of chain complexes $C(\Gamma, 1)$.

PROOF. The chain complexes are filtered: $F_0 \subset F_1 \subset F_2 ...$, where F_m is the sum of symmetric powers of order up through m. The action of Cf preserves this filtration, so something is induced on the associated graded object, call it $\mathrm{gr}(f)$.

Lemma 1. $f : X \to Y$ quasiisomorphism implies that its symmetric powers are quasiisomorphisms.

Lemma 2. If X and Y are filtered complexes with filtration bounded from below, $f : X \to Y$ a filtered morphism such that $\mathrm{gr}(f)$ is a quasiisomorphism, then f is a quasiisomorphism.

Lemma 1 + Lemma 2 implies the theorem above.

PROOF OF LEMMA 1

Define a homotopy between morphisms of complexes as usual. ($[d, h] = f - g$). One writes $f \sim g$. Now one can prove that, for complexes over a field, quasiisomorphism = homotopy equivalence. But one can prove that tensor powers of a homotopy equivalence are homotopy equivalences.

PROOF OF SECOND LEMMA

Usually this is done with spectral sequences, but there is another way.

SUBlemma 1. $f : X \to Y$ is a quasiisomorphism iff its cone is acyclic, where the cone is the total complex of the bicomplex

$0 \to X \to Y \to 0 \to ...$, where X is in degree -1.

SUBlemma 2. If X is filtered bounded below, then if $\mathrm{gr}X$ is acyclic, X is acyclic.

Proof that the sublemmas imply the lemma is straightforward logic.

PROOF OF SUBLEMMAS

For the first , use the standard exact sequence:

$$H^i(X) \to H^i(Y) \to H^i(\mathrm{cone} f) \to H^{i+1}(X) \to ...$$

For the second, the filtration of the complexes induces a filtration on cohomology,....

CONCLUSION. The cohomology of differential graded Lie algebras is invariant under quasiisomorphisms.

DEFORMATION FUNCTOR (revised) for DGLA with negative degree components.

fix $\Gamma, R \supset m$, Artin algebra with nilpotent ideal m.

Result is a groupoid whose objects are elements of $\Gamma^1 \otimes m$ satisfying the Maurer-Cartan equation and whose morphisms comes from the action of the group associated with $\Gamma^0 \otimes m$.

The Lie algebra of the stabilizer of some object Γ is the set of solutions of $[a, \gamma] - da = 0$, which contains as an ideal the set of $a = [\gamma, b] + db$, for $b \in \Gamma^{-1}$. There is a corresponding normal subgroup, which gives rise to a quotient groupoid with the same objects but fewer morphisms. (EXERCISE: check that this is correct.)

THEOREM. If $f : \Gamma^1 \to \Gamma^2$ is a quasiisomoprhism, it induces an equivalence of the modified (as above) deformation functors.

NOTE (A.W.) Is the modified deformation functor related to the "extended moduli spaces" used in gauge theory.

Kontsevich, Lecture 8

September 15, 1994

Notes by K.

Today we will make some essential preparations to the proofs of theorems from the last lecture.

STRONG HOMOTOPY LIE ALGEBRAS

By definition, SHLA is a co-(free commutative associative) Z-graded algebra C without co-unit + co-derivation d of C of degree $+1$, $d^2 = 0$.

Notice that in the definition we don't fix an isomorphism of C with CoComm(V) for some Z-graded space V. We will refer to the choice of such an isomorphism (of Z-graded coalgebras) as a coordinates on C.

In coordinates derivation d is determined by its restriction to co-generators, i.e. by composition

$$\sum_{n>=1} S^n(V) = C \xrightarrow{d} C[1] \stackrel{\text{projection}}{\longrightarrow} V[1] \to V[1].$$

This is just a collection of maps

$$d_n : S^n(V) \to V[1]$$

satisfying an infinite system of quadratic equation (encoded as $d^2 = 0$).

Let $A := V[-1]$, maps d_n lead to "higher brackets"

$$[\, ,\, , \ldots ,\,]_n : \wedge^n(A) \to A[2-n],$$

for $n = 1, 2, \ldots$

Condition $dd = 0$ in explicit form is:

For $n \geq 1$ and homogeneous $v_1, \ldots, v_n$

$$\sum_{\sigma \in S_n} \sum_{k,l \geq 1, k+l=n+1} \pm [[v_{\sigma_1}, \ldots, v_{\sigma_k}]_k, \ldots., v_{\sigma_n}]_l = 0.$$

$n = 1$ equation is just $[[v]_1]_1 = 0$. Hence, $[\,]_1 : A \to A[1]$ can be considered as a differential.

$n = 2$ equation means that $[\, ,\,]_2 : \wedge^2(A) \to A$ is a homomorphism of complexes.

$n = 3$ equation means that $[\, ,\,]_2$ satisfies Jacobi identity up to homotopy given by $[\, ,\, ,\,]_3$.

....

COROLLARY: on $H^*(A, [\,]_1)$ bracket $[\, ,\,]_2$ defines a structure of Z-graded Lie algebra.

We have seen already in Lecture 6 that DGLA = SHLA with coordinates in which $[...]_k = 0$ for $k = 3, 4, \ldots$

MORPHISMS OF SHLA-s

By definition, morphism is morphism of differential graded coalgebras $f : C_1 \to C_2$.

Remark: free algebras are defined by functorial property $\mathrm{Hom}_{\mathrm{algebras}}(\mathrm{Comm}(V), B) = \mathrm{Hom}(V, B)$. Analogously, co-free algebras are defined by $\mathrm{Hom}_{\mathrm{coalgebras}}(B, \mathrm{CoFree}(V)) = \mathrm{Hom}(B, V)$ falgebra.or CONNECTED B.

Thus, morphism of co-free coalgebras in coordinates is an infinite collection of maps

$$f_1 : A_1 \to A_2,\ f_2 : \wedge^2(A_1) \to A_2[-1],\ \text{etc.}$$

Compatibility with d turns into a sequence of equations, meaning that f_1 is a morphism of complexes, compatible with $[\,,\,]_2$ up to homotopy...

Notice that for DGLAs A_1, A_2 there are much more morphisms in the category of SHLA than in DGLA.

GEOMETRIC PICTURE OF SHLA

Dual space to a cofree coalgebra $C = \sum_n S^n(V)$ is an algebra of formal power series $C^* = \prod_n (S^n(V))^*$ (without unit). Adding unit we get formal functions on a formal manifold (may be, infinite-dimensional) with a base point 0. Algebraic "choice of coordinates" corresponds to the identification of $\mathrm{Spec}(C^{*+k_1})$ with the formal neighborhood of zero at the tangent space $T_0(C) := \mathrm{Ker}(\Delta : C \to C \otimes C)$.

SHLA structure defines an odd vector field d, $[d, d] = 0$ vanishing at 0. ($\Longleftrightarrow$ action of algebraic supergroup $G_a^{0|1}$). Morphisms of SHLAs are equivariant mappings.

Thus, SHLA are critical points of $G_a^{0|1}$-actions. What can one say about non-critical points?:

Theorem: non-vanishing odd formal vector field d, $[d, d] = 0$ is equivalent to the vector field with constant coefficients. (In some coordinates (x_i) $d = d/dx_1$). Proof: exercise.

The situation is parallel to the usual theory of ordinary differential equations: vector field is locally equivalent to the constant one near points where it is non-zero, and the classification of critical points is hard.

The next analogy with analysis is

THEOREM ON INVERSE MAPPING: homomorphism $f : C_1 \to C_2$ between two co-free Z-graded coalgebras is isomorphism if and only if the induced map on the level of tangent spaces $Tf_0 : T_0(C_1) \to T_0(C_2)$ is an isomorphism.

PROOF: $C_{1,2}$ are filtered: $F_k(C) = \mathrm{Ker}((\Delta \otimes 1 \otimes ... 1)...(\Delta \otimes 1)\Delta)$ ($k+1$ times, $k \geq 0$). Map f is compatible with filtrations. Using induction as in the last lecture we obtain that F is an isomorphism. QED

If C is a SHLA then on $T_0(C)$ arises differential (from the linear part of d at zero). We consider it as a complex.

Definition: TANGENT QUASIISOMORPHISM between SHLAs is a morphism $f : C_1 \to C_2$ inducing quasi-isomorphism on tangent spaces.

Lemma: Tangent qis induces quasiisomorphism of chain complexes C_*. Proof: the same as of the analogous statement from Lecture 7 on DGLAs.

One of reasons of introducing SHLA: if there exists t-qis: $C_1 \to C_2$ then there exists (not-canonical) t-qis: $C_2 \to C_1$. (Will prove soon). It follows that (existence of t-qis) is an equivalence relation. Call it HOMOTOPY EQUIVALENCE.

Problem: classify SHLAs up to homotopy equivalence. Solution: introduce two basic types of SHLAs:

1) contractible: there are coordinates in which $[\dots]_k = 0$ for $k > 1$ and $\mathrm{Ker}[\,]_1 = \mathrm{Im}[\,]_1$.

2) minimal: $[\,]_1 = 0$ in some ($\Longleftrightarrow$ any) coordinates.

THEOREM ON MINIMAL MODELS: Each SHLA is isomorphic (after adding 1) to the tensor product of a contractible and a minimal SHLA.

Corollary 1: inversion of t-qis:

$$\begin{array}{ccc} \mathrm{Contr}_1 \otimes \mathrm{Min}_1 & \xrightarrow{\text{t-qis}} & \mathrm{Contr}_2 \otimes \mathrm{Min}_2 \\ \uparrow \text{t-qis} & & \downarrow \text{t-qis} \\ \mathrm{Min}_1 & \xrightarrow{\text{composition t-qis}} & \mathrm{Min}_2 \end{array}$$

Last horizontal arrow is t-qis between two minimal SHLA, hence it is an isomorphism (by inverse mapping theorem). Invert it.

Corollary 2: homotopy classes of SHLA = equivalence classes of minimal SHLAs. (Use the same diagram).

RELATION WITH MASSEY PRODUCTS:

If A is DGLA then we construct a structure (up to iso) of minimal SHLA on $H(A)$. That is, $[\ ]_2$ (=usual bracket on $H(A)$) and higher []s. $[\ ]_3, [\ ]_4$ etc. depend on the choice of coordinates. Only leading coefficients are canonically defined.

Example of the simplest Massey operation: $x, y, z \in H(A), [x, y] = [y, z] = [z, x] = 0$. Element in $H(A)$/Lie ideal generated by x, y, z. Degree = $\deg x + \deg y + \deg z - 1$. Pick representatives X, Y, Z of x, y, z in $\mathrm{Ker}d : [X, Y] = d\gamma, [Y, Z] = d\alpha, [Z, X] = d\beta$. By Jacobi identity: $d([\alpha, X] \pm [\beta, Y] \pm [\gamma, Z]) = 0$. Call cohomology class of the expression in brackets by $[x, y, z]$. Exercise: $[x, y, z]$ is well-defined modulo $[H(A), \langle x, y, z\rangle]$ and it is represented by $[x, y, z]_3$ in any coordinate system.

PROOF OF THE MINIMAL MODEL THEOREM: Pick coordinates and try to modify it by higher order corrections getting as result three groups of coordinates (x_i, y_i, z_j) in which $d = \sum_i x_i d/dy_i + \sum_j \mathrm{coeff} * z^{\geq 2} d/dz_j$. First order: split complex $(A, [\ ]_1)$ into the sum of $(..0 \to 0 \to k \to k \to 0 \to 0 \to ...)$ and $(..0 \to k \to 0 \to 0 \to ...)$. Step of induction: we have $d = \sum_i x_i d/dy_i + \sum_j \mathrm{coeff} * z^{2\leq \dots \leq N} d/dz_j +$ higher terms. Denote $(\sum_i x_i d/dy_i)$ by d_0.

Next term in the Taylor expansion is

$$\sum_i A(x, y, z)_i \frac{d}{dx_i} + \sum_i B(x, y, z)_i \frac{d}{dy_i} + \sum_j C(x, y, z) \frac{d}{dz_j},$$

A, B, C are homogeneous polynomials of degree $N + 1$.

Equation $[d, d] = 0$ gives (1) $d_0(A_i) = 0$ (2) $-A_i + d_0(B_i) = 0$ (3) $d_0(C_j) =$ some function $F_j(z)$ ($F_j(z)$ arises from commuting of the middle term in the formula for d with itself).

If we apply a diffeomorphism close identity exp(vector field ξ),

$$\xi = \sum_i A'_i \frac{d}{dx_i} + \sum_i B'_i \frac{d}{dy_i} + \sum_j C'_j \frac{d}{dz_j},$$

where A', B', C' are polynomials of degree $N+1$, the change of d will be:

(1) $A_i \to A_i + d_0(A'_i)$

(2) $B_i \to B_i + A'_i + d_0(B'_i)$

(3) $C_j \to C_j + d_0(C'_j)$

We pose $A'_i := -B_i, B'_i := 0$, killing A and B. Also, we can find C' such that the new C is function in z only. The reason is that on $k[[x, y, z]]$ cohomology of d_0 are equal $k[[z]]$. QED

Kontsevich, Lecture 9

Notes by Alan Weinstein

SOME REFERENCES

W. Goldman, J. Millson, The homotopy invariance of the Kuranishi space, Ill. J. Math. 34 (1990), 337-367.

Goldman-Millson, the deformation theory of representations of π_1 (Kahler manifold), Publ. I.H.E.S. 68 (1988), 43-96. (contains description of functor from Artin algebras to groupoids)

Review article by Feigin-Fuks (1986) in Sovremeenaye Problemy Mathematik, Fund. Napravlenie, vol. 21 (relation of H_0 and moduli space) (maybe not translated).

Deformation of complex manifolds: best reference is

Kodaira, K., Complex manifolds and deformations of complex structures (book)

there is also Kuranishi, Deformation theory (book, pretty old-fashioned)

For algebraic deformation theory, there was a 1979 preprint of Stasheff-Schlesinger, "Deformation and rational homotopy theory", which was never published (but K has a copy).

PLAN FOR TODAY. Finish the abstract nonsense, go on to examples.

Recall that, associated to a deformation problem was a functor from Artin algebras to groupoids. In examples, we went from the deformation problem to a DGLA and from there to a functor. On the other hand, we can also go from DGLA's to chain complexes– differential free coalgebras (SHLA's). Today, we will construct an arrow from SHLA's to functors on Artin algebras to prove the homotopy invariance of deformation theory.

Recall that an SHLA is essentially a formal manifold with a single (base) point, and an odd vector field with $[d, d] = 0$ and vanishing at the base point.

How do we picture such an odd vector field d on a supermanifold M? Let S be the subspace defined by the vanishing of d. It is given by the vanishing of df for all functions f. This can be pretty complicated and singular.

We will construct a sort of foliation of S. The operator $[d,]$ is a differential on the vector fields; consider its kernel. These vector fields commute with d, so they are tangent to S and hence define vector fields on S.

We have $\mathrm{Im}[d,] \to \ker[d,] \to \mathrm{vect}(S)$ inclusions of linear subspaces. In fact these are inclusions of Lie subalgebras (by Jacobi) which are also $O(S)$ submodules (by Leibniz), so they define "singular foliations" of $\mathrm{vect}(S)$. We are particularly interested in the foliation defined by $\mathrm{Im}[d,]$.

We can try to decompose the (even points) of S as a union of leaves, which are submanifolds S_α of various dimensions (something like symplectic leaves of a Poisson structure? AW)

For each point x of S, its formal neighborhood is a SHLA. The SHLA's sitting at different points of the same leaf (for the IMAGE foliation) are isomorphic SHLA's. (Use the flows of the vector fields tangent to the leaf.)

Groupoid associated to this picture: (something like holonomy groupoid of a foliation? AW)

objects=points of S

morphisms are given by paths $f(t)$ in a leaf and vector fields $v(t)$ generating them, modulo some identifications:

$v(t)$ is equivalent to $v(t) + u(t)$ where $u(t)$ vanishes at $f(t)$. (Here we are solving $f'(t) = [d, v(t)](f(t))$ to get a path in the leaf.)

$v(t)$ is equivalent to $v(t) + [d, u(t)]$

we can move everything by diffeomorphisms depending on t such that $D(t)D(t)^{-1} = [d, ?(t)]$.

One can check that the groupoid axioms are satisfied by looking at minimal models for the transverse structure along a leaf. (There is a splitting theorem, where the "trivial" factor is a contractible SHLA.)

(NOTE–The algebra of multivector fields on a manifold makes the cotangent bundle into a supermanifold (with odd fibres). A Poisson structure is an odd vector field on this manifold.) (I don't quite have this right AW.)

SHLA$\Longrightarrow$ FUNCTOR ON ARTIN ALGEBRAS

C coalgebra without counit, $d : C \to C[1]$. Artin algebra R with maximal ideal m.

coints of S (objects of groupoid) will be $\mathrm{Hom}_{\mathrm{coalg}}(m^*, C^0)$ such that the image is contained in the kernel of d.

