CHAPTER 1

L_∞-algebras

1.1. Koszul sign and unshuffles

Given a graded vector space V, the twist map extends naturally, for every $n \geq 0$, to an action of the symmetric group Σ_n on the graded vector space $\bigotimes^n V$:

$$\mathtt{tw}\colon V^{\otimes n} \times \Sigma_n \to V^{\otimes n}.$$

More explicitely, setting $\sigma_{\mathtt{tw}} = \mathtt{tw}(-, \sigma^{-1})\colon V^{\otimes n} \to V^{\otimes n}$, for $v_1, \ldots, v_n$ homogeneous vectors and $\sigma \in \Sigma_n$ we have:

$$\sigma_{\mathtt{tw}}(v_1 \otimes \cdots \otimes v_n) = \pm(v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}),$$

where the sign is the signature of the restriction of σ to the subset of indices i such that v_i has odd degree.

Definition 1.1.1. The **Koszul sign** $\varepsilon(\sigma, V; v_1, \ldots, v_n) = \pm 1$ is defined by the relation

$$\sigma_{\mathtt{tw}}^{-1}(v_1 \otimes \cdots \otimes v_n) = \varepsilon(V, \sigma; v_1, \ldots, v_n)(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)})$$

The **antisymmetric Koszul sign** $\chi(\sigma, V; v_1, \ldots, v_n) = \pm 1$ is the product of the Koszul sign and the signature of the permutation:

$$\chi(\sigma, V; v_1, \ldots, v_n) = (-1)^{\sigma} \varepsilon(\sigma, V; v_1, \ldots, v_n).$$

For notational simplicity we shall write $\varepsilon(\sigma; v_1, \ldots, v_n)$ or $\varepsilon(\sigma)$ for the Koszul sign when there is no possible confusion about V and $v_1, \ldots, v_n$; similarly for $\chi(\sigma; v_1, \ldots, v_n)$ and $\chi(\sigma)$.

Notice that for $\sigma, \tau \in \Sigma_n$ we have

$$\mathtt{tw}(v_1 \otimes \cdots \otimes v_n, \sigma\tau) = \varepsilon(\sigma; v_1, \ldots, v_n)\mathtt{tw}(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}, \tau)$$

and therefore

$$\varepsilon(\sigma\tau; v_1, \ldots, v_n) = \varepsilon(\sigma; v_1, \ldots, v_n)\varepsilon(\tau; v_{\sigma(1)}, \ldots, v_{\sigma(n)}).$$

Lemma 1.1.2. *Given homogeneous vectors $v_1, \ldots, v_n \in V$ and $\sigma \in \Sigma_n$ we have*

$$\chi(\sigma, sV; sv_1, \ldots, sv_n) = (-1)^{\sum_{i=1}^{n}(n-i)(\overline{v_{\sigma(i)}} - \overline{v_i})} \varepsilon(\sigma, V; v_1, \ldots, v_n).$$

Proof. It is sufficient to check for σ a trasposition of two consecutive elements, and this is easy. □

Definition 1.1.3. Let V, W be graded vector spaces, a multilinear map

$$f\colon V \times \cdots \times V \to W$$

is called (graded) **symmetric** if

$$f(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = \varepsilon(\sigma) f(v_1, \ldots, v_n), \qquad \text{for every } \sigma \in \Sigma_n.$$

It is called (graded) **skewsymmetric** if

$$f(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = \chi(\sigma) f(v_1, \ldots, v_n), \qquad \text{for every } \sigma \in \Sigma_n.$$

Definition 1.1.4. The **symmetric powers** of a graded vector space V are defined as

$$V^{\odot n} = \bigodot^n V = \frac{\bigotimes^n V}{I},$$

where I is the subspace generated by the vectors

$$v_1 \otimes \cdots \otimes v_n - \varepsilon(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}, \qquad v_i \in V, \quad \sigma \in \Sigma_n\,.$$

We will denote by $\pi\colon \bigotimes^n V \to \bigodot^n V$ the natural projection and

$$v_1 \odot \cdots \odot v_n = \pi(v_1 \otimes \cdots \otimes v_n).$$

Definition 1.1.5. The **exterior powers** of a graded vector space V are defined as
$$V^{\wedge n} = \bigwedge^n V = \frac{\bigotimes^n V}{J},$$
where J is the subspace generated by the vectors
$$v_1 \otimes \cdots \otimes v_n - \chi(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}, \qquad v_i \in V, \quad \sigma \in \Sigma_n .$$
We will denote by $\pi\colon \bigotimes^n V \to \bigwedge^n V$ the natural projection and
$$v_1 \wedge \cdots \wedge v_n = \pi(v_1 \otimes \cdots \otimes v_n).$$

