Part I

Classical theory

Preliminary version January 17, 2012

Beware: these notes are in a very preliminary form: lots of materials should be added, lots of mistakes sholuld be corrected, lots of proof should be simplified etc.

Chapter I

Smooth families of compact complex manifolds

In this chapter we introduce the notion of a family $f\colon \mathcal{X} \to B$ of compact complex manifolds as a proper holomorphic submersion of complex manifolds. Easy examples will show that in general the fibres $X_t := f^{-1}(t)$ are not biholomorphic each other, see e.g. Example I.1.4. Using integration of vector fields we prove that the family is locally trivial if and only if a certain morphism $\mathcal{KS}$ of sheaves over B is trivial, while the restriction of $\mathcal{KS}$ at a point $b \in B$ is a linear map $\mathrm{KS}\colon T_{b,B} \to H^1(X_b, T_{X_b})$, called the *Kodaira-Spencer map*, which can interpreted as the first derivative at the point b of the map

$$B \to \{\text{isomorphism classes of complex manifolds}\}, \quad t \mapsto X_t.$$

Then, according to Kodaira, Nirenberg and Spencer we define a *deformation* of a complex manifolds X as the data of a family $\mathcal{X} \to B$, of a base point $0 \in B$ and of an isomorphism $X \simeq X_0$. The isomorphism class of a deformation involves only the structure of f in a neighbourhood of X_0.

For every complex manifold M we denote by:

- $\mathcal{O}_M(U)$ the $\mathbb{C}$-algebra of holomorphic functions $f\colon U \to \mathbb{C}$ defined on an open subset $U \subset M$.
- $\mathcal{O}_M$ the trivial complex line bundle $\mathbb{C} \times M \to M$.
- T_M the holomorphic tangent bundle to M. The fibre of T_M at a point $x \in M$, i.e. the complex tangent space at x, is denoted by $T_{x,M}$.

If $x \in M$ is a point we denote by $\mathcal{O}_{M,x}$ the $\mathbb{C}$-algebra of germs of holomorphic functions at a point $x \in M$; a choice of local holomorphic coordinates $z_1, \ldots, z_n$, $z_i(x) = 0$, gives an isomorphism $\mathcal{O}_{M,x} = \mathbb{C}\{z_1, \ldots, z_n\}$, being $\mathbb{C}\{z_1, \ldots, z_n\}$ the $\mathbb{C}$-algebra of convergent power series.

In order to avoid a too heavy notation we sometimes omit the subscript M, when the underlying complex manifold is clear from the context.

I.1 Smooth families of compact complex manifolds

Definition I.1.1. A *smooth family of compact complex manifolds* is a proper holomorphic map $f\colon M \to B$ such that:

1. M, B are nonempty complex manifolds and B is connected.
2. The differential of f, $f_*\colon T_{p,M} \to T_{f(p),B}$ is surjective at every point $p \in M$.

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Two families $f\colon M \to B$, $g\colon N \to B$ over the same base are isomorphic if there exists a holomorphic isomorphism $N \to M$ commuting with f and g.

From now on, when there is no risk of confusion, we shall simply say *smooth family* instead of smooth family of compact complex manifolds.

Note that if $f\colon M \to B$ is a smooth family then f is open, closed and surjective. If $V \subset B$ is an open subset then $f\colon f^{-1}(V) \to V$ is a smooth family; more generally for every holomorphic map of connected complex manifolds $C \to B$, the pull-back $M \times_B C \to C$ is a smooth family.

For every $b \in B$ we denote $M_b = f^{-1}(b)$: M_b is a regular submanifold of M.

Definition I.1.2. A smooth family $f\colon M \to B$ is called *trivial* if it is isomorphic to the product $M_b \times B \to B$ for some (and hence all) $b \in B$. It is called *locally trivial* if there exists an open covering $B = \cup U_a$ such that every restriction $f\colon f^{-1}(U_a) \to U_a$ is trivial.

Lemma I.1.3. *Let $f\colon M \to B$ be a smooth family, $b \in B$. The normal bundle $N_{M_b/M}$ of M_b in M is trivial.*

Proof. Let $E = T_{b,B} \times M_b \to M_b$ be the trivial bundle with fibre $T_{b,B}$. The differential $f_*\colon T_{x,M} \to T_{b,B}$, $x \in M_b$ induces a surjective morphism of vector bundles $(T_M)_{|M_b} \to E$ whose kernel is exactly T_{M_b}.
By definition $N_{M_b/M} = (T_M)_{|M_b}/T_{M_b}$ and then $N_{M_b/M} = T_{b,B} \times M_b$. $\square$

The following examples of families show that, in general, if $a, b \in B$, $a \neq b$, then M_a is not biholomorphic to M_b and therefore that non every family is locally trivial.

Example I.1.4. Consider $B = \mathbb{C} - \{0, 1\}$,

$$M = \{([x_0, x_1, x_2], \lambda) \in \mathbb{P}^2 \times B \mid x_2^2 x_0 = x_1(x_1 - x_0)(x_1 - \lambda x_0)\},$$

and $f\colon M \to B$ the projection. Then f is a family and the fibre M_λ is a smooth plane cubic with j-invariant

$$j(M_\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$

(Recall that two elliptic curves are biholomorphic if and only if they have the same j-invariant.)

Example I.1.5. (The universal family of hypersurfaces)

For fixed integers $n, d > 0$, consider the projective space $\mathbb{P}^N$, $N = \binom{d+n}{n} - 1$, with homogeneous coordinates $a_{i_0,\dots,i_n}$, $i_j \geq 0$, $\sum_j i_j = d$, and denote

$$X = \left\{ ([x], [a]) \in \mathbb{P}^n \times \mathbb{P}^N \;\middle|\; \sum_{i_0+\dots+i_n=d} a_{i_0,\dots,i_n} x_0^{i_0} \cdots x_n^{i_n} = 0 \right\}.$$

X is a smooth hypersurface of $\mathbb{P}^n \times \mathbb{P}^N$, the differential of the projection $X \to \mathbb{P}^N$ is not surjective at a point $([x], [a])$ if and only if $[x]$ is a singular point of X_a.
The subset $B = \{[a] \in \mathbb{P}^N \mid X_a \text{ is smooth }\}$ is open; if $M = f^{-1}(B)$ then $f\colon M \to B$ is a family and every smooth hypersurface of degree d of $\mathbb{P}^n$ is isomorphic to a fibre of f.

Example I.1.6 (Hopf surfaces)**.** Let $A \in GL(2, \mathbb{C})$ be a matrix with eigenvalues of norm > 1 and let $\langle A \rangle \simeq \mathbb{Z} \subset GL(2, \mathbb{C})$ be the subgroup generated by A. The action of $\langle A \rangle$ on $X = \mathbb{C}^2 - \{0\}$ is free and properly discontinuous: in fact a linear change of coordinates $C\colon \mathbb{C}^2 \to \mathbb{C}^2$ changes the action of $\langle A \rangle$ into the action of $\langle C^{-1}AC \rangle$ and therefore it is not restrictive to assume A is a lower triangular matrix.
Therefore the quotient $S_A = X/\langle A \rangle$ is a compact complex manifold called *Hopf surface*: the holomorphic map $X \to S_A$ is the universal cover and then for every point $x \in S_A$ there exists a natural isomorphism $\pi_1(S_A, x) \simeq \langle A \rangle$. We have already seen that if A, B are conjugated matrix then S_A is biholomorphic to S_B. Conversely if $f\colon S_A \to S_B$ is a biholomorphism then f lifts to a biholomorphism $g\colon X \to X$ such that $gA = B^k g$; since f induces an isomorphism of fundamental groups $k = \pm 1$.
By Hartogs' theorem g extends to a biholomorphism $g\colon \mathbb{C}^2 \to \mathbb{C}^2$ such that $g(0) = 0$; since for every $x \neq 0$ $\lim_{n\to\infty} A^n(x) = +\infty$ and $\lim_{n\to\infty} B^{-n}(x) = 0$ it must be $gA = Bg$. Taking the differential at 0 of $gA = Bg$ we get that A is conjugated to B.

Example I.1.7 (Complete family of Hopf surfaces)**.** Denote $B = \{(a, b, c) \in \mathbb{C}^3 \mid |a| > 1,\ |c| > 1\}$, $X = B \times (\mathbb{C}^2 - \{0\})$ and let $\mathbb{Z} \simeq G \subset \mathrm{Aut}(X)$ be the subgroup generated by

$$(a, b, c, z_1, z_2) \mapsto (a, b, c, az_1, bz_1 + cz_2)$$

The action of G on X is free and properly discontinuous, let $M = X/G$ be its quotient and $f\colon M \to B$ the projection on the first coordinates: f is a family whose fibres are Hopf surfaces. Every Hopf surface is isomorphic to a fibre of f, this motivate the adjective "complete". In particular all the Hopf surfaces are diffeomorphic to $S^1 \times S^3$ (to see this look at the fibre over $(2, 0, 2)$).

Theorem I.1.8. *Let $M \xrightarrow{f} B$ be a smooth family of compact complex manifolds. Then for every point $0 \in B$ there exist an open neighbourhood $0 \in U \subset B$ and a diffeomorphism $\phi\colon M_0 \times U \to f^{-1}(U)$, where $M_0 = f^{-1}(0) \subset M$, such that:*

1. *$\phi(x, 0) = x$ and $f\phi(x, t) = t$ for every $x \in M_0$ and $t \in U$.*
2. *ϕ is transversely holomorphic, i.e. for every $x \in M_0$ the map $\phi\colon \{x\}\times U \to M$ is holomorphic.*

In particular, if B is connected, then the diffeomorphism type of the fibre M_b is independent from $b \in B$.

Proof. (cf. also [13], [111]) It is not restrictive to assume $B \subset \mathbb{C}^n$ a polydisk with coordinates $t_1, \ldots, t_n$ and $0 \in B$ the base point of the deformation. After a possible shrinking of B there exist a finite open covering $M = \cup W_a$, $a = 1, \ldots, r$, and holomorphic projections $p_a\colon W_a \to U_a = W_a \cap M_0$ such that $(p_a, f)\colon W_a \to U_a \times B$ is a biholomorphism for every a and U_a is a local chart with coordinates $z_i^a\colon U_a \to \mathbb{C}$, $i = 1, \ldots, m$.
Let $\rho_a\colon M_0 \to [0, 1]$ be a C^∞ partition of unity subordinate to the covering $\{U_a\}$ and denote $V_a = \rho_a^{-1}(]0, 1])$; we note that $\{V_a\}$ is a covering of M_0 and $\overline{V_a} \subset U_a$. After a possible shrinking of B we may assume $p_a^{-1}(\overline{V_a})$ closed in M.
For every subset $C \subset \{1, \ldots, r\}$ and every $x \in M_0$ we denote

$$H_C = \left(\bigcap_{a\in C} W_a - \bigcup_{a\notin C} p_a^{-1}(\overline{V_a})\right) \times \left(\bigcap_{a\in C} U_a - \bigcup_{a\notin C} \overline{V_a}\right) \subset M \times M_0,$$

$$C_x = \{a \mid x \in \overline{V_a}\,\}, \qquad H = \bigcup_C H_C.$$

Clearly $(x, x) \in H_{C_x}$ and then H is an open subset of $M \times M_0$ containing the graph G of the inclusion $M_0 \to M$. Since the projection $pr\colon M \times M_0 \to M$ is open and M_0 is compact, after a possible shrinking of B we may assume $pr(H) = M$.
Moreover if $(y, x) \in H$ and $x \in \overline{V_a}$ then $(y, x) \in H_C$ for some C containing a and therefore $y \in W_a$.
For every a consider the C^∞ function $q_a\colon H \cap (M \times U_a) \to \mathbb{C}^m$,

$$q_a(y, x) = \sum_b \rho_b(x) \frac{\partial z^a}{\partial z^b}(x)(z^b(p_b(y)) - z^b(x)).$$

By the properties of H, q_a is well defined and separately holomorphic in the variable y. If $(y, x) \in H \cap (M \times (U_a \cap U_c))$ then

$$q_c(y, x) = \frac{\partial z^c}{\partial z^a}(x) q_a(y, x)$$

and then

$$\Gamma = \{(y, x) \in H \mid q_a(y, x) = 0 \text{ whenever } x \in U_a\}$$

is a well defined closed subset of H.
If $y \in V_a \subset M_0$ and x is sufficiently near to y then $x \in (\bigcap_{b\in C_y} U_b - \bigcup_{b\notin C} \overline{V_b})$ and, for every $b \in C_y$,

$$z^b(y) = z^b(x) + \frac{\partial z^b}{\partial z^a}(x)(z^a(y) - z^a(x)) + o(\|z^a(y) - z^a(x)\|).$$

Therefore

$$q_a(y, x) = z^a(y) - z^a(x) + o(\|z^a(y) - z^a(x)\|).$$

In particular the map $x \mapsto q_a(y,x)$ is a local diffeomorphism at $x = y$.
Denote $K \subset H$ the open subset of points (y,x) such that, if $y \in p_a^{-1}(V_a)$ then $u \mapsto q_a(y,u)$ has maximal rank at $u = x$; note that K contains G.
Let Γ_0 be the connected component of $\Gamma \cap K$ that contains G; Γ_0 is a C^∞-subvariety of K and the projection $pr\colon \Gamma_0 \to M$ is a local diffeomorphism. Possibly shrinking B we may assume that $pr\colon \Gamma_0 \to M$ is a diffeomorphism.
By implicit function theorem Γ_0 is the graph of a C^∞ projection $\gamma\colon M \to M_0$.
After a possible shrinking of B, the map $(\gamma, f)\colon M \to M_0 \times B$ is a diffeomorphism, take $\phi = (\gamma, f)^{-1}$.
To prove that, for every $x \in M_0$, the map $t \mapsto \phi(x,t)$ is holomorphic we note that $f\colon \phi(\{x\}\times B) \to B$ is bijective and therefore $\phi(x,-) = f^{-1}pr\colon \{x\} \times B \to \phi(\{x\} \times B)$.
The map $f^{-1}\colon B \to \phi(\{x\} \times B)$ is holomorphic if and only if $\phi(\{x\} \times B) = \gamma^{-1}(x)$ is a holomorphic subvariety and this is true because for x fixed every map $y \mapsto q_a(y,x)$ is holomorphic. $\square$

I.2 Čech and Dolbeault cohomology

We assume that the reader is familiar with cohomology theory of sheaves and with Dolbeault theorem. In this section we fix some notation used in the rest of the book.

If M is a complex manifold and E is a holomorphic vector bundle on M, we denote:

- $E^\vee$ the dual bundle of E.
- $\Gamma(U, E)$ the space of holomorphic sections $s\colon U \to E$ on an open subset $U \subset M$.
- $\Omega^1_M = T^\vee_M$ the holomorphic cotangent bundle of M.
- $\Omega^p_M = \bigwedge^p T^\vee_M$ the bundle of holomorphic differential p-forms.

For every open subset $U \subset M$ we denote by $\Gamma(U, \mathcal{A}^{p,q}_M)$ the $\mathbb{C}$-vector space of differential (p,q)-forms on U. If $z_1, \dots, z_n$ are local holomorphic coordinates, then $\phi \in \Gamma(U, \mathcal{A}^{p,q}_M)$ is written locally as $\phi = \sum \phi_{I,J} dz_I \wedge d\bar{z}_J$, where $I = (i_1, \dots, i_p)$, $J = (j_1, \dots, j_q)$, $dz_I = dz_{i_1} \wedge \dots \wedge dz_{i_p}$, $d\bar{z}_J = d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$ and the $\phi_{I,J}$ are C^∞ functions.
Similarly, if $E \to M$ is a holomorphic vector bundle we denote by $\Gamma(U, \mathcal{A}^{p,q}(E))$ the space of differential (p,q)-forms on U with value in E; locally, if $e_1, \dots, e_r$ is a local frame for E, an element of $\Gamma(U, \mathcal{A}^{p,q}(E))$ is written as $\sum_{i=1}^r \phi_i e_i$, with $\phi_i \in \Gamma(U, \mathcal{A}^{p,q})$. Note that there exist natural isomorphisms $\Gamma(U, \mathcal{A}^{p,q}(E)) \simeq \Gamma(U, \mathcal{A}^{0,q}(\Omega^p_M \otimes E))$.

The Dolbeault's cohomology of a holomorphic vector bundle E, denoted by $H^{p,*}_{\bar\partial}(M,E)$ is the cohomology of the Dolbeault complex

$$0 \longrightarrow \Gamma(M, \mathcal{A}^{p,0}(E)) \xrightarrow{\bar\partial} \Gamma(M, \mathcal{A}^{p,1}(E)) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \Gamma(U, \mathcal{A}^{p,q}(E)) \xrightarrow{\bar\partial} \cdots$$

Notice that $H^{p,0}_{\bar\partial}(M,E) = \Gamma(M, \Omega^p_M \otimes E)$ is the space of holomorphic p-forms with values in E.

From now on, for simplicity of notation, we denote $H^q(M,E) = H^{0,q}_{\bar\partial}(M,E)$, $h^q(M,E) = \dim_{\mathbb{C}} H^q(M,E)$, $H^q(M, \Omega^p(E)) = H^{p,q}_{\bar\partial}(M,E)$.

The *Hodge numbers* of a fixed compact complex manifold M are by definition

$$h^{p,q} = \dim_{\mathbb{C}} H^{p,q}_{\bar\partial}(M, \mathcal{O}) = \dim_{\mathbb{C}} H^{0,q}_{\bar\partial}(M, \Omega^p).$$

The *Betti numbers* of M are the dimensions of the spaces of the De Rham cohomology of M, i.e.

$$b_p = \dim_{\mathbb{C}} H^p_d(M, \mathbb{C}), \quad H^p_d(M, \mathbb{C}) = \frac{d\text{-closed } p\text{-forms}}{d\text{-exact } p\text{-forms}}.$$

Let $\mathcal{U} = \{U_a\}$, $a \in \mathcal{I}$, be an open covering of a complex manifold M; for every $a_0, \dots, a_k \in \mathcal{I}$ we denote $U_{a_0,\dots,a_k} = U_{a_0} \cap \dots \cap U_{a_k}$. For every sheaf of abelian groups $\mathcal{F}$ on M we denote by

$$C^*(\mathcal{U}, \mathcal{F}): \qquad C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{d} C^2(\mathcal{U}, \mathcal{F}) \xrightarrow{d} \cdots$$

the corresponding alternating Čech complex. Here $C^k(\mathcal{U}, E)$ is the group of alternating sequences $\{f_{a_0,a_1,\ldots,a_k}\}$, $a_0,\ldots,a_k \in \mathcal{I}$, $f_{a_0,a_1,\ldots,a_k} \in \mathcal{F}(U_{a_0,\ldots,a_k})$. Alternating means that for every permutation $\sigma \in \Sigma_{k+1}$ we have $f_{a_{\sigma(0)},a_{\sigma(1)},\ldots,a_{\sigma(k)}} = (-1)^\sigma f_{a_0,a_1,\ldots,a_k}$ and $f_{a_0,a_1,\ldots,a_k} = 0$ whenever $a_i = a_j$ for some $i \neq j$. The cohomology of the above complex is denoted $H^*(\mathcal{U}, \mathcal{F})$.

Proposition I.2.1. *For every holomorphic vector bundle E and every open covering $\mathcal{U} = \{U_a\}$, $a \in \mathcal{I}$, there exists a natural morphism of $\mathbb{C}$-vector spaces $\theta\colon H^k(\mathcal{U}, E) \to H^{0,k}_{\overline{\partial}}(M, E)$.*

Proof. Let $t_a\colon M \to \mathbb{C}$, $a \in \mathcal{I}$, be a differentiable partition of unity subordinate to the covering $\{U_a\}$: this means that $supp(t_a) \subset U_a$, the family of supports $\{supp(t_a)\}$ is locally finite and $\sum_a t_a = 1$.

Given $f \in C^k(\mathcal{U}, E)$ and $a \in \mathcal{I}$ we consider

$$\phi_a(f) = \sum_{c_1,\ldots,c_k} f_{a,c_1,\ldots,c_k} \overline{\partial} t_{c_1} \wedge \cdots \wedge \overline{\partial} t_{c_k} \in \Gamma(U_a, \mathcal{A}^{0,k}(E)),$$

$$\phi(f) = \sum_a t_a \phi_a(f) \in \Gamma(M, \mathcal{A}^{0,k}(E)).$$

Since every $f_{a,c_1,\ldots,c_k}$ is holomorphic we have $\overline{\partial}\phi_a = 0$ and then

$$\overline{\partial}\phi(f) = \sum_a \overline{\partial} t_a \wedge \phi_a(f) = \sum_{c_0,\ldots,c_k} f_{c_0,\ldots,c_k} \overline{\partial} t_{c_0} \wedge \cdots \wedge \overline{\partial} t_{c_k}.$$

We claim that ϕ is a morphism of complexes; in fact

$$\phi(df) = \sum_a t_a \sum_{c_0,\ldots,c_k} df_{a,c_0,\ldots,c_k} \overline{\partial} t_{c_0} \wedge \cdots \wedge \overline{\partial} t_{c_k} =$$

$$\sum_a t_a \left(\overline{\partial}\phi(f) - \sum_{i=0}^k \sum_{c_i} \overline{\partial} t_{c_i} \wedge \sum_{c_0,\ldots,\widehat{c_i},\ldots,c_k} f_{a,c_0,\ldots,\widehat{c_i},\ldots,c_k} \overline{\partial} t_{c_0} \wedge \cdots \wedge \widehat{\overline{\partial} t_{c_i}} \wedge \cdots \wedge \overline{\partial} t_{c_k} \right) =$$

$$= \sum_a t_a \overline{\partial}\phi(f) = \overline{\partial}\phi(f).$$

Setting θ as the morphism induced by ϕ in cohomology, we need to prove that θ is independent from the choice of the partition of unity. We first note that, if $df = 0$ then, over $U_a \cap U_b$, we have

$$\begin{aligned}
\phi_a(f) - \phi_b(f) &= \textstyle\sum_{c_1,\ldots,c_k} (f_{a,c_1,\ldots,c_k} - f_{b,c_1,\ldots,c_k}) \overline{\partial} t_{c_1} \wedge \cdots \wedge \overline{\partial} t_{c_k} \\
&= \textstyle\sum_{c_1,\ldots,c_k} \sum_{i=1}^k (-1)^{i-1} f_{a,b,c_1,\ldots,\hat{c}_i,\ldots,c_k} \overline{\partial} t_{c_1} \wedge \cdots \wedge \overline{\partial} t_{c_k} \\
&= \textstyle\sum_{i=1}^k (-1)^{i-1} \sum_{c_1,\ldots,c_k} f_{a,b,c_1,\ldots,\hat{c}_i,\ldots,c_k} \overline{\partial} t_{c_1} \wedge \cdots \wedge \overline{\partial} t_{c_k} \\
&= \sum_{i=1}^k \sum_{c_i} \overline{\partial} t_{c_i} \wedge \sum_{c_1,\ldots,\widehat{c_i},\ldots,c_k} f_{a,b,c_1,\ldots,\widehat{c_i},\ldots,c_k} \overline{\partial} t_{c_1} \wedge \cdots \wedge \widehat{\overline{\partial} t_{c_i}} \wedge \cdots \wedge \overline{\partial} t_{c_k} \\
&= 0.
\end{aligned}$$

Let v_a be another partition of 1, $\eta_a = t_a - v_a$, and denote, for $f \in Z^k(\mathcal{U}, E)$,

$$\tilde{\phi}_a = \sum_{c_1,\ldots,c_k} f_{a,c_1,\ldots,c_k} \overline{\partial} v_{c_1} \wedge \cdots \wedge \overline{\partial} v_{c_k},$$

$$\psi_a^j = \sum_{c_1,\ldots,c_k} f_{a,c_1,\ldots,c_k} \overline{\partial} t_{c_1} \wedge \cdots \wedge \overline{\partial} t_{c_{j-1}} \wedge v_{c_j} \overline{\partial} v_{c_{j+1}} \wedge \cdots \wedge \overline{\partial} v_{c_k}, \quad j = 1,\ldots,k.$$

The same argument as above shows that $\tilde{\phi}_a = \tilde{\phi}_b$ and $\psi_a^j = \psi_b^j$ for every a, b, j. Therefore all the ψ_a^j come from a global section $\psi^j \in \Gamma(M, \mathcal{A}^{0,k-1}(E))$; moreover $\phi - \tilde{\phi} = \sum_j (-1)^{j-1} \overline{\partial}\psi^j$ and then ϕ, $\tilde{\phi}$ determine the same cohomology class. □

A well known theorem of Leray asserts that if $\mathcal{U}$ is a open Stein covering of M then the above morphism θ is an isomorphism. The explicit description of θ^{-1} is rather easy for $k = 1$. Assume that $\mathcal{U} = \{U_a\}$ be an open covering of a complex manifold M such that $H^1_{\overline{\partial}}(U_a, E) = 0$ for every a. Given $\phi \in \Gamma(M, \mathcal{A}^{0,1}(E))$ a $\overline{\partial}$-closed form, then for every a there exists $\psi_a \in \Gamma(U_a, \mathcal{A}^{0,0}(E))$ such that $\overline{\partial}\psi_a = \phi$. Setting $f_{a,b} = \psi_a - \psi_b$ we have $f = \{f_{a,b}\} \in C^1(\mathcal{U}, E)$ and $df = 0$. The cohomology class of f in $H^1(\mathcal{U}, E)$ is well defined and then we have defined a map

$$\sigma\colon H^{0,1}_{\overline{\partial}}(M, E) \to H^1(\mathcal{U}, E), \qquad \phi \mapsto [f].$$

We left to the reader the easy verification that $\sigma = \theta^{-1}$.

Example I.2.2. Let $T \to \mathbb{P}^1$ be the holomorphic tangent bundle, x_0, x_1 homogeneous coordinates on $\mathbb{P}^1$, $U_i = \{x_i \neq 0\}$. Since the tangent bundle of $U_i = \mathbb{C}$ is trivial, by Dolbeault's lemma, $H^1(U_i, T) = 0$ and by Leray's theorem $H^i(\mathbb{P}^1, T) = H^i(\{U_0, U_1\}, T)$, $i = 0, 1$.
Consider the affine coordinates $s = x_1/x_0$, $t = x_0/x_1$, then the holomorphic sections of T over U_0, U_1 and $U_{0,1} = U_0 \cap U_1$ are given respectively by convergent power series

$$\sum_{i=0}^{+\infty} a_i s^i \frac{\partial}{\partial s}, \qquad \sum_{i=0}^{+\infty} b_i t^i \frac{\partial}{\partial t}, \qquad \sum_{i=-\infty}^{+\infty} c_i s^i \frac{\partial}{\partial s}.$$

Since, over $U_{0,1}$, $t = s^{-1}$ and $\dfrac{\partial}{\partial t} = -s^2 \dfrac{\partial}{\partial s}$, the Cech differential is given by

$$d\left(\sum_{i=0}^{+\infty} a_i s^i \frac{\partial}{\partial s}, \sum_{i=0}^{+\infty} b_i t^i \frac{\partial}{\partial t}\right) = \sum_{i=0}^{+\infty} a_i s^i \frac{\partial}{\partial s} + \sum_{i=-\infty}^{2} b_{2-i} s^i \frac{\partial}{\partial s},$$

and then $H^1(\{U_0, U_1\}, T) = 0$ and

$$H^0(\{U_0, U_1\}, T) = \left\langle \left(\frac{\partial}{\partial s}, -t^2 \frac{\partial}{\partial t}\right), \left(s\frac{\partial}{\partial s}, -t\frac{\partial}{\partial t}\right), \left(s^2\frac{\partial}{\partial s}, -\frac{\partial}{\partial t}\right) \right\rangle.$$

Example I.2.3. If $X = \mathbb{P}^1 \times \mathbb{C}^n_t$ then $H^1(X, T_X) = 0$. If $\mathbb{C} \subset \mathbb{P}^1$ is an affine open subset with affine coordinate s, then $H^0(X, T_X)$ is the free $\mathcal{O}(\mathbb{C}^n)$-module generated by

$$\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_n}, \frac{\partial}{\partial s}, s\frac{\partial}{\partial s}, s^2\frac{\partial}{\partial s}.$$

The proof is essentially the same (replacing the constant terms a_i, b_i, c_i with holomorphic functions over $\mathbb{C}^n$) of Example I.2.2.