In coordinates $C = \mathrm{Sym}(a)[1]$, an object is a $\gamma \in m \otimes A^1$ satisfying the generalized Maurer-Cartan equation:

$$[\gamma]_1 + \frac{1}{2}[\gamma, \gamma]_2 + \frac{1}{6}[\gamma, \gamma, \gamma]_3 + ... = 0.$$

WHICH OBJECTS ARE EQUIVALENT? (full definition of morphism would involve "nasty formulas"; see further remark below)

Consider differential equations for $\gamma(t)$ (polynomial in t)

$$\gamma'(t) = [a(t)]_1 + [a(t), \gamma(t)]_2 + \frac{1}{2!}[a(t), \gamma(t), \gamma(t)]_3 + ...$$

where $a(t)$ is a polynomial in t with values in $A^0 \otimes m$.

We say here that $\gamma(0)$ is equivalent to $\gamma(1)$.

Morphisms are equivalence classes of such differential equations under an equivalence relation like the one above.

LEMMA. For DGLA'S, the deformation functor constructed a few lectures ago agrees with the functor just constructed for SHLA's.

(Straightforward to check.)

LEMMA. The two maps (inclusion and projection) minimal $\longrightarrow$ minimal $\otimes$ contractible induce equivalence of deformation functors.

COROLLARY. Quasiisomorphisms between SHLA's (DGLA's) induce equivalences of their deformation functors. (Theorem promised 1 week ago.)

(Application: Goldman-Millson) the moduli space of unitary representations of the fundamental group of a compact Kahler manifold is locally quadratic when the H^0 is zero.

THEOREM. If A is a SHLA with all nonpositive cohomology zero, then

1. all automorphisms in the values of the deformation functor are the identity.

2. π_0 (deformation functor) is represented by the coalgebra $H_0(C)$.

PROOF (pretty garbled, I'm afraid AW)

1. Since $H^0(A) = 0$, any homomorphism $m^* \to \ker d \subset C^0$ Lie algebra of automorphisms of object $= H^0$(same complex filtered)

quotients have zero cohomology at degree zero.

2. The minimal model has no morphisms. Look at the Maurer-Cartan equations....

STANDARD STATEMENTS OF DEFORMATION THEORY

1. $H^1(\Gamma) = 0 \implies$ no deformations 2. $H^0(\gamma) = 0, H^2(\gamma) = 0 \implies$ smooth moduli space whose tangent space is H^1.

3. $\dim H^1 - \dim H^2 \leq \dim$ moduli space $\leq \dim H^1$.

ACTUAL MODULI SPACES

Theorem (Kuranishi) X compact complex manifold. There exists a miniversal deformation over a germ of analytic space.

Theorem (Goldman-Millson) The formal completion of this germ can be defined through the formal theory related with vector valued forms. (Assuming $H^0 = 0$; otherwise the statement is more complicated.)

Theorem (Artin) If two germs of analytic spaces are formally equivalent, then they are analytically equivalent.

EXAMPLES.

CURVES. Let X be a complex curve with no holomorphic vector fields (genus at least 2). Then the germ of moduli space is smooth, with tangent space $H^1(X, TX)$. Its dimension is $3g - 3$. (This is not actually the moduli space, for which we have to divide as well by morphisms far from the identity, giving a orbifold structure.)

SURFACES. Consider a surface X of degree d (at least 4) in CP^3. We have the cohomological bounds on the dimension of the moduli space of complex structures on X.

Miracle: the dimension of the moduli space is always equal to the dimension of H^1, even though H^2 is nontrivial for d at least 5.

PROOFS: For degree at least 5, the dimension of H^1 is the dimension of the space of hypersurfaces modulo linear transformations.

For degree 4, $\dim H^1(X, T) = 20, \dim H^2(X, T) = 0$, but we have only a 19-dimensional family of quartics. The remaining family are the K3 surfaces.

Kontsevich, Lecture 10

September 22, 1994

Notes by K.

HARMONIC DECOMPOSITION

Let (C^*, d) be a complex of pre-Hilbert spaces (i.e. we fix a positive hermitian scalar product on each C^k). We assume that (1) conjugate operators d^* to d are defined (we don't assume that d are bounded) (2) C^k with Laplacian $\Delta := dd^* + d * d$ is orthogonal direct sum of a finite-dimensional space $\mathbf{H}^k$ on which $\Delta = 0$ and a space on which Δ is invertible.

Then (C^*, d) decomposes canonically into the orthogonal direct sum of complexes

$$\ldots \to 0 \to 0 \to \mathbf{H}^k \to 0 \to \ldots$$

and contractible complexes of length 2. Spaces $\mathbf{H}^*$ are canonically isomorphic to cohomology $H^k(C^*)$.

Denote by G Green operator on C^* acting by zero on $\mathbf{H}^k$ and by Δ^{-1} on the rest.

KURANISHI SPACE

X - compact complex manifold.

Lie algebra controlling deformations of complex structures on X: $\Gamma^* := \Gamma(X, \Omega^{0,*} \otimes T^{1,0})$, differential $= \bar{d}$ (vector valued forms).

Choose hermitian metric h on TX (not Kahler!). Induce L_2-norms on Γ^*. Then the harmonic decomposition appear because Green operator exists by the theory of pseudo-differential operators.

We will construct a germ of analytic space in $\mathbf{H}^1$.

Define map

$$M : \Gamma^1 \to \text{orthogonal complement to } \mathbf{H}^1 \text{ in } \Gamma^1 :$$

$$\phi \mapsto \text{projection along } \mathbf{H}^1 \text{ of } \phi + \frac{1}{2}\bar{d} * G([\phi, \phi]).$$

On the level of tangent spaces at zero M is surjection. We expect that the germ at 0 of $M^{-1}(0)$ is a germ of manifold of dim=dimH^1. To prove: introduce norms on Γ^1 in which M is analytic (at least continuous!)

Naive counting: if ϕ has n derivatives than $[\phi, \phi]$ has $n-1$ derivatives, $G([\phi, \phi])$ has $(n-1)+2$ derivatives, $\bar{d}$ of ... has $n-1+2-1 = n$ derivatives.

(1) C_n-norms (maximum of derivatives up to n-th order) are not good because the Green operator in dimension larger then 1 can make unbounded function from C_2 function.

(2)Sobolev norms are not good because they give spaces not closed under the product of functions which appear as a part of $[\phi, \phi]$.

Nirenberg's idea: use Hoelder norms. Parameters $n \geq 0$ (integer), $0 < a < 1$. In coordinates: f – function in R^d with support in a fixed compact.

$$|f|_{n+a} := \sum_{k=0,\ldots,n} (\sup|D^k f| + \sup_{x,y}(|D^k f(x) - D^k f(y)|/|x-y|^a)).$$

Spaces C_{n+a} are strictly between C_n and C_{n+1}.

Properties of Hoelder norms:

(1) $|fg|_{n+a} < \text{Const}|f|_{n+a}|g|_{n+a}$;

(2) $|f'|_{n+a-1} < \text{Const}|f|_{n+a}$;

(3) $|f|_{n+a-1} < |f|_{n+a}$;

(4) $|f|_{n+a} < \text{Const}|\text{Laplacian of f}|_{n+a-2} + \text{Const}|f|_0$.

The only non-trivial property is (4). We will not prove it, just use.

After that we get analytic germ $M^{-1}(0)$ consisting of smooth forms. We can identify the germ of $M^{-1}(0)$ with H^1 using orthogonal projection to $\mathbf{H}^1$.

Kuranishi map: $k : M^{-1}(0) \to \mathbf{H}^2$, $\phi \to$ harmonic part of $[\phi, \phi]$. Germ of analytic map, Kuranishi space:= $K := k^{-1}(0)$ - germ of analytic space.

LEMMA: $\phi \in K \iff \bar{d}(\phi) + [\phi, \phi]/2 = 0$ and ϕ is orthogonal to $\mathrm{Im}(\bar{d})$.

PROOF: $\Longrightarrow$: we want to prove that $R := \bar{d}(\phi) + [\phi, \phi]/2$ is equal to zero. Because harmonic part of $[\phi, \phi]$ is harmonic we have $[\phi, \phi] = \Delta G([\phi, \phi])$. Substitute it into the formula for R: $R = \bar{d}(\phi + \bar{d}^* G([\phi, \phi])/2) + \bar{d}^* \bar{d} G([\phi, \phi]) = \bar{d}^* \bar{d} G([\phi, \phi])/2$ (the first summand is in $\bar{d}(\mathbf{H}^1) = 0) = \bar{d}^* G \bar{d}([\phi, \phi])/2 = \bar{d}^* G([\bar{d}\phi, \phi]) = \bar{d}^* G([\bar{d}\phi + [\phi, \phi], \phi]) = \bar{d}^* G([R, \phi])$.

We use Jacobi identity $[[\phi, \phi], \phi] = 0$. Hence $R = \bar{d}^* G([R, \phi])$. For ϕ small enough operator $?? \to \bar{d}^* G([??, \phi])$ has norm less than 1 with respect to Hoelder norms.$\Longrightarrow R = 0$.

$\Longleftarrow$: leave as an exercise. QED

It is not trivial to prove that we get an actual miniversal deformation (see formal version in Lecture 3). We omit the proof of this fact.

Formalization (Goldman-Millson):

Definition: ANALYTIC DGLA is a DGLA with norms $| \ |_i$ on Γ^i (in our example $| \ |_i$ will be Hoelder norm $| \ |_{N+a-i}, N$ is large).

Axioms: (1) d^i are bounded operators,

(2) complex Γ^* of pre-Banach spaces is continuously isomorphic to the sum of pre-Banach complexes of length 1 and continuously contractible pre-Banach complexes of length 2,

(3) $\dim H^1$, $\dim H^2 < +\infty$,

(4) for $x, y \in \Gamma^1 \quad |[x, y]|_2 <= \mathrm{Const}|x|_1 |y|_1$.

One can repeat Kuranishi's arguments and get a germ of analytic space. To prove that it is an actual miniversal deformation one needs extra properties of Γ^0. It was not developed accurately by Goldman-Millson. Nevertheless one can check that we get miniversal deformations for the case of flat/holomorphic bundles too.

FORMAL VERSION OF KURANISHI SPACE

Γ - DGLA/ any field of char=0. Choose subspace Γ'^1 in Γ^1 complementary to $d(\Gamma^0)$. Construct a new DGLA Γ':

degree	-1	0	1	2	3	$\dots$
	0	0	Γ'^1	Γ^2	Γ^3	$\dots$

with brackets and differential induced from Γ^*. Formal moduli space for Γ' is well-defined because $H^{\leq}(\Gamma') = 0$ and co-functions on it are $H_0(\Gamma', 1)$. Call it formal Kuranishi space of Γ (or formal miniversal deformation). It is not canonical.

Exercise:(1) equivalence class of fKS does not depend on the choice of Γ'^1,

(2) equivalence class of fKS is invariant under qis of DGLAs,

(3) if $H^0(\Gamma) = 0$ then fKS is formal moduli space,

(4) for analytic DGLA formal completion of KS is fKS.

KAEHLER METHODS

$\partial - \bar{\partial}$–Lemma: Let C^{**} be a bicomplex of pre-Hilbert spaces, differentials (unbounded) $\delta : C^{ij} \to C^{i+1,j}, \bar{\delta} : C^{ij} \to C^{i+1,j}$. Assume that Laplacian for δ = Laplacian for $\bar{\delta}$ and satisfies properties as in the harmonic decomposition lemma. Then $C^{*,*}$ can be decomposed into the direct sum of bicomplexes looking like

$$\begin{matrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{matrix}$$

all the differentials are zero, and

$$\begin{array}{ccc} C & \xrightarrow{1} & C \\ \uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle -1} \\ C & \xrightarrow{1} & C \end{array}$$

all other components are zero.

Proof:this is the tensor product of Harmonic decomposition lemma with itself.QED

Basic examples: X - Kaehler manifold: Dolbeault bicomplex of differential forms, generalization: forms with coefficients in unitary local system.

Other examples: $N = (2, 2)$ supersymmetric field theories.

We will show three applications of $\partial - \bar{\partial}$–Lemma in all of which X will be a compact complex manifold such that there exists a Kaehler metric on X. We will not fix it.

1. Moduli of complex representations of $\pi_1(X)$.

Fix a unitary representation $\rho : \pi_1(X) \to U(N) \to GL(N, C)$. Denote by ξ the associated local system of vector spaces. Controlling DGLA Γ is $\Omega^*(X, \mathrm{End}\xi)$. Because X is complex we have two extra differentials δ and $\bar{\delta}$. Consider diagram

$$\Gamma^* \stackrel{\text{inclusion}}{\longleftarrow} \mathrm{Ker}\delta \stackrel{\text{projection}/\mathrm{Im}\delta}{\longrightarrow} H(X, \mathrm{End}\xi).$$

Both arrows are qis of DGLA. Differential on the last DGLA is zero.

Conclusion: Γ is formal (i.e. qis to its cohomology with zero differential, $\Longleftrightarrow$ on a minimal model only $[\ ,\ ]_2$ is non-vanishing). Corollary (Goldman-Millson): Moduli space has singularity at $[\rho]$ isomorphic to an intersection of homogeneous quadratic cones. Number of quadratic equations is $\mathrm{dim}H^2$.

In fact, we have more than that: we have an identification of germs. There is a germ of holomorphic vector field on moduli space corresponding to the Euler vector field on vector space H^1.

Question: How to write down explicitly this germ of vector field? What kind of transcendental functions we have to use?

There is a bunch of theorems proven by C.Simpson few years ago about moduli spaces of representations of π_1 of Kahler manifolds. He constructed a real-analytic action of C^* on moduli. Presumably, our vector field is a holomorphic component of Simpson's. Still I don't know what kind of functions appear (do they satisfy a non-linear algebraic differential equations?, how to continue them analytically? etc.)

2. Moduli of holomorphic vector bundles.

Fix again a unitary representation $\rho : \pi_1(X) \to U(N) \to GL(N, C)$. We consider deformation theory of holomorphic vector bundle $\xi \times O$. Controlling DGLA Γ is $\Gamma(X, \Omega^{0,*} \times \mathrm{End}\xi), \bar{\delta}$. Consider Lie subalgebra $\mathrm{Ker}\delta$. It has zero differential already. Inclusion of $\mathrm{Ker}\delta$ is qis. Again, we have quadratic singularities and mysterious germ of the vector field.

3. Moduli of complex structures on Calabi-Yau manifolds. Suppose that X admits a holomorphic everywhere non-vanishing N-form where $N = \mathrm{dim}X$. In other words, $c_1(X) = 0$ in $\mathrm{Pic}(X)$ = moduli of line bundles on X. Such manifolds are called Calabi-Yau because they admit Calabi-Yau metrics, that is Kaehler metrics with $c_1(X) = 0$ on the level of differential forms.

TIAN-TODOROV THEOREM: moduli of complex structures on CY manifolds are unobstructed. Moduli space is smooth of dimension=dim$H^1(X,T)$.

Original proof uses Calabi-Yau metrics it was looking as a miraculous cancellation of complicated terms. Again, Goldman and Millson realized that it is a consequence of homotopy invariance of Kuranishi space.

Controlling DGLA Γ is $\Gamma(X,\Omega^{0,*}\times T^{1,0})$. We include it in larger DGLA graded by $Z\times Z:\Gamma'=\Gamma(X,\Omega^{0,*}\times\wedge T^{1,0})$ with differential= $\bar{\delta}$and brackets = wedge product for $\bar{\partial}$ forms times Schouten-Nijenhuis bracket for polyvector fields.

Let us choose a holomorphic volume element vol on X. Using it we can identify $\wedge^k T^{1,0}$ with $\wedge^{N-k}(T^{1,0})^*$. This iso changes by a scalar factor if we change vol. We denote by ∂' operator on Γ' induced from δ on Ω^{**}.

LEMMA (Tian-Todorov): $[f,g]=\partial'(f\wedge g)-\partial'(f)\wedge g\pm f\wedge\partial'(g)$.

Here wedge is natural product on $\Gamma(X\wedge(T^{0,1})^*\times\wedge T^{1,0})$.

This lemma can be obtained by simple direct calculations on coordinates. Next time I'll tell more about it. QED

Consider diagram

$$\Gamma'\overset{\text{inclusion}}{\longleftarrow}\text{Ker}\partial\overset{\text{projection}/\text{Im}\partial'}{\longrightarrow}H^*(X,\wedge^*T).$$

Last DGLA has zero [,] and differential.

It follows from TT lemma that both arrows are qis. (More details in the next lecture). Thus, we have smooth moduli spaces because all quadratic equations are zero, and germs of vector fields on moduli. There is satisfactory understanding of these vector fields in terms of variations of Hodge structures.

Kontsevich Lecture 11

Notes by AW

CALABI-YAU MANIFOLDS

Recall that a Calabi-Yau manifold is one which admits a nowhere vanishing holomorphic volume element (determined up to a constant) and a Kahler metric.

Consider the bigraded space $\Gamma^{\cdot\cdot}$ = holomorphic multivector fields tensor antiholomorphic differential forms. With a fixed volume element, we can identify the multivector fields with holomorphic differential forms and then identify $\Gamma^{\cdot\cdot}$ with all the smooth differential forms. We get a quasi isomorphism, not depending on the choice of constant in the volume element, between cohomology with value in the multivector fields,...... (SORRY, I LOST THE THREAD HERE.)

Now consider $\Gamma^{\cdot\cdot}$ as a supercommutative algebra by Λ.