Definition 1.1.6. The set of **unshuffles** of type (p,q) is the subset $S(p,q) \subset \Sigma_{p+q}$ of permutations σ such that $\sigma(i) < \sigma(i+1)$ for every $i \neq p$. Equivalently
$$S(p,q) = \{\sigma \in \Sigma_{p+q} \mid \sigma(1) < \sigma(2) < \ldots < \sigma(p), \quad \sigma(p+1) < \sigma(p+2) < \ldots < \sigma(p+q)\}.$$

The unshuffles are a set of representatives for the left cosets of the canonical embedding of $\Sigma_p \times \Sigma_q$ inside Σ_{p+q}. More precisely for every $\eta \in \Sigma_{p+q}$ there exists a unique decomposition $\eta = \sigma\tau$ with $\sigma \in S(p,q)$ and $\tau \in \Sigma_p \times \Sigma_q$.

1.2. L_∞-algebras

Let $(L, d, [,])$ be a differential graded Lie algebra. Then we have:

1. $d(d(x_1)) = 0$;
2. $d[x_1, x_2] - \big([dx_1, x_2] - (-1)^{\overline{x_1}\,\overline{x_2}}[dx_2, x_1]\big) = 0$;
3. $[[x_1, x_2], x_3] - (-1)^{\overline{x_2}\,\overline{x_3}}[[x_1, x_3], x_2] + (-1)^{\overline{x_1}(\overline{x_2}+\overline{x_3})}[[x_2, x_3], x_1] = 0$;

Using the formalism of unshuffles we can write Leibniz and Jacobi identities respectively as
$$\sum_{\sigma \in S(2,0)} \chi(\sigma) d[x_{\sigma(1)}, x_{\sigma(2)}] - \sum_{\sigma \in S(1,1)} \chi(\sigma)[dx_{\sigma(1)}, x_{\sigma(2)}] = 0$$
and
$$\sum_{\sigma \in S(2,1)} \chi(\sigma)[[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}] = 0.$$

Definition 1.2.1. An L_∞ **structure** on a graded vector space L is a sequence of skewsymmetric maps
$$l_n\colon \bigwedge^n L \to L, \qquad \deg(l_n) = 2 - n, \quad n > 0,$$
such that for every $n > 0$ and every sequence of homogeneous vectors $x_1, \ldots, x_n \in L$ we have:
$$\sum_{k=1}^{n} (-1)^{n-k} \sum_{\sigma \in S(k,n-k)} \chi(\sigma)\, l_{n-k+1}(l_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}), x_{\sigma(k+1)}, \ldots, x_{\sigma(n)}) = 0.$$
An L_∞**-algebra** is a graded vector space endowed with an L_∞ structure.

Sometimes, L_∞-algebras are also called **strong homotopy Lie algebras**.

Remark 1.2.2.
1. Every DGLA $(L, d, [,])$ is also an L_∞ algebras with $l_1 = d$, $l_2 = [,]$ and $l_n = 0$ for every $n > 2$.
2. For an L_∞-algebra $(L, l_1, l_2, \ldots)$ we have $\deg(l_1) = 1$ and $l_1^2 = 0$ and then (L, l_1) is a DG-vector space.
3. For an L_∞-algebra $(L, l_1, l_2, \ldots)$ the morphism l_2 induces a structure of graded Lie algebra in the cohomology of the complex (L, l_1). In fact if $l_1(x) = l_1(y) = 0$ we have $l_1(l_2(x, y)) = \pm l_2(l_1(x), y) \pm l_2(l_1(y), x) = 0$. The equation
$$\sum_{k=1}^{3} (-1)^{3-k} \sum_{\sigma \in S(k,3-k)} \chi(\sigma)\, l_{4-k}(l_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}), x_{\sigma(k+1)}, \ldots, x_{\sigma(3)}) = 0.$$
implies that if $l_1(x_i) = 0$ then
$$\sum_{\sigma \in S(2,1)} \chi(\sigma)\, l_2(l_2(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)})$$
belongs to the image of l_1 and then
$$[\ ,\ ]\colon H^*(L, l_1) \times H^*(L, l_1) \to H^*(L, l_1), \qquad [x, y] = l_2(x, y),$$

satisfies the graded Jacobi identity.