I.3 The Kodaira-Spencer map

Given a holomorphic map $f\colon X \to Y$ of complex manifolds and complexified vector fields $\eta \in \Gamma(X, \mathcal{A}^{0,0}(T_X))$, $\gamma \in \Gamma(Y, \mathcal{A}^{0,0}(T_Y))$ we write $\gamma = f_*\eta$ if for every $x \in X$ we have $f_*\eta(x) = \gamma(f(x))$, where $f_*\colon T_{x,X} \to T_{f(x),Y}$ is the differential of f.

Let $f\colon M \to B$ be a fixed smooth family of compact complex manifolds, $\dim B = n$, $\dim M = m + n$; for every $b \in B$ we let $M_b = f^{-1}(b)$.

Definition I.3.1. A holomorphic coordinate chart $(z_1, \ldots, z_m, t_1, \ldots, t_n)\colon U \hookrightarrow \mathbb{C}^{m+n}$, $U \subset M$ open, is called *admissible* if $f(U)$ is contained in a coordinate chart $(v_1, \ldots, v_n)\colon V \hookrightarrow \mathbb{C}^n$, $V \subset B$, such that $t_i = v_i \circ f$ for every $i = 1, \ldots, n$.

Since the differential of f has everywhere maximal rank, by the implicit function theorem, M admits a locally finite covering of admissible coordinate charts.

Lemma I.3.2. *Let $f\colon M \to B$ be a smooth family of compact complex manifolds. For every $\gamma \in \Gamma(B, \mathcal{A}^{0,0}(T_B))$ there exists $\eta \in \Gamma(M, \mathcal{A}^{0,0}(T_M))$ such that $f_*\eta = \gamma$.*

Proof. Let $M = \cup U_a$ be a locally finite covering of admissible charts; on every U_a there exists $\eta_a \in \Gamma(U_a, \mathcal{A}^{0,0}(T_M))$ such that $f_*\eta_a = \gamma$. It is then sufficient to take $\eta = \sum_a \rho_a \eta_a$, being $\rho_a \colon U_a \to \mathbb{C}$ a partition of unity subordinate to the covering $\{U_a\}$. $\square$

Let $T_f \subset T_M$ be the holomorphic vector subbundle of tangent vectors v such that $f_*v = 0$. If $z_1, \ldots, z_m, t_1, \ldots, t_n$ is an admissible system of local coordinates then $\dfrac{\partial}{\partial z_1}, \ldots, \dfrac{\partial}{\partial z_m}$ is a local frame of T_f. Note that the restriction of T_f to M_b is equal to T_{M_b}.

For every open subset $V \subset B$ let $\Gamma(V, T_B)$ be the space of holomorphic vector fields on V. For every $\gamma \in \Gamma(V, T_B)$ take $\eta \in \Gamma(f^{-1}(V), \mathcal{A}^{0,0}(T_M))$ such that $f_*\eta = \gamma$. In an admissible system of local coordinates z_i, t_j we have $\eta = \sum_i \eta_i(z,t)\dfrac{\partial}{\partial z_i} + \sum_j \gamma_i(t)\dfrac{\partial}{\partial t_j}$, with $\gamma_i(t)$ holomorphic, $\overline{\partial}\eta = \sum_i \overline{\partial}\eta_i(z,t)\dfrac{\partial}{\partial z_i}$ and then $\overline{\partial}\eta \in \Gamma(f^{-1}(V), \mathcal{A}^{0,1}(T_f))$.

Obviously $\overline{\partial}\eta$ is $\overline{\partial}$-closed and then we can define the *Kodaira-Spencer* map

$$\mathcal{KS}(V)_f \colon \Gamma(V, T_B) \to H^1(f^{-1}(V), T_f), \qquad \mathcal{KS}(V)_f(\gamma) = [\overline{\partial}\eta].$$

Lemma I.3.3. *The map $\mathcal{KS}(V)_f$ is a well-defined homomorphism of $\mathcal{O}(V)$-modules.*

Proof. If $\tilde{\eta} \in \Gamma(f^{-1}(V), \mathcal{A}^{0,0}(T_M))$, $f_*\tilde{\eta} = \gamma$, then $\eta - \tilde{\eta} \in (f^{-1}(V), \mathcal{A}^{0,0}(T_f))$ and $[\overline{\partial}\tilde{\eta}] = [\overline{\partial}\eta] \in H^1(f^{-1}(V), T_f)$. If $g \in \mathcal{O}(V)$ then $f_*(f^*g)\eta = g\gamma$, $\overline{\partial}(f^*g)\eta = (f^*g)\overline{\partial}\eta$. $\square$

If $V_1 \subset V_2 \subset B$ then the Kodaira-Spencer maps $\mathcal{KS}(V_i)_f \colon \Gamma(V_i, T_B) \to H^1(f^{-1}(V_i), T_f)$, $i = 1, 2$, commute with the restriction maps $\Gamma(V_2, T_B) \to \Gamma(V_1, T_B)$, $H^1(f^{-1}(V_2), T_f) \to H^1(f^{-1}(V_1), T_f)$. Therefore we get a well defined $\mathcal{O}_{B,b}$-linear map

$$\mathcal{KS}_f \colon \Theta_{B,b} \to (R^1 f_* T_f)_b,$$

where $\Theta_{B,b}$ and $(R^1 f_* T_f)_b$ are by definition the direct limits, over the set of open neighbourhood V of b, of $\Gamma(V, T_B)$ and $H^1(f^{-1}(V), T_f)$ respectively.

If $b \in B$, then there exists a linear map $\mathrm{KS}_f \colon T_{b,B} \to H^1(M_b, T_{M_b})$ such that for every open subset $b \in V \subset B$ there exists a commutative diagram

$$\begin{array}{ccc} \Gamma(V, T_B) & \xrightarrow{\mathcal{KS}(V)_f} & H^1(f^{-1}(V), T_f) \\ \big\downarrow & & \big\downarrow{\scriptstyle r} \\ T_{b,B} & \xrightarrow{\mathrm{KS}_f} & H^1(M_b, T_{M_b}) \end{array}$$

where the vertical arrows are the natural restriction maps. In fact, if V is a polydisk then $T_{b,B}$ is the quotient of the complex vector space $\Gamma(V, T_B)$ by the subspace $I = \{\gamma \in \Gamma(V, T_B) \mid \gamma(b) = 0\}$; by $\mathcal{O}(V)$-linearity I is contained in the kernel of $r \circ \mathcal{KS}(V)_f$.

The Kodaira-Spencer map has at least two geometric interpretations: obstruction to the holomorphic lifting of vector fields and first-order variation of complex structures (this is a concrete feature of the general philosophy that deformations are a derived construction of automorphisms).

Proposition I.3.4. *Let $f \colon M \to B$ be a family of compact complex manifolds and $\gamma \in \Gamma(V, T_B)$, then $\mathcal{KS}(V)_f(\gamma) = 0$ if and only if there exists $\eta \in \Gamma(f^{-1}(V), T_M)$ such that $f_*\eta = \gamma$.*

Proof. One implication is trivial; conversely let $\eta \in \Gamma(f^{-1}(V), \mathcal{A}^{0,0}(T_M))$ such that $f_*\eta = \gamma$. If $[\overline{\partial}\eta] = 0$ then there exists $\tau \in \Gamma(f^{-1}(V), \mathcal{A}^{0,0}(T_f))$ such that $\overline{\partial}(\eta - \tau) = 0$, $\eta - \tau \in \Gamma(f^{-1}(V), T_M)$ and $f_*(\eta - \tau) = \gamma$. $\square$

To compute the Kodaira-Spencer map in terms of Cech cocycles we assume that V is a polydisk with coordinates $t_1, \ldots, t_n$ and we fix a locally finite covering $\mathcal{U} = \{U_a\}$ of admissible holomorphic coordinates $z_1^a, \ldots, z_m^a, t_1^a, \ldots, t_n^a \colon U_a \to \mathbb{C}$, $t_i^a = f^* t_i$. On $U_a \cap U_b$ we have the transition functions

$$\begin{cases} z_i^b = g_{i,a}^b(z^a, t^a), & i = 1, \ldots, m \\ t_i^b = t_i^a, & i = 1, \ldots, n \end{cases}$$

Consider a fixed integer $h = 1, \ldots, n$ and $\eta \in \Gamma(f^{-1}(V), \mathcal{A}^{0,0}(T_M))$ such that $f_*\eta = \dfrac{\partial}{\partial t_h}$; in local coordinates we have

$$\eta = \sum_i \eta_i^a(z^a, t^a)\frac{\partial}{\partial z_i^a} + \frac{\partial}{\partial t_h^a}, \qquad \eta = \sum_i \eta_i^b(z^b, t^b)\frac{\partial}{\partial z_i^b} + \frac{\partial}{\partial t_h^b}.$$

Since, for every a, $\eta - \dfrac{\partial}{\partial t_h^a} \in \Gamma(U_a, \mathcal{A}^{0,0}(T_f))$ and $\overline{\partial}\left(\eta - \dfrac{\partial}{\partial t_h^a}\right) = \overline{\partial}\eta$, $\mathcal{KS}(V)_f\left(\dfrac{\partial}{\partial t_h}\right) \in H^1(\mathcal{U}, T_f)$ is represented by the cocycle

$$\mathcal{KS}(V)_f\left(\frac{\partial}{\partial t_h}\right)_{b,a} = \left(\eta - \frac{\partial}{\partial t_h^b}\right) - \left(\eta - \frac{\partial}{\partial t_h^a}\right) = \frac{\partial}{\partial t_h^a} - \frac{\partial}{\partial t_h^b} = \sum_i \frac{\partial g_{i,a}^b}{\partial t_h^a}\frac{\partial}{\partial z_i^b}. \tag{I.1}$$

The above formula allows to prove easily the invariance of the Kodaira-Spencer maps under base change; more precisely if $f\colon M \to B$ is a smooth family, $\phi\colon C \to B$ a holomorphic map, $\hat{\phi}$, $\hat{f}$ the pullbacks of ϕ and f,

$$\begin{array}{ccc} M \times_B C & \xrightarrow{\hat{\phi}} & M \\ \big\downarrow{\scriptstyle \hat{f}} & & \big\downarrow{\scriptstyle f} \\ C & \xrightarrow{\phi} & B \end{array}$$

$c \in C$, $b = f(c)$.

Theorem I.3.5. *In the above notation, via the natural isomorphism $M_b = \hat{f}^{-1}(c)$, we have*

$$\mathrm{KS}_{\hat{f}} = \mathrm{KS}_f\phi_*\colon T_{c,C} \to H^1(M_b, T_{M_b}).$$

Proof. It is not restrictive to assume $B \subset \mathbb{C}^n_t$, $C \subset \mathbb{C}^s_u$ polydisks, $c = \{u_i = 0\}$ and $b = \{t_i = 0\}$, $t_i = \phi_i(u)$. If $z^a, t^a\colon U_a \to \mathbb{C}$, $z^b, t^b\colon U_b \to \mathbb{C}$ are admissible local coordinate sets with transition functions $z_i^b = g_{i,a}^b(z^a, t^a)$, then $z^a, u^a\colon U_a \times_B C \to \mathbb{C}$, $z^b, t^b\colon U_b \times_B C \to \mathbb{C}$ are admissible with transition functions $z_i^b = g_{i,a}^b(z^a, \phi(u^a))$. Therefore

$$\mathrm{KS}_{\hat{f}}\left(\frac{\partial}{\partial u_h}\right)_{b,a} = \sum_i \frac{\partial g_{i,a}^b}{\partial u_h^a}\frac{\partial}{\partial z_i^b} = \sum_{i,j}\frac{\partial g_{i,a}^b}{\partial t_j^a}\frac{\partial \phi_j}{\partial u_h^a}\frac{\partial}{\partial z_i^b} = \mathrm{KS}_f\left(\phi_*\frac{\partial}{\partial u_h}\right)_{b,a}.$$

□

I.4 Rigid varieties

For every $0 < R \leq +\infty$ the polydisk of radius R is defined as

$$\Delta_R^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_i| < R,\ i = 1, \ldots, n\}.$$

Lemma I.4.1. *Let $f\colon M \to \Delta_R^n$ be a smooth family of compact complex manifolds and $t_1, \ldots, t_n$ a set of linear coordinates in the polydisk $\Delta_R^n \subset \mathbb{C}^n$. If there exist holomorphic vector fields $\chi_1, \ldots, \chi_n$ on M such that $f_*\chi_h = \dfrac{\partial}{\partial t_h}$ then there exists $0 < r \leq R$ such that $f\colon f^{-1}(\Delta_r^n) \to \Delta_r^n$ is the trivial family.*

Proof. For every $r \leq R$, $h \leq n$ denote

$$\Delta_r^h = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_1| < r, \ldots, |z_h| < r, z_{h+1} = 0, \ldots, z_n = 0\} \subset \Delta_R^n.$$

We prove by induction on h that there exists $R \geq r_h > 0$ such that the restriction of the family f over $\Delta_{r_h}^h$ is trivial. Taking $r_0 = R$ the statement is obvious for $h = 0$. Assume that the family is trivial over $\Delta_{r_h}^h$, $h < n$; shrinking Δ_R^n if necessary it is not restrictive to assume $R = r_h$ and the family trivial over Δ_R^h. The integration of the vector field χ_{h+1} gives an open neighbourhood $M \times \{0\} \subset U \subset M \times \mathbb{C}$ and a holomorphic map $H\colon U \to M$ with the following properties (see e.g. [**?**, Ch. VII]):

1. For every $x \in M$, $\{x\} \times \mathbb{C} \cap U = \{x\} \times \Delta(x)$ with $\Delta(x)$ a disk.

2. For every $x \in M$ the map $H_x = H(x, -)\colon \Delta(x) \to M$ is the solution of the Cauchy problem

$$\begin{cases} \dfrac{dH_x}{dt}(t) = \chi_{h+1}(H_x(t)) \\ H_x(0) = x \end{cases}$$

   In particular if $H(x,t)$ is defined then $f(H(x,t)) = f(x) + (0, \ldots, t, \ldots, 0)$ (t in the $(h+1)$-th coordinate).

3. If $V \subset M$ is open and $V \times \Delta \subset U$ then for every $t \in \Delta$ the map $H(-,t)\colon V \to M$ is an open embedding.

Since f is proper there exists $r \le R$ such that $f^{-1}(\Delta_r^h) \times \Delta_r \subset U$; then the holomorphic map $H\colon f^{-1}(\Delta_r^h) \times \Delta_r \to f^{-1}(\Delta_r^{h+1})$ is a biholomorphism giving a trivialization of the family over Δ_r^{h+1}. $\square$

Example I.4.2. The Lemma I.4.1 is generally false if f is not proper (cf. the exercise in Lecture 1 of [61]). Consider for instance an irreducible polynomial $Q \in \mathbb{C}[x_1, \ldots, x_n, t]$; denote by $f\colon \mathbb{C}_x^n \times \mathbb{C}_t \to \mathbb{C}_t$ the projection on the second factor and

$$V = \left\{ (x,t) \,\middle|\, Q(x,t) = \frac{\partial Q}{\partial x_i}(x,t) = 0,\ i = 1, \ldots, n \right\}.$$

Assume that $f(V)$ is a finite set of points and set

$$B = \mathbb{C} - f(V), \qquad X = \{(x,t) \in \mathbb{C}^n \times B \mid Q(x,t) = 0\}.$$

Then X is a smooth hypersurface, the restriction $f\colon X \to B$ is surjective and its differential is surjective everywhere. Since X is closed in the affine variety $\mathbb{C}^n \times B$, by Hilbert's Nullstellensatz there exist regular functions $g_1, \ldots, g_n \in \mathcal{O}(\mathbb{C}^n \times B)$ such that

$$g := \sum_{i=1}^n g_i \frac{\partial Q}{\partial x_i} \equiv 1 \qquad (\mathrm{mod}\ Q).$$

On the open subset $U = \{g \neq 0\}$ the algebraic vector field

$$\chi = \sum_{i=1}^n \frac{g_i}{g} \left(\frac{\partial Q}{\partial x_i} \frac{\partial}{\partial t} - \frac{\partial Q}{\partial t} \frac{\partial}{\partial x_i} \right) = \frac{\partial}{\partial t} - \sum_{i=1}^n \frac{g_i}{g} \frac{\partial Q}{\partial t} \frac{\partial}{\partial x_i}$$

is tangent to X and lifts $\dfrac{\partial}{\partial t}$.

In general the fibres of $f\colon X \to B$ are not biholomorphic: consider for example the case $Q(x,y,\lambda) = y^2 - x(x-1)(x-\lambda)$. Then $B = \mathbb{C} - \{0,1\}$ and $f\colon X \to B$ is the restriction to the affine subspace $x_0 \neq 0$ of the family $M \to B$ of Example I.1.4. The fibre $X_\lambda = f^{-1}(\lambda)$ is $M_\lambda - \{\text{point}\}$, where M_λ is an elliptic curve with j-invariant $j(\lambda) = 2^8(\lambda^2 - \lambda + 1)^3 \lambda^{-2} (\lambda - 1)^{-2}$. If X_a is biholomorphic to X_b then, by Riemann's extension theorem, also M_a is biholomorphic to M_b and then $j(a) = j(b)$.

Theorem I.4.3. *A family $M \xrightarrow{f} B$ of compact complex manifolds is locally trivial at a point $b_0 \in B$ if and only if $\mathcal{KS}_f\colon \Theta_{B,b_0} \to (R^1 f_* T_f)_{b_0}$ is trivial.*

Proof. One implication is clear; conversely assume $\mathcal{KS}_f = 0$, it is not restrictive to assume B a polydisk with coordinates $t_1, \ldots, t_n$ and f a smooth family. After a possible shrinking of B we have $\mathcal{KS}(B)_f \left(\dfrac{\partial}{\partial t_i} \right) = 0$ for every $i = 1, \ldots, n$. According to I.3.4 there exist holomorphic vector fields ξ_i such that $f_* \xi_i = \dfrac{\partial}{\partial t_i}$; by I.4.1 the family is trivial over a smaller polydisk $\Delta \subset B$. $\square$

Note that if a smooth family $f\colon M \to B$ is locally trivial, then for every $b \in B$ the Kodaira-Spencer map $\mathrm{KS}_f\colon T_{b,B} \to H^1(M_b, T_{M_b})$ is trivial for every $b \in B$. Conversely we have the following result.

Corollary I.4.4. *Let $f\colon M \to B$ a smooth family of compact complex manifolds. If the dimension of $H^1(M_b, T_{M_b})$ is independent on b and $\mathrm{KS}_f = 0$ at every point $b \in B$, then the family is locally trivial.*

Proof. According to base change theorem ([4, Ch. 3, Thm. 4.12], [60, I, Thm. 2.2], [56]) the sheaf $R^1 f_* T_f$ is locally free and for every $b \in B$ the natural map $R^1 f_* T_f \otimes_{\mathcal{O}_B} \mathbb{C}_b \to H^1(M_b, T_{M_b})$ is an isomorphism. Thus the morphism $\mathcal{KS}_f$ is trivial if and only if $\mathrm{KS}_f = 0$ at every point. □

Corollary I.4.5. *Let $f\colon M \to B$ a smooth family of compact complex manifolds. If $H^1(M_0, T_{M_0}) = 0$ then the family is locally trivial in a neighbourhood of $0 \in B$.*

Proof. By semicontinuity theorem we have $H^1(M_b, T_{M_b}) = 0$ for every b in a open neighbourhood of 0. □

Definition I.4.6. A compact complex manifold X is called **rigid** if $H^1(X, T_X) = 0$.

Example I.4.7. Every product of projective speces is a rigid manifold.

The next examples show that Corollary I.4.4 is generally false if the dimension $H^1(M_b, T_{M_b})$ is not constant.

Example I.4.8. Consider the following family of Hopf surfaces $f\colon M \to B$, where $B = \mathbb{C}$, $M = X/G$, $X = B \times (\mathbb{C}^2 - \{0\})$ and $G \simeq \mathbb{Z}$ is generated by $(b, z_1, z_2) \mapsto (b, 2z_1, b^2 z_1 + 2z_2)$. In the notation Example I.1.6 the fibre M_b is the Hopf surface $S_{A(b)}$, where $A(b) = \begin{pmatrix} 2 & 0 \\ b^2 & 2 \end{pmatrix}$ and then M_0 is not biholomorphic to M_b for every $b \neq 0$. This family is isomorphic to $N \times_{\mathbb{C}} B$, where $B \to \mathbb{C}$ is the map $b \mapsto b^2$ and N is the quotient of $\mathbb{C} \times (\mathbb{C}^2 - \{0\})$ by the group generated by $(s, z_1, z_2) \mapsto (s, 2z_1, sz_1 + 2z_2)$. By base-change property, the Kodaira-Spencer map $\mathrm{KS}_f\colon T_{0,B} \to H^1(M_0, T_{M_0})$ is trivial.

On the other hand the family is trivial over $B - \{0\}$, in fact the map

$$(B - \{0\}) \times (\mathbb{C}^2 - \{0\}) \to (B - \{0\}) \times (\mathbb{C}^2 - \{0\}), \qquad (b, z_1, z_2) \mapsto (b, b^2 z_1, z_2)$$

induces to the quotient an isomorphism $(B - \{0\}) \times M_1 \simeq (M - f^{-1}(0))$. Therefore the Kodaira-Spencer map $\mathrm{KS}_f\colon T_{b,B} \to H^1(M_b, T_{M_b})$ is trivial for every b. According to Corollary I.4.4 the dimension of $H^1(M_b, T_{M_b})$ is not constant: in fact it is proved in [60] that $h^1(M_0, T_{M_0}) = 4$ and $h^1(M_b, T_{M_b}) = 2$ for $b \neq 0$.

Example I.4.9. Let $M \subset \mathbb{C}_b \times \mathbb{P}^3_x \times \mathbb{P}^1_u$ be the submanifold defined by the equations

$$u_0 x_1 = u_1(x_2 - b x_0), \qquad u_0 x_2 = u_1 x_3,$$

$f\colon M \to \mathbb{C}$ the projection onto the first factor and $f^*\colon M^* = (M - f^{-1}(0)) \to (\mathbb{C} - \{0\})$ its restriction. We left to the reader the easy verification that f is a smooth family of compact complex manifolds. Here we prove that:

1. f^* is a trivial family.

2. f is not locally trivial at $b = 0$.

Proof of 1. After the linear change of coordinates $x_2 - b x_0 \mapsto x_0$ the equations of $M^* \subset \mathbb{C} - \{0\} \times \mathbb{P}^3 \times \mathbb{P}^1$ become

$$u_0 x_1 = u_1 x_0, \qquad u_0 x_2 = u_1 x_3$$

and there exists an isomorphism of families $\mathbb{C} - \{0\} \times \mathbb{P}^1_s \times \mathbb{P}^1_u \to M^*$, given by

$$(b, [t_0, t_1], [u_0, u_1]) \mapsto (b, [t_0 u_1, t_0 u_0, t_1 u_1, t_1 u_0], [u_0, u_1]).$$

□

Proof of 2. Let $Y \simeq \mathbb{P}^1 \subset M_0$ be the subvariety of equation $b = x_1 = x_2 = x_3 = 0$. Assume f locally trivial, then there exist an open neighbourhood $0 \in U \subset \mathbb{C}$ and a commutative diagram of holomorphic maps

$$\begin{array}{ccc} Y \times U & \xrightarrow{j} & M \\ \Big\downarrow{\scriptstyle pr} & & \Big\downarrow{\scriptstyle f} \\ U & \overset{i}{\hookrightarrow} & \mathbb{C} \end{array}$$

where i is the inclusion, j is injective and extends the identity $Y \times \{0\} \to Y \subset M_0$.
Possibly shrinking U it is not restrictive to assume that the image of j is contained in the open subset $V_0 = \{x_0 \neq 0\}$. For $b \neq 0$ the holomorphic map $\delta\colon V_0 \cap M_b \to \mathbb{C}^3$,

$$\delta(b, [x_0, x_1, x_2, x_3], [u_0, u_1]) = \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right),$$

is injective; therefore for $b \in U$, $b \neq 0$, the holomorphic map $\delta j(-, b)\colon Y \simeq \mathbb{P}^1 \to \mathbb{C}^3$ is injective. This contradicts the maximum principle of holomorphic functions. $\square$

Applying the base change $\mathbb{C} \to \mathbb{C}$, $b \mapsto b^2$, to the family $M \to \mathbb{C}$ of Example I.4.9 we get a family with trivial KS at every point of the base but not locally trivial at 0. It is not difficult to prove that M_0 is the Segre-Hirzebruch surface $\mathbb{F}_2$ and then $H^1(\mathbb{F}_2, T_{\mathbb{F}_2}) = \mathbb{C}$, while $\mathbb{P}^1 \times \mathbb{P}^1$ is rigid.

I.5 Deformations

For every pair of pointed complex manifolds (M, x), (N, y) we denote by $\mathrm{Mor}_{\mathbf{Ger}}((M, x), (N, y))$ the set of germs of holomorphic maps $f\colon (M, x) \to (N, y)$. Every element of $\mathrm{Mor}_{\mathbf{Ger}}((M, x), (N, y))$ is an equivalence class of pairs (U, f), where $x \in U \subset M$ is an open neighbourhood of x, $f\colon U \to N$ is a holomorphic map such that $f(x) = y$ and $(U, f) \sim (V, g)$ if and only if there exists an open subset $x \in W \subset U \cap V$ such that $f_{|W} = g_{|W}$.

Roughly speaking a deformation is a "framed germ" of family; more precisely

Definition I.5.1. Let (B, b_0) be a germ of complex manifold, a *deformation* $M_0 \xrightarrow{i} M \xrightarrow{f} (B, b_0)$ of a compact complex manifold M_0 over (B, b_0) is a pair of holomorphic maps

$$M_0 \xrightarrow{i} M \xrightarrow{f} B$$

such that:

1. $fi(M_0) = b_0$.
2. There exists an open neighbourhood $b_0 \in U \subset B$ such that $f\colon f^{-1}(U) \to U$ is a proper smooth family.
3. $i\colon M_0 \to f^{-1}(b_0)$ is an isomorphism of complex manifolds.

M is called the total space of the deformation and (B, b_0) the base germ space.

Definition I.5.2. Two deformations of M_0 over the same base

$$M_0 \xrightarrow{i} M \xrightarrow{f} (B, b_0), \qquad M_0 \xrightarrow{j} N \xrightarrow{g} (B, b_0)$$

are isomorphic if there exists an open neighbourhood $b_0 \in U \subset B$, and a commutative diagram of holomorphic maps

$$\begin{tikzcd} M_0 \arrow[r, "i"] \arrow[d, "j"'] & f^{-1}(U) \arrow[d, "f"] \arrow[dl] \\ g^{-1}(U) \arrow[r, "g"'] & U \end{tikzcd}$$

with the diagonal arrow a holomorphic isomorphism.

For every germ of complex manifold (B,b_0) we denote by $\mathrm{Def}_{M_0}(B,b_0)$ the set of isomorphism classes of deformations of M_0 with base (B,b_0).