LEMMA (Tian-Todorov). ∂' is an odd second order differential operator, defining Poisson brackets by

$$[f,g]=\partial'(f\wedge g)-\partial' f\wedge g\pm f\wedge\partial' g.$$

MODEL SITUATION. Real smooth manifold with volume element Y. Then we can identify multivector fields with differential forms by interior product. Then d transfers to an operator d' on multivector fields. if we think of the forms as functions on the odd tangent bundle, with d as a vector field, then when we go over to thinking of the multivector fields as functions on the odd cotangent bundle, we can think of going from one to the other by a "Fourier transform in odd variables". If we have

$$d = \sum dy_i \frac{\partial}{\partial y_i},$$

we get

$$d' = \sum \frac{\partial^2}{\partial \xi_i \partial y_i}.$$

The symbol of d' is an odd symmetric bivector field on $\Pi T{*}Y$, which gives the Schouten bracket. (Batalin-Vilkovisky geometry.)

NEW CONSTRUCTION OF CLOSED DIFFERENTIAL FORMS

Let α be an even function on the odd cotangent bundle satisfying two equations:

$d'\alpha = 0$

$[\alpha, \alpha] = 0$

This implies $d'(\alpha^n) = 0$. Using the isomorphism with forms, we get interesting closel of Phid differential forms. (REMEMBER THAT WE ARE CARRYING AROUND A VOLUME ELEMENT.)

Note that $f, g \in \mathrm{Ker}\partial' \Longrightarrow [f, g] \in \mathrm{Im}\partial'$, so $\mathrm{Ker}\partial'$ is a Lie subalgebra containing $\mathrm{Im}\partial'$ as a Lie ideal with the quotient being abelian.

COROLLARY. On the cohomology with values in multivector fields, the bracket [,] induced from the Schouten bracket is zero. In particular, $H^0(X, T)$ is an abelian Lie algebra, so the connected component of the identity in the automorphism group of X is abelian.

Because dimension of the automorphism group $= \frac{1}{2}h^1(X)$ is locally constant, we can construct a good moduli space even when $H^0(X, T)$ is not zero.

QUESTION: Why after deformation do we still have a CY manifold?

We will obtain as a corollary of Kodaira stability theorem in the first part of the next lecture.

FLAT STRUCTURE ON MODULI SPACE OF CY MANIFOLDS

PREPARATIONS: Let M be a Kahler manifold with real-analytic Kahler form omega. Choose a point m. Then we can construct a holomorphic affine structure on a neighborhood of this point.

Look at $M \times \bar{M}$ containing the diagonal as a totally real submanifold. The form omega has an analytic continuation to a holomorphic symplectic form on a neighborhood of the diagonal. the fibres of the projections onto the factors of the product are lagrangian submanifolds (because ω is a 1-1 form). But then these leaves carry flat affine structures. (Learned from a physics paper - Vafa, Cecotti,...)

One can use the same construction also for pseudo-Kahler forms (= nondegenerate closed 1,2-forms without condition of positivity).

QUESTION (AW) Is there a more geometric description of this "exponential mapping"?

WEIL-PETERSSON METRIC ON MODULI OF CY SPACE

there are two descriptions.

First, on the moduli space M we construct a line bundle whose fibre at each point is the space of homomorphic volume elements. This descend to the moduli space because action of $H^0(X,T)$ on $H^0(X,\Lambda^n TX)$ (and, hence, on $H^0(X,\Lambda^n T^*X)$) is trivial by qis in Tian-Todorov theorem.

There is a hermitian metric on L given by $\int_X \mathrm{vol}\wedge\overline{\mathrm{vol}}$,

The Weil-Petersson (pseudo)metric is the curvature of this metric on L. In fact, this is just a non-degenerate 1,1-form which is positive if we restrict it to families of POLARIZED Calabi-Yau (i.e., families of complex structures with fixed Kahler class).

Approach 2. Identify the tangent space to M at $[X]$ with $H^1(X,T_X)$. Using the volume, we identify these with $H^1(X,\Omega^{n-1})$. Now the pairing is given by integrating $\alpha\wedge\bar{\alpha}$.

CLAIM: 1=2, flat structure arising from the WP (pseudo)-metric is the same as the one arising from quasiisomorphisms.

We will prove all this in the next lefcture.

STANDARD FACTS ABOUT CY MANIFOLDS

Theorem (Yau). In the real class represented by the Kahler form, there is another Kahler form whose n-th power is a constant times $\mathrm{vol}\wedge\overline{\mathrm{vol}}$ (equivalently, the metric is Einstein).

Theorem (Bogomolov). Each CY manifold X has a finite covering $\tilde{X}$ which is a product of a complex torus with flat metric and complex structure times a product of indecomposable hyperkahler manifolds times a product of "indecomposable CY manifolds in the proper sense". All the factors of the last two types are simply connected.

Indecomposable hyperkahler is one for which $\dim H^2(X,O)=1$, with the class represented by a complex symplectic structure on X. The CY metric has holonomy $\mathrm{Sp}(\dim X/2)$.

Indecomposable CY in the proper sense means that $n=\dim X>2$, and $\dim H^k(X,O)$ is 1 for $k=0$ and n and 0 otherwise. These manifolds are all algebraic.

Moduli spaces for the first two factors:

for tori–well known $GL(n,C)\backslash GL(2n,R)/GL(2n,Z)$. Polarized tori with integral polarization class are algebraic (called abelian varieties). Moduli space of abelian varieties is $U(n)\backslash Sp(2n,R)/$disrete subgroup.

for hyperkahler manifolds–according to Todorov, moduli space (of polarized hyper-Kahler manifolds) is open and dense in $SO(2)\times O(n)\backslash O(2,n)/O(2,n;Z)$, maybe up to finite covering.

When $\dim X=1$, a Calabi-Yau manifold is an elliptic curve, defined by a lattice parameter τ. The Weil-Petersson metric is $\partial\bar{\partial}\log\mathrm{Im}\tau$, which is the standard upper half plane metric.

When $\dim X=2$, we have the K3 surfaces and C^2/Z^4.

There are a lot of 19 dimensional families of algebraic surfaces, intersecting one another along a complicated locus. Kodaira proposed first to consider nonalgebraic K3 surfaces.

A classification of K3 surfaces was given by Piatetski-Shapiro and Shafarevich, with an error fixed by Looijenga.

CLAIM. For compact complex surfaces X carrying nowhere zero vanishing holomorphic volume element, with $H^1(X,O)=0$, there is always a Kahler metric. (Idea: first show that $\dim H^1(X,T)=20$, by Riemann-Roch. Also, deformations are unobstructed since $H^2(X,T)=0$. The moduli space carries a line bundle given by the second complex cohomology of the surfaces, containing $H^0(X,\Omega^2)$ as a subspace. Its orthogonal space intersects the integer cohomology, so we can find a line bundle L with Chern class $c_1(L)$ in this intersection. We can assume that $(c_1(L),c_1(L))\geq 0$. By Riemann-Roch and Serre duality, $h^0(L)+h^0(L^*)>0$. Thus we get line bundles with a lot of sections and can prove that X can be deformed to an algebraic surafce. Then we have to study limits of Kahler K3 surfaces etc...).

Kontsevich Lecture 12

Notes by AW

MORE DETAILS ABOUT LAST TIME

Recall that a Calabi-Yau manifold is a compact complex manifold which ADMITS a holomorphic volume form (nowhere 0) and a Kahler metric.

Stability Theorem (Kodaira). In an analytic family X_t of compact complex manifolds, the set of t for which X_t has a Kahler form is open. (Proof is nonelementary, using functional analysis.)

FACT (C^*,d) a complex of finite dimensional vector spaces, with d depending continuously on a parameter d. The dimensions of the homology groups are upper semicontinuous functions of t. (Proof is elementary.)

THEOREM. (Kodaira? Grauert?) Given a family X_t of complex manifolds carrying a family E_t of holomorphic vector bundles, then $\dim H^k(X_t,E_t)$ is USC.

Proof. Cohomology is given by the kernel of a family of elliptic operators (laplacian).

Note also that, if the dimension is constant, we get a holomorphic bundle over the parameter space.

PROOF OF THE KODAIRA STABILITY THEOREM

Suppose that X_0 is Kahler. Look at sheaf cohomology with coefficients in differential forms. using the ideas above and a spectral sequence, one concludes that the dimensions of these cohomologies are constant.

LEMMA: The following sequence is exact:

$$\begin{aligned}0 &\to (\ker d\cap\Omega^{1,1}+d\Omega^1)/d\Omega^1 \to H^2(X,C)\\ &\to (\ker\partial:\Omega^{2,0}\to\Omega^{3,0})/\mathrm{Im}\partial:\Omega^{1,0}\to\Omega^{2,0})+\text{another term with }\bar\partial\ .\end{aligned}$$

Proof. Let w be closed. Write it as $w_{20}+w_{11}+w_{02}\ldots.$

FROM THE LEMMA, it follows that the dimension of $((\ker d\cap\Omega^{1,1})+d\Omega^1)/d\Omega^1$ is at least $h^2(X_2)-h^{2,0}(X_t)-h^{0,2}(X_t)$, which equals $h^{1,1}(X_t)=h^{1,1}(X,0)$. Rewrite $((\ker d\cap\Omega^{1,1})+d\Omega^1)/d\Omega^1$ as $(\ker d\cap\Omega^{1,1})/(\ker d\cap\Omega^{1,1}\cap d\Omega^1)$. The last space is a quotient space of $L:=(\ker d\cap\Omega^{1,1})/(\partial\partial'\text{of }\bar\Phi\Omega^{0,0})$.

We have for all small t: $\dim L\geq h^{1,1}$. Then apply the $\partial-\bar\partial$ Lemma: at $t=0$ $\dim L=H^{1,1}$. Now, identify L with $(\ker d\cap\Omega^{1,1}\cap(\text{orthogonal complement to }(\partial\bar\partial\Omega^{0,0}))=$ intersection in $\Omega^{1,1}$ of $\mathrm{Ker}\partial,\mathrm{Ker}\bar\partial$ and $\mathrm{Ker}(\partial\bar\partial)^*$.

This is the same as the kernel of "sum of squares":

L iso to $\mathrm{Ker}((\partial^*\partial)^2+(\bar\partial^*\bar\partial)^2+\partial\bar\partial(\partial\bar\partial)^*)$. R.H.S. is elliptic PDO of order 4 with positive index. $\Longrightarrow$ $\dim L$ is upper semicontinuous. $\Longrightarrow$ $\dim L$ is locally constant. Hence we have a smooth family of harmonic representatives of closed 1,1-forms. They are positive everywhere on X_t for small t. QED of Kodaira theorem.

EXERCISE. Suppose that we have a finite-dimensional bicomplex C_t of vector spaces with differentials depending on a parameter. Suppose that for C_0 we have a decomposition as in the $\partial - \bar\partial$ lemma into sum of trivial and small squares. Also suppose that the dimension of the cohomology of the total complex is constant. Then we have a $\partial - \bar\partial$ decomposition of C_t for t near 0.

Since the dimension of $H^0(X_t, \Omega^n_{X_t})$ is constant equal to 1, we can conclude that the existence of a volume form persists after small deformations. Nevertheless, we develop explicit ...

DEFORMATION THEORY OF COMPLEX MANIFOLDS WITH VOLUME ELEMENTS

$(X, \mathrm{vol}) \Longrightarrow$ DGLA, defined to be

$$\Gamma^k_{\mathrm{vol}} = \text{sections of } \Omega^{0,k} \otimes T^{1,0} \oplus \Omega^{0,k-1}$$

the differential is $\bar\partial + \partial'$ in a suitable way; ∂' is the divergence.

The brackets are given by the bracket of vector fields and the action of vector fields on functions. It is Dolbeaut resolution of the sheaf of DGLA on X, $0 \to T_X \to O_X \to 0$, with the functions in degree 1.

CLAIM. $\gamma_{\mathrm{vol}} \in \Gamma^1_{\mathrm{vol}}$ with $D\gamma_{\mathrm{vol}} + \frac{1}{2}[\gamma_{\mathrm{vol}}, \gamma_{\mathrm{vol}}] = 0$ corresponds to new complex structure on X with holomorphic volume element. Action of Γ^0_{vol} has same orbits as diff$X$.

1. $\gamma_{\mathrm{vol}} = (\gamma, f)$, where γ is a Beltrami differential and f is a function. In coordinates, $\gamma = \sum \gamma_{ij} d\bar z_i \dfrac{\partial}{\partial z_j}$ (vol = product of dz_i).

The new complex structure is such that its antiholomoprhic vector fields are generated by $\dfrac{\partial}{\partial \bar z_i} + \sum \gamma_{ij} \dfrac{\partial}{\partial z_j}$. Let α_j be the dual basis of 1 forms. Then the volume element is given by $1 + f$ in this dual basis.

The Maurer-Cartan equation becomes $\bar\partial\gamma + \frac{1}{2}[\gamma, \gamma] = 0$ (which is integrability of the complex structure), and the equation

$$\partial'\gamma + \bar\partial f + [\gamma, f] = 0,$$

which is equivalent to the equation $d((1+f)(\alpha_1 \wedge \ldots \wedge \alpha_n)) = 0$.

To see this we write $\alpha_j = dz_j - \sum \gamma_{ij} d\bar z_i$, then compute $d\alpha_j$. Using the MC equatiion for γ, one gets $d\alpha_j = \sum \dfrac{\partial \gamma_{ij}}{\partial z_k} d\bar z_i \wedge \alpha_k$.

Now one can compute the differential of the volume element and show that it is zero.

"EXERCISE". (Solution not known to K.) Guess what is the right DGLA associated with the problem of deformations of complex manifolds equipped with Kahler forms.

GAUSS-MANIN CONNECTIONS

X_t locally trivial family of topological spaces. Then we get flat vector bundles on the base given by the cohomology (complex coefficients) of the fibres.

Suppose now that the X_t are complex manifolds which admit Kahler metrics. Then we have on $H^n(X, C)$ a pure Hodge structure of weight n; i.e. a rational lattice within it, and a decomposition into the direct sum of $H^{p,q}$.

Now suppose that X_t depends analytically on parameters in an analytic space. assume for simplicity that this space is again a complex manfiold.

Now it is important that the Hodge decomposition is NOT invariant under parallel translation in the flat connection. In fact, ∇(smooth section of $H^{p,q}$) has components in 3 spaces, but wedged with 1 forms of different types.

$$\Omega^1 \otimes H^{p,q} + \Omega^{0,1} \otimes H^{p+1,q-1} + \Omega^{1,0} \otimes H^{p-1,q+1}.$$

COROLLARY. $F_0^p = \sum_{p \geq p_0} H^{p,q}$ are holomrphic subbundles.

Proof – Look at a family of p, q forms,...

Also, motion of the Hodge component $H^{p,q}$ in direction $H^{p-1,q+1}$ is given by the contraction with the element in $H^1(X,T)$ corresponding to the 1-st order deformation.

BACK TO CALABI-YAU MANIFOLDS

We have proven that there is a miniversal deformation of X over a germ M of analytic manifold of dim $= h^{n-1,1}$ such that the X_t are CY for small t.

We can identify $H^n(X_t, Q)$ with that for X_0 by using the Gauss-Maniin connection. We have the map $t \mapsto H^{n,0}(X_t) = V \otimes C$, the period map of M into the projective space of lines in $V \otimes C$.

The period map is locally an embedding. One can see the motion of the Hodge component $H^{n,0}$ of H^n by using the natural isomorphism from $H^1(X,T) \otimes H^{n,0}(X)$ to $H^{n-1,1}(X)$.

MORE ABOUT THE WEIL-PETERSSON METRIC

On V (as above–middle cohomology) we have a bilinear form given by the Poincare pairing. This gives a metric on an open domain in the tautological line bundle over $P(V \otimes C), v \mapsto (v, \bar{v})$.

the curvature is a 1-1 form on a domain in $P(V \otimes C)$. The induced 1,1 form on moduli space M via the period map is in general pseudo-kahler. To get a positive form, we must restrict to families of POLARIZED CY manifolds. These are such for which there exists $[\omega_t]$ covariantly constant under Gauss-Manin and which give kahler metrics on X_t. Universal family of CY in the proper sense is locally polarized because Kahler cone $\{[\omega] | \omega \text{ is Kahler form}\}$ is open in $H^2(X, R)$ when $h^{2,0}(X) = 0$.

In general, if one has a real-analytic pseudokahler form, one can construct flat structures around each basepoint.

On the other hand, we can choose a holomorphic lift of the period map $P : M \to V \otimes C$. We get

$$m \mapsto (P(m), d\bar{P}(m_0))/(P(m), \bar{P}(m_0)) \in T_{m_0}^{0,1\,*} M.$$

THEOREM. The flat structure arising from the period map (or Weil-Petersson metric) is the same as the one which arises from diagrams of DGLA's.

PROOF: We realize $H^1(X,T)$ as $\dfrac{\mathrm{Ker}\partial'}{\mathrm{Im}\partial'}$. In the proof of Tian-Todorov theorem we have diagram of qis:

$$\gamma \leftarrow \mathrm{Ker}\partial' \to \mathrm{Im}\partial' \to H^1(X,T).$$

For element $g \in H^1(X,T)$ there exists $\gamma \in \mathrm{Ker}\partial'$, $\bar{d}(\gamma) + [\gamma, \gamma]/2 = 0$ and $[\gamma] = g$. Let us construct volume element for complex structure defined by Beltrami differential γ: In explicit formulas of deformation theory of complex varieties with volume elements (see above) we take pair $(g, 0)$.