Example 1.2.3. Let L be a graded vector space and $\phi\colon L^0\times L^0\times L^0\to L^{-1}$ a skewsymmetric map. Then the sequence

$$l_n(x_1,\dots,x_n)=\begin{cases}\phi(x_1,x_2,x_3) & \text{if } n=3 \text{ and } x_i\in L^0,\\ 0 & \text{otherwise,}\end{cases}$$

gives an L_∞ structure on L.

Definition 1.2.4. A **linear morphism** $f\colon (L,l_1,l_2,\dots)\to(H,h_1,h_2,\dots)$ of L_∞-algebras is a linear map $f\colon L\to H$ of degree 0 such that

$$fl_n(x_1,\dots,x_n)=h_n(f(x_1),\dots,f(x_n))$$

for every $n>0$ and every $x_1,\dots,x_n\in L$.

Do not confuse the notion of linear morphism of L_∞-algebras with the notion of L_∞-morphism (that we will give later). Every linear morphism is also an L_∞-morphism but the converse is not true.

Remark 1.2.5. In literature there exist two different (equivalent) definitions of L_∞-algebras depending by different sign conventions. Here we follow [**36, 75, 77**], while in [**80, 81**] the maps l_k differ from ours by the sign $(-1)^{k(k+1)/2}$.

1.3. $L_\infty[1]$-algebras

Definition 1.3.1. An $L_\infty[1]$ **structure** on a graded vector space V is a sequence of symmetric maps

$$q_n\colon V^{\odot n}\to V,\qquad \deg(q_n)=1,\quad n>0,$$

such that for every $n>0$ and every sequence of homogeneous vectors $v_1,\dots,v_n\in V$ we have:

$$\sum_{k=1}^{n}\sum_{\sigma\in S(k,n-k)}\varepsilon(\sigma)\,q_{n-k+1}(q_k(v_{\sigma(1)},\dots,v_{\sigma(k)}),v_{\sigma(k+1)},\dots,v_{\sigma(n)})=0. \tag{1}$$

An $L_\infty[1]$**-algebra** is a graded vector space endowed with an $L_\infty[1]$ structure.

Theorem 1.3.2. *For every graded vector space V there exists a canonical bijection from the set of $L_\infty[1]$ structures on V and the set of L_∞ structures on sV. This bijection is induced by the relations*

$$l_k(sv_1,\dots,sv_k)=-(-1)^{\sum_i(k-i)\overline{v_i}}sq_k(v_1,\dots,v_k).$$

PROOF. Immediate consequence of Lemma 1.1.2. Notice that for every k we have a commutative diagram

$$\begin{array}{ccc} V^{\otimes k} & \xrightarrow{q_k} & V\\ \downarrow{\scriptstyle s^{\otimes k}} & & \downarrow{\scriptstyle s}\\ (sV)^{\otimes k} & \xrightarrow{-l_k} & sV\end{array}$$

□

Remark 1.3.3. Since $V=(sV)[1]$ the above theorem says that there is a bijection between L_∞ structures on L and $L_\infty[1]$ structures on $L[1]$. Very often, in literature an L_∞ structure on a graded vector space L is defined as an $L_\infty[1]$ structures on $L[1]$.

Definition 1.3.4. Given $f\in\operatorname{Hom}^*_{\mathbb{K}}(V^{\odot n+1},V)$ and $g\in\operatorname{Hom}^*_{\mathbb{K}}(V^{\odot m+1},V)$ their **symmetric Gerstenhaber bracket** is defined as

$$[f,g]=f\circ g-(-1)^{\overline{f}\,\overline{g}}g\circ f\in\operatorname{Hom}^*_{\mathbb{K}}(V^{\odot n+m+1},V),$$

where

$$f\circ g(v_0,\dots,v_{n+m})=\sum_{\sigma\in S(m+1,n)}\epsilon(\sigma)f(g(v_{\sigma(0)},\dots,v_{\sigma(m)}),v_{\sigma(m+1)},\dots,v_{\sigma(m+n)}).$$

Thus, a sequence of maps $q_k \in \mathrm{Hom}^1_{\mathbb{K}}(V^{\odot k}, V)$ gives an $L_\infty[1]$ structure on V if and only if for every $n > 0$ we have

$$\sum_{a+b=n+1} [q_a, q_b] = 0.$$

In fact

$$\sum_{k=1}^{n}[q_{n-k+1}, q_k] = \sum_{k=1}^{n} q_{n-k+1} \circ q_k + \sum_{k=1}^{n} q_k \circ q_{n-k+1} = 2\sum_{k=1}^{n}(q_{n-k+1} \circ q_k)$$

and we recover exactly the left side of (1).