If $\xi:\ M_0\xrightarrow{i}M\xrightarrow{f}(B,b_0)$ is a deformation and $g\colon(C,c_0)\to(B,b_0)$ is a morphism of germs of complex manifolds then

$$g^*\xi:\ M_0\xrightarrow{(i,c_0)}M\times_B C\xrightarrow{pr}(C,c_0)$$

is a deformation with base point c_0. It is clear that the isomorphism class of $g^*\xi$ depends only by the class of g in $\mathrm{Mor}_{\mathbf{Ger}}((C,c_0),(B,b_0))$.
Therefore every $g\in\mathrm{Mor}_{\mathbf{Ger}}((C,c_0),(B,b_0))$ induces a well defined pull-back morphism

$$g^*\colon\mathrm{Def}_{M_0}(B,b_0)\to\mathrm{Def}_{M_0}(C,c_0).$$

Given any deformation $M_0\to M\xrightarrow{f}(B,b_0)$, it is well defined the Kodaira-Spencer map $\mathrm{KS}_f\colon T_{b_0,B}\to H^1(M_0,T_{M_0})$ which is invariant under isomorphism of deformations.

Definition I.5.3. A deformation $\xi:\ M_0\xrightarrow{i}M\xrightarrow{f}(B,b_0)$ of a compact complex manifold M_0, with Kodaira-Spencer map $\mathrm{KS}_f\colon T_{b_0,B}\to H^1(M_0,T_{M_0})$, is called:

1. *Versal* if KS_f is surjective and for every germ of complex manifold (C,c_0) the morphism

$$\mathrm{Mor}_{\mathbf{Ger}}((C,c_0),(B,b_0))\to\mathrm{Def}_{M_0}(C,c_0),\qquad g\mapsto g^*\xi$$

   is surjective.

2. *Semiuniversal* if it is versal and KS_f is bijective.

3. *Universal* if KS_f is bijective and for every pointed complex manifolds (C,c_0) the morphism

$$\mathrm{Mor}_{\mathbf{Ger}}((C,c_0),(B,b_0))\to\mathrm{Def}_{M_0}(C,c_0),\qquad g\mapsto g^*\xi$$

   is bijective.

Versal deformations are also called *complete*; semiuniversal deformations are also called *miniversal* or *Kuranishi* deformations.

Note that if ξ is semiuniversal, $g_1,g_2\in\mathrm{Mor}_{\mathbf{Ger}}((C,c_0),(B,b_0))$ and $g_1^*\xi=g_2^*\xi$ then, according to Theorem I.3.5, $dg_1=dg_2\colon T_{c_0,C}\to T_{b_0,B}$.

Definition I.5.4. A deformation $M_0\to M\to(B,b_0)$ is called *trivial* if it is isomorphic to

$$M_0\xrightarrow{Id\times\{b_0\}}M_0\times B\xrightarrow{pr}(B,b_0).$$

Corollary I.5.5. *Let X be a compact complex manifold. If $H^1(X,T_X)=0$ then every deformation of X is trivial.*

Example I.5.6. In the notation of Example I.4.9, the deformation $M_0\to M\xrightarrow{b}(\mathbb{C},0)$ is not universal: in order to see this it is sufficient to prove that M is isomorphic to the deformation g^*M, where $g\colon(\mathbb{C},0)\to(\mathbb{C},0)$ is the holomorphic map $g(b)=b+b^2$. The equation of g^*M is

$$u_0x_1=u_1(x_2-(b+b^2)x_0),\qquad u_0x_2=u_1x_3,$$

and the isomorphism of deformations $g^*M\to M$ is given by

$$(b,[x_0,x_1,x_2,x_3],[u_0,u_1])=(b,[(1+b)x_0,x_1,x_2,x_3],[u_0,u_1]).$$

I.6 Historical survey, I

The deformation theory of complex manifolds began in the years 1957-1960 by a series of papers of Kodaira-Spencer [58], [59], [60] and Kodaira-Nirenberg-Spencer [57].

The main results of these papers were the completeness and existence theorem for versal deformations.

Theorem I.6.1 (Completeness theorem, [59]). *A deformation ξ over a smooth germ $(B,0)$ of a compact complex manifold M_0 is versal if and only if the Kodaira-Spencer map $\mathrm{KS}_\xi\colon T_{0,B} \to H^1(M_0, T_{M_0})$ is surjective.*

Note that if a deformation $M_0 \longrightarrow M \xrightarrow{f} (B,0)$ is versal then we can take a linear subspace $0 \in C \subset B$ making the Kodaira-Spencer map $T_{0,C} \to H^1(M_0, T_{M_0})$ bijective; by completeness theorem $M_0 \to M \times_B C \to (C,0)$ is semiuniversal.

In general, a compact complex manifold does not have a versal deformation over a smooth germ. The problem of determining when such a deformation exists is one of the most difficult in deformation theory. A partial answer is given by

Theorem I.6.2 (Existence theorem, [57]). *Let M_0 be a compact complex manifold. If $H^2(M_0, T_{M_0}) = 0$ then M_0 admits a semiuniversal deformation over a smooth base.*

The condition $H^2(M_0, T_{M_0}) = 0$ is sufficient but it is quite far from being necessary. The "majority" of manifolds having a versal deformation over a smooth germ has the above cohomology group different from 0.

The next problem is to determine when a semiuniversal deformation is universal: a sufficient (and almost necessary) condition is given by the following theorem.

Theorem I.6.3. ([98], [112]) *Let $\xi : M_0 \longrightarrow M \longrightarrow (B,0)$ be a semiuniversal deformation of a compact complex manifold M_0. If $b \mapsto h^0(M_b, T_{M_b})$ is constant (e.g. if $H^0(M_0, T_{M_0}) = 0$) then ξ is universal.*

Remark I.6.4. If a compact complex manifold M has finite holomorphic automorphisms then $H^0(M, T_M) = 0$, while the converse is generally false (take as an example the Fermat quartic surface in $\mathbb{P}^3$, cf. [102]).

Example I.6.5. If M is a compact manifolds with ample canonical bundle then, by a theorem of Matsumura [84], $H^0(M_0, T_{M_0}) = 0$.

One of the most famous theorems in deformation theory (at least in algebraic geometry) is the stability theorem of submanifolds proved by Kodaira in 1963.

Definition I.6.6. Let Y be a closed submanifold of a compact complex manifold X. Y is called *stable* if for every deformation $X \xrightarrow{i} \mathcal{X} \xrightarrow{f} (B,0)$ there exists a deformation $Y \xrightarrow{j} \mathcal{Y} \xrightarrow{g} (B,0)$ and a commutative diagram of holomorphic maps

$$\begin{tikzcd} Y \arrow[r, "j"] \arrow[d, "i_{|X}"'] & \mathcal{Y} \arrow[d, "g"] \arrow[dl] \\ \mathcal{X} \arrow[r, "f"] & B \end{tikzcd}$$

Not every submanifold is stable, for instance consider the submanifold Y of Example I.4.8.

Theorem I.6.7 (Kodaira stability theorem, [55]). *Let Y be a closed submanifold of a compact complex manifold X. If $H^1(Y, N_{Y/X}) = 0$ then Y is stable.*

I.7 Exercises

Exercise I.7.1. In the notation of Example I.1.6, if $A = e^{2\pi i \tau} I \in GL(2, \mathbb{C})$, $\tau = a + ib$, $b < 0$, then the Hopf surface S_A is the total space of a holomorphic G-principal bundle $S_A \to \mathbb{P}^1$, where $G = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$.

Exercise I.7.2. Let $p \geq 0$ be a fixed integer and M a compact complex manifold. For every $0 \leq q \leq p$, denote by $F_q \subset H^p_d(M, \mathbb{C})$ the subspace of cohomology classes represented by a d-closed form $\eta \in \oplus_{i \leq q} \Gamma(M, \mathcal{A}^{p-i,i})$. Prove that there exist injective linear morphisms $F_q/F_{q-1} \to H^{p-q,q}_{\overline{\partial}}(M, \mathcal{O})$. Deduce that $b_p \leq \sum_q h^{p-q,q}$.

Exercise I.7.3. There exists an action of the group $\mathrm{Aut}(M_0)$ of holomorphic isomorphisms of M_0 on the set $\mathrm{Def}_{M_0}(B, b_0)$: if $g \in \mathrm{Aut}(M_0)$ and $\xi : M_0 \xrightarrow{i} M \xrightarrow{f} (B, b_0)$ is a deformation we define

$$\xi^g : M_0 \xrightarrow{ig^{-1}} M \xrightarrow{f} (B, b_0).$$

Prove that $\xi^g = \xi$ if and only if $g \colon f^{-1}(b_0) \to f^{-1}(b_0)$ can be extended to an isomorphism $\hat{g} \colon f^{-1}(V) \to f^{-1}(V)$, $b_0 \in V$ open neighbourhood, such that $f\hat{g} = f$.

Exercise I.7.4. A universal deformation $\xi : M_0 \xrightarrow{i} M \xrightarrow{f} (B, b_0)$ induces a representation (i.e. a homomorphism of groups)

$$\rho \colon \mathrm{Aut}(M_0) \to \mathrm{Aut}_{\mathbf{Ger}}(B, b_0), \qquad \rho(g)^*\xi = \xi^g, \quad g \in \mathrm{Aut}(M_0).$$

Every other universal deformation over the germ (B, b_0) gives a conjugate representation.

Exercise I.7.5. The deformation $M_0 \longrightarrow M \xrightarrow{f} \mathbb{C}$, where f is the family of Example I.4.9, is not universal.

Chapter II

Analytic algebras and singularities

Historically, a major step in deformation theory has been the introduction of deformations of complex manifolds over (possibly non reduced) analytic singularities. This chapter is a short introductory course on analytic algebras and analytic singularities; moreover we give an elementary proof of the Nullstellenstaz for the ring $\mathbb{C}\{z_1,\ldots,z_n\}$ of convergent complex power series. Quite important in deformation theory are the smoothness criterion II.2.3 and the two dimension bounds II.5.3 and II.5.4.

II.1 Analytic algebras

Let $\mathbb{C}\{z_1,\ldots,z_n\}$ be the ring of convergent power series with complex coefficient. Every $f\in\mathbb{C}\{z_1,\ldots,z_n\}$ defines a holomorphic function in a nonempty open neighbourhood U of $0\in\mathbb{C}^n$; for notational simplicity we still denote by $f\colon U\to\mathbb{C}$ this function.

If f is a holomorphic function in a neighbourhood of 0 and $f(0)\neq 0$ then $1/f$ is holomorphic in a (possibly smaller) neighbourhood of 0. This implies that f is invertible in $\mathbb{C}\{z_1,\ldots,z_n\}$ if and only if $f(0)\neq 0$ and therefore $\mathbb{C}\{z_1,\ldots,z_n\}$ is a local ring with maximal ideal

$$\mathfrak{m}=(z_1,\ldots,z_m)=\{f\mid f(0)=0\}.$$

The *multiplicity* of a power series $f\in\mathbb{C}\{z_1,\ldots,z_n\}$ is the biggest integer s such that $f\in\mathfrak{m}^s$. Moreover the following results hold:

- $\mathbb{C}\{z_1,\ldots,z_n\}$ is Noetherian ([40, II.B.9], [33]).
- $\mathbb{C}\{z_1,\ldots,z_n\}$ is a unique factorization domain ([40, II.B.7], [33]).
- $\mathbb{C}\{z_1,\ldots,z_n\}$ is a Henselian ring ([76], [32], [33]).
- $\mathbb{C}\{z_1,\ldots,z_n\}$ is a regular local ring of dimension n (see e.g. [3], [33], [85] for the basics about dimension theory of local Noetherian ring).

We recall, for the reader's convenience, that the dimension of a local Noetherian ring A with maximal ideal $\mathfrak{m}$ is the minimum integer d such that there exist $f_1,\ldots,f_d\in\mathfrak{m}$ with the property $\sqrt{(f_1,\ldots,f_d)}=\mathfrak{m}$. In particular $\dim A=0$ if and only if $\sqrt{0}=\mathfrak{m}$, i.e. if and only if $\mathfrak{m}$ is nilpotent.

We also recall that a morphism of local rings $f\colon(A,\mathfrak{m})\to(B,\mathfrak{n})$ is called local if $f(\mathfrak{m})\subset\mathfrak{n}$.

Definition II.1.1. A $\mathbb{C}$-algebra is called an **analytic algebra** if it is isomorphic to $\mathbb{C}\{z_1,\ldots,z_n\}/I$, for some $n\geq 0$ and some ideal $I\subset(z_1,\ldots,z_n)$. A morphism of analytic algebras is a local morphism of local $\mathbb{C}$-algebras.

Every analytic algebra is a local Noetherian ring. We denote by **An** the category of analytic algebras.

Theorem II.1.2 (Implicit function theorem). *Let $f_1, \ldots, f_m \in \mathbb{C}\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ be power series of positive multiplicity such that* $\det\left(\dfrac{\partial f_i}{\partial y_j}(0)\right) \neq 0$. *Then there exist m convergent power series $\psi_1(x), \ldots, \psi_m(x) \in \mathbb{C}\{x_1, \ldots, x_n\}$ of positive multiplicity such that*

$$f_i(x_1, \ldots, x_n, \psi_1(x), \ldots, \psi_m(x)) = 0 \qquad \textit{for every} \quad i = 1, \ldots, m.$$

The morphisms of analytic algebras

$$\mathbb{C}\{x_1, \ldots, x_n\} \xrightarrow{x_i \mapsto x_i} \frac{\mathbb{C}\{x_1, \ldots, x_n, y_1, \ldots, y_m\}}{(f_1, \ldots, f_m)} \xrightarrow{y_i \mapsto \psi_i(x)} \mathbb{C}\{x_1, \ldots, x_n\}$$

are isomorphisms and each one the inverse of the other.

Proof. See e.g. [40]. □

Corollary II.1.3. *Every analytic algebra is isomorphic to $\dfrac{\mathbb{C}\{z_1, \ldots, z_k\}}{I}$ for some $k \geq 0$ and some ideal $I \subset (z_1, \ldots, z_k)^2$.*

Proof. Let $A = \mathbb{C}\{z_1, \ldots, z_n\}/I$ be an analytic algebra. If the ideal I is not contained in $(z_1, \ldots, z_n)^2$, then there exists $f \in I$ and an index i such that $\dfrac{\partial f}{\partial z_i}(0) \neq 0$; up to permutation of indices we may suppose $i = n$. Therefore A is isomorphic to $\mathbb{C}\{z_1, \ldots, z_{n-1}\}/J$, where J is the kernel of the surjective (by implicit function theorem) morphism

$$\mathbb{C}\{z_1, \ldots, z_{n-1}\} \to \frac{\mathbb{C}\{z_1, \ldots, z_{n-1}, z_n\}}{(f)} \to A.$$

The conclusion follows by induction on n. □

Lemma II.1.4. *Let $(R, \mathfrak{m})$ be an analytic algebra. For every finite sequence $r_1, \ldots, r_n \in \mathfrak{m}$ there exists a unique morphism of analytic algebras*

$$f \colon \mathbb{C}\{z_1, \ldots, z_n\} \to R$$

such that $f(z_i) = r_i$.

Proof. We first note that, by the lemma of Artin-Rees ([3, 10.19]), $\cap_n \mathfrak{m}^n = 0$ and then every local homomorphism $f \colon \mathbb{C}\{z_1, \ldots, z_n\} \to R$ is uniquely determined by its factorizations

$$f_s \colon \frac{\mathbb{C}\{z_1, \ldots, z_n\}}{(z_1, \ldots, z_n)^s} \to \frac{R}{\mathfrak{m}^s}.$$

Since $\mathbb{C}\{z_1, \ldots, z_n\}/(z_1, \ldots, z_n)^s$ is a $\mathbb{C}$-algebra generated by $z_1, \ldots, z_n$, every f_s is uniquely determined by $f(z_i)$; this proves the unicity.

In order to prove the existence it is not restrictive to assume $R = \mathbb{C}\{u_1, \ldots, u_m\}$; the convergent power series r_i gives a germ of holomorphic map

$$r = (r_1, \ldots, r_n) \colon (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0),$$

and $r^*(z_i) = r_i$, where $r^* \colon \mathbb{C}\{z_1, \ldots, z_n\} \to \mathbb{C}\{u_1, \ldots, u_m\}$ is the induced morphism of analytic algebras. □

Lemma II.1.5. *Given an analytic algebra R and an integer $n \geq 0$ there exists an analytic algebra $R\{z_1, \ldots, z_n\}$ and a morphism $i \colon R \to R\{z_1, \ldots, z_n\}$ having the following universal property. For every morphism of analytic algebras $f \colon R \to A$ and every sequence $a_1, \ldots, a_n \in \mathfrak{m}_A$ there exists an unique morphism of analytic algebras $g \colon R\{z_1, \ldots, z_n\} \to A$ such that $gi = f$ and $g(z_i) = a_i$.*

Proof. If $R = \mathbb{C}\{y_1, \ldots, y_m\}/I$, define $i \colon C\{z_1, \ldots, z_n\} \to \mathbb{C}\{y_1, \ldots, y_m, z_1, \ldots, z_n\}$ as the natural inclusion and $R\{z_1, \ldots, z_n\} = \mathbb{C}\{y_1, \ldots, y_m, z_1, \ldots, z_n\}/(i(I))$, where $(i(I))$ is the ideal generated by $i(I)$. The proof of the universal property is an easy consequence of Lemma II.1.4 and it is left to the reader. □

Notice that $(R\{z_1, \ldots, z_n\})\{z_{n+1}, \ldots, z_m\} = R\{z_1, \ldots, z_m\}$.

II.2 Tangent space and smoothness

da fare: spazio tangente di Zariski, criterio di surgettivita'

Definition II.2.1. An analytic algebra is called **smooth** if it is isomorphic to the power series algebra $\mathbb{C}\{z_1,\ldots,z_k\}$ for some $k\geq 0$. A morphism of analytic algebras $f\colon R\to S$ is **smooth** if there exists an isomorphism $\phi\colon S\to R\{z_1,\ldots,z_k\}$ such that ϕf is the natural inclusion.

Notice that an analytic algebra R is smooth if and only if the morphism $\mathbb{C}\to R$ is smooth. Composition of smooth morphisms is a smooth morphism.

Proposition II.2.2. *Let R be an analytic algebra. The following conditions are equivalent:*

1. *R is smooth.*
2. *For every surjective morphism of analytic algebras $B\to A$, the morphism*
$$\mathrm{Mor}_{\mathbf{An}}(R,B)\to\mathrm{Mor}_{\mathbf{An}}(R,A)$$
is surjective.
3. *For every $n\geq 2$ the morphism*
$$\mathrm{Mor}_{\mathbf{An}}\left(R,\frac{\mathbb{C}\{t\}}{(t^n)}\right)\to\mathrm{Mor}_{\mathbf{An}}\left(R,\frac{\mathbb{C}\{t\}}{(t^2)}\right)$$
is surjective.

Proof. The implication [$1\Rightarrow 2$] is an immediate consequence of Lemma II.1.4 and [$2\Rightarrow 3$] is trivial. In order to prove [$3\Rightarrow 1$] we write $R=\mathbb{C}\{z_1,\ldots,z_k\}/I$, where $I\subset(z_1,\ldots,z_k)^2$, and we denote by $\pi\colon\mathbb{C}\{z_1,\ldots,z_k\}\to R$ the projection. Assume $I\neq 0$ and let $s\geq 2$ be the the greatest integer such that $I\subset(z_1,\ldots,z_k)^s$: we claim that $\mathrm{Mor}_{\mathbf{An}}(R,\mathbb{C}\{t\}/(t^{s+1}))\to\mathrm{Mor}_{\mathbf{An}}(R,\mathbb{C}\{t\}/(t^2))$ is not surjective. Choosing $f\in I-(z_1,\ldots,z_k)^{s+1}$, after a possible generic linear change of coordinates of the form $z_i\mapsto z_i+a_iz_1$, $a_2,\ldots,a_k\in\mathbb{C}$, we may assume that f contains the monomial z_1^s with a nonzero coefficient, say $f=cz_1^s+\cdots$; let $\alpha\colon R\to\mathbb{C}\{t\}/(t^2)$ be the morphism defined by $\alpha(z_1)=t$, $\alpha(z_i)=0$ for $i>1$. Assume that there exists $\beta\colon R\to\mathbb{C}\{t\}/(t^{s+1})$ that lifts α, then $\beta(z_1)-t,\beta(z_2),\ldots,\beta(z_k)\in(t^2)$ and therefore $\beta\pi(f)\equiv ct^s \pmod{t^{s+1}}$. □

Proposition II.2.3. *Let $f\colon R\to S$ be a morphism of analytic algebras. The following conditions are equivalent:*

1. *f is smooth.*
2. *For every surjective morphism of analytic algebras $\alpha\colon B\to A$, the morphism*
$$\mathrm{Mor}_{\mathbf{An}}(S,B)\to\mathrm{Mor}_{\mathbf{An}}(R,B)\times_{\mathrm{Mor}_{\mathbf{An}}(R,A)}\mathrm{Mor}_{\mathbf{An}}(S,A)$$
is surjective.
3. *For every surjective morphism of local Artinian $\mathbb{C}$-algebras $\alpha\colon B\to A$, the morphism*
$$\mathrm{Mor}_{\mathbf{An}}(S,B)\to\mathrm{Mor}_{\mathbf{An}}(R,B)\times_{\mathrm{Mor}_{\mathbf{An}}(R,A)}\mathrm{Mor}_{\mathbf{An}}(S,A)$$
is surjective.

Proof. The only non trivial implication is [$3\Rightarrow 1$]. We first note that an element of the fibered product
$$\mathrm{Mor}_{\mathbf{An}}(R,B)\times_{\mathrm{Mor}_{\mathbf{An}}(R,A)}\mathrm{Mor}_{\mathbf{An}}(S,A)$$
is nothing else than a commutative diagram of morphisms of analytic algebras

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d] & S \arrow[d] \\ B \arrow[r, "\alpha"] & A . \end{tikzcd}$$

Da finire.

□

Example II.2.4. The morphism of analytic algebras

$$R = \frac{\mathbb{C}\{x,y\}}{(x^3,y^3)} \to S = \frac{\mathbb{C}\{x,y\}}{(x^3,x^2y^2,y^3)}$$

is not smooth. However for every $n \geq m$ the natural morphism

$$\mathrm{Mor}_{\mathbf{An}}\left(S, \frac{\mathbb{C}\{t\}}{(t^n)}\right) \to \mathrm{Mor}_{\mathbf{An}}\left(R, \frac{\mathbb{C}\{t\}}{(t^n)}\right) \times_{\mathrm{Mor}_{\mathbf{An}}\left(R, \frac{\mathbb{C}\{t\}}{(t^m)}\right)} \mathrm{Mor}_{\mathbf{An}}\left(S, \frac{\mathbb{C}\{t\}}{(t^m)}\right)$$

is surjective

II.3 Analytic singularities

Here we briefly recall the basic definition and some properties: we refer to [21, 32, 33, 73] for proofs and more details.

Let $U \subset \mathbb{C}^n$ be an open subset and $\mathcal{I} = (g_1, \dots, g_m) \subset \mathcal{O}_U$ a finitely generated ideal sheaf, $g_1, \dots, g_m \in \Gamma(U, \mathcal{I})$. Setting $V = \{u \in U \mid g_1(u) = \cdots = g_m(u) = 0\}$ and $\mathcal{O}_V = \mathcal{O}_U/\mathcal{I}$, we have that the pair $(V, \mathcal{O}_V)$ is a $\mathbb{C}$-ringed space called *local model*.

Definition II.3.1. A **complex space** is a Hausdorff $\mathbb{C}$-ringed space locally isomorphic to a local model.

In particular, if $(X, \mathcal{O}_X)$ is a complex space, then for every $x \in X$ the stalk $\mathcal{O}_{X,x}$ is an analytic algebra. Conversely, every analytic algebra $R = \mathbb{C}\{z_1, \dots, z_n\}/I$ is isomorphic to the stalk of the structure sheaf of a complex space: in fact, if $f_1, \dots, f_m$ is a set of generatorsof the the ideal I and then we have $R = \mathcal{O}_{X,0}$, where $(X, 0) \subset (\mathbb{C}^n, 0)$ is the complex analytic subspace defined by the equations $f_1 = \cdots = f_m = 0$.

Definition II.3.2. An **analytic singularity** is a germ (X, x) of a complex space at a point. A morphism of analytic singularities is a germ of morphism of complex spaces. The category of analytic singularities will be denoted by **Ger**.

Theorem II.3.3 (Grothendieck, [35])**.** *The functor*

$$\mathbf{Ger} \to \mathbf{An}^{opp}, \qquad (X,x) \mapsto \mathcal{O}_{X,x},$$

is an equivalence of categories.

Proof. See [35, 21]. □

Definition II.3.4. The *Zariski tangent space* $T_{x,X}$ of an analytic singularity (X, x) is the $\mathbb{C}$-vector space $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,x}, \mathbb{C})$.

Notice that $T_{x,X}$ is the dual vector space of $\dfrac{\mathfrak{m}}{\mathfrak{m}^2}$ where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,x}$; in particular $T_{X,x}$ is finite dimesional and its dimension is equal the minimum number of generators of $\mathfrak{m}$. Every morphism of singularities $(X, x) \to (Y, y)$ induces a linear morphism of Zariski tangent spaces $T_{x,X} \to T_{y,Y}$.

The dimension of an analytic singularity (X, x) is by definition the dimension of the analytic algebra $\mathcal{O}_{X,x}$: in particular we have $\dim(X, x) \leq \dim T_{x,X}$.

Definition II.3.5. A **fat point** is an analytic singularity of dimension 0.

Lemma II.3.6. *An analytic singularity (X, x) is a fat point if and only if the analytic algebra $\mathcal{O}_{X,x}$ is Artinian.*

Proof. By definition of dimension, an analytic algebra has dimension 0 if and only if its maximal ideal is nilpotent. □

II.4 The curve selection lemma

The aim of this section is to give, following [76], an elementary proof the following theorem.

Theorem II.4.1 (curve selection lemma). *Let $I \subset \mathbb{C}\{z_1, \ldots, z_n\}$ be a proper ideal and $h \notin \sqrt{I}$. Then there exists a morphism of analytic algebras $\phi\colon \mathbb{C}\{z_1, \ldots, z_n\} \to \mathbb{C}\{t\}$ such that $\phi(I) = 0$ and $\phi(h) \neq 0$.*

Before proving Theorem II.4.1 we need a series of results that are of independent interest. We recall the following

Definition II.4.2. A power series $p \in \mathbb{C}\{z_1, \ldots, z_n, t\}$ is called a *Weierstrass polynomial* in t of degree $d \geq 0$ if

$$p = t^d + \sum_{i=0}^{d-1} p_i(z_1, \ldots, z_n)t^i, \qquad p_i(0) = 0.$$

In particular if $p(z_1, \ldots, z_n, t)$ is a Weierstrass polynomial in t of degree d then $p(0, \ldots, 0, t) = t^d$.