Thus, in local coordinates $\alpha_1 \wedge \ldots \wedge \alpha_n$ is a holomorphic n-form. It homogeneous components with respect to the initial complex structure are vol (in degree $n, 0$), γ contracted with vol (in degree $n-1, 1$), etc. It is clear that $(n-1, 1)$ component is ∂- closed. Pairing of this form with harmonic (for the initial structure) $(1, n-1)$-forms is linear on g, because it depends only on ∂-cohomology class of γ contracted with vol. QED

Kontsevich Lecture 13

Notes by AW

MORE ON K3.

Take a complex surface X with vanishing $H^1(X, O)$ and a holomorphic volume element.

Theorem: Such a surface is Kahler.

Hodge table:

$$\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 20 & 0 \\ 1 & 0 & 1 \end{array}$$

H^2 is an even unimodular lattice with index 3,19 for the Poincare pairing.

By the theory of quadratic forms over Z, this is $-E_8 + E_8 + 3$ copies of $\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$.

....

More discussion here on the relation between the integer lattice and the subspace $H^0(X, \Omega^2) \subset H^2(X, C)$. The aim seems be to give a description of the moduli space of K3 surfaces.

THEOREM S a complex space. Consider the following groupoid. Objects are families of K3 surfaces over S, morphisms are isomorphisms of families.

This groupoid is equivalent to the groupoid of local systems Λ over S with Z-valued scalar product and extra structure:

$L =$ holomorphic subbundle of complexified Λ $C =$ open subset in the total space.....> ? >? >? >? >

ALGEBRAIC K3 SURFACES

By Kodaira, its necessary and sufficient for the Kahler cone to contain an integral class. The degree of an algebraic K3 is defined to be the minimum of the $(v, v)/2$ for v in the Kahler cone $C \cap \Lambda$.

As a set, we can introduce the set M_d of equivalence classes of K3 with fixed $(v, v)/2 = d$.

It is a 19-dimensional quasi-projective variety.

M_1 has an open part which consists of K3 surfaces which are double coverings of CP^2, ramified along curves of degree 6.

$M_2 =$ quartics in CP^3.

Such elementary descriptions exist up to M_5.

On each M_d the WP metric is positive and locally looks like $SO(2, 19)/SO(2) \times SO(19)$.

MILES REID–Analogous picture in dimension 3.

Consider 3d CY in the proper sense. X simply connected, Hodge numbers

$$\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & b & a & 0 \\ 0 & a & b & 0 \\ 1 & 0 & 0 & 1 \end{array}$$

Dimension of moduli space M is b.

The period mapping maps M to $P(X^3(X,C))$ (map symplectic structure to the space of volume elements). The target space is symplectic of dimension $2b+2$.

By Griffiths transversality, the period mapping is an embedding. The cone over M is a lagrangian cone in $H^3(X,C)$, so M itself is legendrian in the projective space.

CONJECTURE (Reid) All the lagrangian cones which arise from moduli spaces are degenerations of one infinite dimensional cone.

Idea (Clemens). To connect moduli spaces for CY manifolds with different a and b. Let X be a 3d CY. $j : CP^1 \to X$ a rational nonsingular curve. These curves should be isolated. In fact. first order deformations are given by global sections of the normal bundle. This is a 2d bundle with $C_1 = -2$.

THEOREM. (Grothendieck) Any holomorphic vector bundle on CP^1 is isomorphic to a direct sum of line bundles which are tensor powers of the tautological bundle. The sum of the (negatives of the) powers is the Chern class.

Thus the typical normal bundle should be $O(a) \oplus O(-2-a)$.

The first order deformations of this bundle E are $H^1(CP^1, \operatorname{End} E)$.

Deformation arguments show that $(-1,-1)$ curves are preserved under deformation of X.....

Theorem. If C is a $(-1,-1)$ curve, X/C is an analytic space with it's singularity at the contracted point isomorphic to sum of squares $= 0$.

Clemens idea is to deform X/C in the category of analytic spaces.

FLAT DEFORMATION: Deform the singular part like sum of four squares = epsilon, where epsilon is a function on the parameter space.

What happens to $H^i(X)$ if we deform X/C to a smooth variety? More generally, we could deform several $(-1,-1)$ curves C_α.

$$H^2(X_{\text{new}}) = H^2(X)/\langle [C_\alpha] \rangle.$$

$$\text{rank} H^3(X_{\text{new}}) = \text{old rank} + 2 + \text{linear relations between} [c_\alpha].$$

When the $[c_\alpha]$ generate H^2, we get a complex manifold with $H^2 = 0$. In this case, by a theorem of Wall, we have a connected sum of $S^3 \times S^3$'s.

There is also a theorem of Tian which says that the deformations are unobstructed.

Now introduce the moduli space M_g of complex structures on the connected sum of g copies of $S^3 \times S^3$ ("quaternionic curves"). This space has dimension $g-1$. Now take a limit as g goes to infinity.

MODULI SPACES OF OTHER (NOT CY) MANIFOLDS

In almost all examples, the moduli space is smooth and of dimension equal to that of $H^1(X,T)$, despite the fact that $H^2(X,T)$ may be zero.

For example, in CP^n, look at complete intersections $P_1 ... P_k = 0$, where the degree of P_i is $d_i > 1$,

Deform by varying the coefficients of the P_i.

THEOREM (Kodaira Spencer for hypersurfaces, Palamodov)

this deformation is a versal deformation except in K3 surfaces and the following cases:

$$n = 3, k = 1, d_1 = 4$$
$$n = 4, k = 2, d_1 = 3, d_3 = 2$$
$$n = 5, k = 3, d_1 = d_2 = d_3 = 2$$

The deformations are unobstructed.

QUESTION. For CY we have homotopy equivalence of the deformation DGLA with its cohomology (with zero differential and zero bracket). Is the same true for other manifolds?

CONJECTURE. (A. Todorov) Suppose that X is a projective algebraic variety with canonical bundle K_X very ample. (sections separate points...) Then there are no obstructions to deformation.

BACK TO GENERAL ALGEBRA–HOMOTOPICAL ALGEBRA

(There is a book by Quillen–1971 on this subject, containing some axioms and examples, but the situation of this subject is currently very unsatisfactory)

GENERAL PRINCIPLE. Suppose that we have a functor F or more general construction from some algebraic structures to some other category of algebraic structures. Then we can construct a derived functor RF from the same initial structures to a category of differential graded algebraic structures modulo homotopy equivalence.

Assume that F is defined in terms of operations in a tensor category over characteristic zero.

First step: F is applicable to any tensor category, hence it is applicable to tensor category of complexes.

Also, we need to prove that $F(\text{qis}) = \text{qis}$.

Second step: replace algebras by free resolutions. Then apply the functor to the free resolution to get RF.

CLAIM. Deformation theory, as a construction from certain kinds of algebras to DGLA's is the derived functor of the "functor of derivations" from algebras to Lie algebras. (Actually it's a construction rather than a functor.)

NEXT TIME: examples.

Lecture 14.

Notes by M.K.

EXAMPLES OF DERIVED FUNCTORS

We start from standard additive functors.

Example 1. Fix associative algebra A. Functor $(A-\text{mod})^{\text{opposite}} \times A-\text{mod} \longrightarrow$ vector spaces $P, Q \longrightarrow \text{Hom}_{A-\text{mod}}(P, Q)$.

Pick free resolutions $P^* : ... \to P^{-2} \to P^{-1} \to P^0$ (qis $P[0]$) and Q^*. Apply functor interior Hom to complexes P^*, Q^* get complex $\text{Hom}(P^*, Q^*)$; k-th component $\text{Hom}(P^*, Q^*)^k$ is equal to the direct product $\prod_i \text{Hom}_{A-\text{mod}}(P^i, Q^{i+k})$.

Lemma: $\text{Hom}(P^*, Q^*)$ is qis to $\text{Hom}(P^*, Q[0])$ (no need to choose resolution of Q).

Proof: We want to prove that the cone of morphism $\mathrm{Hom}(P^*, Q^*) \to \mathrm{Hom}(P^*, Q[0])$ is contactible. Notice that $\mathrm{Hom}(P^*, Q^*)$ is *not* a total complex of a bicomplex because we use infinite products instead of sums. $\mathrm{Hom}(P^*, Q^*)$ is filtered by degree in Q-component. This filtration is DECREASING and COMPLETE. The same is true for the cone. It is easy to see that if the associated graded complex is contarctible then the total complex is contarctible. Associated graded factors are complexes

$$... \to \mathrm{Hom}_{A-\mathrm{mod}}(P^k, Q^{-1}) \to \mathrm{Hom}_{A-\mathrm{mod}}(P^k, Q^0) \to \mathrm{Hom}_{A-\mathrm{mod}}(P^k, Q) \to 0.$$

We can replace $\mathrm{Hom}_{A-\mathrm{mod}}(P^k, ???)$ by $\mathrm{Hom}_{\text{vector spaces}}(G^k, ???)$ where G^k denotes a space of generators of free A-module P^k. Hence asoociated graded factors are contractible. QED

Cohomology of comples $\mathrm{Hom}(G_k, Q)$ are called Ext-groups.

INDEPENDENCE OF EXTs of the choice of resolution P^*:

Scheme of proof is quite general:

Step 1. For any two free resolutions P_1^*, P_2^* there exists qis $f : P_1^* \to P_2^*$ which is a morphism of complexes of A-modules. Construct f by induction: $f_0 : P_1^0 \to P_2^0$ will be any lift of the map $P_1^0 \to P$ to P_2^0 (using freeness of P_1^0); $f_0 d : P_1^{-1} \to P_2^0$ has image in $d(P_2^{-1})$. Pick a lift to P_2^{-1}. Et cetera.

Step 2. For two maps $f, g : P_1 \to P_2$ of free complexes in degrees ≤ 0 if f, g induce the same map on cohomology then f is homotopic to g. Proof: again by induction.

Step 3. From Steps 1,2: If P_1 and P_2 are two free resolutions, then there are two qis: $f : P_1 \to P_2$ and $g : P_2 \to P_1$ and both compositions fg and gf are homotopic to Id. Hence between $\mathrm{Hom}(P_1, Q[0])$ and $\mathrm{Hom}(P_2, Q[0])$ there is a homotopy equivalence and they have the same cohomology groups. QED

Example 2. $A^{\text{opposite}} - \text{modules} \times A - \text{modules} \to \text{vector spaces } P, Q \mapsto P \otimes_A Q$.

Again, we pick two free resolutions P^*, Q^*; k-th component of $P^* \otimes Q^*$ is finite sum $P^i \otimes Q^j$ over $i + j = k$. It is enough to choose free resolution only for one module P or Q. The same scheme gives derived functor with cohomology $\mathrm{Tor}_A(P, Q)$ independing on the choice of resolutions.

Remark: free modules are 1) projective (for first example): $\mathrm{Hom}_A(\text{Free}, ???)$ is exact, 2) flat (for 2nd example): Free$\otimes$??? is exact. Of course, the replacement of Q by a free resolution in Ex 1 was a wrong procedure, one has to use injective resolutions...

NON-ADDITIVE CATEGORIES AND FUNCTORS

Example 3. $F :$ Lie algebras $longrightarrow$ vector spaces, $g \mapsto g/[g, g] = H_1(g, 1)$.

Free resolutions are DGLAs in degrees ≤ 0 which are free as GLAs and are qis to $g[0]$. Simple induction shows that there exists at least one free resolution. Functor F applied to free resolution g^* gives the complex of generators of g^*.

THEOREM: cohomology groups of the derived functor are independent on the choice of free resolution and are isomporphic to homology $\tilde{H}_*(g, 1)[-1]$. ($\tilde{H}$ denotes reduced homology, i.e. remove $H_0(g, 1) = 1$).

Proof:

Lemma 1. If g is free then $H_k(g, 1) = 0$ for $k > 1$.

Proof of lemma 1: It is enough to prove that $H^k(g,1)=0$ for $k>1$ because for arbitray Lie algebra g its cohomology are dual to its homology. We have an interpretation of $H^k(g,1)$ as Ext-groups: $H^k(g,1)=\mathrm{Ext}^k_{g-\text{modules}}(1,1)$. It follows from the free resolution of 1 as Ug-module:

$$\ldots \to Ug\otimes\wedge^2(g)\to Ug\otimes g\to Ug\to 0.$$

Now we will use independence of Exts on the choice of resolutions: g is free, hence Ug is free as an associative algebra. $Ug=1+G+G\otimes G+\ldots$ where G is the space of generators of g. Another free resolution of 1:

$$0\to Ug\otimes G\to Ug\to 0.$$

It has length 2. $\mathrm{Ext}^k(1,1)=0$ for $k>1$. QED

Lemma 1 means that the chain complex of $\mathrm{Lie}(G)$ is qis to G for any vector space G. The chain complex as a space is a sum of tensors in G with some symmetry conditions. Hence it is defined in terms of tensor algebra, and its contractibility is purely formal property. It means that Lemma 1 is applicable to arbitrary tensor category in characteristic 0. In particular, it is applicable to the category of Z-graded spaces.

Let g^* be a free resolution of g.

Construct the reduced chain complex of g^*:

$$\begin{array}{cccccccc}
\text{degree} & -3 & & -2 & & -1 & & 0\\
\ldots & g^{-2} & \to & g^{-1} & \to & g^0 & \to & 0\\
 & & \nearrow & & \nearrow & & & \\
\ldots & g^{-1}\otimes g^0 & \to & \wedge^2 g^0 & \to & 0 & & \\
 & & \nearrow & & & & & \\
\ldots & \wedge^3 g^0 & \to & 0 & & & & \\
 & & & & & & & \\
\ldots & 0 & & & & & &
\end{array}$$

Differential is the sum of arrows $\to$ and $\nearrow$.

Lemma 2. $\tilde{C}_*(g^*,1)$ is qis to $\tilde{C}_*(g,1)$.

Proof: Cone of morphism $\tilde{C}_*(g^*,1)\to\tilde{C}_*(g,1)$ is contractible because it is filtered (horizontal lines) with contractible quotients. (Cones of $\wedge^k(g^*)\to\wedge^k(g)$. Functor $\wedge^k$ from complexes of vector spaces to complexes preserve qis by the argument with homotopies).

Lemma 3. $\tilde{C}_*(g^*,1)$ is qis to $F(g^*)$.

Proof: Cone of morphism $\tilde{C}_*(g^*,1)\to F(g^*)$ is contractible because it is filtered (sloppy lines) with contractible quotients (by lemma 1).

¿From Lemmas 2,3 follows the Theorem. QED

Theorem suggests that there exists a CANONICAL free resolution of g with generators equal to $\tilde{C}_*(g)[-1]$. In fact, this is the case.

Introduce on $\mathrm{Lie}(\tilde{C}_*(g)[-1])$ differential equal to the sum of the differential arising from the differential on $\tilde{C}_*(g)[-1]$ and of the differential arising from co-commutative co-associative co-product on $\tilde{C}_*(g)$. (See Lecture 6).

Theorem: cohomology of $\mathrm{Lie}(\tilde{C}_*(g)[-1])$ with the differential as above is equal to $g[0]$.

We will prove it in the next lecture.

Lecture 15,

Notes by M.K.

At the end of the last lecture we formulated theorem (D.Quillen):

Let g be a Lie algebra, then $\mathrm{Lie}(\tilde{C}_*(g)[-1])$ with natural differential is a free resolution of g.

It will be THEOREM 1 of today's lecture. In the proof we will use important general criterium allowing homotopy inversion of some functors:

THEOREM 2. Let Γ_1 and Γ_2 be SHLA, and $f : \mathrm{CoComm}(\Gamma_1[1]) \to \mathrm{CoComm}(\Gamma_2[1])$ a morphism of differential graded coalgebras (= morphism of SHLAs). Assume that f is qis. Then f is tangent qis (i.e. induce qis of Γ_1 and Γ_2) if (1) both Γ_1 and Γ_2 are concentrated in degrees < 0, or (2) both Γ_1 and Γ_2 are concentrated in degrees > 0.

In lecture 8 we proved the inverse implication: tangent qis is a qis.

PROOF OF THEOREM 2:

First of all, by minimal model theorem we can replace Γ's by minimal models. We want to prove that f is an isomorphism.

Case (1): Chain complex for $\Gamma_{1\,\mathrm{or}\,<2}$ is

$$\begin{array}{lcccc} \text{degree} & -4 & -3 & -2 & -1 \\ & \Gamma^{-3} & \Gamma^{-2} & \Gamma^{-1} & 0 \\ & \oplus S^2(\Gamma^{-1}) & & & \end{array}$$

Differential maps $S^2(\Gamma^{-1})$ to Γ^{-2}. Hence $H_1(\Gamma, 1) = \Gamma^{-1} \implies \Gamma_1^{-1} = \Gamma_2^{-1}$.

Next step $\implies$ qis is is on Γ^{-2} et cetera.

Case(2): exercise (it differs a bit from Case (1)). QED

WHY WE EXLUDED DEGREE 0?