1.4. Extension of scalars

Given a graded vector space V and a DG-algebra (A, d) we have natural scalar extension maps

$$\mathrm{Hom}^*_{\mathbb{K}}(V^{\otimes n}, V) \xrightarrow{\tilde{\ }} \mathrm{Hom}^*_{\mathbb{K}}((V \otimes A)^{\otimes n}, V \otimes A)$$

defined in the following way:

$$\tilde{f}(v \otimes a) = f(v) \otimes a + (-1)^{\overline{v}} v \otimes d(a), \qquad \text{for } n = 1,$$

$$\tilde{f}(v_1 \otimes a_1, \ldots, v_n \otimes a_n) = (-1)^{\sum_{i<j} \overline{a_i}\,\overline{v_j}} f(v_1, \ldots, v_n) \otimes a_1 a_2 \cdots a_n, \qquad \text{for } n > 1.$$

Notice that $(-1)^{\sum_{i<j} \overline{a_i}\,\overline{v_j}}$ is the Koszul sign relating the sequences $v_1, a_1, v_2, \ldots, v_n, a_n$ and $v_1, \ldots, v_n, a_1, \ldots, a_n$. Moreover scalar extension preserves symmetry and skewsymmetry.

Lemma 1.4.1. *Scalar extension commutes with the bijection described in Theorem 1.3.2.*

PROOF. Easy and straightforward. □

Proposition 1.4.2. *Scalar extension preserves L_∞ and $L_\infty[1]$ structures.*

PROOF. It is sufficient to prove the theorem only for $L_\infty[1]$ structures. This is an easy consequence of the Leibniz rule in the DG-algebra A and it is left as an exercise. □

It is obvious that for every L_∞-algebra L and every morphism of DG-algebras $f\colon A \to B$, the morphism $\mathrm{Id}_L \otimes f\colon L \otimes A \to L \otimes B$ is a linear morphism of L_∞-algebras.

1.5. Maurer-Cartan and deformation functors associated to L_∞-algebras

Most of the notions concernings differential graded Lie algebras extends to this more general framework. For instance, the descending central series $L^{[n]}$ of an L_∞-algebra $(L, l_1, l_2, \ldots)$ is defined recursively as $L^{[1]} = L$ and

$$L^{[n]} = \mathrm{Span}\{l_k(x_1, \ldots, x_k) \mid k \geq 2,\ x_i \in L^{[n_i]},\ 0 < n_i < n,\ n_1 + \cdots + n_k \geq n\}.$$

An L_∞-algebra $(L, l_1, l_2, \ldots)$ is called **nilpotent** if $L^{[n]} = 0$ for $n >> 0$; notice that this implies in particular that $l_n = 0$ for $n >> 0$.

Definition 1.5.1. A Maurer-Cartan element in a nilpotent L_∞-algebra $(L, l_1, l_2, \ldots)$ is a vector $x \in L^1$ that satisfies the **Maurer-Cartan equation**:

$$\sum_{n>0} \frac{1}{n!} l_n(x, x, \ldots, x) = 0.$$

The subset of Maurer-Cartan elements will be denoted $\mathrm{MC}(L)$.

Thus for every L_∞-algebra L it makes sense to consider the Maurer-Cartan functor

$$\mathrm{MC}_L\colon \mathbf{Art} \to \mathbf{Set}, \qquad \mathrm{MC}_L(A) = \mathrm{MC}(L \otimes \mathfrak{m}_A),$$

where the L_∞ structure on $L \otimes \mathfrak{m}_A$ is given by scalar extension.