Theorem II.4.3 (Weierstrass preparation theorem). *Let $f \in \mathbb{C}\{z_1, \ldots, z_n, t\}$ be a power series such that $f(0, \ldots, 0, t) \neq 0$. Then there exists a unique $e \in \mathbb{C}\{z_1, \ldots, z_n, t\}$ such that $e(0) \neq 0$ and ef is a Weierstrass polynomial in t.*

Proof. For the proof we refer to [32], [33], [38], [56], [40], [76]. □

Corollary II.4.4. *Let $f \in \mathbb{C}\{z_1, \ldots, z_n\}$ be a power series of multiplicity d. Then, after a possible generic linear change of coordinates there exists $e \in \mathbb{C}\{z_1, \ldots, z_n\}$ such that $e(0) \neq 0$ and ef is a Weierstrass polynomial of degree d in z_n.*

Proof. After a generic change of coordinates of the form $z_i \mapsto z_i + a_i z_n$, $a_i \in \mathbb{C}$, the series $f(0, \ldots, 0, z_n)$ has multiplicity d. □

Lemma II.4.5. *Let $f, g \in \mathbb{C}\{x_1, \ldots, x_n\}[t]$ with g a Weierstrass polynomial. If $f = hg$ for some $h \in \mathbb{C}\{x_1, \ldots, x_n, t\}$ then $h \in \mathbb{C}\{x_1, \ldots, x_n\}[t]$.*

Proof. Write $g = t^s + \sum g_i(x)t^{s-i}$, $g_i(0) = 0$, $f = \sum_{i=0}^{r} f_i(x)t^{r-i}$ $h = \sum_i h_i(x)t^i$, we need to prove that $h_i = 0$ for every $i > r - s$. Assume the contrary and choose an index $j > r - s$ such that the multiplicity of h_j takes the minimum among all the multiplicities of the power series h_i, $i > r - s$. From the equality $0 = h_j + \sum_{i>0} g_i h_{j+i}$ we get a contradiction. □

Notice that if g is not a Weierstrass polynomial, then the above result is false: consider for instance the case $n = 0$, $f = t^3$ and $g = t + t^2$.

Lemma II.4.6. *Let $f \in \mathbb{C}\{x_1, \ldots, x_n\}[t]$ be an irreducible monic polynomial of degree d. Then the polynomial $f_0(t) = f(0, \ldots, 0, t) \in \mathbb{C}[t]$ has a root of multiplicity d.*

Proof. Let $c \in \mathbb{C}$ be a root of $f_0(t)$. If the multiplicity of c is $l < d$ then the multiplicity of the power series $f_0(t + c) \in \mathbb{C}\{t\}$ is exactly l and therefore $f(x_1, \ldots, x_n, t + c)$ is divided in $\mathbb{C}\{x_1, \ldots, x_n\}[t]$ by a Weierstrass polynomial of degree l. □

Lemma II.4.7. *Let $p \in \mathbb{C}\{x\}[y]$ be a monic polynomial of positive degree d in y. Then there exists a homomorphism $\phi\colon \mathbb{C}\{x\}[y] \to \mathbb{C}\{t\}$ such that $\phi(p) = 0$ and $\phi(x) = t^s$ for some integer $s > 0$.*

Proof. If $d = 1$ then $p(x, y) = y - p_1(x)$ and we can consider the morphism ϕ given by $\phi(x) = t$, $\phi(y) = p_1(t)$. By induction we can assume that the theorem holds for monic polynomials of degree $< d$.

If p is reducible we have done, otherwise, writing $p = y^d + p_1(x)y^{d-1} + \cdots + p_d(x)$, after the coordinate change $x \mapsto x$, $y \mapsto y - p_1(x)/d$ we can assume $p_1 = 0$. For every $i \geq 2$ denote by $\mu(p_i) = \alpha_i > 0$ the multiplicity of p_i (we set $\alpha_i = +\infty$ if $p_i = 0$). Let $j \geq 2$ be a fixed index such that $\dfrac{\alpha_j}{j} \leq \dfrac{\alpha_i}{i}$ for every i. Setting $m = \alpha_j$, we want to prove that the monic polynomial $p(\xi^j, y)$

is not irreducible. In fact $p(\xi^j, y) = y^d + \sum_{i\geq 2} h_i(\xi)y^{d-i}$, where $h_i(\xi) = p_i(\xi^j)$. For every i the multiplicity of h_i is $j\alpha_i \geq i\alpha_j = im$ and then

$$q(\xi, y) = p(\xi^j, \xi^m y)\xi^{-dm} = t^d + \sum \frac{h_i(\xi)}{\xi^{mi}} y^{d-i} = y^d + \sum \eta_i(\xi)y^{d-i}$$

is a well defined element of $\mathbb{C}\{\xi, y\}$. Since $\eta_1 = 0$ and $\eta_j(0) \neq 0$ the polynomial q is not irreducible and then, by induction there exists a nontrivial morphism $\psi\colon \mathbb{C}\{\xi\}[y] \to \mathbb{C}\{t\}$ such that $\psi(q) = 0$, $\psi(\xi) = t^s$ and we can take $\phi(x) = \psi(\xi^j) = t^{js}$ and $\phi(y) = \psi(\xi^m y)$. □

Theorem II.4.8 (Weierstrass division theorem)**.** *Let $p \in \mathbb{C}\{z_1, \dots, z_n, t\}$, $p \neq 0$, be a Weierstrass polynomial of degree $d \geq 0$ in t. Then for every $f \in \mathbb{C}\{z_1, \dots, z_n, t\}$ there exists a unique $h \in \mathbb{C}\{z_1, \dots, z_n, t\}$ such that $f - hp \in \mathbb{C}\{z_1, \dots, z_n\}[t]$ is a polynomial of degree $< d$ in t.*

Proof. For the proof we refer to [32], [33], [38], [56], [40], [76]. □

We note that an equivalent statement for the division theorem is the following:

Corollary II.4.9. *If $p \in \mathbb{C}\{z_1, \dots, z_n, t\}$, $p \neq 0$, is a Weierstrass polynomial of degree $d \geq 0$ in t, then $\mathbb{C}\{z_1, \dots, z_n, t\}/(p)$ is a free $\mathbb{C}\{z_1, \dots, z_n\}$-module with basis $1, t, \dots, t^{d-1}$.*

Proof. Clear. □

Theorem II.4.10 (Newton-Puiseux)**.** *Let $f \in \mathbb{C}\{x, y\}$ be a power series of positive multiplicity. Then there exists a nontrivial morphism of analytic algebras $\phi\colon \mathbb{C}\{x, y\} \to \mathbb{C}\{t\}$ such that $\phi(f) = 0$. Moreover, if f is irreducible then $\ker\phi = (f)$.*

In the above statement nontrivial means that $\phi(x) \neq 0$ or $\phi(y) \neq 0$.

Proof. After a linear change of coordinates we can assume $f(0, y)$ a non zero power series of multiplicity $d > 0$; by preparation theorem there exists an invertible power series e such that $p = ef$ is a Weierstrass polynomial of degree d in y. According to Lemma II.4.7 there exists a homomorphism $\phi\colon \mathbb{C}\{x\}[y] \to \mathbb{C}\{t\}$ such that $\phi(p) = 0$ and $0 \neq \phi(x) \in (t)$. Therefore $\phi(p(0, y)) \in (t)$ and, being p a Weierstrass polynomial we have $\phi(y) \in (t)$ and then ϕ extends to a local morphism $\phi\colon \mathbb{C}\{x, y\} \to \mathbb{C}\{t\}$.

Assume now f irreducible, up to a possible change of coordinates and multiplication for an invertible element we may assume that $f \in \mathbb{C}\{x\}[y]$ is an irreducible Weierstrass polynomial of degree $d > 0$. Let $\phi\colon \mathbb{C}\{x, y\} \to \mathbb{C}\{t\}$ be a nontrivial morphism such that $\phi(f) = 0$, then $\phi(x) \neq 0$ (otherwise $\phi(y)^d = \phi(f) = 0$) and therefore $\ker(\phi) \cap \mathbb{C}\{x\} = 0$ Let $g \in \ker(\phi)$, by division theorem there exists $r \in \mathbb{C}\{x\}[y]$ such that $g = hf + r$ and then $r \in \ker(\phi)$.

Let $R(f, r) \in \mathbb{C}\{x\}$ be the resultant of the elimination of y on the polynomials f, r. By general properties of the resultatnt we have $R(f, r) \in (f, r) \subset \ker(\phi)$ and then $R(f, r) \in \ker(\phi) \cap \mathbb{C}\{x\} = 0$; since $\mathbb{C}\{x\}$ is a unique factorization domain, this implies that f divides r. □

We recall that, if A is a commutative ring and $p, q \in A[x]$ with p monic polynomial, then the resultant $R(p, q)$ is equal to the determinant of the morphism of free A-modules

$$\text{multiplication by } q\colon \frac{A[x]}{(p)} \to \frac{A[x]}{(p)}.$$

Lemma II.4.11. *Let A be an integral domain and $0 \neq \mathfrak{p} \subset A[t]$ a prime ideal such that $\mathfrak{p} \cap A = 0$. Denote by K the fraction field of A and by $\mathfrak{p}^e \subset K[x]$ the ideal generated by $\mathfrak{p}$. Then:*

1. *$\mathfrak{p}^e$ is a prime ideal of the euclidean ring $K[x]$.*
2. *$\mathfrak{p}^e \cap A[x] = \mathfrak{p}$.*
3. *There exists $q \in \mathfrak{p}$ such that for every monic polynomial $p \notin \mathfrak{p}$ we have $R(p, q) \neq 0$.*

Proof. [1] We have $\mathfrak{p}^e = \left\{ \frac{p}{a} \,\middle|\, p \in \mathfrak{p},\, a \in A - \{0\} \right\}$. If $\frac{p_1}{a_1}\frac{p_2}{a_2} \in \mathfrak{p}^e$ with $p_i \in A[x]$ and $a_i \in A$, then there exists $a \in A - \{0\}$ such that $ap_1p_2 \in \mathfrak{p}$. Since $\mathfrak{p} \cap A = 0$ it must be $p_1 \in \mathfrak{p}$ or $p_2 \in \mathfrak{p}$. This shows that $\mathfrak{p}^e$ is prime.
[2] If $q \in \mathfrak{p}^e \cap A[x]$, then there exists $a \in A$, $a \neq 0$ such that $aq \in \mathfrak{p}$ and, since $\mathfrak{p} \cap A = 0$ we have $q \in \mathfrak{p}$.
[3] Let $q \in \mathfrak{p} - \{0\}$ be of minimal degree. Since $K[t]$ is an Euclidean ring, $\mathfrak{p}^e = qK[t]$ and, since $\mathfrak{p}^e$ is prime, q is irreducible in $K[t]$. If $p \in A[t] \setminus \mathfrak{p}$ is a monic polynomial then $p \notin \mathfrak{p}^e = qK[t]$ and then $R(p, q) \neq 0$. □

Theorem II.4.12. *Let A be a unitary ring, $\mathfrak{p} \subset A[t]$ a prime ideal and denote $\mathfrak{q} = A \cap \mathfrak{p}$. If $\mathfrak{p} \neq \mathfrak{q}[t]$ (e.g. if $\mathfrak{p}$ is proper and contains a monic polynomial) then there exists $q \in \mathfrak{p}$ such that for every monic polynomial $p \notin \mathfrak{p}$ we have $R(p, q) \notin \mathfrak{q}$. If moreover A is a unique factorization domain we can choose q irreducible.*

Proof. The ideal $\mathfrak{q}$ is prime and $\mathfrak{q}[t] \subset \mathfrak{p}$, therefore the image of $\mathfrak{p}$ in $(A/\mathfrak{q})[t] = A[t]/\mathfrak{q}[t]$ is still a prime ideal satisfying the hypothesis of Lemma II.4.11. It is therefore sufficient to take q as any lifting of the element described in Lemma II.4.11 and use the functorial properties of the resultant.

If A is UFD and q is not irreducible we can write $q = hg$ with $g \in \mathfrak{p}$ irreducible; using the bilinearity relations of resultant $R(p, f) = R(p, h)R(p, g)$ we get $R(p, g) \notin \mathfrak{q}$. □

The division theorem allows to extend the definition of the resultant to power series. In fact if $p \in \mathbb{C}\{z_1, \dots, z_n\}[t]$ is a Weierstrass polynomial in t of degree d, for every $f \in \mathbb{C}\{z_1, \dots, z_n, t\}$ we can define the resultant $R(p, f) \in \mathbb{C}\{z_1, \dots, z_n\}$ as the determinant of the morphism of free $\mathbb{C}\{z_1, \dots, z_n\}$-module

$$f\colon \frac{\mathbb{C}\{z_1, \dots, z_n, t\}}{(p)} \to \frac{\mathbb{C}\{z_1, \dots, z_n, t\}}{(p)}$$

induced by the multiplication with f. It is clear that $R(p, f) = R(p, r)$ whenever $f - r \in (p)$.

Lemma II.4.13. *Let $p \in \mathbb{C}\{z_1, \dots, z_n, t\}$ be a Weierstrass polynomial of positive degree in t and $V \subset \mathbb{C}\{z_1, \dots, z_n, t\}$ a $\mathbb{C}$-vector subspace. Then $R(p, f) = 0$ for every $f \in V$ if and only if there exists a Weierstrass polynomial q of positive degree such that:*

1. *q divides p in $\mathbb{C}\{z_1, \dots, z_n\}[t]$*
2. *$V \subset q\mathbb{C}\{z_1, \dots, z_n, t\}$*

Proof. One implication is clear, in fact if $p = qr$ then the multiplication by q in not injective in $\mathbb{C}\{z_1, \dots, z_n, t\}/(p)$; therefore $R(p, q) = 0$ and by Binet's theorem $R(p, f) = 0$ for every $f \in (q)$.
For the converse let $p = p_1p_2 \dots p_s$ be the irreducible decomposition of p in the UFD $\mathbb{C}\{z_1, \dots, z_n\}[t]$. If $R(p, f) = 0$ and $r = f - hp \in \mathbb{C}\{z_1, \dots, z_n\}[t]$ is the remainder of the division then $R(p, r) = 0$ and then there exists a factor p_i dividing r and therefore also dividing f. In particular, setting $V_i = V \cap (p_i)$, we have $V = \cup_i V_i$ and therefore $V = V_i$ for at least one index i and we can take $q = p_i$. □

Theorem II.4.14. *Let $\mathfrak{p} \subset \mathbb{C}\{z_1, \dots, z_n\}$ be a prime ideal and $h \notin \mathfrak{p}$. Then there exists a morphism of analytic algebras $\phi\colon \mathbb{C}\{z_1, \dots, z_n\} \to \mathbb{C}\{t\}$ such that $\phi(\mathfrak{p}) = 0$ and $\phi(h) \neq 0$.*

Proof. We first consider the easy cases $n = 1$ and $\mathfrak{p} = 0$. If $\mathfrak{p} = 0$ then, after a possible change of coordinates, we may assume $h(0, \dots, 0, t) \neq 0$ and therefore we can take $\phi(z_i) = 0$ for $i = 1, \dots, n-1$ and $\phi(z_n) = t$. If $n = 1$ the only prime nontrivial ideal is (z_1) and therefore the trivial morphism $\phi\colon \mathbb{C}\{z_1\} \to \mathbb{C} \subset \mathbb{C}\{t\}$ satisfies the statement of the theorem.

Assume then $n > 1$, $\mathfrak{p} \neq 0$ and fix a nonzero element $g \in \mathfrak{p}$. After a possible linear change of coordinates and multiplication by invertible elements we may assume both h and g Weierstrass polynomials in the variable z_n. Denoting

$$\mathfrak{r} = \mathfrak{p} \cap \mathbb{C}\{z_1, \dots, z_{n-1}\}[z_n], \qquad \mathfrak{q} = \mathfrak{p} \cap \mathbb{C}\{z_1, \dots, z_{n-1}\} = \mathfrak{r} \cap \mathbb{C}\{z_1, \dots, z_{n-1}\},$$

according to Theorem II.4.12, there exists $\hat{f} \in \mathfrak{r}$ such that $R(h, \hat{f}) \notin \mathfrak{q}$. On the other hand, since $g \in \mathfrak{p}$, we have $R(g, f) \in \mathfrak{q}$ for every $f \in \mathfrak{p}$. By induction on n there exists a morphism

$\tilde{\psi}\colon \mathbb{C}\{z_1,\dots,z_{n-1}\}\to\mathbb{C}\{x\}$ such that $\tilde{\psi}(\mathfrak{q})=0$ and $\tilde{\psi}(R(h,\hat{f}))\neq 0$. Denoting by $\psi\colon \mathbb{C}\{z_1,\dots,z_n\}\to\mathbb{C}\{x,z_n\}$ the natural extension of $\tilde{\psi}$ we have $R(\psi(h),\psi(\hat{f}))\neq 0$ and $R(\psi(g),\psi(f))=0$ for every $f\in\mathfrak{p}$. Applying Lemma II.4.13 to the Weierstrass polynomial $\psi(g)$ and the vector space $V=\psi(\mathfrak{p})$ we prove the existence of an irreducible factor p of $\psi(g)$ such that $\psi(\mathfrak{p})\subset p\mathbb{C}\{x,z_n\}$. In particular p divides $\psi(\hat{f})$, therefore $R(\psi(h),p)\neq 0$ and $\psi(h)\notin p\mathbb{C}\{x,z_n\}$.

By Newton-Puiseux' theorem there exists $\eta\colon\mathbb{C}\{x,z_n\}\to\mathbb{C}\{t\}$ such that $\eta(p)=0$ and $\eta(\psi(h))\neq 0$. It is therefore sufficient to take ϕ as the composition of ψ and η. □

Proof of Theorem II.4.1. If $h\notin\sqrt{I}$ then there exists (cf. [3]) a prime ideal $\mathfrak{p}$ such that $I\subset\mathfrak{p}$ and $h\notin\mathfrak{p}$. □

II.5 Curvilinear obstructions and lower dimension bounds of analytic algebras

As an application of Theorem II.4.1 we give some bounds for the dimension of an analytic algebra; this bounds will be very useful in deformation and moduli theory. The first bound (Lemma II.5.3) is completely standard and the proof is reproduced here for completeness; the second bound (Theorem II.5.4, communicated to the author by H. Flenner) finds application in the "T^1-lifting" approach to deformation problems.

We need the following two results of commutative algebra.

Lemma II.5.1. *Let $(A,\mathfrak{m})$ be a local Noetherian ring and $J\subset I\subset A$ two ideals. If $J+\mathfrak{m}I=I$ then $J=I$.*

Proof. This a special case of Nakayama's lemma [3], [76]. □

Lemma II.5.2. *Let $(A,\mathfrak{m})$ be a local Noetherian ring and $f\in\mathfrak{m}$, then $\dim A/(f)\geq\dim A-1$. Moreover, if f is nilpotent then $\dim A/(f)=\dim A$, while if f is not a zerodivisor then $\dim A/(f)=\dim A-1$.*

Proof. [3]. □

Lemma II.5.3. *Let R be an analytic algebra with maximal ideal $\mathfrak{m}$, then $\dim R\leq\dim_{\mathbb{C}}\dfrac{\mathfrak{m}}{\mathfrak{m}^2}$ and equality holds if and only if R is smooth.*

Proof. Let $n=\dim_{\mathbb{C}}\dfrac{\mathfrak{m}}{\mathfrak{m}^2}$ and $f_1,\dots,f_n\in\mathfrak{m}$ inducing a basis of $\dfrac{\mathfrak{m}}{\mathfrak{m}^2}$. If $J=(f_1,\dots,f_n)$ by assumption $J+\mathfrak{m}^2=\mathfrak{m}$ and then by Lemma II.5.1 $J=\mathfrak{m}$, $R/J=\mathbb{C}$ and $0=\dim R/J\geq\dim R-n$. According to Lemma II.1.3 we can write $R=\mathbb{C}\{z_1,\dots,z_n\}/I$ for some ideal contained in $(z_1,\dots,z_n)^2$. Since $\mathbb{C}\{z_1,\dots,z_n\}$ is an integral domain, according to Lemma II.5.2 $\dim R=n$ if and only if $I=0$. □

Theorem II.5.4. *Let $R=P/I$ be an analytic algebra, where $P=\mathbb{C}\{z_1,\dots,z_n\}$ and $I\subset P$ is a proper ideal. Denoting by $\mathfrak{m}=(z_1,\dots,z_n)$ the maximal ideal of P and by $J\subset I$ the ideal*

$$J=\left\{f\in I\;\middle|\;\frac{\partial f}{\partial z_i}\in I,\quad\forall\, i=1,\dots,n\right\}$$

we have $\dim R\geq n-\dim_{\mathbb{C}}\dfrac{I}{J+\mathfrak{m}I}$.

Proof. (taken from [19]) We first introduce the *curvilinear obstruction map*

$$\gamma_I\colon \mathrm{Mor}_{\mathbf{An}}(P,\mathbb{C}\{t\})\to\mathrm{Hom}_{\mathbb{C}}\left(\frac{I}{J+\mathfrak{m}I},\mathbb{C}\right).$$

Given $\phi\colon P\to\mathbb{C}\{t\}$, if $\phi(I)=0$ we define $\gamma_I(\phi)=0$; while if $\phi(I)\neq 0$ and s is the biggest integer such that $\phi(I)\subset(t^s)$ then we define, for every $f\in I$, $\gamma_I(\phi)f$ as the coefficient of t^s in the power series expansion of $\phi(f) = f(\phi(z_1), \dots, \phi(z_n))$. It is clear that $\gamma_I(\phi)(\mathfrak{m}I) = 0$, while if $\phi(I) \subset (t^s)$ and $f \in J$ we have

$$\frac{d\phi(f)}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(\phi(z_1), \dots, \phi(z_n)) \frac{d\phi(z_i)}{dt} \in (t^s)$$

and therefore $\phi(f) \in (t^{s+1})$ (this is the point where the characteristic of the field plays an essential role). The vectors in the image of γ_I are called *curvilinear obstructions* and we will prove the theorem by showing that there exist $n - \dim R$ linearly independent curvilinear obstructions.

The ideal I is finitely generated, say $I = (f_1, \dots, f_d)$, according to Nakayama's lemma we can assume $f_1, \dots, f_d$ a basis of $I/\mathfrak{m}I$. By repeated application of Theorem II.4.1 (and possibly reordering the f_i's) we can assume that there exists an $h \leq d$ such that the following holds:

1. $f_i \notin \sqrt{(f_1, \dots, f_{i-1})}$ for $i \leq h$;
2. for every $i \leq h$ there exists a morphism of analytic algebras $\phi_i : P \to \mathbb{C}\{t\}$ such that $\phi_i(f_i) \neq 0$, $\phi_i(f_j) = 0$ if $j < i$ and the multiplicity of $\phi_i(f_j))$ is bigger than or equal to the multiplicity of $\phi_i(f_i))$ for every $j > i$.
3. $I \subset \sqrt{(f_1, \dots, f_h)}$.

Condition 3) implies that $\dim R = \dim P/(f_1, \dots, f_h) \geq n - h$, hence it is enough to prove that $\gamma_I(\phi_1), \dots, \gamma_I(\phi_h)$ are linearly independent in $\mathrm{Hom}_{\mathbb{C}}\left(\dfrac{I}{J + \mathfrak{m}I}, \mathbb{C}\right)$ and this follows immediately from the fact that the matrix $a_{ij} = \gamma_I(\phi_i)f_j$, $i, j = 1, \dots, h$, has rank h, being triangular with nonzero elements on the diagonal. $\square$

Example II.5.5. Here it is an example where the dimension bound of Theorem II.5.4 is better than the standard one. Let $I \subset \mathbb{C}\{x, y\}$ be the ideal generated by the polynomial $f = x^5 + y^5 + x^3y^3$ and by its partial derivatives $f_x = 5x^4 + 3x^2y^3$, $f_y = 5y^4 + 3x^3y^2$. Clearly $f \in J = \{g \in I \mid g_x, g_y \in I\}$ and, in order to prove that $J + \mathfrak{m}I \neq \mathfrak{m}I$ it is sufficient to prove that $f \notin \mathfrak{m}I$. Assume $f - cf \in (f_x, f_y)$ for some $c \in \mathfrak{m}$, then since $1 - c$ is invertible we have $f \in (f_x, f_y)$, say $f = af_x + bf_y$. Looking at the homogeneous components of degree ≤ 5 we get

$$a \in \frac{x}{5} + \mathfrak{m}^2, \qquad b \in \frac{y}{5} + \mathfrak{m}^2$$

and then

$$f - \frac{x}{5}f_x - \frac{y}{5}f_y = -\frac{1}{5}x^3y^3 \in \mathfrak{m}^2(f_x, f_y)$$

which is not possible.

II.6 Artin's theorem on the solution of analytic equations

All the algebraic results of this chapter that make sense also for the ring of formal power series $\mathbb{C}[[z_1, \dots, z_n]]$ and their quotients, remain true: in particular the Weierstrass preparation and division theorem holds with the same statement [68]. In many cases, especially in deformation theory, we seek for solutions of systems of analytic equations but we are able to solve these equation only formally; in this situation a great help comes from the following theorem, proved by M. Artin in 1968.

Theorem II.6.1 (Artin [1])**.** *Let n, m, N, c be non negative integers and let*

$$f_i(x, y) = f_i(x_1, \dots, x_n, y_1, \dots, y_N) \in \mathbb{C}\{x_1, \dots, x_n, y_1, \dots, y_N\}, \quad i = 1, \dots, m.$$

be convergent power series. Assume that there exist N formal power series $\overline{y}_i(x) \in \mathbb{C}[[x_1, \dots, x_n]]$, $i = 1, \dots, N$, without constant terms such that $f_i(x, \overline{y}(x)) = 0$ for every $i = 1, \dots, m$. Then there exists N convergent power series $y_i(x) \in C\{x_1, \dots, x_n\}$ such that $f_i(x, y(x)) = 0$ for every i and $y_i(x) \equiv \overline{y}_i(x) \pmod{\mathfrak{m}^c}$, where $\mathfrak{m}$ is the maximal ideal of $\mathbb{C}[[x_1, \dots, x_n]]$.

Here we will prove Artin's theorem under the additional assumption that the f_i's are polynomials: the proof of the general case uses the same ideas but it is technically more difficult. More precisely we give a complete proof of the following theorem.

Theorem II.6.2. *Let n, m, N, c be non negative integers and let*

$$f_i(x,y) = f_i(x_1,\dots,x_n,y_1,\dots,y_N) \in \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_N], \quad i = 1,\dots,m.$$

be m polynomials. Assume that there exists N formal power series $\overline{y}_i(x) \in \mathbb{C}[[x_1,\dots,x_n]]$, $i = 1,\dots,N$, such that $f_i(x,\overline{y}(x)) = 0$ for every $i = 1,\dots,m$. Then there exist N convergent power series $y_i(x) \in C\{x_1,\dots,x_n\}$ such that $f_i(x,y(x)) = 0$ for every i and $y_i(x) \equiv \overline{y}_i(x) \pmod{\mathfrak{m}^c}$, where $\mathfrak{m}$ is the maximal ideal of $\mathbb{C}[[x_1,\dots,x_n]]$.

We work by induction on n, being the theorem trivially true for $n = 0$; so we assume $n > 0$ and the theorem true for $n-1$. The proof of inductive step is a consequence of Weierstrass preparation and implicit function theorems.

Lemma II.6.3. *Let $g, f_1,\dots,f_m \in \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_N]$ and c a positive integer. Assume that there exist formal power series $\overline{y}_i(x) \in \mathbb{C}[[x_1,\dots,x_n]]$, $i = 1,\dots,N$, such that*

1. *$g(x,\overline{y}(x)) \neq 0$,*
2. *$g(x,\overline{y}(x))$ divides $f_i(x,\overline{y}(x))$ in the ring $\mathbb{C}[[x_1,\dots,x_n]]$ for every $i = 1,\dots,m$.*

Then there exist N convergent power series $y_i(x) \in C\{x_1,\dots,x_n\}$ such that $y_i(x) \equiv \overline{y}_i(x)$ $\pmod{\mathfrak{m}^c}$ and:

1. *$g(x,y(x)) \neq 0$,*
2. *$g(x,y(x))$ divides $f_i(x,y(x))$ in the ring $\mathbb{C}\{x_1,\dots,x_n\}$ for every $i = 1,\dots,m$.*

Proof. Let's denote by r the multiplicity of $g(x,\overline{y}(x))$; it is not restrictive to assume $c > r$. Up to a linear change of the coordinates x_i we have, by Weierstrass preparation theorem

$$g(x,\overline{y}(x)) = (x_n^r + \overline{a}_1(x)x_n^{r-1} + \cdots + \overline{a}_r(x))\overline{e}(x),$$

where $\overline{e}(x)$ is invertible and every $\overline{a}_i(x)$ is a formal power series in $x_1,\dots,x_{n-1}$.