There are contrexamples: One can construct non-trivial Lie algebras g with trivial homology groups: $H_*(G, 1) = 0$ for $* > 0$. There is no such finite-dimensional Lie algebra (Hint: compute Euler characteristic of the chain complex). One of infinite-dimensional examples: polynomial vector fields in infinite-dimensional space – {finite linear combinations of monomial in $x_* \times d/dx_*$, where $x_1, x_2, \ldots$ are formal variables}.

PROOF OF THEOREM 1:

LEMMA: for Lie algebra g with trivial bracket Theorem 1 is true.

PROOF OF LEMMA: the statement of this lemma is purely formal about cancellations of spaces of tensors with some symmetries. If it is true in one sufficiently representative object in a tensor category, then it holds for all tensor categoies. So, it is enough to prove it for example for g graded sitting in degree -1.

Let L^* be a DGLA about which we want to prove that it is a resolution of g. Chain complex of L^* is looking like

$$\begin{array}{ccc} L^{-3} & L^{-2} & L^{-1} \\ \oplus S^2(L^{-1}) & & \end{array}$$

with differentials in directions East and North-East. This chain complex maps to the chain complex of g. We want to prove that it is qis. Use filtration in direction North-West. Associated graded complex computes homology of L in which we forgot that L was diffrential. Thus it computes homology of a free Lie algebra which is the space of generators(see Lecture 14). This is chain complex of g. Chain complex of L^* is qis to the chain complex of g.

Applying Theorem 2 we conclude that L^* is qis to g. QED

Noe we are able to prove Theorem 1: DGLA Lie$(\tilde{C}_*(g)[-1])$ is looking like

degree	-2	-1	0	1
	$\wedge^3(g)$	$\wedge^2(g)$	g	0

Its chain complex is:

degree	-2	-1	0	1
	$\wedge^3(g)$	$\wedge^2(g)$	g	0
		$g \otimes \wedge^2(g)$	$\wedge^2(g)$	0
			Complicated Thing	0

Complicated Thing here is component of degree 3 in Lie(g). Differentials go in directions East and South-East. Use filtration in direction South-West. Associated graded complex computes cohomology spaces of Lie algebra for trivial bracket on g. This is the stiuation of LEMMA. QED

EXERCISE: mimic all this story and construct functorial free resolution of commuttaive associative algebras (without unit).

Construct derived functor of $A \mapsto A/A^2$, (Comm assoc algebras without 1) $\longrightarrow$ vector spaces.

Next example of derived functor: functor $A \mapsto A/A^2$ from associative algebras without 1 to vector spaces. Cohomology of the derived functor are computed by the following complex:

$$... \to A \otimes A \otimes A \to A \otimes A \to A \to 0.$$

As the graded space it is Coassociative coalgebra cogenerated by A, differential comes form the product on A:

$$d(a_1 \otimes ... \otimes a_n) = \sum_{i=1}^{n-1} (-1)^i (a_1 \otimes ...(a_i a_i + 1)... \otimes a_n).$$

Usually people don't consider this complex because:

FACT: for A with unit this complex is contractible.

PROOFS:we will give two separate proofs.

1) explicit homotopy: $H(a_1 \otimes ... \otimes a_n) = (1 \otimes a_1 .... \otimes a_n); Hd + dH =$ Identity map. QED

2) For algebra A without 1 define A' as A with added unit: $A' = A + k1$.

Lemma: dual to the complex as above computes $\mathrm{Ext}_{A'-\mathrm{mod}}(1,1)$ (with exeption $\mathrm{Ext}^0(1,1) = 1$).

Proof of Lemma: free resolution of 1 as A'-module:

$$... \to A' \otimes A \otimes A \to A' \otimes A \to A' \to 0.$$

It is contractible because of cancellations:

$$... \to A^3 \oplus A^2 \to A^2 \oplus A \to A \oplus 1 \to 0.$$

End of proof of lemma

If now A is already with 1, then A' is as algebra equal to the direct sum of A and k (ground field). We use another free resolution of 1:

$$..- \to A' \to A' \to A' \to 0,$$

cancellations of $...A \oplus 1 \to A \oplus 1 \to A \oplus 1 \to 0$. $\mathrm{Hom}_{A'-\mathrm{mod}}(\text{resolution}, 1)$ is complex

$$....1 \xrightarrow{\mathrm{id}} 1 \xrightarrow{0} 1 \xrightarrow{\mathrm{id}} 1 \xrightarrow{0} 0.$$

QED

If we want to repeat all the story in the beginning of today's lecture for associative algebras, we have to prove a fact analogous to the LEMMA in the proof of theorem 1:

For free associative algebra without unit

$$A = V \oplus (V \otimes V) \oplus ...$$

cohomology of the derived functor $(A \mapsto A/A^2)$ are equal to $V[1]$.

Proof: For such algebra A we have a resolution of 1 of length 2: $0 \to A' \otimes V to A' \to 0$
QED

EXAMPLE: Deformation theory: Fix a kind of algebraic structures (like Lie algebras, Modules, etc.) Construction (not a functor): algebraic structures $\to$ Lie algebras $A \mapsto \mathrm{Der}(A)$.

Derived construction: replace A by a free resolution A^* in degrees < 0, $\mathrm{Der}(A^*)$ is DGLA.

META-THEOREM: (we will prove in the next lecture);

1) $\mathrm{Der}(A^*)$ has cohomology only in degrees ≥ 0,

2) $H^0(\mathrm{Der}(A^*)) = \mathrm{Der}(A)$,

3) Kuranishi space constructed from $\mathrm{Der}(A^*)$ is the miniversal deformation of A.

4) qis type of $\mathrm{Der}(A^*)$ as DGLA is independent on the choice of resolution A^*.

This theorem gives the universal point of view on deformation theory of algebraic structures. For classical algebraic structures (Commutative, associative, Lie algebras) we have standard deformation complexes which are DGLA (see lecture 6).

COROLLARY: Standard complexes give DGLA quasi-isomorphic to the universal ones from the meta-theorem. In fact, one has to modify a little bit "universal deformation theory" for Lie and associative algebras: instead of construction $A \mapsto \mathrm{Der}(A)$ with values in Lie algebras use $A \mapsto$ 2-term complex $(A \to \mathrm{Der}(A))$ with values in DGLAs.

PROOF of the corollary: we explain it in example of deformations of Lie algberas. Other cases are completely parallel.

Standard deformation DGLA for Lie algebra g is $\mathrm{Der}(\tilde{C}_*(g, 1))$ where $\tilde{C}_*(g, 1)$ is truncated chain complex of G considered as differential graded co-commuttaive coalgebra.

Universal DGLA contsructed by the canonical free resolution of g is $\mathrm{Der}(\mathrm{Lie}(\tilde{C}_*(g, 1)))$ and consists of derivations of the differential graded Lie algebra constructed functorially from $\tilde{C}_*(g, 1)$.

Hence, by functoriality, we get a morphism of DGLAs:

$$\mathrm{Der}(\tilde{C}_*(g,1)) \to \mathrm{Der}(\mathrm{Lie}(\tilde{C}_*(g,1))).$$

Let us prove that it is a qis of complexes.

By definition, $\mathrm{Der}(\mathrm{Lie}(\tilde{C}_*(g,1)))$ as a space is equal to

$$\mathrm{Hom}(\tilde{C}_*(g,1), \mathrm{Lie}(\tilde{C}_*(g,1))).$$

Spectral sequence type arguments show that it is qis to $\mathrm{Hom}(\tilde{C}_*(g,1), g)$ because $\mathrm{Lie}(\tilde{C}_*(g,1))$ is qis to g.

Again, by definition of derivations, complex $\mathrm{Hom}(\tilde{C}_*(g,1), g)$ is equal to $\mathrm{Der}(\tilde{C}_*(g,1))$. QED

REMARK: we have seen a remarkable duality between calssical algebraic structures: Lie algebras are dual to commutative associatiev (without 1), Associative algebras (without 1) are dual to associative. If we want to construct functorial free resolution of some algebras, we use co-algebras odf the dual type and then we get a pretty small representative of qis type of deformation DGLA.

There was a theory developed rescently by Ginzburg-Kapranov of certain "Koszul duality' between algebraic structures which generalizes 3 classical examples. It is clear now that there are many other algebraic structures which admit dual and have nice canonical deformation complexes.

Examples: Poisson algebras (like functions on Poiison varieties), again without units, Vertex Operator algebras, Gravity algebras (essentially solutions of associativity equations in toplogical string theory) ...

Lecture 16,

Notes by M.K.

Today we will prove META-THEOREM from the last lecture about deformations of algebraic structures.

Precise meaning of words "algebraic structure" (on vector spaces):

1)set of basic operations F_i. Each operation has some number of arguments: inetger $n_i \geq 0$.

Algebras are vector spaces V endowed with maps $F_i : V^{\otimes n_1} \to V$, satisfying a set of identities:

2) Identities between opertaions. finite polylinear expressions in variables $v_1, ..., v_k$, for some k looking like: Sum of coefficient times $F_*(..., F_*(..., F_*(..., F_*(...)...))) = 0$. Inside we put some permutations of $v_1, ..., v_k$.

Modern name for it is OPERAD, algebras are algebras over Operads. I will describe it some time later.

Examples:

1) Fix associative algebra A with unit. A-modules are algebras with basic operations: F_a, for $a \in A, n_a = 1$. Relations: $F_{a+b}(v) = F_a(v) + F_b(v), F_{\lambda a}(v) = \lambda F_a(v), F_1(v) = v, F_{ab}(v) = F_a(F_b(v))$.

2) Associative algebras with units: Basic operations are Product, $n = 2$, and Unit, $n = 0$. Relations are evident.

3) Modules over non-fixed algebras: a mix of two previous examples. More natural to describe it as two vector spaces A, V plus 3 basic operations: Product: $A \otimes A \to A$, Unit: $A^0 = 1 \to A$, Action: $A \otimes V \to V$. One can also to describe it as one vector space $A \oplus V$ with two commuting projectors (on A and on V) sum of which is equal Id. So, will be 5 basic operations.

It is clear that one can express a lot in such a way.

For each kind of algebraic structures one can consider the category Algebras of algberas of this type. There is an evident forgetful functor: Algebras $\longrightarrow$ vector spaces (ususally we don't denote it at all) and adjoint functor: Free: vector spaces $\longrightarrow$ algebras. Any morphism from a free algebra is the same as a linear map from the space of generators. Analogously, any derivation of a free algebra is defined by its restriction to generators.

There is an evident extension of algberaic structures to any tensor category. Hence, there are always Differential Graded versions of algebraic structures.

Also, if A is an algebra of some kind and C is a commutative associative algebra with 1 then on tensor product $(A \otimes C)$ arised structure of the same kind as of A.

Everything what I'm going to tell is true for arbitary algebraic structure. It is reasonable to imagine that I'm talking about something familiar, like associative algebras.

Proof of the main theorem will consist of several elementary steps.

FREE RESOLUTIONS. Definition: free resolution A^* is a differential graded algebra in degrees ≤ 0 which is

(1) free graded algebra (forgetting diferential),

(2) its cohomology as of a complex sits in degree 0.

$A := H^0(A^*)$ is an algebra. We say that A^* is a resolution of A.

FREE RESOLUTIONS EXIST. For algebra A we can construct an epimorphism from a free algebra A^0 to A. For example, Free(A) maps onto A. In the next step, construct free GLA generated by A^0 and some space G^{-1} in degree -1, and introduce differential $d : G^{-1} \to A^0$ with the image equal to the Kernel of the epi: $A^0 \to A$. Extend d by Lebniz rule to whole GLA. Proceed by induction, adding new generators and defining differential of new generators to be closed elements in the previous step. Why $d^2 = 0$? By construction, $d^2 = 0$ on generators. For any odd derivation d $d^2 = [d, d]/2$ is again a derivation. So, by Leibniz rule D^2 vanishes.

QUASI-ISOMORPHISMS BETWEEN FREE RESOLUTIONS. Let A_1^* and A_2^* be two free resolutions of A. Then there exists a qis of DGalgebras $f : A_1^* \to A_2^*$ over A. Proof: Denote by G^* graded space of generators of A_1^*. We have $A_1^0 \to A \xleftarrow{\text{epi}} A_2^0$. Because A_1^0 is free we can lift it to A_1^0 using arbitrary lift on generators G^0. Again, by induction, we construct dg-map from A_1^* to A_2^*. It will be automatically qis, because cohomology of A^* sitting in degree 0.

So, the qis-type of the resolution as DG-algebra is independent of the choice of resolution. It will be convenient to introduce a notion of homotopy in algebraic situation and mimic usual constructions in homotopy theory of topological spaces.

DEFORMATION COMPLEX OF A MORPHISM. Let $f : A^* \to B^*$ be a dg-morphism of two dg-algberas (not necessarily resolutions). We define complex Def($f : A^* \to B^*$) as following: its N-th component consists of 1-st order deformations in degree N of F as a graded (not differential) morphism. In other words, it is the space of graded morphisms $A^* \to B^* \otimes k[\varepsilon_N]/(\varepsilon_N^2)$, where ε_N is a variable in degre $-N$, morphism should be equal to F modulo ε_N. We can write this morphism as $f + H \times \varepsilon_N$. $H : A^*[N] \to B^*$ is called a deformation of f. It satisfies a kind of Leibniz rule. Differential in $\mathrm{Def}(f : A^* \to B^*)$ is defined by supercommutator with d. It comes from the action of supergroup $A^{0|1}$ on the whole picture.

Deformation complex of a morphism behaves well if A^* is free as a graded algebra. In such a case, if G^* denotes the space of generators of A^* then graded morphisms of A^* to B^* can be identified with k-points (even, in degree 0) of a graded vector space $\mathrm{Hom}(G^*, B^*)$. This (infinite-dimensional) graded vector space we can consider as a graded manifold (just an affine space). Differentials in A^*, B^* give an odd vector field on this manifold with square equal to 0. The standard picture (lecture 8) is that we have a singular foliation on the space of fixed points, and at each fixed point we have a differential on the tangent space. Fixed points in the superspace of morphisms are exactly differntial garded morphisms, and the tangent complex is Deformation complex.

Morphism sitiing on the same leaf of foliation are called homotopic, more precisely...

HOMOTOPY OF MORPHISMS. Let f_0, f_1 be DGmorphisms from A^* to B^*. Homotopy between f_0 and f_1 is, by definition

(1) a family of dg morphisms $f_t : A^* \to B^*$, and (2) a family of graded linear maps $H_t : A^* \to B^*[-1]$ depending locally polynomially on t, i.e. $f_t(a), H_t(a)$ are polynomials in t for each homogeneous a; f_t and N_t should satisfy conditions:

1) values of f_t at $t = 0, 1$ are our original f_0, f_1, 2) H_t belongs to $\mathrm{Def}(f_t : A^* \to B^*)^{-1}$ for each t and 3) $d(H_t) : d_B H_t + H_t d_A : A^* \to B^*$ is equal to $\frac{d}{dt} f_t$.

Notice that for any family f_t of dg-morphisms its derivative $\frac{d}{dt} f_t$ belongs to $\mathrm{Def}(f_t : A^* \to B^*)^0$ and it is closed.

This definition of homotopy is the translation of the geometric picture into algebraic language. It also can be reformulated as one DG-morphism $F : A \to B \otimes k[t, dt]$, where $\deg(t) = 0$, such that composition of F with two maps to B^* which arise from $k[t, dt] \to k$, $t \mapsto 0$ or 1, $dt \mapsto 0$.

Exercise: prove that define F is equivalent to homotopy in the definition above. Notice that it is not clear from the definition whether existence of a homotopy defines an equivalence relation on the set of dg-morphisms. Of course, we can formally close it to an equivalence relation.

HOMOTOPY EQUIVALENCE OF MORPHISMS BETWEEN FREE RESOLUTIONS. Theorem: Let $f_0, f_1 : A^* \to B^*$ are two DGmorphisms of free resolutions inducing the same map on H^0. Then f_0 is homotopic to f_1. Proof: Denote by G^* the space of generators of A^*. Define f_t on G^0 by: $f_t(x) = f_0(x) + t(f_1(x) - f_0(x))$. Composition $A^0 \xrightarrow{f_t} B^0 \to H^0(B^*) = B^0/dB^{-1}$ is independent of t, because it is so on generators. It follows that $\frac{d}{dt} f_t(x)$ is represented zero at $H^0(B^*)$ and we can choose $H^t(x)$ such that $dH_t(x) = \frac{d}{dt} f_t(x)$. Then we procede with induction: we want to define $f_t(x)$ and $H_t(x)$ on new generators of A^*. $df_t(x)$ should be equal to $f_t(dx)$ and we know it already by previous steps. Moreover, $f_t(dx)$ is closed by assumptions. Thus, we can choose some $f_t(x)$ for $\deg x < -1$ because cohomology of B^* vanishes. Also, in $\deg x = -1$ element $f_t(dx)$ is zero in $H^0(B^*)$ because f_t induces map $A^0 \to H^0(B^*)$ independing on t, and vanishes on dA^{-1}. Also, we can choose $f_t(x)$ as a polynomial in t with fixed values at $t = 0, 1$. Analogously, we define $H_t(x)$ as solutions of equations $dH_t(x)+H_t(dx)=\frac{d}{dt}f_t(dx)$. There will be no problems at all because $H^{<0}(B^*)=0$. QED

HOMOTOPY EQUIVALENCE OF FREE RESOLUTIONS. As in topology, we can call two DGLAs A^*,B^* homotopy equivalent if there exist dg-morphisms $f:A^*\to B^*$ and $g:B^*\to A^*$ such that fg is homotopic to Id_B and gf is homotopic to Id_A.