Lemma 1.5.2. *The tangent space of MC_L is $Z^1(L, l_1)$ and there exists a canonical complete obstruction theory with values in $H^2(L, l_1)$.*

PROOF. The first part is clear, since if $\mathfrak{m}_A^2 = 0$, then the Maurer-Cartan equation reduces to $l_1(x) = 0$. Assume that $0 \to I \to A \to B \to 0$ is a small extension in **Art**, let $y \in \mathrm{MC}(L \otimes \mathfrak{m}_B)$ and choose a lifting $x \in L^1 \otimes \mathfrak{m}_A$ of it. Denoting

$$h(x) = \sum_{n>0} \frac{1}{n!} l_n(x, x, \ldots, x) \in L^2 \otimes I,$$

for every $s \in L^1 \otimes I$ we have $h(x + s) = h(x) + l_1(s)$ and then y admits a lifting to $\mathrm{MC}_L(A)$ if and only if $h(x)$ is a coboundary in the complex $(L \otimes I, l_1)$. For every $n > 0$ we have

$$0 = \sum_{k=1}^{n} (-1)^{n-k} \sum_{\sigma \in S(k,n-k)} \chi(\sigma)\, l_{n-k+1}(l_k(x, \ldots, x), x, \ldots, x)$$

and then

$$\sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} l_{n-k+1}(l_k(x, \ldots, x), x, \ldots, x) = 0\ .$$

Dividing for $n!$ we get

$$\sum_{k=1}^{n} (-1)^{n-k} \frac{l_{n-k+1}}{(n-k)!} \left(\frac{l_k}{k!}(x, \ldots, x), x, \ldots, x \right) = 0$$

and summing over all $n > 0$ we obtain (setting $a = n - k + 1$)

$$\sum_{a,k>0} (-1)^{a-1} \frac{l_a}{(a-1)!} \left(\frac{l_k}{k!}(x, \ldots, x), x, \ldots, x \right) = 0\ ,$$

$$\sum_{a>0} (-1)^{a-1} \frac{l_a}{(a-1)!} (h(x), x, \ldots, x) = l_1(h(x)) = 0.$$

Thus $h(x)$ is a cocycle in $(L \otimes I, l_1)$ and its cohomology class is well defined. □

The notions of linear morphism and nilpotency for $L_\infty[1]$-algebras are defined similarly to the L_∞ case.

Definition 1.5.3. The Maurer-Cartan equation on a nilpotent $L_\infty[1]$-algebra $(V, q_1, q_2, \ldots)$ is defined as

$$\sum_{n>0} \frac{1}{n!} q_n(v, v, \ldots, v) = 0, \qquad v \in V^0.$$

It is clear that the the bijection $s\colon V^0 \to (sV)^1$ preserves solutions of Maurer-Cartan equations.

Definition 1.5.4. Let $(L, l_1, l_2, \ldots)$ be a nilpotent L_∞-algebra and $x, y \in \mathrm{MC}(L)$. We shall say that x and y are **homotopy equivalent** if there is some $\xi \in \mathrm{MC}(L \otimes \mathbb{K}[t, dt])$ such that $e_0(\xi) = x$ and $e_1(\xi) = y$, where $e_0, e_1 \colon L \otimes \mathbb{K}[t, dt] \to L = L \otimes \mathbb{K}$ are the evaluation maps at $t = 0$ and $t = 1$ respectively.

We will denote by $\mathrm{Def}(L)$ the quotient of $\mathrm{MC}(L)$ under the equivalence relation generated by homotopy.

We will prove later that homotopy is already an equivalence relation: here we dont need this result.

The construction of $\mathrm{Def}(L)$ is functorial and then we may define a functor

$$\mathrm{Def}_L \colon \mathbf{Art} \to \mathbf{Set}, \qquad \mathrm{Def}_L(A) = \mathrm{Def}(L \otimes \mathfrak{m}_A).$$

It is easy to see that the tangent space of the functor Def_L is $H^1(L, l_1)$. This follows from Lemma 1.5.2, from the fact the morphisms of DG-vector spaces

$$e_1, e_0 \colon L[t, dt] \to L$$

are homotopic via the homotopy $\mathrm{Id}_L \otimes \int_0^1$ and observing that for $z \in Z^1(L)$ and $u \in L^0$ we have $z + l_1(u)t + udt \in Z^1(L[t, dt])$.

Proposition 1.5.5. *Let L be a differential graded Lie algebra. Then for every For every $A \in$ **Art** the homotopy equivalence in $\mathrm{MC}_L(A)$ is the same as gauge equivalence and then the above definition of Def_L coincides with the one given in Section* **??**.