By Weierstrass division theorem we have

$$\overline{y}_i(x) = (x_n^r + \overline{a}_1(x)x_n^{r-1} + \cdots + \overline{a}_r(x))u_i(x) + \sum_{j=0}^{r-1} \overline{y}_{ij}(x)x_n^j$$

where the $\overline{y}_{ij}(x)$'s are formal power series in $x_1,\dots,x_{n-1}$. Now, it follows easily from Taylor formula that we may replace the power series $\overline{y}_i(x)$ with any power series of the form

$$\overline{y}_i(x) + h_i(x_n^r + \overline{a}_1(x)x_n^{r-1} + \cdots + \overline{a}_r(x)), \qquad h_i \in \mathfrak{m}^{c-r},$$

and the assumption of the lemma remains satisfied. In particular we may choose the h_i's in such a way that every $u_i(x)$ is a polynomial. Let's introduce new variables A_i, Y_{ij} and set

$$y_i = (x_n^r + A_1x_n^{r-1} + \cdots + A_r)u_i(x) + \sum_{j=0}^{r-1} Y_{ij}x_n^j\ .$$

We obtain a new set of polynomials

$$G(x,A,Y) = g(x,y), \qquad F_i(x,A,Y) = f_i(x,y),$$

and by Euclidean division with respect to the variable x_n in the ring $\mathbb{C}[x_i, A_j, Y_{ij}]$ we have:

1. $G(x,A,Y) = (x_n^r + A_1x_n^{r-1} + \cdots + A_r)Q + \sum_{k=0}^{r-1} G_kx_n^k$
2. $F_i(x,A,Y) = (x_n^r + A_1x_n^{r-1} + \cdots + A_r)R_i + \sum_{k=0}^{r-1} F_{ik}x_n^k$.

Since $G(x, \overline{a}_i(x), \overline{y}_{ij}(x)) = g(x, \overline{y}(x))$, the unicity of remainder in Weierstrass division theorem implies $Q(0) \neq 0$ and $G_k(x, \overline{a}_i(x), \overline{y}_{ij}(x)) = 0$ for every k. Since $(x_n^r + \overline{a}_1(x)x_n^{r-1} + \cdots + \overline{a}_r(x))$ divides $f_i(x, \overline{y}(x))$, we also have $F_{hk}(x, \overline{a}_i(x), \overline{y}_{ij}(x)) = 0$ for every h, k.

By the inductive assumption we can find $a_i(x), y_{ij}(x) \in \mathbb{C}\{x_1, \ldots, x_{n-1}\}$, arbitrarily near to $\overline{a}_i(x), \overline{y}_{ij}(x)$ in the $\mathfrak{m}$-adic topology and such that

$$G_k(x, a_i(x), y_{ij}(x)) = F_{hk}(x, a_i(x), y_{ij}(x)) = 0, \qquad \forall\, h, k\ .$$

Therefore

1. $g(x, y(x)) = G(x, a(x), y_{ij}(x)) = (x_n^r + a_1(x)x_n^{r-1} + \cdots + a_r(x))Q$
2. $f_i(x, y(x)) = F_i(x, a(x), y_{ij}(x)) = (x_n^r + a_1(x)x_n^{r-1} + \cdots + a_r(x))R_i$

and then $g(x, y(x))$ divides $f_i(x, y(x))$ for every i. □

Lemma II.6.4. *The Theorem II.6.2 holds under the following additional assumption:*

1. $m \leq N$,
2. *if* $\delta(x, y) = \det \left(\dfrac{\partial f_i}{\partial y_j}\right)_{i,j=1,\ldots,m}$, *then* $\delta(x, \overline{y}(x)) \neq 0$.

Proof. According to Lemma II.6.3, there exist convergent power series $u_i(x) \in C\{x_1, \ldots, x_n\}$, $i = 1, \ldots, N$, such that $u_i(x) \equiv \overline{y}_i(x) \pmod{\mathfrak{m}^c}$ and

1. $\delta(x, u(x)) \neq 0$,
2. $\delta^2(x, u(x))$ divides $f_i(x, y(x))$ for every $i = 1, \ldots, m$,
3. $f_i(x, u(x)) \in \mathfrak{m}^{c+r}$ for every $i = 1, \ldots, m$, where r is the multiplicity of $\delta(x, u(x))$.

Let's write $f_i(x, u(x)) = \delta^2(x, u(x))F_i(x)$, $F_i(x) \in \mathbb{C}\{x_1, \ldots, x_n\}$, and denote by $M_{ij}(x, y)$ the coefficients of the adjoint matrix of $\left(\dfrac{\partial f_i}{\partial y_j}\right)_{i,j=1,\ldots,m}$. We have the Laplace identities:

$$\sum_j \frac{\partial f_i}{\partial y_j} M_{jk} = \begin{cases} 0 & \text{for } i \neq k\,, \\ \delta & \text{for } i = k\,. \end{cases}$$

We are able to prove that there exist $v_1(x), \ldots, v_m(x) \in \mathbb{C}\{x_1, \ldots, x_n\}$ such that the convergent power series

$$y_j(x) = u_j(x) + \delta(x, u(x)) \sum_k M_{jk}(x, u(x))v_k(x), \qquad j = 1, \ldots, m\ ,$$

$$y_j(x) = u_j(x) \qquad i = m+1, \ldots, N\ ,$$

satisfy the requirement of Theorem II.6.2. For every index i, by Taylor formula, we have:

$$f_i(x, y(x)) = f_i(x, u(x)) + \delta(x, u(x)) \sum_{j=1}^m \frac{\partial f_i}{\partial y_j}(x, u(x)) \sum_k M_{jk}(x, u(x))v_k(x) + \delta^2(x, u(x))Q_i$$

$$f_i(x, y(x)) = \delta^2(x, u(x))(F_i(x) + v_i(x) + Q_i)$$

where every Q_i is a polynomial in v_i containing only monomials of degree ≥ 2. By implicit function theorem II.1.2 we may solve the system of m equations

$$F_i(x) + v_i(x) + Q_i(x, v) = 0, \qquad i = 1, \ldots, m,$$

in the ring $\mathbb{C}\{x_1, \ldots, x_n\}$. Moreover, since $F_i(x) \in \mathfrak{m}^{c-r}$ we also have $v_i(x) \in \mathfrak{m}^{c-r}$ and then $y_i(x) \equiv u_i(x) \pmod{\mathfrak{m}^c}$. □

Proof of Theorem II.6.2. The power series $\overline{y_i}(x)$ give a morphism of rings

$$\phi\colon \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_N]\to\mathbb{C}[[x_1,\dots,x_n]],\qquad y_i\mapsto\overline{y_i}(x),$$

whose kernel P contains the polynomials f_i. Possibly enlarging the set of equations it is not restrictive to assume that the polynomials f_i generate P. Let $X=V(P)\subset\mathbb{C}^{n+N}$ the affine irredible variety defined by P and denote by $r=n+N-\dim X$ its codimension. Since $P\cap\mathbb{C}[x_1,\dots,x_n]=0$, the projection $X\to\mathbb{C}^n$, $(x,y)\mapsto x$, is a domonan morphism of algebraic varieties.

According to Bertini's theorem, for e generic point $p\in X$, the Zariski tangent space T_pX has codimension r and the projection $T_pX\to\mathbb{C}^n$ is surjective. Up to a permutation of indices we may assume that T_pX is the annihilator ot the r differentials $df_1,\dots,df_r$. Then the matrix

$$\left(\frac{\partial f_i}{\partial y_j}\right)_{i=1,\dots,r;j=1,\dots,N}$$

has rank r and, up to permutaion of indices, we may assume that the determinant

$$\delta(x,y)=\det\left(\frac{\partial f_i}{\partial y_j}\right)_{i,j=1,\dots,r}$$

does not vanish in p. In particular $\delta(x,y)\notin P$ and then $\delta(x,\overline{y}(x))\neq 0$.

According to Lemma II.6.4 there exist convergent power series $y_1(x),\dots,y_N(x)$ such that $y_i(x)\equiv\overline{y}_i(x)\pmod{\mathfrak{m}^c}$ for every i and

$$f_j(x,y(x))=0\qquad j=1,\dots,r\,.$$

Let $Y=V(f_1,\dots,f_r)\subset\mathbb{C}^{n+N}$ be the zero locus of $f_1,\dots,f_r$; clearly $X\subset Y$ and, since X,Y have the same dimension in p we have that X is an irreducible component of Y and then there exists a polynomial $h(x,y)\notin P$ such that $Y\subset X\cup V(h)$.

Since $h(x,\overline{y}(x))\neq 0$, possibly increasing the integer c, we have $h(x,y(x))\neq 0$. Let Q be the kernel of the morphism of rings

$$\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_N]\to\mathbb{C}\{x_1,\dots,x_n\},\quad y_i\mapsto y_i(x);$$

we want to prove that $P\subset Q$. We have already proved that $f_1,\dots,f_r\in Q$, $h\notin Q$, and therefore $V(Q)\subset Y$ and $V(Q)\not\subset V(h)$. Since $V(Q)$ is irreducible we have $V(Q)\subset X=V(P)$ and then $P\subset Q$. □

II.7 Exercises

Exercise II.7.1. Prove that the ring of entire holomorphic functions $f\colon\mathbb{C}\to\mathbb{C}$ is an integral domain but it is not factorial (Hint: consider the sine function $\sin(z)$).

Exercise II.7.2 (Cartan's Lemma, [10])**.** Let R be an analytic algebra with maximal ideal $\mathfrak{m}$ and G a finite group of automorphisms of R.

Prove that there exists an integer n, an injective homomorphism of groups $G\to GL(\mathbb{C}^n)$ and a G-isomorphism of analytic algebras $R\cong\mathcal{O}_{\mathbb{C}^n,0}/I$ for some G-stable ideal $I\subset\mathcal{O}_{\mathbb{C}^n,0}$. (Hint: there exists a direct sum decomposition $\mathfrak{m}=V\oplus\mathfrak{m}^2$ such that $gV\subset V$ for every $g\in G$.)

Exercise II.7.3. Prove that $f,g\in\mathbb{C}\{x,y\}$ have a common factor of positive multiplicity if and only if the $\mathbb{C}$-vector space $\mathbb{C}\{x,y\}/(f,g)$ is infinite dimensional.

Exercise II.7.4. If $f\colon(X,x)\to(Y,y)$ is a morphism of analytic singularities define the schematic fibre $(f^{-1}(y),x)$ as the subgerm of (X,x) defined by equations $f^*(g)=0$, where g varies on a set of geberators of the maximal ideal of $\mathcal{O}_{Y,y}$. Prove that the dimension of a singularity (X,x) is the minimum integer d such that there exists a morphism $f\in\mathrm{Mor}_{\mathbf{Ger}}((X,x),(\mathbb{C}^d,0))$ such that $(f^{-1}(0),x)$ is a fat point.

Exercise II.7.5 (Rückert's nullstellensatz)**.** Let $I, J \subset \mathbb{C}\{z_1, \ldots, z_n\}$ be proper ideals. Prove that

$$\mathrm{Mor}_{\mathbf{An}}\left(\frac{\mathbb{C}\{z_1,\ldots,z_n\}}{I}, \mathbb{C}\{t\}\right) = \mathrm{Mor}_{\mathbf{An}}\left(\frac{\mathbb{C}\{z_1,\ldots,z_n\}}{J}, \mathbb{C}\{t\}\right) \qquad \iff \qquad \sqrt{I} = \sqrt{J},$$

where the left equality is intended as equality of subsets of $\mathrm{Mor}_{\mathbf{An}}(\mathbb{C}\{z_1, \ldots, z_n\}, \mathbb{C}\{t\})$.

Exercise II.7.6. Prove that for every analytic algebra R with maximal ideal $\mathfrak{m}$ there exist natural isomorphisms

$$\mathrm{Hom}_{\mathbb{C}}\left(\frac{\mathfrak{m}}{\mathfrak{m}^2}, \mathbb{C}\right) = \mathrm{Der}_{\mathbb{C}}(R, \mathbb{C}) = \mathrm{Mor}_{\mathbf{An}}\left(R, \frac{\mathbb{C}[t]}{(t^2)}\right).$$

Exercise II.7.7. In the notation of Theorem II.5.4 prove:

1. $I^2 \subset J$.
2. $I = J + \mathfrak{m}I$ if and only if $I = 0$.
3. If $I \subset \mathfrak{m}^2$ then
$$\mathrm{Hom}_{\mathbb{C}}\left(\frac{I}{J + \mathfrak{m}I}, \mathbb{C}\right) = \mathrm{Ext}^1_R(\Omega_R, \mathbb{C}),$$
where Ω_R is the R-module of separated differentials.
4. For every short exact sequence $0 \to E \to F \to G \to 0$ of R-modules of finite length (i.e. annihilated by some power of $\mathfrak{m}$) it is defined a map
$$ob\colon \mathrm{Der}_{\mathbb{C}}(R, G) \to \mathrm{Hom}_R\left(\frac{I}{J}, E\right)$$
with the property that $ob(\phi) = 0$ if and only if ϕ lifts to a derivation $R \to F$. Moreover, if $\mathfrak{m}_R E = 0$ then $\mathrm{Hom}_R\left(\frac{I}{J}, E\right) = \mathrm{Hom}_{\mathbb{C}}\left(\frac{I}{J + \mathfrak{m}I}, E\right)$.

Exercise II.7.8. Use Theorem II.6.1 to prove:

1. Every irreducible convergent power series $f \in \mathbb{C}\{z_1, \ldots, z_n\}$ is also irreducible in $\mathbb{C}[[z_1, \ldots, z_n]]$.
2. $\mathbb{C}\{z_1, \ldots, z_n\}$ is integrally closed in $\mathbb{C}[[z_1, \ldots, z_n]]$.

Chapter III

Flatness and infinitesimal deformations

III.1 Flatness and relations

In this section $A \in \mathbf{Art}$ is a fixed local artinian $\mathbb{K}$-algebra with residue field $\mathbb{K}$.

Lemma III.1.1. *Let M be an A-module, if $M \otimes_A \mathbb{K} = 0$ then $M = 0$.*

Proof. If M is finitely generated this is Nakayama's lemma. In the general case consider a filtration of ideals $0 = I_0 \subset I_1 \subset \cdots \subset I_n = A$ such that $I_{i+1}/I_i = \mathbb{K}$ for every i. Applying the right exact functor $\otimes_A M$ to the exact sequences of A-modules

$$0 \longrightarrow \mathbb{K} = \frac{I_{i+1}}{I_i} \longrightarrow \frac{A}{I_i} \longrightarrow \frac{A}{I_{i+1}} \longrightarrow 0$$

we get by induction that $M \otimes_A (A/I_i) = 0$ for every i. □

The following is a special case of the *local flatness criterion* [85, Thm. 22.3]

Theorem III.1.2. *For an A-module M the following conditions are equivalent:*

1. *M is free.*
2. *M is flat.*
3. *$\mathrm{Tor}_1^A(M, \mathbb{K}) = 0$.*

Proof. The only nontrivial assertion is 3) $\Rightarrow$ 1). Assume $\mathrm{Tor}_1^A(M, \mathbb{K}) = 0$ and let F be a free module such that $F \otimes_A \mathbb{K} = M \otimes_A \mathbb{K}$. Since $M \to M \otimes_A \mathbb{K}$ is surjective there exists a morphism $\alpha\colon F \to M$ such that its reduction $\overline{\alpha}\colon F \otimes_A \mathbb{K} \to M \otimes_A \mathbb{K}$ is an isomorphism. Denoting by K the kernel of α and by C its cokernel we have $C \otimes_A \mathbb{K} = 0$ and then $C = 0$; $K \otimes_A \mathbb{K} = \mathrm{Tor}_1^A(M, \mathbb{K}) = 0$ and then $K = 0$. □

Corollary III.1.3. *Let $h\colon P \to L$ be a morphism of flat A-modules, $A \in \mathbf{Art}$. If $\overline{h}\colon P \otimes_A \mathbb{K} \to L \otimes_A \mathbb{K}$ is injective (resp.: surjective) then also h is injective (resp.: surjective).*

Proof. Same proof of Theorem III.1.2. □

Corollary III.1.4. *Let $0 \to M \to N \to P \to 0$ be an exact sequence of A-modules with N flat. Then:*

1. *$M \otimes_A \mathbb{K} \to N \otimes_A \mathbb{K}$ injective $\Rightarrow$ P flat.*
2. *P flat $\Rightarrow$ M flat and $M \otimes_A \mathbb{K} \to N \otimes_A \mathbb{K}$ injective.*

Proof. Take the associated long $\mathrm{Tor}_*^A(-, \mathbb{K})$ exact sequence and apply III.1.2 and IV.1. □

Corollary III.1.5. *Let*

$$P \xrightarrow{f} Q \xrightarrow{g} R \xrightarrow{h} M \longrightarrow 0 \tag{III.1}$$

be a complex of A-modules such that:

1. *P, Q, R are flat.*
2. *$Q \xrightarrow{g} R \xrightarrow{h} M \longrightarrow 0$ is exact.*
3. *$P \otimes_A \mathbb{K} \xrightarrow{\overline{f}} Q \otimes_A \mathbb{K} \xrightarrow{\overline{g}} R \otimes_A \mathbb{K} \xrightarrow{\overline{h}} M \otimes_A \mathbb{K} \longrightarrow 0$ is exact.*

Then M is flat and the sequence (III.1) is exact.

Proof. Denote by $H = \ker h = \operatorname{Im} g$ and $g = \phi\eta$, where $\phi \colon H \to R$ is the inclusion and $\eta \colon Q \to H$; by assumption we have an exact diagram

$$\begin{tikzcd}
P \otimes_A \mathbb{K} \arrow[r, "\overline{f}"] & Q \otimes_A \mathbb{K} \arrow[rr, "\overline{g}"] \arrow[dr, bend left, "\overline{\eta}"'] & & R \otimes_A \mathbb{K} \arrow[r, "\overline{h}"] & M \otimes_A \mathbb{K} \arrow[r] & 0 \\
& & H \otimes_A \mathbb{K} \arrow[ur, bend left, "\overline{\phi}"] \arrow[dr, bend right] & & & \\
& & & 0 & &
\end{tikzcd}$$

which allows to prove, after an easy diagram chase, that $\overline{\phi}$ is injective. According to Corollary III.1.4 H and M are flat modules. Denoting $L = \ker g$ we have, since H is flat, that also L is flat and $L \otimes_A K \to Q \otimes_A \mathbb{K}$ injective. This implies that $P \otimes_A \mathbb{K} \to L \otimes_A \mathbb{K}$ is surjective. By Corollary IV.1 $P \to L$ is surjective. □

Corollary III.1.6. *Let $n > 0$ and*

$$0 \longrightarrow I \longrightarrow P_0 \xrightarrow{d_1} P_1 \longrightarrow \cdots \xrightarrow{d_n} P_n,$$

a complex of A-modules with $P_0, \ldots, P_n$ flat. Assume that

$$0 \longrightarrow I \otimes_A \mathbb{K} \longrightarrow P_0 \otimes_A \mathbb{K} \xrightarrow{\overline{d_1}} P_1 \otimes_A \mathbb{K} \longrightarrow \cdots \xrightarrow{\overline{d_n}} P_n \otimes_A \mathbb{K}$$

is exact; then $I, \operatorname{coker}(d_n)$ are flat modules and the natural morphism $I \to \ker(P_0 \otimes_A \mathbb{K} \to P_1 \otimes_A \mathbb{K})$ is surjective.

Proof. Induction on n and Corollary III.1.5. □

III.2 Deformations of analytic algebras

Let R be a fixed analytic algebra.

Definition III.2.1. A deformation of R over an analytic algebra A is the data of a flat morphism of analytic algebras $f \colon A \to S$ and an isomorphism $S \otimes_A \mathbb{C} \xrightarrow{\phi} R$.

Two deformations $A \xrightarrow{f} S \to S \otimes_A \mathbb{C} \xrightarrow{\phi} R$ and $A \xrightarrow{g} T \to T \otimes_A \mathbb{C} \xrightarrow{\psi} R$ are equivalent if there exists an isomorphism $h \colon S \to T$ making the following diagram commutative

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g"'] & S \arrow[r] \arrow[d, "h"] & S \otimes_A \mathbb{C} \arrow[d, "h \otimes Id"'] \arrow[dr, "\phi"] & \\
& T \arrow[r] & T \otimes_A \mathbb{C} \arrow[r, "\psi"] & R
\end{tikzcd}$$

Definition III.2.2. An **infinitesimal deformation** is a deformation over a local local artinian $\mathbb{C}$-algebra A. A **first order deformation** is a deformation over the $\mathbb{C}$-algebra $A = \mathbb{C}[\varepsilon]/(\varepsilon^2)$.

Example III.2.3. Given an analytic algebra R and $A \in \mathbf{Art}$, a deformation of R over A is called **trivial** if it is isomorphic to $S = R \otimes_{\mathbb{C}} A$, with $f\colon A \to S$, $f(a) = 1 \otimes a$ and $\phi\colon R \otimes_{\mathbb{C}} A \otimes_A \mathbb{C} = R \to R$ equal to the identity.

Lemma III.2.4. *Every infinitesimal deformation of* $\mathbb{C}\{z_1, \dots, z_n\}$ *is trivial.*

Proof. Let $f\colon A \to S$ and $S \otimes_A \mathbb{C} \xrightarrow{\phi} \mathbb{C}\{z_1, \dots, z_n\}$ be an infinitesimal deformation. Then the composite map $S \to S \otimes_A \mathbb{C} \xrightarrow{\phi} \mathbb{C}\{z_1, \dots, z_n\}$ is surjective and we may choose $s_1, \dots, s_n \in \mathfrak{m}_S$ such that $\phi(s_i \otimes 1) = z_i$. Then we have a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[dr, "f"'] & A\{z_1, \dots, z_n\} \arrow[d, "h"] \arrow[dr] & \\
& S \arrow[r, "\phi"] & \mathbb{C}\{z_1, \dots, z_n\}
\end{tikzcd}$$

and since S is flat over A and $h \otimes Id\colon A\{z_1, \dots, z_n\} \otimes_A \mathbb{C} \to S \otimes_A \mathbb{C}$ is an isomorphism, also h is an isomorphism. $\square$

III.3 Infinitesimal deformations of complex manifolds

Let X be a complex manifold of dimension n and A a local artinian $\mathbb{C}$-algebra.

Definition III.3.1. A deformation of X over A is the data of a flat morphism of complex spaces $f\colon X_A \to \operatorname{Spec}(A)$ and a closed embedding $i\colon X \to X_A$ inducing an isomorphism $i\colon X \xrightarrow{\simeq} f^{-1}(0)$, where $0 = \operatorname{Spec}(\mathbb{C})$ is the closed point of $\operatorname{Spec}(A)$.

Two deformations $X \xrightarrow{i} X_A \xrightarrow{f} \operatorname{Spec}(A)$ and $X \xrightarrow{j} \tilde{X}_A \xrightarrow{g} \operatorname{Spec}(A)$ are isomorphic if there exists an isomorphism of complex spaces $h\colon X_A \to \tilde{X}_A$ such that $f = gh$ and $j = hi$. The set of isomorphism classes of deformations of X over a local artinian $\mathbb{C}$-algebra will be denoted $\operatorname{Def}_X(A)$.

Since A is local artinian, the complex space X_A is supported on X and then for a deformation $X \to \mathcal{X} \to \operatorname{Spec}(A)$ we have that $\mathcal{O}_{\mathcal{X}}$ is a sheaf of flat A-algebras such that $\mathcal{O}_{\mathcal{X}} \otimes_A \mathbb{C} = \mathcal{O}_X$. By Lemma III.2.4 for every $x \in X$ the stalk $\mathcal{O}_{\mathcal{X},x}$ is isomorphic to $A\{z_1, \dots, z_n\}$ and then, by Theorem II.3.3 the germ $(\mathcal{X}, x)$ is isomorphic to $(X \times \operatorname{Spec}(A), x)$. In particular the sheaf $\mathcal{O}_{\mathcal{X}}$ is locally isomorphic to $\mathcal{O}_X \otimes_{\mathbb{C}} A$.

Conversely every sheaf of A-algebras on X locally isomorphic to $\mathcal{O}_X \otimes_{\mathbb{C}} A$ gives a deformation of X over $\operatorname{Spec}(A)$. Thus we have the following useful description of infinitesimal deformations of a complex manifold X.

Lemma III.3.2. *For a complex manifold X and a local artinian $\mathbb{C}$-algebra we have*

$$\operatorname{Def}_X(A) = \frac{\textit{sheaves of } A\textit{-algebras on } X \textit{ locally isomorphic to } \mathcal{O}_X \otimes_{\mathbb{C}} A}{\textit{isomorphisms of sheaves of } A\textit{-algebras}}$$

Recall that a complex manifold is called rigid if $H^1(X, T_X) = 0$ (Definition I.4.6); for instance, every Stein manifold is rigid.

Proposition III.3.3. *Every first order deformation of a rigid complex manifold is trivial.*

Proof. Let $X \to \mathcal{X}$ be a deformation of a complex manifold X over $\mathbb{C}[\varepsilon]$, $\varepsilon^2 = 0$, given by a sheaf $\mathcal{O}_{\mathcal{X}}$ of flat $\mathbb{C}[\varepsilon]$-algebras. Applying the functor $\mathcal{O}_{\mathcal{X}} \otimes_{\mathbb{C}[\varepsilon]} -$ to the short exact sequence

$$0 \to \mathbb{C} \xrightarrow{\varepsilon} \mathbb{C}[\varepsilon] \to \mathbb{C} \to 0$$

we get an exact sequence of sheaves

$$0 \to \mathcal{O}_X \xrightarrow{\varepsilon} \mathcal{O}_{\mathcal{X}} \xrightarrow{\pi} \mathcal{O}_X \to 0.$$

Since $\mathcal{O}_{\mathcal{X}}$ is locally isomorphic to $\mathcal{O}_X[\varepsilon]$ the sheaf $\Omega^1_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_X$ (see next Remark III.3.4) is locally isomorphic to $\Omega^1_X \oplus \mathcal{O}_X d\varepsilon$ and then we have a commutative diagram witn exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r, "\varepsilon"] \arrow[d, equal] & \mathcal{O}_{\mathcal{X}} \arrow[r] \arrow[d, "d"] & \mathcal{O}_X \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r, "d\varepsilon"] & \Omega^1_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_X \arrow[r] & \Omega^1_X \arrow[r] & 0
\end{tikzcd}$$

If X is rigid, then $\operatorname{Ext}^1(\Omega^1_X, \mathcal{O}_X) = H^1(X, T_X) = 0$, the second row is a trivial extension and therefore there exists

$$h \in \operatorname{Hom}_{\mathcal{O}_X}(\Omega^1_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_X, \mathcal{O}_X) = \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(\Omega^1_{\mathcal{X}}, \mathcal{O}_X) = \operatorname{Der}_{\mathbb{C}}(\mathcal{O}_{\mathcal{X}}, \mathcal{O}_X)$$

such that $h(d\varepsilon) = 1$. The map

$$\phi\colon \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_X \oplus \varepsilon\mathcal{O}_X, \qquad \phi(u) = \pi(u) + \varepsilon h(u),$$

is an isomorphism of sheaves of $\mathbb{C}[\varepsilon]$-algebras. □

Remark III.3.4. Every complex space X carries the sheaf Ω^1_X of Kähler differentials which is a coherent $\mathcal{O}_X$-module. If $Y \subset X$ is an open embedding, then $\Omega^1_Y = (\Omega^1_X)_{|Y}$ and, for the above local model $(V, \mathcal{O}_V)$ we have

$$\Omega^1_V = \frac{\Omega^1_U}{(dg_1, \ldots, dg_m)} \otimes \mathcal{O}_V = \frac{\Omega^1_U}{(dg_1, \ldots, dg_m) + \mathcal{I}\Omega^1_U}.$$

For every morphism $f\colon X \to Y$ of complex spaces we have an exact sequence

$$f^*\Omega^1_Y \to \Omega^1_X \to \Omega^1_{X/Y} \to 0,$$

where $\Omega^1_{X/Y}$ is the space of relatice differentials. Similarly for every coherent ideal sheaf $\mathcal{J} \subset \mathcal{O}_X$, if $Z \subset X$ is the support of $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{J}$, then the pair $(Z, \mathcal{O}_Z)$ is a complex space and we have an exact sequence of sheaves

$$\frac{\mathcal{J}}{\mathcal{J}^2} \xrightarrow{d} \Omega^1_X \otimes \mathcal{O}_Z \longrightarrow \Omega^1_Z \longrightarrow 0.$$

Chapter IV

Functors of Artin rings

In this chapter we collect some definitions and main properties of deformation functors.