COROLLARY: any two free resolutions of the same algebra are homotopy equivalent.

CONSTRUCTION OF DERIVED FUNCTORS "Reasonable" functors between algebraic structures usually can be formulated in terms of tensor categories, and have extensions to DG-algebras. Also, ususally the notion of homotopy of DG- morphisms is preserved by such an extension (as a family of morphisms parametrized by DG-affine scheme $\mathrm{Spec}(k[t,dt])$). Hence, the homotopy type of image of the functor applied to a free resolution is independent on the choice of resolution.

If we want some cohomology theories as a result, then we get derived functor with values in complexes. Lemma: our 'fancy" notion of homotopy between morphisms of complexes gives the same equivalence relation as the usual one. Proof: if f_t is a polynomial family of morphisms and H_t are homotopies then $dH_t+H_td=\frac{d}{dt}f_t,\ f_1-f_0=dH+Hd$, where $H=\int_0^1 H_t dt$.

DERIVED CONSTRUCTION OF DEFORMATIONS OF MORPHISMS. The idea is that also Derivations do not form a functor, it can be written as $\mathrm{Der}(A)=\mathrm{Def}(\mathrm{Id}:A\to A)$. Morphisms in any category can be considered as objects of a new category with morphisms between $f:A\to B$ and $f':A'\to B'$ be sets of commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \uparrow & & \downarrow \\ A' & \xrightarrow{f'} & B' \end{array}$$

Applying general scheme with homotopies one get for two free resolutions A^* and B^* of the same algebra morphisms of complexes Def(Id: $A^*\to A^*)\to\mathrm{Def}(f:A^*\to B^*)\to\mathrm{Def}(gf:B^*\to B^*)$ qis Def(Id: $B^*\to B^*$) and analogously Def(Id: $B^*\to B^*)\to\mathrm{Def}(\mathrm{Id}:A^*\to A^*)$. Compositions in both orders are qis, hence all arrows are qis and cohomology of $\mathrm{Der}(A^*)$ and $\mathrm{Der}(B^*)$ are the same. In fact, there is a sall problem here because $\mathrm{Def}(f:A^*\to B^*)$ is an infinite product and we get non-polynomial families of maps. One can check that the integrals (in passing to the homotopy of morphissm of complexes in the usual sense) are still well-defined because by spectral sequence-type arguments $\mathrm{Def}(f:A^*\to B^*)$ is qis to a complex with the total space Hom(Generators of A^*,B).

The problem is that we used as intermediate steps complexes $\mathrm{Def}(f:A^*\to B^*)$ which don't carry natural DGLA structure.

QIS BETWEEN DGLA-s. Now we construct qis between $\mathrm{Der}(A_1^*)$ and $\mathrm{Der}(A_2^*)$ for any two resolutions A_1^* and A_2^* of the same algebra A. First of all, we reduce the problem to the case when one resolution is generated by some subspace of generators of another resolution. Denote by C^* DGLA freely generated by A_1^* and A_2^*. It maps to A because its degree zero generators (generators of A_1^0 cup generators of A_2^0) maps to A. Moreover, it is map onto, because it is so for subalgebra A_1^*. Then we can add more and more generators to C^* killing cohomology classes. What we get is a new free resolution B^* containing both A_1^* and A_2^* as free subalgebras generated by subspace in generators.

Let us denote one of A_i^* simply by A^*. Its generators we denote by $\{x\}$, generators of B^* denote by $\{x, y\}$. Consider the following commuttaive diagram of complexes:

$$\begin{array}{ccc}
\text{sums of } ?(x)\dfrac{d}{dx} & \overset{\text{onto}}{\longleftarrow} & \text{sums of } ?(x)\dfrac{d}{dx}+?(x,y)\dfrac{d}{dy} \\
\Big\downarrow \text{inclusion} & & \Big\downarrow \text{inclusion} \\
\text{sums of } ?(x,y)\dfrac{d}{dx} & \overset{\text{inclusion}}{\longrightarrow} & \text{sums of } ?(x,y)\dfrac{d}{dx}+?(x,y)\dfrac{d}{dy}
\end{array}$$

In abstract terms, we have

$$\begin{array}{ccc}
\begin{array}{c}\mathrm{Der}(A^*) = \\ \mathrm{Def}(\mathrm{Id} : A^* \to A^*)\end{array} & \longleftarrow & \begin{array}{c}\text{Derivations of } B^* \\ \text{preserving subalgebra} A^*\end{array} \\
\Big\downarrow & & \Big\downarrow \\
\mathrm{Def}(\text{inclusion} : A^* \to B^*) & \longrightarrow & \mathrm{Def}(\mathrm{Id} : B^* \to B^*) = \mathrm{Der}(B^*)
\end{array}$$

By homotopy invariance we conclude that both vertical arrows and lower horizontal arrow are qis. Hence the upper horizontal arrow is qis. All complexes in this diagram except Def(inclusion: $A^* \to B^*$) are DGLAs and morphisms are DGLA morphisms. Hence, $\mathrm{Der}(A^*)$ is qis to $\mathrm{Der}(B^*)$. QED

In order to finish the proof of the main theorem we have to establish relations between actual deformations and derivations of algebras with abstract versions arising from DGLA $\mathrm{Der}(A^*)$. We will do it next time.

Lecture 18,

Notes by M.K.

Topic of today's and the next lecture:

ANALOGY BETWEEN ASSOCIATIVE ALGEBRAS AND ISOLATED SINGULARITIES OF FUNCTIONS

MORE ABOUT SINGULARITIES:

Let f be a holomorphic function in a neighborhood of closed ball $\bar{B}$ in C^n. Assume that f has no critical points on the boundary $d\bar{B}$. Then f has finitely many isolated critical points inside B (if f has a holomorphic curve of critical points then this curve meets boundary somewhere).

We construct a germ of a manifold M_f. Consider space O_{good} of functions g in $O(\bar{B})$ close enough to f. Action of Lie algebra $T(\bar{B})$ defines a subspace at the tangent space to O_{good}. We claim that it is a subbundle of finite codimension of TO_{good}. It defines an integrable foliation on O_{good} because it comes from Lie algebra action. Define M_f as a germ of the space of leaves of this foliation near f.

Subspace in $T_g O_{\text{good}}$ at a point g is $\sum_k v_k(x) df/dx_k \subset O(\bar{B})$. It is just the ideal generated by derivatives of f. Quotient space is CoKer of the map

$$T(\bar{B}) \xrightarrow{\wedge df} O(\bar{B}).$$

It is zero cohomology of the complex (Koszul):

$$\ldots \to \wedge^2 T(\bar{B}) \xrightarrow{\wedge df} T(\bar{B}) \xrightarrow{\wedge df} O(\bar{B}).$$

This complex we can consider as

1) DGLA of polyvector fields with differential $[f,]$,

2) super-commutative algebra $O(\bar{B}) \otimes C[\xi_i]$, where ξ_i have degree -1 (and generate a exterior algebra) with differential $d : d\xi_i = df/dx_i(X), dx_i = 0$.

We will use both points of view. The second description is essentially free algebra (algebras $O(\bar{B})$ have properties analogous to polynomial algebras).

Next fact has elementary functional analytic nature and I will omit its proof: FACT: cohomology groups of Koszul complex are finite-dimensional and its Euler characteristic is locally constant on the space O_{good}.

LEMMA (de Rham): cohomology of this complex vanishes at degree < 0.

Thus, it is a version of a free resolution.

De Rham lemma is a corollary of the general criterium (Serre) for complete intersections.

THEOREM. Let $\phi_j, j = 1, ..., m$ be holomorphic functions in a ball B, or polynomials. Then the associated Koszul complex Functions $\otimes C[\xi_j]$, $\deg\xi_j = -1$ with differential $D(\xi_j) = \phi_j$ has cohomology in degree 0 if and only if $\dim\{x : \phi_*(x) = 0\}$ is equal to $n - m$.

Proof of the theorem: in one direction (opposite direction is analogous). Assume that $\dim\{x : f_*(x) = 0\}$ is equal to $n - m$. We will show by induction that for $k \leq m$ Koszul complex $K(k)$ associated with $\phi_j, j = 1, ..., k$ has cohomology only at degree zero. If it is true for k, then $K(k+1)$ can be considered as a total complex of the short bicomplex

$$\begin{array}{ccccc} \deg & -1 & & 0 \\ & K(k) & \to & K(k) \end{array}$$

(or,equivalently, as a cone of morphism $K(k) \to K(k)$ given by multiplication by ϕ_{k+1}). Spectral sequence degenerates, $K(k+1)$ has cohomology in degree -1 equal to Ker(multiplicaion by ϕ_{k+1} in $\{$functions$\}$/ideal generated by $\phi_1, .., \phi_k$). Cohomology of $K(k+1)$ in degrees not equal to $0, -1$ vanishes.

If $\phi_{k+1}\psi \neq 0$ is zero in $\{$functions$\}$/ideal generated by $\phi_1, .., \phi_k$ it means that ϕ_{k+1} vanishes on a generic point of variety given by euations $\phi_1, .., \phi_k$. Thus, dimension of some component of variety given by euations $\phi_1, .., \phi_k, \phi_{k+1}$ is equal to the dimension of some component for k, i.e. it is greater than or equal to $n - k$. Adding new equations we can drop the dimension only by one by each equation. Thus, there is a component of $\{x : \phi_*(x) = 0\}$ of dimension $> n - m$. Contradiction with the assumption. QED

Thus, we see that $T(\bar{B})$ defines an integrable foliation on O_{good} of finite codimension $\mu = \dim H^0$(Koszul complex).

We assume now that f has only ONE critical point in B, mu is called Milnor number of the singularity.

We get a germ of μ-dimensional manifold M_f. It is independent on the choice of ball B. In fact, formal completion of M_f is purely algebraic construction: we can replace in Koszul complex by formal power series at the critical point of f.

On formal completion of M_f acts infinite-dimensional pro-algebraic group AUT (formal completion of f at the critical point). This group is projective limit of finite-dimensional affine algebraic groups AUT=lim (AUT_k), where AUT_k is the image of AUT in the group Diff_k of k-jets of formal diffeomorphisms. AUT_{k+1} maps onto AUT_k with the kernel which is a subgroup in Ker($\mathrm{Diff}_{k+1} \to \mathrm{Diff}_k$). The last group is equivalent to the product of several copies of G_a (affine group). It is known in algebraic geometry that in characteristic zero any algebraic subgroup of $(G_a)^N$ is isomorphic to $(G_a)^M$ for some M. Hence, in our case it is contractible, AUT is homotopy equivalent to AUT_1 which is an algebraic subgroup of GL(dim of space,C). Any affine algebraic group over C has finite fundamental group. Connected component of identity of AUT acts trivially on M_f by construction.

CONCLUSION: "Actual moduli space" of singularities near f is a quotient space of a germ of manifold by an action of a finite group (i.e., an orbifold). We will see later that very often this finite group is non-trivial.

DIFFERENTIAL-GEOMETRIC STRUCTURES ON M_f.

1) On tangent bundle TM_f there is a canonical structure of commutative associative algebra with unit, linear over O_{M_f}. Explanation: if g is close to f, $T_[g](M_f)$ = functions/ideal generated by derivatives of g. It is clear that this gives a structure of algebra, independing on the choice of representative g.

2)On M_f acts Lie algebra $C[x]\frac{d}{dx}$ of polynomial vector fields on the standard line. Field $L_n := x^{n+1}\frac{d}{dx}$, $n > -2$, maps to variation of $f = f^{n+1}$. In other words, consider functions as maps to the standard line. Diffeomorphism of the line acts on euivalence classes of functions. Commutators of $L_* : [L_n, L_m] = (m-n)L_{n+m}$.

Relation between structures 1) and 2): L_n = product of $n+1$ copies of L_0 in the commutative algebra 1).

In open dense part of M_f consisting of g only with Morse singularities we have the following universal picture: there are local coordinates $t_i, i = 1, ..., \mu$ (critical values of f), product in T_M is

$$\frac{d}{dt_i} * \frac{d}{dt_j} = \delta_{ij}\frac{d}{dt_i}$$

(diagonal product). Action of $L_n = \sum_i (t_i)^(n+1)\frac{d}{dt_i}$.

FIXED POINTS OF L_0 ON M_f:

More precisely, L_0 vanishes at the base point of M_f iff f belongs to the ideal generated by its derivatives. In this case we have a germ of fixed points of L_0 in M_f.

Theorem (M.Saito): $f \in$ ideal generated by $f' \iff f$ is quasi-homogeneous in some coordinates.

Quasi-homogeneity means that coordinates x_i have weights $w_i, 0 < w_i < 1, w_i$ is rational, and f has weight 1 ($\Longrightarrow$ it is a polynomial). AUT is not trivial: it contains a cyclic subgroup generated by $x_j \mapsto \exp(2piw_j)x_j$.

Spectrum of the linear part of the action of L_0 on the tangent space to M_f at fixed points consists of several positive rational numbers and 0 with multiplicity one.

That's all for the moment what I want to tell about singularities.

ASSOCIATIVE ALGEBRAS

Let A be an associative algebra with unit. A priori we have TWO deformation theories of A: 1) as an algebra with unit, 2) forget about unit.

CLAIM: These two theories coincide.

On the level of plain deformation theory over Artin algebras it is clear:

1) if algbera has a unit, then it is unique. Hence, all automorphisms preserve the unit.

2) Small deformation of an algbera which has a unit still has a unit: $a*b = ab + h\bar{f}(a,b)$ is associative iff f is a cocycle:

$$af(b,c) - f(ab,c) + f(a,bc) - f(a,b)c = 0.$$

Substitute $a = b = 1 : f(1,c) = f(1,1)c$. We can apply gauge transformation $f(a,b) \mapsto f(a,b) + ag(b) - g(ab) + g(a)b$ where $g : A \to A$ is arbitrary linear map. Choose g such that $g(1) = -f(1,1)$. Then new $f(1,c) = f(1,c) + 1g(c) - g(c) + g(1)c = 0$. Thus new $g(1,1) = 0$. Also, using cocyle for $b = c = 1$ we have new $f(a,1) = a$, new $f(1,1) = 0$. 1 is a unit for new f... QED

EXERCISE: prove that for algebra A with unit DGLAs $\mathrm{Der}(A^*)$ and $\mathrm{Der}(A_1^*)$ are homotopy equivalent. Here A^* is a free resolution of A as an algebra without unit, A_1^* is a free resolution with unit.

DGLA controlling deformations of A is truncated Hochschild complex: we remove from $A \to \mathrm{Hom}(A,A) \to \mathrm{Hom}(A \otimes A, A) \to ...$ the first term A. It looks very unreasonable to do it because for almost all A we will have no-trivial Lie algebra of derivations and cannot construct moduli space, only a miniversal deformation.

We will denote by Γ whole DGLA $C*(A,A)[1]$.

ASSUME THAT Γ IS HOMOTOPY ABELIAN, i.e. that it is qis to a abelian SHLA (=in minimal model all brackets are zero).WE have met already homotopy abelian SHLAs = related with moduli of Calabi-Yau etc. For homotopy abelian SHLA Γ one can conctruct an EXTENDED MODULI SPACE which is a formal graded manifold (may be, infinite-dimensional). This space M is the spectrum of the total cohomology groups $H^*(\Gamma, 1)$ and can be identified with each minimal model.

We consider M as just a $Z/2$-graded manifold. Z-Grading on $O(M)$ means that algebraic group G_m (=multiplicative group) acts on M.

THEOREM: 1) There is a natural structurte of commutative associative algebra with unit on T_M, linear over O_M,

2) Let L_0 be vector field on $M =$ generator of G_m action. Define L_n for $n \geq -1$ as $(n+1)$-st power of L_0. Fields L_* satisfy identity $[L_n, L_m] = (m-n)L_{n+m}$.

(We will prove it on the next lecture).

Fixed points of L_0 are just points of the ordinary moduli space. Grading on the tangent space at fixed points is $k-2$ on $HH^k(A)$. Spectrum is integral.

So, we have a striking parallel between quasi-homogeneous singularities and algebras. There is no direct correspondence because the spectrum of L_0 behave differently.

I can see two possibilities to explain all this:

1) modify somehow the situation with singularities using cyclic automorphism group and get a germ of manifold with product on Tangent space and $\mathrm{Diff}(A^1)$ action with integral spectrum. Then try to guess which alegbras are related with it. Or,

2) construct a large connected moduli space containing as open dense submanifolds Moduli of singularities and Extended Moduli of algebras. {Fixed points of L_0} should have two components (quasi-homogeneous singularities and associative algebras).

Also, in topological sigma-model arise spaces with product on the tangent space and a vector field L_0. (One starts from a (symplectic or algberaic) compact manifold and counts rational curves on it $\Longrightarrow$ Gromov-Witten invariants. For details see paper of Manin and me). I was able to prove in this case analog of the statement 2) in the Theorem. In string theory quasihomogeneous singularities give so called Landau-Ginzburg models.

Kontsevich Lecture 19

Notes by AW

A associative algebra $\Longrightarrow$ DGLA $\Gamma = C(A, A)[1]$.