Proof. It is sufficient to prove that $x, y \in \mathrm{MC}_L(A)$ are gauge equivalent if and only if they are homotopy equivalent. So first, assume $e^a * x = y$ for some $a \in L^0$; then we can consider $z(t) = e^{ta} * x \in \mathrm{MC}_{L[t,dt]}(A)$ and therefore $z(0) = x$ and $z(1) = y$. Conversely assume that $z(0) = x$ and $z(1) = y$ for some $z(t) \in \mathrm{MC}_{L[t,dt]}(A)$; by Corollary **??** there exists $p(t) \in L^0[t]$ such that $p(0) = 0$ and $z(t) = e^{p(t)} * x$. Then $y = z(1) = e^{p(1)} * x$ and this imply that y is gauge equivalent to x. □

Remark 1.5.6. As a consequence of Proposition 1.5.5 we have that the bifunctor

$$\mathrm{Def}\colon \mathbf{DGLA} \times \mathbf{Art} \to \mathbf{Set}$$

is completely determined by the Maurer-Cartan bifunctor

$$\mathrm{MC}\colon \mathbf{DGLA} \times \mathbf{Art} \to \mathbf{Set}.$$

1.6. Construction of $L_\infty[1]$ structures via derived brackets

Assume it is given a graded Lie algebra L and a decomposition $L = V \oplus A$ as graded vector spaces, with V, A graded Lie subalgebra and A abelian. Denoting by $P \in \mathrm{Hom}^0_{\mathbb{K}}(L, L)$ the projection on A with kernel equal to V we have:

(1) $P^2 = P$,
(2) $[Pf, Pg] = 0$ for every $f, g \in L$,
(3) $P[f, g] = P[Pf, g] + P[f, Pg] = P[f, Pg] - (-1)^{\overline{f}\,\overline{g}} P[g, Pf]$ for every $f, g \in L$.

The second item is clearly equivalent to the abelianity of A. The third item is equivalent to the equality $P[f - Pf, g - Pg] = 0$ and then it is equivalent to the fact that $V = \ker P$ is a graded Lie subalgebra of L.

Lemma 1.6.1. *Let $f \in \mathrm{Der}^*_{\mathbb{K}}(L, L)$. Then $f(V) \subset V$ if and only if $Pf = PfP$.*

Proof. Assume $Pf = PfP$, then for every $v \in V$ we have $Pfv = PfPv = 0$. Conversely assume $f(V) \subset V$, then for every $x \in L$ we have

$$Pfx = Pf(Px + (x - Px)) = PfPx + Pf(x - Px) = PfPx,$$

where the last equality follow from the fact that $f(x - Px) \in V$. □

Remark 1.6.2. For every $v \in V$, the inner derivation $f = [v, -]$ satisfies the assumption of Lemma 1.6.1.

Example 1.6.3. Let A be a unitary graded commutative algebra and $L = \mathrm{Der}^*_A(A[t], A[t])$. Denote by $\partial\colon A[t] \to A[t]$ be the usual derivation operator $\partial = \dfrac{d}{dt} \in \mathrm{Der}^0_A(A[t], A[t])$; we may consider A as an abelian graded subalgebra of L, where every $a \in A$ is identified with the operator $a\partial$. Then we have a decomposition

$$L = A[t]\partial = \mathfrak{g}_A \oplus A$$

where $\mathfrak{g}_A = \oplus_{n>0} At^n\partial$ is the subalgebra of derivations vanishing fot $t = 0$: the operator P is therefore given by

$$P(q(t)\partial) = q(0)\partial, \qquad q(t) \in A[t].$$

Example 1.6.4. Let A be a unitary graded commutative algebra and $L = \mathrm{Hom}^*_{\mathbb{K}}(A, A)$. We may consider A as an abelian graded subalgebra of L, where every $a \in A$ is identified with the operator

$$a\colon A \to A, \qquad a(b) = ab.$$

Then we have

$$L = V \oplus A, \qquad V = \{f \in L \mid f(1) = 0\},$$

and therefore we have

$$P\colon L \to L, \qquad P(f)(a) = f(1)a.$$

For every $f \in \operatorname{Der}^*(L,L)$ end every integer $n>0$ denote:

$$f_n\colon \bigodot^n A \to L, \qquad f_1(a)=f(a),$$

$$f_n(a_1,\ldots,a_n)=[f_{n-1}(a_1,\ldots,a_{n-1}),a_n].$$

The fact that f is a derivation implies that every f_n is graded symmetric (easy to prove). Since A is abelian, if $f(A)\subset A$ then $f_n=0$ for every $n>1$.