In the first section, we introduce the notions of functor of Artin rings, of deformation functor and of the associated tangent and obstruction spaces.

The main references for this chapter are [18], [98] and [103].

IV.1 Deformation functors

Let $\mathbb{K}$ be a field, **Set** the category of sets in a fixed universe and by $\{*\}$ a fixed set of cardinality 1. Let $\mathbf{Art} = \mathbf{Art}_{\mathbb{K}}$ be the category of local Artinian $\mathbb{K}$-algebras with residue field $\mathbb{K}$ $(A/\mathfrak{m}_A = \mathbb{K})$; the morphisms in **Art** are local morphism.

We shall say that a morphism $\alpha\colon B \to A$ in **Art** is a *small surjection* if α is surjective and its kernel is annihilated by the maximal ideal $\mathfrak{m}_B$. The artinian property implies that every surjective morphism in **Art** can be decomposed in a finite sequence of small surjections and then a functor F is smooth if and only if $F(B) \to F(A)$ is surjective for every small surjection $B \to A$.

A *small extension* is a small surjection together a framing of its kernel. More precisely a small extension e in **Art** is an exact sequence of abelian groups

$$e\colon \quad 0 \longrightarrow M \longrightarrow B \xrightarrow{\alpha} A \longrightarrow 0,$$

such that α is a morphism in the category **Art** and M is an ideal of B annihilated by the maximal ideal $\mathfrak{m}_B$. In particular M is a finite dimensional vector space over $B/\mathfrak{m}_B = \mathbb{K}$. A small extension as above is called *principal* if $M = \mathbb{K}$.

Definition IV.1.1. A *functor of Artin rings* is a covariant functor $F\colon \mathbf{Art} \to \mathbf{Set}$ such that $F(\mathbb{K}) = \{*\}$.

The functors of Artin rings are the objects of a category whose morphisms are the natural transformations of functors. For simplicity of notation, if $\phi\colon F \to G$ is a natural transformation, we denote by $\phi\colon F(A) \to G(A)$ the corresponding morphism of sets, for every $A \in \mathbf{Art}$.

Example IV.1.2. The *trivial* functor $*$ is the functor defined by $*(A) = \{*\}$, for every $A \in \mathbf{Art}$.

Example IV.1.3. Let V be a $\mathbb{K}$-vector space. Then, $F, G\colon \mathbf{Art} \to \mathbf{Set}$, defined by

$$F(A) = V \otimes \mathfrak{m}_A, \qquad G(A) = \operatorname{Hom}_{\mathbb{K}}(V, V \otimes \mathfrak{m}_A)$$

are functors of Artin rings. Notice that $G(A)$ is the kernel of the morphism

$$\operatorname{Hom}_A(V \otimes A, V \otimes A) = \operatorname{Hom}_{\mathbb{K}}(V, V \otimes A) \to \operatorname{Hom}_{\mathbb{K}}(V, V \otimes \mathbb{K}) = \operatorname{Hom}_{\mathbb{K}}(V, V)$$

and then $G(A)$ is the set of A-linear endomorphism of $V \otimes A$ that are trivial modulus $\mathfrak{m}_A$.

Example IV.1.4. Let R be a local complete $\mathbb{K}$-algebra with residue field $\mathbb{K}$. The functor

$$\mathrm{h}_R\colon \mathbf{Art} \to \mathbf{Set}, \qquad \mathrm{h}_R(A) = \operatorname{Hom}_{\mathbb{K}-\mathrm{alg}}(R, A),$$

is a functor of Artin rings.

The category **Art** is closed under fiber products, i.e., every pair of morphisms $C \to A$, $B \to A$ may be extended to a commutative diagram

$$\begin{array}{ccc} B \times_A C & \longrightarrow & C \\ \big\downarrow & & \big\downarrow \\ B & \longrightarrow & A, \end{array} \tag{IV.1}$$

such that the natural map

$$\mathrm{h}_R(B \times_A C) \to \mathrm{h}_R(B) \times_{\mathrm{h}_R(A)} \mathrm{h}_R(C)$$

is bijective, for every R.

Definition IV.1.5. Let $F\colon \mathbf{Art} \to \mathbf{Set}$ be a functor of Artin rings; for every fiber product

$$\begin{array}{ccc} B \times_A C & \longrightarrow & C \\ \big\downarrow & & \big\downarrow \\ B & \xrightarrow{\beta} & A \end{array}$$

in **Art**, consider the induced map

$$\eta\colon F(B \times_A C) \to F(B) \times_{F(A)} F(C).$$

The functor F is *homogeneous* if η is bijective whenever β is surjective [96, Definition 2.5]. The functor F is a *deformation functor* if:

1. η is surjective, whenever β is surjective;
2. η is bijective, whenever $A = \mathbb{K}$.

The name deformation functor comes from the fact that almost all functors arising in deformation theory satisfy the conditions of Definition IV.1.5. Every prorepresentable functor is a homogeneous deformation functor.

Remark IV.1.6. Our definition of deformation functors involves conditions that are slightly more restrictive than the classical Schlessinger conditions H1, H2 of [98] and the semi-homogeneity condition of [96]. The main motivations of this change are:

1. Functors of Artin rings satisfying Schlessinger condition H1, H2 and H3 do not necessarily have a “good”obstruction theory (see [18, Example 6.8]).
2. The definition of deformation functor extends naturally in the framework of derived deformation theory and extended moduli spaces [78].

IV.2 Smooth and prorepresentable functors

Definition IV.2.1. (cf. [98]) A functor $F\colon \mathbf{Art} \to \mathbf{Set}$ is *prorepresentable* if it is isomorphic to h_R, for some local complete $\mathbb{K}$-algebra R with residue field $\mathbb{K}$. F is *representable* if it is isomorphic to h_R, for some $R \in \mathbf{Art}$.

The formal smoothness of $\mathrm{Spec}(R)$ is equivalent to the property that $A \to B$ surjective implies $\mathrm{h}_R(A) \to \mathrm{h}_R(B)$ surjective. This motivate the following definition.

Definition IV.2.2. A natural transformation $\phi\colon F \to G$ of functors of Artin rings is called *smooth* if, for every surjective morphism $A \to B$ in **Art**, the map $F(A) \to G(A) \times_{G(B)} F(B)$ is also surjective. A functor of Artin rings F is called *smooth* if $F(A) \to F(B)$ is surjective, for every surjective morphism $A \to B$ in **Art**, i.e., the natural transformation $F \to *$ is smooth.

Remark IV.2.3. If $\phi\colon F \to G$ is a smooth natural transformation, then $\phi\colon F(A) \to G(A)$ is surjective for every A (take $B = \mathbb{K}$).

Exercise IV.2.4. 1. If $F \to G$ and $G \to H$ are smooth, then the composition $F \to H$ is smooth.

2. If $u\colon F \to G$ and $v\colon G \to H$ are natural transformations of functors such that u is surjective and vu is smooth. Then, v is smooth.

3. If $F \to G$ and $H \to G$ are natural transformations of functor such that $F \to G$ is smooth, then $F \times_G H \to H$ is smooth.

Lemma IV.2.5. *Let R be a local complete noetherian $\mathbb{K}$-algebra with residue field $\mathbb{K}$. The following conditions are equivalent:*

1. *R is isomorphic to a power series ring $\mathbb{K}[[x_1, \ldots, x_n]]$.*

2. *The functor h_R is smooth.*

3. *For every $s \geq 2$ the morphism*

$$\mathrm{h}_R\left(\frac{\mathbb{K}[t]}{(t^{s+1})}\right) \to \mathrm{h}_R\left(\frac{\mathbb{K}[t]}{(t^2)}\right)$$

is surjective.

Proof. The only nontrivial implication is [3 ⇒ 1]. Let n be the embedding dimension of R, then we can write $R = \mathbb{K}[[x_1, \ldots, x_n]]/I$ for some ideal $I \subset (x_1, \ldots, x_n)^2$; we want to prove that $I = 0$. Assume therefore $I \neq 0$ and denote by $s \geq 2$ the greatest integer such that $I \subset (x_1, \ldots, x_n)^s$: we claim that

$$\mathrm{h}_R\left(\frac{\mathbb{K}[t]}{(t^{s+1})}\right) \to \mathrm{h}_R\left(\frac{\mathbb{K}[t]}{(t^2)}\right)$$

is not surjective. Choosing $f \in I - (x_1, \ldots, x_n)^{s+1}$, after a possible generic linear change of coordinates of the form $x_i \mapsto x_i + a_i x_1$, with $a_2, \ldots, a_k \in \mathbb{K}$, we may assume that f contains the monomial x_1^s with a nonzero coefficient, say $f = cx_1^s + \ldots$; let $\alpha\colon R \to \mathbb{K}[t]/(t^2)$ be the morphism defined by $\alpha(x_1) = t$, $\alpha(x_i) = 0$ for $i > 1$. Assume that there exists $\tilde{\alpha}\colon R \to \mathbb{K}[t]/(t^{s+1})$ that lifts α and denote by $\beta\colon \mathbb{K}[[x_1, \ldots, x_n]] \to \mathbb{K}[t]/(t^{s+1})$ the composition of $\tilde{\alpha}$ with the projection $\mathbb{K}[[x_1, \ldots, x_n]] \to R$. Then $\beta(x_1) - t, \beta(x_2), \ldots, \beta(x_n) \in (t^2)$ and therefore $\beta(f) \equiv ct^s \not\equiv 0 \pmod{t^{s+1}}$. □

Definition IV.2.6. Given a functor of Artin rings $F\colon \mathbf{Art} \to \mathbf{Set}$ and a group functor of Artin rings $G\colon \mathbf{Art} \to \mathbf{Grp}$, by a G-action on F we shall mean a natural transformation $G \times F \to F$ such that

$$G(A) \times F(A) \to F(A)$$

is a $G(A)$-action on $F(A)$ in the usual sense for every $A \in \mathbf{Art}$. Then one can clearly define in the obvious way the quotient functor F/G.

Proposition IV.2.7. *In the situation of Definition IV.2.6, if F and G are deformation functors and G is smooth, then F/G is a deformation functor and the natural projection $F \to F/G$ is smooth.*

Proof. Easy exercise. □

Later we will give lots of examples where F and G are homogeneous and F/G is not homogeneous. Moreover it is possible to prove that over a field of characteristic 0 every group deformation functor is smooth.

IV.3 Examples of deformation functors

Automorphisms functor

In this section every tensor product is intended over $\mathbb{K}$, i.e $\otimes = \otimes_{\mathbb{K}}$. Let $S \xrightarrow{\alpha} R$ be a morphism of commutative unitary $\mathbb{K}$-algebras, for every $A \in \mathbf{Art}$, we have natural morphisms $S \otimes A \xrightarrow{\alpha} R \otimes A$ and $R \otimes A \xrightarrow{p} R$, $p(x \otimes a) = x\overline{a}$, where $\overline{a} \in \mathbb{K}$ is the class of a in the residue field of A.

Lemma IV.3.1. *Given $A \in \mathbf{Art}$ and a commutative diagram of morphisms of $\mathbb{K}$-algebras*

$$\begin{tikzcd} S\otimes A \arrow[r, "\alpha"] \arrow[d, "\alpha"] & R\otimes A \arrow[d, "p"] \\ R\otimes A \arrow[r, "p"] \arrow[ur, "f"] & R \end{tikzcd}$$

we have that f is an isomorphism and $f(R\otimes J) \subset R\otimes J$ for every ideal $J \subset A$.

Proof. f is a morphism of A-algebras, in particular for every ideal $J \subset A$, $f(R\otimes J) \subset Jf(R\otimes A) \subset R\otimes J$. In particular, if $B = A/J$, then f induces a morphism of B-algebras $\overline{f}\colon R\otimes B \to R\otimes B$. We claim that, if $\mathfrak{m}_A J = 0$, then f is the identity on $R\otimes J$; in fact for every $x \in R$, $f(x\otimes 1) - x\otimes 1 \in \ker p = R\otimes \mathfrak{m}_A$ and then if $j \in J$, $x \in R$.

$$f(x\otimes j) = jf(x\otimes 1) = x\otimes j + j(f(x\otimes 1) - x\otimes 1) = x\otimes j.$$

Now we prove the lemma by induction on $n = \dim_{\mathbb{K}} A$, being f the identity for $n = 1$. Let

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

be a small extension with $J \neq 0$. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R\otimes J & \longrightarrow & R\otimes A & \longrightarrow & R\otimes B & \longrightarrow & 0 \\ & & \big\downarrow Id & & \big\downarrow f & & \big\downarrow \overline{f} & & \\ 0 & \longrightarrow & R\otimes J & \longrightarrow & R\otimes A & \longrightarrow & R\otimes B & \longrightarrow & 0 \end{array}$$

By induction $\overline{f}$ is an isomorphism and by snake lemma also f is an isomorphism. □

Definition IV.3.2. For every $A \in \mathbf{Art}$ let $\mathrm{Aut}_{R/S}(A)$ be the set of commutative diagrams of graded $\mathbb{K}$-algebra morphisms

$$\begin{tikzcd} S\otimes A \arrow[r] \arrow[d] & R\otimes A \arrow[d] \\ R\otimes A \arrow[r] \arrow[ur, "f"] & R \end{tikzcd}$$

According to Lemma IV.3.1, $\mathrm{Aut}_{R/S}$ is a functor from the category $\mathbf{Art}$ to the category of groups $\mathbf{Grp}$. Here we consider $\mathrm{Aut}_{R/S}$ as a functor of Artin rings (just forgetting the group structure).

Let $\mathrm{Der}_S(R, R)$ be the space of S-derivations $R \to R$ of. If $A \in \mathbf{Art}$ and $J \subset \mathfrak{m}_A$ is an ideal then, since $\dim_{\mathbb{K}} J < \infty$ there exist natural isomorphisms

$$\mathrm{Der}_S(R, R)\otimes J = \mathrm{Der}_S(R, R\otimes J) = \mathrm{Der}_{S\otimes A}(R\otimes A, R\otimes J),$$

where a given derivation $d = \sum_i d_i \otimes j_i \in \mathrm{Der}_S(R, R)\otimes J$ corresponds to the $S\otimes A$-derivation

$$d\colon R\otimes A \to R\otimes J \subset R\otimes A, \qquad d(x\otimes a) = \sum_i d_i(x)\otimes j_i a.$$

For every $d \in \mathrm{Der}_{S\otimes A}(R\otimes A, R\otimes A)$, denote $d^n = d\circ \ldots \circ d$ the iterated composition of d with itself n times. The generalized Leibniz rule gives

$$d^n(uv) = \sum_{i=0}^{n} \binom{n}{i} d^i(u) d^{n-1}(v), \qquad u, v \in R\otimes A.$$

In particular, note that, if $d \in \mathrm{Der}_S(R, R)\otimes \mathfrak{m}_A$, then d is a nilpotent endomorphism of $R\otimes A$ and

$$e^d = \sum_{n\geq 0} \frac{d^n}{n!}$$

is a morphism of $\mathbb{K}$-algebras belonging to $\mathrm{Aut}_{R/S}(A)$.

Proposition IV.3.3. *For every $A \in \mathbf{Art}_{\mathbb{K}}$ the exponential*

$$\exp\colon \mathrm{Der}_S(R,R) \otimes \mathfrak{m}_A \to \mathrm{Aut}_{R/S}(A)$$

is a bijection.

Proof. This is obvious if $A = \mathbb{K}$; by induction on the dimension of A we may assume that there exists a nontrivial small extension

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

such that $\exp\colon \mathrm{Der}_S(R,R) \otimes \mathfrak{m}_B \to \mathrm{Aut}_{R/S}(B)$ is bijective. We first note that if $d \in \mathrm{Der}_S(R,R) \otimes \mathfrak{m}_A$, $h \in \mathrm{Der}_S(R,R) \otimes J$ then $d^i h^j = h^j d^i = 0$ whenever $j > 0$, $j + i \geq 2$ and then $e^{d+h} = e^d + h$; this easily implies that exp is injective.
Conversely take a $f \in \mathrm{Aut}_{R/S}(A)$; by the inductive assumption there exists $d \in \mathrm{Der}_S(R,R) \otimes \mathfrak{m}_A$ such that $\overline{f} = \overline{e^d} \in \mathrm{Aut}_{R/S}(B)$; denote $h = f - e^d\colon R \otimes A \to R \otimes J$. Since $h(ab) = f(a)f(b) - e^d(a)e^d(b) = h(a)f(b) + e^d(a)h(b) = h(a)\overline{b} + \overline{a}h(b)$ we have that $h \in \mathrm{Der}_S(R,R) \otimes J$ and then $f = e^{d+h}$. □

The same argument works also if $S \to R$ is a morphism of sheaves of $\mathbb{K}$-algebras over a topological space and $\mathrm{Der}_S(R,R)$, $\mathrm{Aut}_{R/S}(A)$ are respectively the vector space of S-derivations of of R and the $S \otimes A$-algebra automorphisms of $R \otimes A$ lifting the identity on R.

Infinitesimal deformations of projective varieties

Let X be a projective variety over $\mathbb{K}$. An *infinitesimal deformation* of X over $\mathrm{Spec}(A)$ is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & X_A \arrow[d, "\pi"] \\ \mathrm{Spec}(\mathbb{K}) \arrow[r, "a"] & \mathrm{Spec}(A), \end{tikzcd}$$

where π is a proper and flat morphism, $a \in \mathrm{Spec}(A)$ is the closed point, i is a closed embedding and $X \cong X_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\mathbb{K})$. If $A = \mathbb{K}[\epsilon]$ we call it a *first order* deformation of X.

Remark IV.3.4. Let X_A be an infinitesimal deformation of X. By definition, it can be interpreted as a morphism of sheaves of algebras $\mathcal{O}_A \to \mathcal{O}_X$, such that $\mathcal{O}_A$ is flat over A and $\mathcal{O}_A \otimes_A \mathbb{K} \to \mathcal{O}_X$ is an isomorphism.

Given another deformation X'_A of X over $\mathrm{Spec}(A)$, we say that X_A and X'_A are *isomorphic* if there exists an isomorphism $\phi : X_A \to X'_A$ over $\mathrm{Spec}(A)$, that induces the identity on X, that is, the following diagram is commutative

$$\begin{tikzcd} & X \arrow[dl, "i"'] \arrow[dr, "i'"] & \\ X_A \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & X'_A \arrow[dl, "\pi'"] \\ & \mathrm{Spec}(A). & \end{tikzcd}$$

An infinitesimal deformation of X over $\mathrm{Spec}(A)$ is called *trivial* if it is isomorphic to the infinitesimal product deformation, i.e., to the deformation

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & X \times \mathrm{Spec}(A) \arrow[d] \\ \mathrm{Spec}(\mathbb{K}) \arrow[r, "a"] & \mathrm{Spec}(A). \end{tikzcd}$$

X is called *rigid* if every infinitesimal deformation of X over $\mathrm{Spec}(A)$ (for each $A \in \mathbf{Art}$) is trivial.

For every deformation X_A of X over $\mathrm{Spec}(A)$ and every morphism $A \to B$ in $\mathbf{Art}$ ($\mathrm{Spec}(B) \to \mathrm{Spec}(A)$), there exists an associated deformation of X over $\mathrm{Spec}(B)$, called *pull-back deformation*, induced by a base change:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B) \arrow[d] \\ \mathrm{Spec}(\mathbb{K}) \arrow[r] & \mathrm{Spec}(B). \end{tikzcd}$$

Definition IV.3.5. The *infinitesimal deformation functor* Def_X of X is defined as follows:

$$\mathrm{Def}_X : \mathbf{Art} \to \mathbf{Set},$$

$$A \longmapsto \mathrm{Def}_X(A) = \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{infinitesimal deformations} \\ \text{of } X \text{ over } \mathrm{Spec}(A) \end{array} \right\}.$$

Proposition IV.3.6. Def_X *is a deformation functor, i.e., it satisfies the conditions of Definition IV.1.5*

Proof. See [98, Section 3]. □

Infinitesimal deformations of locally free sheaves

Let X be a projective scheme and $\mathcal{E}$ a locally free sheaf of $\mathcal{O}_X$-modules on X. An *infinitesimal deformation of* $\mathcal{E}$ over $\mathrm{Spec}(A)$ is a locally free sheaves of $\mathcal{O}_X \otimes A$-modules $\mathcal{E}_A$ on $X \times \mathrm{Spec}(A)$, together with a morphism $\pi_A : \mathcal{E}_A \to \mathcal{E}$, such that $\pi_A : \mathcal{E}_A \otimes_A \mathbb{K} \to \mathcal{E}$ is an isomorphism.

Given another deformation $\mathcal{E}'_A$ of $\mathcal{E}$ over $\mathrm{Spec}(A)$, we say that $\mathcal{E}_A$ and $\mathcal{E}'_A$ are *isomorphic* if there exists an isomorphism of shaeves of $\mathcal{O}_X \otimes A$-modules $\phi : \mathcal{E}_A \to \mathcal{E}'_A$ over $\mathrm{Spec}(A)$, that commutes with the morphisms $\pi_A : \mathcal{E}_A \otimes_A \mathbb{K} \to \mathcal{E}$ and $\pi'_A : \mathcal{E}'_A \otimes_A \mathbb{K} \to \mathcal{E}$, i.e., $\pi_A \circ \phi = \pi_A$.

For every deformation $\mathcal{E}_A$ of $\mathcal{E}$ over $\mathrm{Spec}(A)$ and every morphism $A \to B$ in $\mathbf{Art}$ ($\mathrm{Spec}(B) \to \mathrm{Spec}(A)$), there exists an associated deformation of $\mathcal{E}$ over $\mathrm{Spec}(B)$, called *pull-back deformation*, induced by a base change:

$$\begin{tikzcd} p^*\mathcal{E}_A \arrow[r] \arrow[d] & \mathcal{E}_A \arrow[d] \\ X \times \mathrm{Spec}(B) \arrow[r, "p"] & X \times \mathrm{Spec}(A). \end{tikzcd}$$

Definition IV.3.7. The *infinitesimal deformation functor* $\mathrm{Def}_{\mathcal{E}}$ of $\mathcal{E}_A$ is defined as follows:

$$\mathrm{Def}_{\mathcal{E}} : \mathbf{Art} \to \mathbf{Set},$$

$$A \longmapsto \mathrm{Def}_{\mathcal{E}}(A) = \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{infinitesimal deformations} \\ \text{of } \mathcal{E} \text{ over } \mathrm{Spec}(A) \end{array} \right\}.$$

Proposition IV.3.8. $\mathrm{Def}_{\mathcal{E}}$ *is a deformation functor, i.e., it satisfies the conditions of Definition IV.1.5.*

Proof. See [98, Section 3]. □

Remark IV.3.9. Given a projective scheme X, we have defined a deformation fo a locally free sheaf $\mathcal{E}$ over $\mathrm{Spec}(A)$, as a sheaf $\mathcal{E}_A$ on $X \times \mathrm{Spec}(A)$, i.e., we are considering the trivial deformations of X. More generally, we can define infinitesimal deformations of the pair $(X, \mathcal{E})$ whenever we allow deformations of X too.

Infinitesimal deformations of maps

Definition IV.3.10. Let $f : X \to Y$ be a holomorphic map and $A \in \mathbf{Art}$. An *infinitesimal deformation of f over* $\operatorname{Spec}(A)$ is a commutative diagram of complex spaces

$$\begin{tikzcd} X_A \arrow[rr, "\mathcal{F}"] \arrow[dr, "\pi"'] & & Y_A \arrow[dl, "\mu"] \\ & S, & \end{tikzcd}$$

where $S = \operatorname{Spec}(A)$, (X_A, π, S) and (Y_A, μ, S) are infinitesimal deformations of X and Y, respectively (Definition IV.3.5), $\mathcal{F}$ is a holomorphic map that restricted to the fibers over the closed point of S coincides with f.

If $A = \mathbb{K}[\epsilon]$ we have a *first order deformation* of f.

Definition IV.3.11. Let

$$\begin{tikzcd} X_A \arrow[rr, "\mathcal{F}"] \arrow[dr, "\pi"'] & & Y_A \arrow[dl, "\mu"] \\ & S & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} X'_A \arrow[rr, "\mathcal{F}'"] \arrow[dr, "\pi'"'] & & Y'_A \arrow[dl, "\mu'"] \\ & S & \end{tikzcd}$$

be two infinitesimal deformations of f. They are *equivalent* if there exist bi-holomorphic maps $\phi : X_A \to X'_A$ and $\psi : Y_A \to Y'_A$ (that are equivalence of infinitesimal deformations of X and Y, respectively) such that the following diagram is commutative:

$$\begin{tikzcd} X_A \arrow[r, "\mathcal{F}"] \arrow[d, "\phi"'] & Y_A \arrow[d, "\psi"] \\ X'_A \arrow[r, "\mathcal{F}'"] & Y'_A. \end{tikzcd}$$

Definition IV.3.12. The *functor of infinitesimal deformations* of a holomorphic map $f : X \to Y$ is

$$\operatorname{Def}(f) : \mathbf{Art} \to \mathbf{Set},$$

$$A \longmapsto \operatorname{Def}(f)(A) = \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{infinitesimal deformations of} \\ f \text{ over } \operatorname{Spec}(A) \end{array} \right\}.$$

Proposition IV.3.13. $\operatorname{Def}(f)$ *is a deformation functor, since it satisfies the conditions of Definition IV.1.5.*

Proof. It follows from the fact that the functors Def_X and Def_Y of infinitesimal deformations of X and Y are deformation functors. □

IV.4 Tangent space

Definition IV.4.1. Let $F\colon \mathbf{Art} \to \mathbf{Set}$ be a deformation functor. The set

$$T^1F = F\left(\frac{\mathbb{K}[t]}{(t^2)}\right)$$

is called the *tangent space* of F.

Proposition IV.4.2. *The tangent space of a deformation functor has a natural structure of vector space over $\mathbb{K}$. For every natural transformation of deformation functors $F \to G$, the induced map $T^1F \to T^1G$ is linear.*

Proof. (See [98, Lemma 2.10]) Since $F(\mathbb{K})$ is just one point, by Condition 2. of Definition IV.1.5, there exists a bijection $F\left(\frac{\mathbb{K}[t]}{(t^2)} \times_{\mathbb{K}} \frac{\mathbb{K}[t]}{(t^2)}\right) \cong F\left(\frac{\mathbb{K}[t]}{(t^2)}\right) \times F\left(\frac{\mathbb{K}[t]}{(t^2)}\right)$.