Assume that Γ is homotopy abelian (in minimal model, all brackets are zero)

Then we get an extended moduli space M which is a formal Z-graded manifold, the functions on M being $H^*(\Gamma, 1)$.

Consider M just as a $Z/2$ graded manifold with a G_m action, where G_m is the multiplicative group of a field, considered as an algebraic group.

THEOREM 1. On TM there is a natural associative commutative product. 2. L_0 the generator of the G_m action, $L_n = (n+1)$-st power of L_0, $[L_n, L_m] = (m-n)L_{n+m}$.

HOCHSCHILD COMPLEX

$C^{\cdot}(A, A)$. Assume that A has an identity.

LEMMA. Hochschild cohomology is $\mathrm{Ext}^*_{A-\mathrm{bimodules}}(A, A)$.

PROOF. Construct an explicit free resolution, with homotopy operator given by tensor product with 1.

MODULI OF MODULES

Let R be an associative algebra with unit, M an R module. (For us, R will be $A \otimes A^{\mathrm{op}}$ and M will be A).

By general principles, we need to choose a free resolution $M^{\cdot}$ of M and then a DGLA $\mathrm{Hom}(M^{\cdot}, M^{\cdot})$. It is also a DGAA, with brackets the usual commutators of the associative product. This gives on the groups $\mathrm{Ext}_{A-\mathrm{modules}}(M, M)$ an associative product (Yoneda product).

There is an explicit, smaller, free resolution given as follows. Look at $... \to R \otimes R \otimes M \to R \otimes M \to M$.

Differential and homotopy operators are given by the same formulas as in the Hochschild complex. Look on $R^{\cdot}[1] \otimes M$ as a free comodule over the free coalgebra $\otimes^{\cdot} R[1]$ cogenerated by M.

Then the complex $\mathrm{Hom}(M, M) \to \mathrm{Hom}(R \otimes M, R) \to ...$ is quasiisomorphic to $\mathrm{Hom}(M, M)$. (Note that here and above $\mathrm{Hom}(M, M)$ is "underlined Hom", which is a huge functor much bigger than ordinary Hom.

Structure of DGAA on the complex:

"Composition product" as in Hochschild complex.

EXERCISE. Check that this DGAA structure is quasiisomorphic to

$$\mathrm{Hom}_{R-\mathrm{modules}}(M^{\cdot}, M^{\cdot}).$$

IN THE SPECIAL CASE where $R = A \otimes A^{\mathrm{op}}$ and $M = A$, we have a different resolution.

After some work, we get on $C^{\cdot}(A, A)$ a structure of DGAA (usual formulas in Hochschild cohomology).

EXERCISE. This DGAA is qis to HOM$(A^\cdot, A^\cdot)$, where $A^\cdot$ is free resolution of A.

We get a second structure of DGLA on $C^\cdot(A, A)$ given by the Gerstenhaber bracket.

More precisely, the deformation theory of A itself is given by a DGLA structure on $C^\cdot(A, A)[1]$.

CLAIM. The DGLA structure obtained from commutators in the DGAA structure is homotopy abelian.

COROLLARY. The Yoneda product on $\mathrm{Ext}_{A-\mathrm{bimodules}}(A, A)$ is graded commutative. (These measure deformations of A as a bimodule.)

The picture above is quite general.

Given a homomorphism $f : A \to A$, we can construct a bimodule structure M_f on A with $amb = amf(b)$. As an A-module, M_f is free with 1 generator.

There is a 1–1 correspondence between End(A) and bimodules which are free as A-modules with fixed generator. A lot of our discussion was based on the fact that free modules could not be deformed. On the other hand, deformations of Id: $A \to A$ in endomorphisms (i.e. derivations) correspond (up to a small difference arising from generators) to deformations of A as a bimodule.

It is true for plain deformation theory (functors on Artin algebras), and also for the extended deformation theory.

So two languages for the same problem give rise to two different DGLA's, but they turn out to be quasiisomorphic.

Now consider arbitrary algebraic structures, not necessarily associative. Let A and B be two algebras, $f : A \to B$ a morphism.

Deformations of $(f : A \to B)$ are found by replacing A by a free resolution $A^\cdot$.

We consider Hom$(A^\cdot, B)$ as (the functions on) an infinite dimensional manifold, with the $G^{0|1}$ action generated by an odd vector field. its fixed points are homomorphisms of a formal neighborhood into SHLA...

VERY GENERAL STATEMENT–Deformations of the identity map form a homotopy abelian space.

CONSIDER Hom$(A^\cdot, A^\cdot)$ as (functions on) an infinite dimensional monoid with $G^{0|1}$ action. A formal neighborhood of the identity is a formal Lie group G with an odd vector field. Now we can consider the map log from this formal neighborhood to the Lie algebra g. This is a diffeomorphism of formal manifolds. Now the $G^{0|1}$ action is linear, being the lift of group automorphisms by the exponential map which implies that in these coordinates all the higher brackets are zero. Thus there is a deep reason for the homotopy commutativity mentioned above.

Now the DGLA controlling deformations of A is truncated $C^\cdot(A, A)$. One then misses the difference between all derivations and inner derivations.

Now use the correspondence bimodules and generators $\leftrightarrow$ endomorphisms. How to get rid of the generators.

A bimodule gives a functor from A-modules to A-modules given by tensoring on the left with the bimodule.

One should develop the notion of deformations of an abelian category and get from there back to Hochschild cohomology.

EXTENDED MODULI SPACE

We deform the differential in several situations:

1. free resolution
2. free coalgebra with counit cogenerated by A

Definition. (Stasheff). An A_∞ algebra (strong homotopy associative algebra) is a Z-graded vector space V with maps

$$\begin{aligned} m_0 &: 1 \to V[-2] \\ m_1 &: V \to V[-1] \\ m_2 &: V \otimes V \to V \\ &\cdots\cdots\cdots\cdots \\ m_n &: V \otimes V ... \otimes V \to V[n-2] \end{aligned}$$

satisfying some higher associativity conditions. These conditions are equivalent to saying that the differential in $\mathrm{CoAssoc}_1(A)$ is really a differential.

In the special case where $m_0 = 0$, we get the conditions that m_1 is a differential, m_2 is associative up to homotopy, etc.....

the extended moduli space = supermoduli space of A_∞ structures

...deformations of Artin Z-graded algebras.

Let F be a free coalgebra, F^* = formal power series in noncommutative variables (free complete associative algebra).

Construct a product between derivations of F^* which will be a second order differential operator moduli derivations. This will eventually lead to the bracket on Hochschild cohomology.

PICTURES. Think of derivation of F^* as a linear combination of monomials times $\partial/\partial x^i$'s. This acts on a word by replacing each occurrence of x^i by the monomial.

Now we define $v * u$ for derivations v and u by

$v * u(x_1, ..., x_N)$ by applying v to the left of u in each term.

LEMMA 1. $u * v$ is, modulo derivations, independent of the choice of coordinates. In other words,

$$[w, u * v] - [w, u] * v - u * [w, v]$$

is a derivation.

HERE FOLLOWS A "PICTORIAL" PROOF.

Now let d be an odd derivation of $F^*, [d, d] = 0$. This gives an A_∞ algebra. d defines Hochschild cohomology.. The condition that $[d + h\bar{v}, d + h\bar{v}] = 0$ mod H^2 means that v is a cocycle.

We get a product on Hochschild cohomology is given by $v * u$.

THEOREM. this product is associative and commutative. (5 pages of pictures.)

There is also a pictorial proof of the bracket relation on the L_n's. A conceptual proof is still lacking.

NEXT TIME. Will explain a conjecture of Deligne related to these matters.

Lecture 21

Notes by Alan Weinstein

HOCHSCHILD HOMOLOGY COMPUTATIONS

EXAMPLE $A = \mathrm{Mat}_n(k)$. Compute $HH(A)$ as $\mathrm{Ext}_{A-\mathrm{bimodules}}(A, A)$.

In fact, note that A-A-bimodules are the same as Mat_{n^2}-modules.

GENERAL REMARK. Mat_N modules are equivalent as a category to vector spaces (tensor with k^N).

So we can get $HH^*(A) = \mathrm{Ext}_{\mathrm{Vect}}(k, k) = k$ in degree 0, 0 elsewhere.

CONCLUSION. The matrix algebra has no deformations, and all its derivations are inner.

EXERCISE. 1. Suppose that $A = \mathrm{Mat}_n(B)$, B another associative algebra. Then $HH^*(A) = HH^*(B)$. In fact, the DGLA's $C^{\cdot}(.,.)$ are qis.

MORITA EQUIVALENCE. Definition. Algebras A and B are called Morita equivalent if their categories of modules are equivalent as categories.

THEOREM (Morita). A and B are Morita equivalent iff there exists a B-module-A M such which is finitely generated and projective from each side, with each algebra being the commutant of the other. In this case, the equivalence of categories is equivalent as a functor to $\otimes_A$, from A-modules to B-modules.

FACT. Morita equivalent algebras have isomorphic $HH*$ and homotopy equivalent DGLA's.

EXAMPLES OF M.E. ALGEBRAS

Consider a smooth manifold X. Then then endomorphism algebras of all vector bundles over X are Morita equivalent.

We have a Kunneth formula $HH^*(A \otimes B) = HH*(A) \otimes HH^*(B)$.

To verify this, using the Ext picture, it is enough to use projective, not necessarily free resolutions. In fact, the tensor products of projective resolutions are again projective resolutions.

At the level of Hochschild cochains, there is no tensor product between the complexes!

COMMUTATIVE ALGEBRAS

If A is a commutative algebra, we have the Hodge decomposition (Barr, Gerstenhaber, Schack).

$C^*(A, A)[1] = \mathrm{Der}(\mathrm{CoAssoc}_1(A[1]))$, the differential is $[m,]$, where m is the product.

For commutative deformations, the DGLA is the Harrison complex $\mathrm{Der}(\mathrm{CoLie}(A[1]))$.

Let $g = \mathrm{Lie}(A[1]*)$, a free Lie superalgebra.

$(C^*(A, A)[1])^* > \mathrm{Ass}(A[1]*)) = Ug$, the enveloping algebra.

By the PBW theorem, we get an isomorphism of vector spaces (and g modules) with $S*(g)$.

The dual of the differential maps generators of g to g. The symmetric powers of g are subcomplexes of Ug. Passing to the dual, we find that the Hochschild complex is a direct sum of components $C_p(A, A)$, and $HH^*(A) = \oplus HH^*_p(A)$. What are these subcomplexes?

The part where $p = 0$ gives A in degree 0 and 0 elsewhere.

Also, $HH^0(A) = HH^0_0(A) = A, C_1(A, A)$ is the Harrison complex.

NOTE. The Hodge decomposition on the level of cochains is not compatible with bracket. This complicates the discussion of quantization which follows later today.

EXAMPLES. $A = k[x]$. It is a free associative algebra, from which it follows that $HH^0(A) = A, HH^1(A) = k[x]\partial/\partial x$, and higher cohomology is zero.

For $A = k[x_1, ..., x_n]$, we get the the cohomology is the multivector fields.

2 proofs. The first is to take tensor products of polynomials in one variable. The second identifies A-Mod-A as modules over polynomials is $2n$ variables, and to use an explicit resolution of A given by the Koszul cohomology.

GENERALIZATION. If A is the algebra of functions on any smooth affine algebraic variety X. Then (Hochschild-Kostant-Rosenberg), the Hochschild cohomology is the multivector fields (with polynomial coefficients, of course).

To prove, we embed X as the diagonal in $Y = X \times X$. Then one uses the fact that $\mathrm{Ext}_{O_Y}(O_X, O_X)$ is given by the sections of the exterior powers of the normal bundle, whenever Y is a submanifold of X.

On $HH^*(A)$, we have the structure of a Gerstenhaber algebra – in this case the product and bracket become the wedge and Schouten-Nijenhuis brackets.

EXPLICIT COCYCLES. Given a multivector field $v_1 \wedge v_2 ... \wedge v_k$. Then it acts on k-functions by pairing with the wedge product of their differentials.

EXERCISE. Check that this is really a Hochschild cocycle.

HODGE DECOMPOSITION in this case is just in one component in each dimension.

Note that a skew symmetric cochain becomes "symmetric" because we have shifted grading by 1.

COROLLARY. For the algebra A of smooth functions on X, the Harrison cohomology in degrees > 1 is zero. So, for deformation theory, we have only derivations. So these smooth affine algebras behave like free algebras. This means that we could use them instead of free algebras is resolutions.

For commutative algebras, we have a clear geometric picture, introduced by Grothendieck. (Affine schemes). For noncommutative algebras, the geometric picture is not so clear.

Thus, it is interesting to study "quantization", which for our purposes refers to algebras which are close to commutative.

Let us consider the commutative algebras of functions on smooth (analytic, algebraic) manifolds. Then $C^*(A, A)$ contains an important subcomplex consisting of the local cochains, which are given by multidifferential operators. (Grothendieck: notion of differential operator is purely algebraic–multiple commutator with multiplication operators is eventually zero.)

The spaces of local cochains are countable-dimensional.

CLAIM (proof next time). The inclusion of local cochains in all cochains is a quasi-isomorphism. The complex of local cochains also makes sense in the smooth and analytic cases, where the relation with the full Hochschild complex is not so clear.

$*$-products (Berezin, BFFLS, ...)

Work in the category of smooth manifolds $X, A = C^\infty$ functions. Consider formal paths in the space of associative products on A, starting from the usual product. They are formal power series in $\hbar$ with coefficients which are bidifferential operators. We have $\gamma = \sum \gamma_i \hbar^i$, with

$$d\gamma + \frac{1}{2}[\gamma, \gamma] = 0.$$

Represent γ_1 by a bivector field. The next equation for associativity gives the fact that the Schouten square of γ_1 is zero; i.e we have a Poisson structure on X.

So we get a product $f * \hbar g = fg + \hbar f, g + \hbar^2 \gamma_2(f,g) + ...$

BASIC EXAMPLE. Differential operators in a vector space. Consider $F(p_i, q_i) = \sum F_{ab} p^a q^b$. Associate to it the operator in which p is replaced by $h\partial/\partial x$. (Vector fields to the right of functions.) Define the star product of functions by pulling back the product on differential operators. You get the formula

$$*\hbar = \exp(\hbar \leftarrow \frac{\partial}{\partial p} \frac{\partial}{\partial q} \rightarrow).$$

This does not have the good symmetry properties.

More generally, if V is a vector space and we have an element a of its tensor square. We can consider a as a differential operator of second order on $V \oplus V$. Then we can define the product

$$f * \hbar g = \exp(ha) f \otimes g,$$

restricted to the diagonal.

EXERCISE. This is always associative. The underlying Poisson structure is the skew symmetric part of a.

If V is symplectic and a is the Poisson structure inverse to the symplectic form, we get the so-called Moyal product, which is invariant under the action of the (affine) symplectic group.

APPLICATION. Quantum products on modular forms (Zagier). Recall that a modular form of weight k is a holomorphic function on the upper half plane such that $f(t)(d\tau)^k/2$ is invariant under the action of $SL(2,Z)$. We usually assume that f is bounded as $\mathrm{Im}\tau \to +\infty$.

The modular forms are generated by E_4 and E_6, Eisenstein series in degrees 4 and 6.

Now consider the region U in C^2 consisting of those z_1, z_2 such that $\mathrm{Im}(z_1/z_2) > 0$. The modular forms can be thought of as functions on U of various homogenity degrees, invariant under $SL(2,Z)$ sitting in $SL(2,C)$. Now the (complex) Moyal product on C^2 gives a noncommutative product on the modular forms, which looks very complicated in terms of the original Eisenstein coordinates. (The complicated structure was originally found by Zagier, who had a hard time proving associativity.)

QUANTIZATION OF SYMPLECTIC MANIFOLDS

THEOREM (deWilde Lecomte) For any C^∞ symplectic manifold, there exists a quantization. (Simpler, more recent proof by Fedosov.)

QUESTION. Is there a "canonical" quantization in any sense. (We know it only up to equivalence.)

QUESTION. What is going on in the complex analytic case.

THEOREM (Simpler) $H^3(X,R) = 0 \implies$ quantization exists.

Proof of the simpler case. Cover the domain by a Darboux covering with all intersections contractible and put Moyal products there. On the other hand, for this standard example, all quantizations are equivalent. On intersections, choose isomorphisms between the algebras. On triple intersections, we get a 2 cocycle with values in the derivations of the Moyal algebra, which are all interior, equal to the algebra modulo constants. The obstruction to gluing consistently lies in $H^2(X, \text{functions}/R)$, which is isomorphic to $H^3(X,R)$.

Kontsevich Lecture 23

Notes by Alan Weinstein

More on Fedosov quantization

(X, ω) symplectic manifold (C^∞ for now) We will construct a canonical abelian category cal$A$ and an equivalence class G of objects in it such that for each object in G, End(object) is an algebra A, which cal$A \sim A$ modules.

Analogously, if we are given a field, we have a groupoid of algebraic closures; if we are given a space, we have a fundamental groupoid.

To first approximation, we want to associate a Hilbert space canonically to a symplectic manifold. Already in the linear case, we see that the linear symplectic group acts only projectively on the naturally associated Hilbert space.