Definition 1.6.5. In the notation above, the morphism

$$\Phi^n_f\colon A^{\odot n}\to A, \qquad \Phi^n_f = Pf_n,$$

is called the n-th **derived bracket** of $f\in \operatorname{Der}^*(L,L)$. By convention, we set $\Phi^0_f=0$.

Example 1.6.6. Let A be a unitary graded commutative algebra over a field of characteristic 0, then the multiplication maps

$$\mu_n\colon A^{\odot n+1}\to A, \qquad \mu_n(a_0,\ldots,a_n)=a_0a_1\cdots a_n,$$

can be interpreted as derived brackets. In fact, in the set-up of Example 1.6.3 denote

$$h_m=\frac{t^{m+1}\partial}{(m+1)!}\in L.$$

For a fixed integer $n\ge 0$ consider the inner derivation $f=[h_n,-]$. Then $f_1(a)=[h_n,a\partial]=-ah_{n-1}$ and more generally

$$f_m(a_1,\ldots,a_m)=(-1)^m a_1\cdots a_m h_{n-m},$$

giving

$$\Phi^{n+1}_f=(-1)^{n+1}\mu_n, \qquad \Phi^{i+1}_f=0 \text{ for every } i\neq n.$$

Theorem 1.6.7 (**[6, 126, 127]**). *In the above set-up, for every $f,g\in \operatorname{Der}^*(L,L)$ such that $f(V)\subset V, g(V)\subset V$ and every $n>0$ we have*

$$\Phi^n_{[f,g]}=\sum_{a+b=n+1}[\Phi^a_f,\Phi^b_g].$$

Proof. For notational simplicity, in the next formulas we denote by $\pm_K$ the correct Koszul sign. The first step of the proof is to prove that, for every $f,g\in\operatorname{Der}^*(L,L)$ and $n>0$ we have

$$[f,g]_n(a_1,\ldots,a_n)=\sum_{k=0}^n\sum_{\sigma\in S(k,n-k)}\pm_K\,[f_k(a_{\sigma(1)},\ldots,a_{\sigma(k)}),g_{n-k}(a_{\sigma(k+1)},\ldots,a_{\sigma(n)})],$$

where we intend that

$$[f_0(\emptyset),a]=f(a), \qquad [b,g_0(\emptyset)]=-(-1)^{\bar b\,\bar g}g(b).$$

For $n=1$ we have

$$[f,g]_1(a)=f(g(a))-(-1)^{\bar f\,\bar g}g(f(a))=[f_0,g_1(a)]-(-1)^{\bar f\,\bar g}[g_0,f_1(a)]=[f_0,g_1(a)]+(-1)^{\bar a\,\bar g}[f_1(a),g_0].$$

For $n>1$ we have

$$[f,g]_n(a_1,\ldots,a_n)=[[f,g]_{n-1}(a_1,\ldots,a_{n-1}),a_n]$$

$$=\sum_{k=0}^{n-1}\sum_{\sigma\in S(k,n-1-k)}\pm_K\,[[f_k(a_{\sigma(1)},\ldots,a_{\sigma(k)}),g_{n-1-k}(a_{\sigma(k+1)},\ldots,a_{\sigma(n-1)})],a_n]$$

$$=\sum_{k=0}^{n-1}\sum_{\sigma\in S(k,n-1-k)}\pm_K\,[f_{k+1}(a_{\sigma(1)},\ldots,a_{\sigma(k)},a_n),g_{n-1-k}(a_{\sigma(k+1)},\ldots,a_{\sigma(n-1)})]+$$

$$+\sum_{k=0}^{n-1}\sum_{\sigma\in S(k,n-1-k)}\pm_K\,[f_k(a_{\sigma(1)},\ldots,a_{\sigma(k)}),g_{n-k}(a_{\sigma(k+1)},\ldots,a_{\sigma(n-1)},a_n)]$$

$$= \sum_{k=1}^{n} \sum_{\sigma\in S(k,n-1-k),\sigma(n)<n} \pm_K \,[f_k(a_{\sigma(1)},\ldots,a_{\sigma(k)}), g_{n-k}(a_{\sigma(k+1)},\ldots,a_{\sigma(n)})]+$$
$$+\sum_{k=0}^{n-1} \sum_{\sigma\in S(k,n-k),\sigma(n)=n} \pm_K \,[f_k(a_{\sigma(1)},\ldots,a_{\sigma(k)}), g_{n-k}(a_{\sigma(k+1)},\ldots,a_{\sigma(n)})]$$