Consider the map

$$+ : \frac{\mathbb{K}[t]}{(t^2)} \times_{\mathbb{K}} \frac{\mathbb{K}[t]}{(t^2)} \to \frac{\mathbb{K}[t]}{(t^2)},$$
$$(a + bt, a + b't) \longmapsto a + (b + b')t.$$

Then using the previous bijection, the map $+$ induces the addition on $F\left(\frac{\mathbb{K}[t]}{(t^2)}\right)$:

$$F\left(\frac{\mathbb{K}[t]}{(t^2)}\right) \times F\left(\frac{\mathbb{K}[t]}{(t^2)}\right) \overset{\cong}{\to} F\left(\frac{\mathbb{K}[t]}{(t^2)} \times_{\mathbb{K}} \frac{\mathbb{K}[t]}{(t^2)}\right) \overset{F(+)}{\to} F\left(\frac{\mathbb{K}[t]}{(t^2)}\right).$$

Analogously, for the multiplication by a scalar $k \in \mathbb{K}$ we consider the map:

$$k : \frac{\mathbb{K}[t]}{(t^2)} \to \frac{\mathbb{K}[t]}{(t^2)},$$
$$a + bt \longmapsto a + (kb)t.$$

It is an easy exercise to prove that the axioms of vector space are satisfied. The linearity of the map $T^1F \to T^1G$ induced by a natural transformation of deformation functors $F \to G$ follows by the definition of the $\mathbb{K}$-vector space structure on T^1F and T^1G. $\square$

It is notationally convenient to reserve the letter ϵ to denote elements of $A \in \mathbf{Art}$ annihilated by the maximal ideal $\mathfrak{m}_A$, and in particular of square zero.

Example IV.4.3. The tangent space of the functor h_R, defined in Example IV.1.4, is

$$T^1\,\mathrm{h}_R = \mathrm{Hom}_{\mathbb{K}-\mathrm{alg}}(R, \mathbb{K}[\epsilon]) = \mathrm{Hom}_{\mathbb{K}}\left(\frac{\mathfrak{m}_R}{\mathfrak{m}_R^2}, \mathbb{K}\right).$$

Therefore $T^1\,\mathrm{h}_R$ is isomorphic to the Zariski tangent space of $\mathrm{Spec}(R)$ at its closed point.

Definition IV.4.4. Given a functor F and R a local complete $\mathbb{K}$-algebra, R is said to be an *hull* for F if we are given a morphism $h_R \to F$ which is smooth and bijective on tangent spaces.

Remark IV.4.5. (Exercise) An hull, if it exists, is unique up to non-canonical isomorphism.

The notion of hull is a weaker version of prorepresentability and it is related to the notion of semiuniversal deformation. The majority of deformation functors arising in concrete cases are not proprepresentable but they admit an hull as it shown in the following theorem.

Theorem IV.4.6 (Schlessinger, [98])**.** *Let F be a deformation functor with finite dimensional vector space. Then, there exists a local complete noetherian $\mathbb{K}$-algebra R with residue field $\mathbb{K}$ and a smooth natural transformation $\mathrm{h}_R \to F$ inducing an isomorphism on tangent spaces $T^1\,\mathrm{h}_R = T^1F$. Moreover R is unique up to non-canonical isomorphism.*

Proof. We will prove later as a consequence of a more general statement (the factorization theorem). $\square$

Lemma IV.4.7. *Let $\eta\colon F \to G$ be a natural tranformation of deformation functors.*

1. *If G is homogeneous and $\eta\colon T^1F \to T^1G$ is injective, then $\eta\colon F(A) \to G(A)$ is injective for every A and F is homogeneous.*

2. *If F is smooth and $\eta\colon T^1F \to T^1G$ is surjective, then G is a smooth functor and η is a smooth morphism.*

Proof. Every small principal extension

$$0 \to \mathbb{K} \xrightarrow{\alpha} B \xrightarrow{\beta} A \to 0,$$

there exists an isomorphism

$$B \times_{\mathbb{K}} \mathbb{K}[\varepsilon] \to B \times_A B, \qquad (b, \bar{b} + k\varepsilon) \mapsto (b, b + k\alpha(\varepsilon))$$

and then, for every deformation functor G a surjective map

$$\theta \colon G(B) \times T^1G = G(B \times_{\mathbb{K}} \mathbb{K}[\epsilon]) \to G(B) \times_{G(A)} G(B)$$

commuting with the projection on the first factor and such that $\theta(x, 0) = (x, x)$. If G is homogeneous, then θ is bijective.

Assume now G homogeneous and $\eta \colon T^1F \to T^1G$ injective. We will prove by induction on the length of $B \in \mathbf{Art}$ that $\eta \colon F(B) \to G(B)$ is injective. Let $x, y \in F(B)$ such that $\eta(x) = \eta(y) \in G(B)$ and let

$$0 \to \mathbb{K} \xrightarrow{\alpha} B \xrightarrow{\beta} A \to 0$$

be a principal small extension. By induction $\beta(x) = \beta(y) \in F(A)$ and then there exists $v \in T^1F$ such that $\theta(x, v) = (x, y)$. Thus $\theta(\eta(x), \eta(v)) = (\eta(x), \eta(y))$ and, since G is homogeneous this implies $\eta(v) = 0$ and then $v = 0, x = y$. This proves that η is always injective the homogeneity of F is trivial.

Assume now F smooth and $\eta \colon T^1F \to T^1G$ surjective. We need to prove that for every principal small extension as above, the map

$$(\beta, \eta) \colon F(B) \to F(A) \times_{G(A)} G(B)$$

is surjective. Let $(x, y) \in F(A) \times_{G(A)} G(B)$, since F is smooth there exists $z \in F(B)$ such that $\beta(z) = x$; denoting $w = \eta(z)$ we have $(w, y) \in G(B) \times_{G(A)} G(B)$ and then there exists $v \in T^1G$ such that $\theta(w, v) = (w, y)$. Now $\eta \colon T^1F \to T^1G$ is surjective and then $v = \eta(u)$ and $\theta(z, u) = (z, r)$ with $\beta(r) = \beta(z) = x$ and $\eta(r) = y$. □

IV.5 Obstructions

In the set-up of functors of Artin rings, with the term obstructions we intend *obstructions for a deformation functor to be smooth.*

Definition IV.5.1. Let F be a functor of Artin rings. An *obstruction theory* (V, v_e) for F is the data of a $\mathbb{K}$-vector space V and for every small extension in $\mathbf{Art}$

$$e\colon \quad 0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

of an *obstruction map* $v_e \colon F(A) \to V \otimes M$ satisfying the following properties:

1. If $A = \mathbb{K}$ then $v_e(F(\mathbb{K})) = 0$.
2. (base change) For every commutative diagram

$$\begin{array}{ccccccccc}
e_1\colon & 0 & \longrightarrow & M_1 & \longrightarrow & B_1 & \longrightarrow & A_1 & \longrightarrow & 0 \\
& & & \downarrow{\scriptstyle \alpha_M} & & \downarrow{\scriptstyle \alpha_B} & & \downarrow{\scriptstyle \alpha_A} & & \\
e_2\colon & 0 & \longrightarrow & M_2 & \longrightarrow & B_2 & \longrightarrow & A_2 & \longrightarrow & 0.
\end{array}$$

with e_1, e_2 small extensions and α_A, α_B morphisms in $\mathbf{Art}$, we have

$$v_{e_2}(\alpha_A(a)) = (Id_V \otimes \alpha_M)(v_{e_1}(a)) \qquad \text{for every} \quad a \in F(A_1).$$

Remark IV.5.2. It has to be observed that, to give a morphism of sets $v_e \colon F(A) \to V \otimes M$ is the same that to give a map $v_e \colon F(A) \times M^\vee \to V$ such that $v_e(a, -) \colon M^\vee \to V$ is linear for every $a \in F(A)$.

The name *obstruction theory* is motivated by the following result.

Lemma IV.5.3. *Let (V, v_e) be an obstruction theory for a functor of Artin rings F, let*

$$e\colon \quad 0 \longrightarrow M \xrightarrow{\alpha} B \xrightarrow{\beta} A \longrightarrow 0$$

be a small extension and $x \in F(A)$. If x lifts to $F(B)$, i.e. if $x \in \beta(F(B))$, then $v_e(x) = 0$.

Proof. Assume $x = \beta(y)$ for some $y \in F(B)$ and consider the morphism of small extension

$$\begin{array}{cccccccccc}
e'\colon & 0 & \longrightarrow & M & \xrightarrow{(0,\alpha)} & B \times_A B & \xrightarrow{p_1} & B & \longrightarrow & 0 \\
& & & \Big\downarrow{\scriptstyle Id} & & \Big\downarrow{\scriptstyle p_2} & & \Big\downarrow{\scriptstyle \beta} & & \\
e\colon & 0 & \longrightarrow & M & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & A & \longrightarrow & 0.
\end{array}$$

where p_1 and p_2 are the projections. By base change property $v_e(x) = v_{e'}(y)$. Now consider the morphism of small extensions

$$\begin{array}{cccccccccc}
e'\colon & 0 & \longrightarrow & M & \xrightarrow{(0,\alpha)} & B \times_A B & \xrightarrow{p_1} & B & \longrightarrow & 0 \\
& & & \Big\downarrow{\scriptstyle Id} & & \Big\downarrow{\scriptstyle \gamma} & & \Big\downarrow{\scriptstyle \pi} & & \\
e''\colon & 0 & \longrightarrow & M & \xrightarrow{(0,\alpha)} & \mathbb{K} \oplus \alpha(M) & \xrightarrow{\beta} & \mathbb{K} & \longrightarrow & 0.
\end{array}$$

where $\pi\colon B \to \mathbb{K}$ is the projection and $\gamma(a, b) = (\pi(a), a - b)$. Again by base change property $v_{e'}(y) = v_{e''}(\pi(y)) = 0$. □

Definition IV.5.4. An obstruction theory (V, v_e) for F is called *complete* if the converse of Lemma IV.5.3 holds; i.e., the lifting exists if and only if the obstruction vanishes.

Clearly, if F admits a complete obstruction theory then it admits infinitely ones; it is in fact sufficient to embed V in a bigger vector space. One of the main interest (and problem) is to look for the "smallest"complete obstruction theory.

Remark IV.5.5. Let $e\colon\ 0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$ be a small extension and $a \in F(A)$; the obstruction $v_e(a) \in V \otimes M$ is uniquely determined by the values $(Id_V \otimes f)v_e(a) \in V$, where f varies along a basis of $\operatorname{Hom}_{\mathbb{K}}(M, \mathbb{K})$. On the other hand, by base change we have $(Id_V \otimes f)v_e(a) = v_\epsilon(a)$, where ϵ is the small extension

$$\epsilon: \quad 0 \longrightarrow \mathbb{K} \longrightarrow \frac{B \oplus \mathbb{K}}{\{(m, -f(m)) \mid m \in M\}} \longrightarrow A \longrightarrow 0.$$

This implies that every obstruction theory is uniquely determined by its behavior on principal small extensions.

Definition IV.5.6. A morphism of obstruction theories $(V, v_e) \to (W, w_e)$ is a linear map $\theta\colon V \to W$ such that $w_e = (\theta \otimes Id)v_e$, for every small extension e.
An obstruction theory (O_F, ob_e) for F is called *universal* if, for every obstruction theory (V, v_e), there exists a unique morphism $(O_F, ob_e) \to (V, v_e)$.

Theorem IV.5.7 ([18])**.** *Let F be a deformation functor, then:*

1. *There exists the universal obstruction theory (O_F, ob_e) for F, and such obstruction theory is complete.*

2. *Every element of the universal obstruction target O_F is of the form $ob_e(a)$, for some principal extension*

$$e\colon \quad 0 \longrightarrow \mathbb{K} \longrightarrow B \longrightarrow A \longrightarrow 0$$

   and some $a \in F(A)$.

Proof. The proof is quite long and it is postponed to Section IV.6 □

It is clear that the universal obstruction theory (O_F, ob_e) is unique up to isomorphism and depends only by F and not by any additional data.

Definition IV.5.8. The *obstruction space* of a deformation functor F is the universal obstruction target O_F.

Corollary IV.5.9. *Let (V, v_e) be a complete obstruction theory for a deformation functor F. Then, the obstruction space O_F is isomorphic to the vector subspace of V generated by all the obstructions arising from principal extensions.*

Proof. Denote by $\theta\colon O_F \to V$ the morphism of obstruction theories. Every principal obstruction is contained in the image of θ and, since V is complete, the morphism θ is injective. □

Remark IV.5.10. The majority of authors use Corollary IV.5.9 as a definition of obstruction space.

Example IV.5.11. Let R be a local complete $\mathbb{K}$-algebra with residue field $\mathbb{K}$ and $n = \dim T^1 \mathrm{h}_R = \dim \mathfrak{m}_R/\mathfrak{m}_R^2$ its embedding dimension. Then, we can write $R = P/I$, where $P = \mathbb{K}[[x_1, \dots, x_n]]$ and $I \subset \mathfrak{m}_P^2$. We claim that

$$T^2 \mathrm{h}_R := \mathrm{Hom}_P(I, \mathbb{K}) = \mathrm{Hom}_{\mathbb{K}}(I/\mathfrak{m}_P I, \mathbb{K})$$

is the obstruction space of h_R. In fact, for every small extension

$$e\colon \quad 0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

and every $\alpha \in \mathrm{h}_R(A)$, we can lift α to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & P & \longrightarrow & R & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle ob_a(\alpha)} & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & M & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

with β a morphism of $\mathbb{K}$-algebras. It is easy to verify that

$$ob_e(\alpha) = \beta_{|I} \in \mathrm{Hom}_{\mathbb{K}}(I/\mathfrak{m}_P I, M) = T^2 \mathrm{h}_R \otimes M$$

is well defined, it is a complete obstruction and that $(T^2 \mathrm{h}_R, ob_e)$ is the universal obstruction theory for the functor h_R (see [18, Prop. 5.3]).

Let $\phi\colon F \to G$ be a natural transformation of deformation functors. Then, $(O_G, ob_e \circ \phi)$ is an obstruction theory for F; therefore, there exists an unique linear map $ob_\phi\colon O_F \to O_G$ which is compatible with ϕ in the obvious sense.

Theorem IV.5.12 (Standard smoothness criterion)**.** *Let $\phi\colon F \to G$ be a morphism of deformation functors. The following conditions are equivalent:*

1. *ϕ is smooth.*
2. *$T^1\phi\colon T^1F \to T^1G$ is surjective and $ob_\phi\colon O_F \to O_G$ is bijective.*
3. *$T^1\phi\colon T^1F \to T^1G$ is surjective and $ob_\phi\colon O_F \to O_G$ is injective.*

Proof. In order to avoid confusion we denote by ob_e^F and ob_e^G the obstruction maps for F and G respectively.

[1 ⇒ 2] Every smooth morphism is in particular surjective; therefore, if ϕ is smooth then the induced morphisms $T^1F \to T^1G$, $O_F \to O_G$ are both surjective.
Assume that $ob_\phi(\xi) = 0$ and write $\xi = ob_e^F(x)$, for some $x \in F(A)$ and some small extension $e\colon 0 \longrightarrow \mathbb{K} \longrightarrow B \longrightarrow A \longrightarrow 0$. Since $ob_e^G(\phi(x)) = 0$, the element x lifts to a pair $(x, y') \in F(A) \times_{G(A)} G(B)$ and then the smoothness of ϕ implies that x lifts to $F(B)$.
[3 ⇒ 1] We need to prove that for every small extension $e\colon 0 \longrightarrow \mathbb{K} \longrightarrow B \longrightarrow A \longrightarrow 0$ the map

$$F(B) \to F(A) \times_{G(A)} G(B)$$

is surjective. Fix $(x, y') \in F(A) \times_{G(A)} G(B)$ and let $y \in G(A)$ be the common image of x and y'. Then $ob_e^G(y) = 0$ because y lifts to $G(B)$, hence $ob_e^F(x) = 0$ by injectivity of ob_ϕ. Therefore x lifts to some $x'' \in F(B)$. In general $y'' = \phi(x'')$ is not equal to y'. However, $(y'', y') \in G(B) \times_{G(A)} G(B)$ and therefore there exists $v \in T^1G$ such that $\theta(y'', v) = (y'', y')$ where

$$\theta\colon G(B) \times T^1G = G(B \times_{\mathbb{K}} \mathbb{K}[\epsilon]) \to G(B) \times_{G(A)} G(B)$$

is induced by the isomorphism

$$B \times_{\mathbb{K}} \mathbb{K}[\epsilon] \to B \times_A B, \qquad (b, \bar{b} + \alpha\epsilon) \mapsto (b, b + \alpha\epsilon).$$

By assumption $T^1F \to T^1G$ is surjective, v lifts to a $w \in T^1F$ and setting $\theta(x'', w) = (x'', x')$ we have that x' is a lifting of x which maps to y', as required. □

Remark IV.5.13. In most concrete cases, given a natural transformation $F \to G$ it is very difficult to calculate the map $O_F \to O_G$, while it is generally easy to describe complete obstruction theories for F and G and a compatible morphism between them. In this situation, only the implication $[3 \Rightarrow 1]$ of the standard smoothness criterion holds.

Corollary IV.5.14. *Let F be a deformation functor and $\mathrm{h}_R \to F$ a smooth natural transformation. Then, the dimension of O_F is equal to the minimum number of generators of an ideal I defining R as a quotient of a power series ring, i.e., $R = \mathbb{K}[[x_1, \ldots, x_n]]/I$.*

Proof. Apply Nakayama's lemma to the $\mathbb{K}[[x_1, \ldots, x_n]]$-module I and use Example IV.5.11. □

IV.6 Proof of Theorem IV.5.7

We need some care to avoid set theoretic difficulties. First of all, we work on a fixed universe. For every $n \geq 0$, choose a $\mathbb{K}$-algebra $\mathcal{O}_n$ isomorphic to the power series ring $\mathbb{K}[[x_1, \ldots, x_n]]$ and consider the category **Art** whose objects are Artinian quotients of the $\mathcal{O}_n$'s and morphisms are local morphisms of $\mathbb{K}$-algebras. Let **Fvsp** the category whose objects are $\{0, \mathbb{K}, \mathbb{K}^2, \ldots\}$ and morphisms are linear maps.

For a $V \in$ **Fvsp**, we denote by $V^\vee$ its $\mathbb{K}$-dual. If $A \in$ **Art**, we will denote by $\mathfrak{m}_A$ its maximal ideal.

By ε and ε_i we will always mean indeterminates annihilated by the maximal ideal, and in particular of square zero (e.g., the algebra $\mathbb{K}[\varepsilon]$ has dimension 2 and $\mathbb{K}[\varepsilon_1, \varepsilon_2]$ has dimension 3 as a $\mathbb{K}$-vector space).

Definition IV.6.1. A *small extension* e in **Art** is a short exact sequence

$$e: \quad 0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

where $B \to A$ is a morphism in **Art**, $M \in$ **Fvsp** and the image of $M \to B$ is annihilated by the maximal ideal of B. In the sequel of the paper, for every small extension e as above, we shall let $K(e) = M$, $S(e) = B$, $T(e) = A$ (the letters should be a reminder of kernel, source, target).

Definition IV.6.2. We denote by **Smex** the category whose objects are small extensions in **Art**. A morphism of small extensions $\alpha\colon e_1 \to e_2$ is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \longrightarrow & B_1 & \longrightarrow & A_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_M} & & \downarrow{\scriptstyle \alpha_B} & & \downarrow{\scriptstyle \alpha_A} & & \\
0 & \longrightarrow & M_2 & \longrightarrow & B_2 & \longrightarrow & A_2 & \longrightarrow & 0.
\end{array}$$

The category **Smex** is small, in the sense that the class of its objects is a set.

For $A \in$ **Art** and $M \in$ **Fvsp** let $\mathrm{Ex}(A, M)$ be the set isomorphism classes of small extensions of A with kernel M. Denote by $0 \in \mathrm{Ex}(A, M)$ the trivial extension

$$0: \quad 0 \longrightarrow M \longrightarrow A \oplus M \longrightarrow A \longrightarrow 0$$

where the product in $A \oplus M$ is $(a, m)(a', m') = (aa', a_0m' + a'_0m)$, and $a \to a_0$ is the quotient map $A \to \mathbb{K}$. A small extension is trivial if and only if it splits.

If $f : M \to N$ is a morphism in **Fvsp** and $\pi : C \to A$ is a morphism in **Art**, we shall denote by

$$f_* \colon \operatorname{Ex}(A, M) \to \operatorname{Ex}(A, N), \qquad \pi^* \colon \operatorname{Ex}(A, M) \to \operatorname{Ex}(C, M)$$

the induced maps, defined as follows:
Given an extension $e\colon\ 0 \to M \to B \to A \to 0$ in $\operatorname{Ex}(A, M)$, define f_*e as the extension

$$0 \to N \to \frac{B \oplus N}{\{(m, f(m)) \mid m \in M\}} \to A \to 0.$$

Define π^*e as the extension

$$0 \to M \to B \times_A C \to C \to 0.$$

Exercise IV.6.3. In the above set-up prove that $f_*\pi^* = \pi^* f_* \colon \operatorname{Ex}(A, M) \to \operatorname{Ex}(C, N)$.

Exercise IV.6.4. In the notation of Definition IV.6.2, prove that $\alpha_{M*}(e_1) = \alpha_A^*(e_2) \in \operatorname{Ex}(M_2, A_1)$.

Given two small extension

$$e_1 :\quad 0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0,$$

$$e_2 :\quad 0 \longrightarrow N \longrightarrow C \longrightarrow A \longrightarrow 0,$$

with the same target, we define $e_1 \oplus e_2 \in \operatorname{Ex}(A, M \times N)$ as

$$e_1 \oplus e_2 :\quad 0 \to M \times N \to B \times_A C \to A \to 0.$$

We have a natural structure of vector space on $\operatorname{Ex}(M, A)$ where the sum is defined as

$$e_1 + e_2 = +_*(e_1 \oplus e_2), \qquad \text{where} \quad +\colon M \times M \to M$$

and the scalar multiplication is induced by the corrsponding operation on M.
Let F be a deformation functor; for $A \in$ **Art**, $M \in$ **Fvsp** and $a \in F(A)$ define

$$F(A, M, a) = \{e \in \operatorname{Ex}(A, M) \mid a \text{ lifts to } F(S(e))\}.$$

Lemma IV.6.5. *Let F be a deformation functor, then:*

1. *For $A \in$ **Art** and $a \in F(A)$ we have*

$$F(A, M, a) \oplus F(A, N, a) \subset F(A, M \oplus N, a)$$

*for every $M, N \in$ **Fvsp**.*

2. *For $A \in$ **Art**, $a \in F(A)$ and $f\colon M \to N$*

$$f_*F(A, M, a) \subset F(A, N, a).$$

3. *For $A \in$ **Art**, $a \in F(A)$ and $\pi\colon A \to B$*

$$\pi^*F(B, M, \pi(a)) = F(A, M, a).$$

In particular, $F(A, M, a)$ is a vector subspace of $\operatorname{Ex}(A, M)$.

Proof. Immediate from the definition of deformation functors. □

Lemma IV.6.6. *Let F be a deformation functor, $A \in$ **Art**, $M \in$ **Fvsp**, $e \in \operatorname{Ex}(A, M)$ and $a \in F(A)$. Then*

$$e \in F(A, M, a) \text{ if and only if } \quad f_*e \in F(A, \mathbb{K}, a) \text{ for every } f \in M^\vee.$$

Proof. Let e be the small extension

$$0 \longrightarrow M \longrightarrow B \xrightarrow{\pi} A \longrightarrow 0$$

and assume $f_*e \in F(A, \mathbb{K}, a)$ for every $f \in M^\vee$. We prove that a lifts to $F(B)$ by induction on $\dim_{\mathbb{K}} M$; If $\dim M = 1$ there is nothing to prove.
Assume $\dim M > 1$ and let $f \in M^\vee$ with proper kernel $N \subset M$. Consider the following small extensions and morphisms:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & B & \xrightarrow{\pi'} & A' & \longrightarrow & 0 \\
 & & \Big\downarrow i & & \| & & \Big\downarrow \delta & & \\
0 & \longrightarrow & M & \longrightarrow & B & \xrightarrow{\pi} & A & \longrightarrow & 0 \\
 & & \Big\downarrow f & & \Big\downarrow \pi' & & \| & & \\
0 & \longrightarrow & \mathbb{K} & \longrightarrow & A' & \xrightarrow{\delta} & A & \longrightarrow & 0
\end{array}$$

where the bottom row is $f_*(e)$; call e' the top row. We have $i_*e' = \delta^*e$ and then, for every $h \in M^\vee$ we have $\delta^*h_*e = h_*\delta^*e = h_*i_*e'$.
By assumption a lifts to some $a' \in F(A')$ and, since $M^\vee \to N^\vee$ is surjective, we may apply the inductive assumption to the small extension e' and then a' lifts to $F(B)$. □

Lemma IV.6.7. *Let F be a deformation functor, and let $f, g : B \to A$ be morphisms in* **Art**. *Assume that $b \in F(B)$ and $f(b) = g(b) = a \in F(A)$. Then*

$$f^* = g^* : \frac{\mathrm{Ex}(A, M)}{F(A, M, a)} \hookrightarrow \frac{\mathrm{Ex}(B, M)}{F(B, M, b)}.$$

Proof. The injectivity is clear since $f^*F(A, M, a) = g^*F(A, M, a) = F(B, M, b)$. Let

$$e: \qquad 0 \xrightarrow{i} M \to C \xrightarrow{p} A \to 0$$

be a small extension. We want to prove that $f^*e - g^*e \in F(B, M, b)$.
Consider the small extension

$$\nabla: \quad 0 \longrightarrow M \xrightarrow{(i,0)=(0,-i)} D = \frac{C \times_{\mathbb{K}} C}{\{(m, m) \mid m \in M\}} \xrightarrow{(p,p)} A \times_{\mathbb{K}} A \longrightarrow 0$$

and the morphism $\phi\colon B \to A \times_{\mathbb{K}} A$, $\phi(x) = (f(x), g(x))$. Then $f^*e - g^*e = \phi^*\nabla$ and then it is sufficient to prove that $\nabla \in F(A \times_{\mathbb{K}} A, M, \phi(b))$, i.e. that $\phi(b)$ lifts to D. Since $F(A \times_{\mathbb{K}} A) \to F(A) \times F(A)$ is bijective, we must have $\phi(b) = \delta(a)$, where $\delta\colon A \to A \times_{\mathbb{K}} A$ is the diagonal. It is now sufficient to observe that δ lifts to a morphism $A \to D$. □

Definition IV.6.8. Let F be a deformation functor. For every $A \in$ **Art** and $a \in F(A)$ denote by

$$H(A, a) = \frac{\mathrm{Ex}(A, \mathbb{K})}{F(A, \mathbb{K}, a)}.$$

Denote also by $\mathbf{O}_F$ the subcategory of $\mathbf{Vect}_{\mathbb{K}}$ with objects the $H(A, a)$'s, for $A \in$ **Art** and $a \in F(A)$, and morphisms the injective linear maps $f^*\colon H(A, a) \to H(B, b)$, where $f\colon B \to A$ is a morphism in **Art** such that $f(b) = a$.

The category $\mathbf{O}_F$ is filtrant. This means that [51, Def. 1.11.2]:

1. Given morphisms $H(A, a) \to H(B, b)$ and $H(A, a) \to H(C, c)$, there exist morphisms $H(B, b) \to H(S, s)$ and $H(C, c) \to H(S, s)$ such that the resulting diagram is commutative.
2. Given two morphisms $f^*, g^*\colon H(A, a) \to H(B, b)$ there exist a morphism $H(B, b) \to H(C, c)$ such that the composed morphisms $H(A, a) \to H(C, c)$ coincide.