As a second approximation, we want to associate an associative algebra to a symplectic manifold, but the symplectomorphisms do not act on this algebra.

So this category is the third approximation.

PREPARATION. (Lie algebroid, essentially). Given a Lie group G and a Lie algebra L, and homomorphisms $f_1 : G \to \mathrm{Aut}L, f_2 : g = \mathrm{Lie}G \to L$ which are compatible in the sense that $\mathrm{ad}_L f_2 = \mathrm{Lie}(f_1)$ as maps from Lie$G$ to der(L).

Now given a principal G-bundle E over X, define an L connection ∇ in E as follows:

Over trivializing open domain U, trivialize E, then ∇ will be represented by a 1-form with values in L; when we change trivializations by $g : U \to G$, the gauge transformation is:

$$A \mapsto f_1(g^{-1})A + f_2(g^{-1}dg).$$

LEMMA. This is a well defined notion – check consistency for three trivializations.

There is a notion of curvature for an L-connection. Let calL be the bundle of Lie algebras associated with the principal bundle. The curvature is a 2-form on X with values in cal$L$. In a local trivialization, the curvature is given by $R = dA + \frac{1}{2}[A, A]$.

APPLY THIS TO THE FOLLOWING DATA. $G = Sp(2n, R).W =$ the Weyl algebra $=$ associative algebra with identity generated by coordinates on R^{2n} and h, with the commutation relation $[y_i, y_j] = \hbar\omega_{ij}, \hbar$ commuting with everything. The grading is given by letting each y_i have degree 1 and hbar have degree 2. The completed algebra of formal power series with the same relations will be denoted $\hat{W}$ and will be considered as an $R[\,[\hbar]\,]$ module.

Now let $L = (1/\hbar)\hat{W}$; it is closed under brackets because all brackets in $\hat{W}$ contain $\hbar$. Also $[L, \hat{W}]$ is contained in $\hat{W}$.

L is graded as well, starting with L^{-2}.

The action of G as automorphisms of L is the evident one, and the map $f_2 \colon sp(2n) \to L$ (actually L^0) is given by the quadratic functions, with image the expressions $(y_i y_j - y_j y_i)/\hbar$.

Now let X be a symplectic $2n$-dimensional manifold, $E \to X$ the symplectic frame bundle. Look at L connections on E. First consider the class of connections which in symplectic coordinates x_i (and the corresponding trivialization of E) are given by $A = \sum y_i dx_i/\hbar +$ terms of positive degree. This is a well-defined notion. (It fixes the -2 component of A is zero and the -1 component as the solder form.)

The class of such connections is not empty. Locally, these connections form a principal homogeneous space over the group of sections of the vector bundle of 1 forms with values in cal$L^{\geq 0}$. On smooth manifolds, $H^1(X$, sheaf of sections of a vector bundle$) = 0$, which guarantees the existence of connections.

Now consider the subclass of connections for which we impose the additional condition that the component R^{-1} of the curvature is vanishing.

LEMMA: The sheaf of such connections is again locally a principal homogeneous space over the sections of a vector bundle.

To see this, we write $A = \sum y_i dx_i/\hbar + \Gamma_{ijk}/\hbar y_i y_j dx_k + \sum \alpha_k(x) dx_k +$ terms of positive degree. Here Γ is symmetric in the first two indices.

We compute $R^{-1} = dA^{-1} + [A^0, A^{-1}]$

$$
\begin{aligned}
&= [A^0, A^{-1}] = [\sum y_l dx_l/\hbar, \sum \Gamma ...] \\
&= ...
\end{aligned}
$$

and the vanishing condition is equivalent to the linear algebraic equation that Γ is symmetric in the last two indices when it is made covariant by contraction with the symplectic form – this is just the torsion zero connection.

Now we can prove by induction that the set of connections with $A^{-2} = 0, A^{-1}$ is solder form, $R^{-1} = ... = R^{k-1} = 0$, is nonempty.

Say $R = R^{-2} + R^k + ...$ Try to kill R^k.

By the Bianchi identity $dR + [A, R] = 0$. The contribution of R^2 is zero because R^2 is closed and central. So the Bianchi identity tells us that $[A^{-1}, R^k] = 0$.

Lemma. If F_2 is a 2-form on X with values in cal$L$ such that $[A^{-1}, F_2] = 0$, then there exists F_1 such that $F_2 = [A^{-1}, F_1]$.

It is enough to prove this locally using trivializations, since we are dealing with algebraic equations, whose solutions can be patched together by partitions of unity.

In fact, locally, we can identify the relevant forms with the deRham complex of $R[[y_i]]$ in which A^{-1} becomes the usual d, so we can apply the Poincare lemma.

REMARK. If F_2 has degree k, we can choose F_1 to have degree $k+1$.

Now the lemma implies that locally there is a 1-form B^{k+1} such that $[A^{-1}, B^{k+1}] = R^k$, and the set of solutions forms an affine space locally. Now let $\nabla' = \nabla - B^{k+1}$ to kill R^k for the new connection.

COROLLARY. The set of connections with $A^{-2} = 0, A^{-1} =$ solder form, $R = R^{-2}$ is nonempty. Moreover, it is the projective limit of spaces of such connections modulo terms of order geq k, and the successive quotients are affine spaces over spaces of sections of vector bundles. Thus the space of all these "admissible connections" is contractible.

LEMMA. All these connections are gauge equivalent, with gauge group the Lie group corresponding to the pronilpotent Lie algebra $L^{\geq 1}$. (Gauge transformations are sections of a bundle whose fibre is this group.)

Now let A_0 and A_1 be two such "Fedosov" connections. There is a path A_t connecting these two connections. Then $\frac{d}{dt}A_t$ is a 1-form with values in cal$L^{\geq 0}$. The derivative of curvature, which is zero, equals $[d + A, \frac{d}{dt}A_t]$, so that locally there exists g in cal$L^{\geq 1}$ such that $[d+A, g] = \frac{d}{dt}A_t$, by the Poincare lemma used previously (but now for 1 forms instead of 2 forms). A global such g can be build as before.

This gives us a family g_t of sections of cal$L^{\geq 1}$, and we can solve the equations $g(t)^{-1}dg(t) = g_t$ to get a section of exp(cal$L^{\geq 1}$) which realizes the gauge transformation from A_0 to A_1.

Let cal$W$ be the bundle of $\hat{W}$'s associated with the tangent bundle of the symplectic manifold X. An L connection defines a connection on this bundle of algebras, since L acts by derivations on the associative algebra $\hat{W}$. If the connection is a Fedosov connection, this associated connection is flat because $R = R^2$ is central.

CLAIM: the space of parallel sections of this associated bundle is canonically isomorphic as a vector space to $C^\infty(X)[[\hbar]]$.

To see this, we can use the fact that the connection is locally gauge equivalent to one of the form $d + A^{-1}$. (Standard connection.) Why is the isomorphism canonical? – Because it is given by restriction to the zero section.

Now the isomorphism gives us a deformation of the multiplication of $C^\infty(X)$.

More generally, we can construct central connections whose curvatures have the form $\omega/\hbar + \omega_1 + \omega_2\hbar + ... = \gamma(\hbar)$.

We can invert this to get a series $\hbar\omega^{-1} + ...$ which is a path in the space of Poisson structures with value 0 and nondegenerate derivative at $\hbar = 0$.

THE CANONICAL ABELIAN CATEGORY

(X, ω) symplectic. We construct a groupoid C whose objects are Fedosov L-connections (those satisfying all the conditions above). The automorphisms of each object will consist of the invertible elements of the algebra of parallel sections of the bundle of Weyl algebras for the given connection.

Given TWO connections, a morphisms consists of a path between them and the corresponding γ_t in Γ(cal$L^{\geq 1}$). Composition of paths corresponds to composition of morphisms.

For insistency, we must associate with each loop of connections and invertible element in the algebra of parallel sections for the endpoint of the loop. We lift the loop of connections to a path of gauge transformations. We can write $g(1) = \exp(f)$ because exp(cal$L^{\geq 1}$) is a nilpotent Lie group. Now f is a parallel section. CLAIM. f is divisible by $\hbar$, so it is in fact an element of our algebra.

Locally, by gauge transformation, we may assume that A is the standard form in flat space. Then we show by looking term by term that if f is in cal$L^{\geq 1}$, it must be in $\hbar$cal$L^{\geq 1}$.

SO WE HAVE A CANONICAL GROUPOID in which the automorphisms of each object are the invertible elements. Now we can define the notion of module of this groupoid.

ALL OF THE ABOVE WAS BASED ON the vanishing of H^1 = existence of connections.

On more general "manifolds" (algebraic, analytic), we associated to a symplectic structure a canonical shear of abelian categories and equivalence classes of generators.

Next time – we'll apply this to K^3 surfaces.

Kontsevich Lecture 27

Notes by Alan Weinstein

HIDDEN SMOOTHNESS–CONCLUSION

Thesis: every space arising naturally in geometry comes in some sense from a differential graded manifold. Thus we have a structure sheaf O_X, but also a sequence O_X^{-k} of sheaves which form a negatively graded commutative algebra (Also an element t_X in

$K^0(X)$, a finite formal linear combination of vector bundles This is the virtual tangent bundle).

More precisely, there should exist a finitely dimension differential graded manifold $\widehat{X}$ and an odd vector field d such that X is the zero set of d, and $t_X = [T^X_{\text{even}}] - [T^X_{\text{odd}}]$.

These extra data should be "unique up to homotopy".

MAIN EXAMPLE: moduli spaces.

There are 3 situations where an actual moduli space exists (not just a formal one).

1) deformations of algebraic structures (operads) with finite # of generating operations on finite dimensional vector spaces.

2) nonlinear systems of pseudodifferential equations with Fredholm property on compact manifolds (e.g. conformal structures) $\Longrightarrow$ topological field theories.

3) deformation problems on projective schemes.

Essentially, 2 and 3 can be reduced to 1. For example, let X in P^N be a projective scheme. $O(-1)$ is the tautological line bundle, $O(k)$ is its $(-k)$-th tensor power, A_k its space of sections. These have the properties:

1. A_k is finite dimensional;
2. the dimension is "computable" for large k;
3. their direct sum is a commutative associative algebra.

Finiteness theorems tell us that knowing a finite (but large) subsequence A_k of these spaces (k in an interval) with its partially defined multiplication implies a complete description of X. In all these situations, for each p in our moduli space M we can associate a homotopy type of SHLA, usually in nonnegative degrees, with all graded components of finite dimension. The absence of associate a homotopy type of SHLA, usually in nonnegative degrees, with all graded components of finite dimension. The absence of automorphisms $(H^0(g^\cdot) = 0)$ implies the existence of a formal moduli space $\mathrm{Spec}((H_0(g^\cdot, 1))^*)$, which a formal completion of the actual M at p. Very often, $H^i(g)$ is zero for large (positive and negative) i.

EXAMPLES: moduli spaces of complex structures, vector bundles, holomorphic maps)

One can construct locally on M vector bundles $\tilde{g}$ which have structures of SHLA equivalent to g.

Using the standard resolution, on gets over M a bundle of formal DG manifolds.

CONJECTURALLY: there exists a flat connection on $(\mathrm{CoCom}_1 g[1])$ preserving the structure of DG coalgebra (like in Fedosov quantization) This implies a flat connection on thee dual bundle of complete algebras. Take the flat (parallel) sections, which can be quantization). This implies a flat connection on thee dual bundle of complete algebras. Take the flat (parallel) sections, which can be considered as functions on a supermanifold $\widehat{X}$.

REMARK. flat connection on vector bundle E over non-smooth M means a trivialization for the pull back to any tiny space (spec of a local Artin algebra)...with some compatibility conditions, of course.

DEFORMATIONS OF MAPS

Let S and V be complex manifolds, S compact. $X = \mathrm{Map}(S, V)$, a finite dimensional complex space (via Douady, identifying maps with their graphs). We need to construct the structure sheaf O_X.

1. ALGEBRAIC DESCRIPTION

First, from the manifold V, we construct $A = D_v/O_v$, the sheaf of differential operators modulo multiplication operators. We consider A as a sheaf of left O_v-modules. This sheaf of differential operators modulo multiplication operators. We consider A as a sheaf of left O_v-modules. This gives us an infinite dimensional vector bundle in which each fibre is a coalgebra without counit. ... (Its dual space is the algebra (maximal ideal) of formal power series vanishing at a point.) Let L be the free Lie algebra over O_v generated by $A[-1]$. The coalgebra structure in A gives rise to a differential in L. L is a sheaf of DGLA's; as a sheaf of complexes L_V is qis $T_V[-1]$. Now for $f : S \to V$, where S is compact (non necc. smooth), we take the pulled back sheaf f^*L of DGLA's (and consider it as a sheaf of DGLAs's over C !!!! on S.)

EXERCISE. Check that the deformation functor on Artin algebras associated with this sheaf is equivalent to deformations of maps. Look at the universal map $f : X \times S \to V, \pi : X \times S \to S$ the projection. Then define t_X to be $\pi_*(f^*T_V)$.

2. ANALYTIC DESCRIPTION. S now compact complex manifold $\Longrightarrow$ $\tilde{S}$ the C^∞ supermanifold whose functions are the algebra $\Omega^{0,*}(S)$, with the Dolbeault operator. Then look at $\widehat{X}$ = the supermanifold $\underline{\{\text{maps}\}(\widehat{S}, V)}$ as C^∞ manifold). (Here underline means "considered as a supermanifold".) The underlying topological space consists of the ordinary C^∞ maps from S to V. There is an odd vector field on $\widehat{X}$ whose zeroes are the complex analytic maps. The complex structure on V gives a complex analytic structure on $\widehat{X}$.

PROBLEM: construct the sheaf of analytic functions on $\widehat{X}$. (A sheaf of DG commutative algebras).

CONJECTURE: The cohomology of this complex would be the same as one gets via the algebraic approach. (This would be a realization of ideas in "BRST cohomology") One can imitate the analytic construction of higher structure sheaves in other cases.

1) M =complex structures on a manifold V – assume no holomorphic vector fields. For $m \in M$, we have a DGLA, the Kodaira-Spencer algebra (part of Dolbeault). ...... Also, one can consider moduli of holomorphic vector bundles, or moduli of flat connections on finite CW complexes.

BASIC IDEA: We always have a manifold, but it looks singular because we have passed to the 0th cohomology.

INTERSECTION. $Y_1, Y_2 \subset Z$ (complex) submanifolds. $X = Y_1 \cap Y_2$ is singular. How to construct higher structure sheaves on X? Locally, Y_2 is given by transversal equations $f_j = 0$ in Z. We can restrict these functions to Y_1. these restrictions give a Koszul complex which is a DG comm ass algebra: Let's add coordinates ξ_j to Y_1 in degree -1. Define the differential to be $d(\text{functions on } Y_1) = 0$, $d(\xi_j) = f_j$. This construction is not very symmetric. Claim, the cohomology (as sheaves) in Koszul cohomology are $\mathrm{Tor}^Z_{-1}(O_{Y_1}, O_{Y_2})$. Proof: the Koszul complex with f_j as an O_Z module is a free resolution of O_{Y^2}. .... we take t_X to be $[T_{Y_1}] + [T_{Y_2}] - [T_Z]$, all restricted to X.

A GENERALIZATION. Given several submanifolds $Y_1, ..., Y_k$, X their intersection, one can reduce to the previous case by looking at the intersection with the main diagonal in Z^k.

EXERCISE. Locally, a DG manifold such that the coordinates are in degrees 0 and -1 is isomorphic to the Koszul complex for some intersection.

COROLLARY. If we have a moduli problem $g^{\cdot}$, and the cohomology is zero except in degree 1 and 2, then it is locally an intersection of two manifolds.

COMPARE: A Lie algebroid is a dg manifold (SHLA $g^{\cdot}$ with all cohomology just in degree 0 and 1). If a moduli space is locally an intersection, $\dim X \geq \mathrm{rank}(t_X)$. One can define a virtual fundamental class of the moduli space, which is an element $[X]_{\mathrm{virtual}}$ in an element of $H^{\mathrm{closed}}_{2\mathrm{rank}(t_x)}(X, Z)$.

EXAMPLE: say X is globally the intersection of Y_1 and Y_2 in Z. Then X is homotopy equivalent to its tubular neighborhood in Z. Perturb Y_1 and Y_2 as C^∞ manifolds to make their intersection transverse; their intersection will be oriented smooth manifold, sitting in the tubular neighborhood. One can take the fundamental class of this perturbed intersection, which gives a homology class in $H_*(Z)$. In general, when the cohomology for the moduli problem has only 2 components, the theory of Baum Fulton Macpherson gives a formulation of what should play the role of the fundamental class of the moduli space.

WHY ARE THESE FUNDAMENTAL CLASSES SO INTERESTING? There are several problems in geometry $\Longrightarrow$ SHLA's in degree 1 and 2. For example:

1. moduli spaces of complex curves (2d topological gravity);
2. moduli spaces of vector bundles on curves (2d Yang-Mills);
3. moduli of complex structures on complex surfaces (self dual 4d gravity);
4. moduli of vector bundles on surfaces (self-dual Yang-Mills in 4d));
5. maps from non-fixed curves to manifolds (Gromov-Witten invariants);

This example (not yet finished!) was motivation for everything in the course!

END OF COURSE