If $f(V), g(V) \subset V$, then the equality

$$P[f_n(a_1,\ldots,a_n), g_m(b_1,\ldots,b_m)] =$$
$$= P[\Phi_f^n(a_1,\ldots,a_n), g_m(b_1,\ldots,b_m)] + P[f_n(a_1,\ldots,a_n), \Phi_g^m(b_1,\ldots,b_m)]$$

holds for every $n, m \geq 0$. For $n, m > 0$ this follows from the properties of the projection operator P, while for $n = 0$ we have:

$$P[f_0, g_m(b_1,\ldots,b_m)] = Pf(g_m(b_1,\ldots,b_m)) = PfP(g_m(b_1,\ldots,b_m)) = P[f_0, \Phi_g^m(b_1,\ldots,b_m)].$$

Since

$$P[f_n(a_1,\ldots,a_n), \Phi_g^m(b_1,\ldots,b_m)] = Pf_{n+1}(a_1,\ldots,a_n, \Phi_g^m(b_1,\ldots,b_m)).$$

we have

$$P[f_n(a_1,\ldots,a_n), g_m(b_1,\ldots,b_m)] =$$
$$= \Phi_f^{n+1}(a_1,\ldots,a_n, \Phi_g^m(b_1,\ldots,b_m)) - \pm_K \Phi_g^{m+1}(b_1,\ldots,b_m, \Phi_f^n(a_1,\ldots,a_n)),$$

where the Koszul sign relates the sequences

$$f_n, a_1,\ldots,a_n, g_m, b_1,\ldots,b_m \quad \text{and} \quad g_m, b_1,\ldots,b_m, f_n, a_1,\ldots,a_n$$

and then it is equal to

$$\pm_K = (-1)^{(\overline{f_n}+\overline{a_1}+\cdots+\overline{a_n})(\overline{g_m}+\overline{b_1}+\cdots+\overline{b_m})} = (-1)^{\overline{f_n(a_1,\ldots,a_n)}\;\overline{g_m(b_1,\ldots,b_m)}}.$$

We are now ready to prove the theorem, i.e. to prove the formula

$$P[f,g]_n = \sum_{k=1}^{n} [Pf_k, Pg_{n-k+1}].$$

By the previous computation we have

$$P[f,g]_n(a_1,\ldots,a_n) = \sum_{k=0}^{n} \sum_{\sigma\in S(k,n-k)} \pm_K \, P[f_k(a_{\sigma(1)},\ldots,a_{\sigma(k)}), g_{n-k}(a_{\sigma(k+1)},\ldots,a_{\sigma(n)})],$$
$$= \sum_{k=0}^{n} \sum_{\sigma\in S(k,n-k)} \pm_K \, Pf_{k+1}(a_{\sigma(1)},\ldots,a_{\sigma(k)}, Pg_{n-k}(a_{\sigma(k+1)},\ldots,a_{\sigma(n)}))$$
$$- \sum_{k=0}^{n} \sum_{\sigma\in S(k,n-k)} \pm_K \, Pg_{n-k+1}(a_{\sigma(1)},\ldots,a_{\sigma(n-k)}, Pf_k(a_{\sigma(n-k+1)},\ldots,a_{\sigma(n)}))$$
$$= \sum_{k=1}^{n} Pf_k \circ Pg_{n-k+1}(a_1,\ldots,a_n) - (-1)^{\overline{f}\,\overline{g}} \sum_{k=1}^{n} Pg_{n-k+1} \circ Pf_k(a_1,\ldots,a_n)$$
$$= \sum_{k=1}^{n} [Pf_k, Pg_{n-k+1}](a_1,\ldots,a_n).$$

□

Corollary 1.6.8. *In the above set-up if $f \in \mathrm{Der}^1(L,L)$, $f(V) \subset V$ and $f^2 = 0$, then the sequence $q_k = \Phi_f^k$ gives an $L_\infty[1]$ structure on A.*