Moreover, it is required that I is nonempty and connected.
The Lemma IV.6.7 says that 2) holds in the stronger sense that, given two objects, there is at most one morphism between them. In view of this, 1) is equivalent to saying that, given any two objects, there is a third to which they both map (the commutativity of the diagram is ensured by IV.6.7). Given B, C in **Art** and elements $b \in F(B)$, $c \in F(C)$, take $S = B \times_{\mathbb{K}} C$; since F is a deformation functor there exists $s \in F(S)$ mapping to $b \in F(B)$ and to $c \in F(C)$.

Since $\mathbf{O}_F$ is filtrant, the colimit construction interchanges with the forgetful functor $\mathbf{Vect}_{\mathbb{K}} \to \mathbf{Set}$, i.e., the set

$$O_F := \operatorname{colim} \mathbf{O}_F = \bigcup_{\mathbf{O}_F} H(A,a)/\sim,$$

where $\sim$ is the equivalence relation generated by $v \sim f^*v$, is a vector space over $\mathbb{K}$ and the natural maps $H(A,a) \to O_F$ are injective morphisms of vector spaces.
The space O_F is the obstruction space of an obstruction theory (O_F, ob_e), where for every small extension

$$e: \quad 0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

we define $ob_e\colon F(A) \times M^{\vee} \to O_F$ by $ob_e(a,f) = \theta(e)$, where θ is the composition map

$$\theta\colon \operatorname{Ex}(A,M) \xrightarrow{f_*} \operatorname{Ex}(A,\mathbb{K}) \to H(A,a) \hookrightarrow O_F.$$

It is straightforward to verify that (O_F, ob_e) is an obstruction theory, while Lemma IV.6.6 tell us that it is a complete obstruction.
Finally, the universal property of colimits gives the universality of (O_F, ob_e).

Chapter V

Lie algebras

Let A be a commutative ring, by a nonassociative (= not necessarily associative) A-algebra we mean a A-module R endowed with a A-bilinear map $R \times R \to R$. Keep in mind that, even if R is not commutative, we have $ar = ra$ for every $a \in A$ and $r \in R$.

- A nonassociative algebra R is called **unitary** if there exist a "unity" $1 \in R$ such that $1r = r1 = r$ for every $r \in R$.
- A **left ideal** (resp.: **right ideal**) of R is a submodule $I \subset R$ such that $RI \subset I$ (resp.: $IR \subset I$). A submodule is called an **ideal** if it is both a left and right ideal.
- A morphism of A-modules $d\colon R \to R$ is called an A-**derivation** if:
  1. $d(ar) = ad(r)$ for $a \in A$ and $r \in R$;
  2. $d(rs) = d(r)s + rd(s)$ for $r, s \in R$ (Leibniz rule).
- An algebra R is **associative** if $(ab)c = a(bc)$ for every $a, b, c \in R$. Unless otherwise specified, we reserve the simple term **algebra** only to associative algebra.

V.1 Lie algebras

From now on $\mathbb{K}$ will be a fixed field of characteristic $\neq 2$.

Definition V.1.1. A vector space L over $\mathbb{K}$, with an operation $L \times L \to L$, denoted $(x, y) \mapsto [x, y]$ and called the **bracket** of x and y, is called a **Lie algebra** if the following axioms are satisfied:

1. The bracket operation is bilinear.
2. $[x, y] = -[y, x]$ for all x, y in L.
3. (Jacobi identity) $[[x, y], z]] = [x, [y, z]] - [y, [x, z]]$ $(x, y, z \in L)$.

A Lie algebra L is called **abelian** if $[x, y] = 0$ for every $x, y \in L$.

Notice that, since $\operatorname{char}\mathbb{K} \neq 2$, we have $[x, x] = -[x, x] = 0$ for every $x \in L$. A linear subspace $H \subset L$ is called a **subalgebra** if $[x, y] \in H$ whenever $x, y \in K$; in particular, K is a Lie algebra in its own right relative to the inherited operations. Note that any nonzero element $x \in L$ defines a one dimensional abelian subalgebra $H = \mathbb{K}x$.

Example V.1.2. The space $\operatorname{End}_{\mathbb{K}}(V)$ of all linear endomorphisms of a vector space V is a Lie algebra with bracket $[f, g] = fg - gf$. If V is finite dimensional then the subspace $\operatorname{sl}(V) \subset \operatorname{End}(V)$ of endomorphisms with trace equal to 0 is a Lie subalgebra. For any $n > 0$ we denote $\operatorname{sl}_n(\mathbb{K}) = \operatorname{sl}(\mathbb{K}^n)$

Definition V.1.3. A morphism of Lie algebras $f\colon L \to M$ is a linear map commuting with brackets. An isomorphims of Lie algebras is a morphism of Lie algebras which is also an isomorphims of vector spaces.

For every associative $\mathbb{K}$-algebra R we denote by R_L the associated Lie algebra, with bracket equal to the commutator, $[a,b] = ab - ba$; the verification of the following properties is easy and left as an exercise:

1. if $I \subset R$ is an ideal then I_L is a Lie ideal of R_L.

2. if $f\colon R \to R$ is a derivation, then also $f\colon R_L \to R_L$ is a derivation.

As expected not every Lie bracket is the commutator of an associative product.

Proposition V.1.4. *Let $\mathbb{K}$ be a field of characteristic $\neq 2$. Then does not exist any associative product in $\mathrm{sl}_2(\mathbb{K})$ such that $[x,y] = xy - yx$.*

Proof. The canonical basis of the Lie algebra $\mathrm{sl}_2(\mathbb{K})$ is given by the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and we have

$$[A,B] = H, \qquad [H,A] = 2A, \qquad [H,B] = -2B.$$

Assume that there exists an associative product, then

$$AB - BA = H, \qquad HA - AH = 2A, \qquad HB - BH = -2B.$$

Writing

$$H^2 = \gamma_1 A + \gamma_2 B + \gamma H$$

we have

$$0 = [H^2, H] = \gamma_1[A,H] + \gamma_2[B,H]$$

and therefore $\gamma_1 = \gamma_2 = 0$, $H^2 = \gamma H$. Possibly acting with the Lie automorphism

$$A \mapsto B, \qquad B \mapsto A, \qquad H \mapsto -H,$$

it is not restrictive to assume $\gamma \neq -1$.

Since $[AH,H] = [A,H]H = -2AH$, writing $AH = xA + yB + zH$ for some $x,y,z \in \mathbb{K}$ we have

$$0 = [AH,H] + 2AH = x[A,H] + y[B,H] + 2xA + 2yB + 2zH = 4yB + 2zH$$

giving $y = z = 0$ and $AH = xA$. Moreover $2A^2 = A[H,A] = [AH,A] = [xA,A] = 0$ and then $A^2 = 0$.

Since

$$0 = A(H^2) - (AH)H = \gamma AH - xAH = (\gamma x - x^2)A$$

we have either $x = 0$ or $x = \gamma$. In both cases $x \neq -1$ and then $AH + HA = (2x+2)A \neq 0$. This gives a contradiction since

$$-AH = A(AB - H) = ABA = (BA + H)A = HA.$$

□

Example V.1.5. Let R be a nonassociative algebra over $\mathbb{K}$. Then the vector space

$$\mathrm{Der}_{\mathbb{K}}(R,R) = \{d \in \mathrm{Hom}_{\mathbb{K}}(R,R) \mid d(rs) = (dr)s + r(ds),\ \forall\, r,s \in R\}$$

of $\mathbb{K}$-derivations of R is a Lie subalgebra of $\mathrm{End}_{\mathbb{K}}(V) = \mathrm{Hom}_{\mathbb{K}}(R,R)$.

Example V.1.6. Let L be a Lie algebra over a field $\mathbb{K}$ and A a commutative and associative $\mathbb{K}$-algebra. Then the tensor product $L \otimes_{\mathbb{K}} A$ is a Lie algebra with bracket equal to the bilinear extension of

$$[u \otimes a, v \otimes b] = [u,v] \otimes ab.$$

A **representation** of a Lie algebra L on a vector space V is a morphism of Lie algebras $\phi\colon L \to \mathrm{End}(V)$.

Example V.1.7. The adjoint representation of a Lie algebra L is the homomorphism

$$\mathrm{ad} : L \to \mathrm{End}(L), \qquad \mathrm{ad}\, x(y) = [x, y].$$

It is clear that ad is a linear transformation. To see that it preserves the bracket, we calculate:

$$[\mathrm{ad}\, x, \mathrm{ad}\, y](z) = \mathrm{ad}\, x(\mathrm{ad}\, y(z)) - \mathrm{ad}\, y(\mathrm{ad}\, x(z)) = \mathrm{ad}\, x([y, z]) - \mathrm{ad}\, y([x, z]) =$$

$$= [x, [y, z]] - [y, [x, z]] = [[x, y], z] = \mathrm{ad}\, [x, y](z).$$

It is useful to recall the notion of **universal enveloping algebra**: for proofs and more details we refer to [43, 50]. For every Lie algebra L, there exists an associative algebra $U(L)$ together a Lie embedding $L \subset U(L)$ with the following universal property: for every associative algebra R and every Lie morphism $f\colon L \to R$ there exists a unique morphism of associative algebras $F\colon U(L) \to R$ extending f.

V.2 Nilpotent Lie algebras

For a Lie algebra H we denote $[a, b, c] = [a, [b, c]]$ and more generally

$$[a_1, \dots, a_n] = [a_1, [a_2, \dots, a_n]] = [a_1, [a_2, [a_3, \dots, [a_{n-1}, a_n] \dots] \,.$$

Definition V.2.1. The **descending central series** $H^{[n]}$ of a Lie algebra H is defined as

$$H^{[n]} = \mathrm{Span}\{[a_1, \dots, a_n]\}, \qquad a_1, \dots, a_n \in H, \quad n \geq 1.$$

Clearly $H^{[1]} = H$ and $H^{[n]} = [H, H^{[n-1]}]$, where we have used the notation that

$$[U, V] = \mathrm{Span}\{[u, v] \mid u \in U,\ v \in V\}.$$

Lemma V.2.2. *In the notation above we have.*

1. $H^{[n+1]} \subseteq H^{[n]}$ *for every* $n > 0$*;*
2. $H^{[n]}$ *is a Lie ideal of* H *for every* n*;*
3. $[H^{[n]}, H^{[m]}] \subset H^{[n+m]}$ *for every* n, m.

Proof. All the above item are trivially true for $n = 1$: assume then $n > 1$ and proceed by induction on n. We have $H^{[n+1]} = [H, H^{[n]}] \subset [H, H^{[n-1]}] = H^{[n]}$. In particular for every n we have $[H, H^{[n]}] \subset H^{[n]}$ and then $H^{[n]}$ is a Lie ideal.

Assume now already proved that $[H^{[a]}, H^{[m]}] \subset H^{[a+m]}$ for every $a < n$. The vector space $[H^{[n]}, H^{[m]}]$ is generated by the vectors

$$[[x, y], z], \qquad x \in I, y \in H^{[n-1]}, z \in H^{[m]}.$$

By Jacobi identity

$$[[x, y], z] = [x, [y, z]] - [y, [x, z]] \in [H, [H^{[n-1]}, H^{[m]}]] + [H^{[n-1]}, [H, H^{[m]}]]$$

and by induction

$$[[x, y], z] \in [H, H^{[n+m-1]}] + [H^{[n-1]}, H^{[m+1]}] \subset H^{[n+m]}.$$

□

Definition V.2.3. A Lie algebra H is called **nilpotent** if $H^{[n]} = 0$ for $n >> 0$.

Example V.2.4. If $\mathfrak{m}$ is the maximal ideal of a local Artinian ring, then $L \otimes \mathfrak{m}$ is nilpotent for every Lie algebra L; moreover it is a Lie subalgebra of the nilpotent associative algebra $U(L) \otimes \mathfrak{m}$.

Given two elements a, b in a nilpotent Lie algebra L over a field of characteristic 0, we define a sequence of elements $Z_n = Z_n(a, b) \in L$ by the recursive equation

$$Z_0 = b, \qquad Z_{r+1} = \frac{1}{r+1}\sum_{m\geq 0}\frac{B_m}{m!}\sum_{i_1+\cdots+i_m=r}[Z_{i_1}, Z_{i_2}, \ldots, Z_{i_m}, a]\,,$$

where $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, \ldots$ are the Bernoulli numbers, i.e. the rational numbers defined by the generating function

$$\sum_{n\geq 0}\frac{B_n}{n!}t^n = \frac{t}{e^t - 1}.$$

Definition V.2.5. For a nilpotent Lie algebra L over a field of characteristic 0 the map

$$L \times L \xrightarrow{\bullet} L, \qquad a \bullet b = \sum_{n\geq 0} Z_n(a, b),$$

is called **Baker-Campbell-Hausdorff** (BCH) product.

It is easy to check that the first terms of BCH product are

$$a \bullet b = a + b + \frac{1}{2}[a, b] + \frac{1}{12}[a, [a, b]] + \frac{1}{12}[b, [a, b]] + \cdots$$

It is plain that $\bullet$ commutes with morphisms of Lie algebras and that $a \bullet b - a - b$ belongs in the Lie ideal generated by $[a, b]$. The geometric meaning of the BCH product will be described in next section.

V.3 Exponential and logarithm

From now on, $\mathbb{K}$ will be a field of characteristic 0. Let R be a unitary associative $\mathbb{K}$-algebra and $I \subset R$ a nilpotent ideal. We may define the **exponential**

$$e\colon I \to 1 + I \subset R, \quad e^a = \sum_{n\geq 0}\frac{a^n}{n!},$$

and the **logarithm**

$$\log\colon 1 + I \to I, \quad \log(1 + a) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{a^n}{n}.$$

Lemma V.3.1. *Exponential and logarithm are one the inverse of the other, i.e. for every $a, b \in I$ we have*

$$\log(e^a) = a, \qquad e^{\log(1+b)} = 1 + b.$$

Proof. Using the morphism of associative algebras

$$\mathbb{Q}[[t]] \to R, \qquad p(t) \mapsto p(a)$$

and the embedding $\mathbb{Q}[[t]] \subset \mathbb{R}[[t]]$ the proof is reduced to well known facts of calculus. $\square$

Proposition V.3.2. *In the notation above:*

1. *for every $a, b \in R$ and $n \geq 0$*

$$(\operatorname{ad} a)^n b = \sum_{i=0}^{n}(-1)^i\binom{n}{i}a^{n-i}ba^i = \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b(-a)^i.$$

2. *If a is nilpotent in R then also $\operatorname{ad} a$ is nilpotent in $\operatorname{End}(R)$ and therefore it is well defined the invertible operator*
$$e^{\operatorname{ad} a} = \sum_{n\ge 0} \frac{(\operatorname{ad} a)^n}{n!} \in \operatorname{End}(R).$$

3. *For every $a \in I$ and $b \in R$*
$$e^{\operatorname{ad} a} b := \sum_{n\ge 0} \frac{(\operatorname{ad} a)^n}{n!} b = e^a b e^{-a}.$$

4. *For every $a \in I$ and $b \in R$ we have $ab = ba$ if and only if $e^a b = b e^a$.*

5. *For every $a, b \in I$ we have $e^a b = b e^a$ if and only if $e^a e^b = e^b e^a$.*

6. *Given $a, b \in I$ such that $ab = ba$, then*
$$e^{a+b} = e^a e^b = e^b e^a, \qquad \log((1+a)(1+b)) = \log(1+a) + \log(1+b).$$

Proof. [1] We have
$$(\operatorname{ad} a)^n b = a(\operatorname{ad} a)^{n-1}(b) - (\operatorname{ad} a)^{n-1}(b)a$$
By induction
$$(\operatorname{ad} a)^n b = \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} a^{n-i} b a^i - \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{i} a^{n-1-j} b a^{j+1}.$$
Setting $j = i-1$ on the second summand we get
$$(\operatorname{ad} a)^n b = \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} a^{n-i} b a^i + \sum_{i=1}^{n} (-1)^i \binom{n-1}{i-1} a^{n-i} b a^i =$$
$$= \sum_{i=0}^{n} (-1)^i \left(\binom{n-1}{i} + \binom{n-1}{i-1} \right) a^{n-i} b a^i = \sum_{i=0}^{n} (-1)^i \binom{n}{i} a^{n-i} b a^i.$$
[2] If $a^n = 0$ then $(\operatorname{ad} a)^{2n} = 0$.
[3] Using item 1 we get
$$e^{\operatorname{ad} a} b := \sum_{n\ge 0} \frac{(\operatorname{ad} a)^n}{n!} b = \sum_{n\ge 0} \sum_{i=0}^{n} \frac{1}{n!} \binom{n}{i} a^{n-i} b (-a)^i$$
Setting $j = n - i$ we get
$$e^{\operatorname{ad} a} b := \sum_{i,j\ge 0} \frac{1}{i!j!} a^j b (-a)^i = e^a b e^{-a}.$$
[4] We have $e^a b = b e^a$ if and only if $e^a b e^{-a} - b = 0$ and by the above formula
$$e^a b e^{-a} - b = e^{\operatorname{ad} a} b - b = \frac{e^{\operatorname{ad} a} - 1}{\operatorname{ad} a}([a,b]).$$
[5] Setting $x = e^b$ we have by item 4 applied twice
$$e^a e^b = e^b e^a \iff x e^a = e^a x \iff ax = xa \iff a e^b = e^b a \iff ab = ba.$$
[6] Since $ab = ba$ we have for every $n \ge 0$
$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i},$$

$$e^{a+b}=\sum_{n\geq 0}\frac{(a+b)^n}{n!}=\sum_{n\geq 0}\sum_{i=0}^{n}\frac{1}{n!}\binom{n}{i}a^ib^{n-i}=\sum_{n\geq 0}\sum_{i=0}^{n}\frac{1}{i!(n-i)!}a^ib^{n-i}.$$

Setting $j=n-i$ we get

$$e^{a+b}=\sum_{i,j\geq 0}\frac{1}{i!j!}a^ib^j=e^ae^b.$$

Setting $x=\log(1+a)$, $y=\log(1+b)$ we have $e^xe^y=e^ye^x$: therefore $xy=yx$ and

$$\log((1+a)(1+b))=\log(e^xe^y)=\log(e^{x+y})=x+y=\log(1+a)+y=\log(1+b).$$

□

Notice that for every $a\in I$ the operator

$$\frac{e^{\operatorname{ad}a}-1}{\operatorname{ad}a}=\sum_{n\geq 0}\frac{(\operatorname{ad}a)^n}{(n+1)!}\in \operatorname{End}(R)$$

is invertible and its inverse is

$$\frac{\operatorname{ad}a}{e^{\operatorname{ad}a}-1}=\sum_{n\geq 0}\frac{B_n}{n!}(\operatorname{ad}a)^n\,.$$

Let t be an indeterminate and denote by $R[t]$ the algebra of polinomials in the central variable t: more explicitely $R[t]=\oplus_{n\geq 0}Rt^n$ and $(at^i)(bt^j)=abt^{i+j}$ for $a,b\in R$ and $i,j\geq 0$.

Denote by $d\colon R[t]\to R[t]$, $d(a)=a'$, the derivation operator. Multiplication on the left give an injective morphism of algebras

$$\phi\colon R[t]\to \operatorname{End}_{\mathbb{K}}(R[t]),\qquad \phi(a)b=ab$$

and Leibniz formula can be written as

$$\phi(a')=[d,\phi(a)],\qquad a\in R[t].$$

Given $a\in I[t]$ we have $\phi(e^a)=e^{\phi(a)}$ and

$$\phi((e^a)')=de^{\phi(a)}-e^{\phi(a)}d$$

By the above proposition

$$-\phi((e^a)'e^{-a})=e^{\phi(a)}de^{-\phi(a)}-d=\frac{e^{\operatorname{ad}\phi(a)}-1}{\operatorname{ad}\phi(a)}([\phi(a),d])=-\frac{e^{\operatorname{ad}\phi(a)}-1}{\operatorname{ad}\phi(a)}(\phi(a')),$$

and then, since ϕ is injective

$$(e^a)'e^{-a}=\frac{e^{\operatorname{ad}a}-1}{\operatorname{ad}a}(a').$$

Now, let $a,b\in I$ and define

$$Z=\log(e^{ta}e^b)\in I[t].$$

We have $Z=Z_0+tZ_1+\cdots+t^nZ_n+\cdots$, with $Z_0=b$ and $Z_n\in I^n$. By derivation formula we have

$$(e^Z)'e^{-Z}=\frac{e^{\operatorname{ad}Z}-1}{\operatorname{ad}Z}(Z'),$$

$$(e^Z)'e^{-Z}=(e^{ta}e^b)'e^{-b}e^{-ta}=(e^{ta})'e^{-ta}=a.$$

Therefore Z is the solution of the Cauchy problem

$$Z'=\sum_{n\geq 0}\frac{B_n}{n!}(\operatorname{ad}Z)^n(a),\qquad Z(0)=Z_0=b.$$

The coefficients Z_n can be computed recursively

$$Z_{r+1}=\frac{1}{r+1}\sum_{m\geq 0}\frac{B_m}{m!}\sum_{i_1+\cdots+i_m=r}(\operatorname{ad}Z_{i_1})(\operatorname{ad}Z_{i_2})\cdots(\operatorname{ad}Z_{i_m})a$$

and, since $e^ae^b=e^{Z(1)}$ we have the following result.

Theorem V.3.3. *Let I be a nilpotent ideal of an associative algebra, then for every $a, b \in I$ we have*

$$e^a e^b = e^{a \bullet b},$$

where $\bullet$ is the BCH product.

Since $(e^a e^b)e^c = e^a(e^b e^c)$ we obtain immediately that the product $I \times I \xrightarrow{\bullet} I$ is associative. The same argument proves that, if L is a Lie subalgebra of a nilpotent ideal of a unitary associative algebra R then $e^{a\bullet b} = e^a e^b$ and $\bullet \colon L \times L \to L$ is associative. Moreover if $L \to M$ is a surjective morphism of nilpotent Lie algebras and $\bullet$ is associative in L, then it is also associative in M.

We will prove later, using free Lie algebras, that **every nilpotent Lie algebra is a quotient of a Lie algebra contained in a nilpotent associative algebra.** This implies that $\bullet$ is always associative.

Definition V.3.4. For a nilpotent Lie algebra L we denote by $\exp(L) = \{e^a \mid a \in L\}$ the set of formal exponents of elements of L. It is a group with product

$$\exp(L) \times \exp(L) \to \exp(L), \qquad e^a e^b = e^{a \bullet b},$$

with unit e^0 and inverse $(e^a)^{-1} = e^{-a}$.

We have the functorial properties:

1. If $f \colon L \to M$ is a morphism of nilpotent Lie algebras, then the map

$$f \colon \exp(L) \to \exp(M), \qquad f(e^a) = e^{f(a)},$$

   is a homomorphism of groups.

2. Let V be a vector space and $f \colon L \to \mathrm{End}(V)$ a Lie algebra morphism. If the image of L is contained in a nilpotent ideal, then the maps

$$\exp(L) \times V \to V, \qquad (e^a, v) \mapsto e^{f(a)}v,$$

$$\exp(L) \times \mathrm{End}(V) \to \mathrm{End}(V), \qquad (e^a, g) \mapsto e^{f(a)} g e^{-f(a)} = e^{\mathrm{ad}\, f}(g),$$

   are right actions.

V.4 (Co)Simplicial and Semi(co)simplicial objects

Let $\mathbf{\Delta}$ be the category of finite ordinals: the objects are objects are $[0] = \{0\}$, $[1] = \{0, 1\}$, $[2] = \{0, 1, 2\}$ ecc. and morphisms are the non decreasing maps.

Finally $\mathbf{\Delta}_{\mathrm{mon}}$ is the category with the same objects as above and whose morphisms are order-preserving injective maps among them.

Every morphism in $\mathbf{\Delta}_{\mathrm{mon}}$, different from the identity, is a finite composition of *face* morphisms:

$$\partial_k \colon [i-1] \to [i], \qquad \partial_k(p) = \begin{cases} p & \text{if } p < k \\ p+1 & \text{if } k \le p \end{cases}, \qquad k = 0, \dots, i.$$

Equivalently ∂_k is the unique strictly monotone map whose image misses k. The relations about compositions of them are generated by

$$\partial_l \partial_k = \partial_{k+1} \partial_l\,, \qquad \text{for every } l \le k.$$

Definition V.4.1 ([17, 113])**.** Let $\mathbf{C}$ be a category:

1. A *cosimplicial* object in $\mathbf{C}$ is a covariant functor $A^\Delta \colon \mathbf{\Delta} \to \mathbf{C}$.
2. A *semicosimplicial* object in $\mathbf{C}$ is a covariant functor $A^\Delta \colon \mathbf{\Delta}_{mon} \to \mathbf{C}$.
3. A *simplicial* object in $\mathbf{C}$ is a contravariant functor $A_\Delta \colon \mathbf{\Delta} \to \mathbf{C}$.

4. A *semisimplicial* object in $\mathbf{C}$ is a contravariant functor $A_\Delta \colon \mathbf{\Delta}_{mon} \to \mathbf{C}$.

Notice that a semicosimplicial object A^Δ is a diagram in $\mathbf{C}$:

$$A_0 \rightrightarrows A_1 \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} A_2 \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots,$$

where each A_i is in $\mathbf{C}$, and, for each $i > 0$, there are $i+1$ morphisms

$$\partial_k \colon A_{i-1} \to A_i, \qquad k = 0, \ldots, i,$$

such that $\partial_l \partial_k = \partial_{k+1}\partial_l$, for any $l \le k$.

Example V.4.2. Let $\mathbb{K}$ be a field. Define the standard n-simplex over $\mathbb{K}$ as the affine space

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{K}^{n+1} \mid t_0 + t_1 + \cdots + t_n = 1\}.$$

The vertices of Δ^n are the points

$$e_0 = (1, 0, \ldots, 0), \quad e_1 = (0, 1, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 1).$$

Then the family $\{\Delta^n\}$, $n \ge 0$, is a cosimplicial affine space, where for every monotone map $f \colon [n] \to [m]$ we set $f \colon \Delta^n \to \Delta^m$ as the affine map such that $f(e_i) = e_{f(i)}$. Equivalently $f(t_0, \ldots, t_n) = \sum t_i e_{f(i)} = (u_0, \ldots, u_m)$, where

$$u_i = \sum_{\{j \mid f(j) = i\}} t_j \qquad \text{(we intend that } \sum_{\emptyset} t_j = 0\text{).}$$

In particular, for $m = n+1$ we have

$$\partial_k(t_0, \ldots, t_n) = (t_0, \ldots, t_{k-1}, 0, t_k, \ldots, t_n),$$

and this explain why ∂_k is called face map.

Given a semicosimplicial abelian group

$$V^\Delta: \qquad V_0 \rightrightarrows V_1 \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} V_2 \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots,$$

the map

$$\delta \colon V_i \to V_{i+1}, \qquad \delta = \sum_i (-1)^i \partial_i$$

satisfies $\delta^2 = 0$.

Definition V.4.3. The **normal complex** of a semicosimplicial abelian group V^Δ is the complex

$$N(V^\Delta): \qquad V_0 \xrightarrow{\delta} V_1 \xrightarrow{\delta} V_2 \to \cdots$$

Example V.4.4. Let $\mathcal{L}$ be a sheaf of abelian groups on a topological space X, and $\mathcal{U} = \{U_i\}$ an open covering of X; it is naturally defined the semicosimplicial abelian group of Čech alternating cochains:

$$\mathcal{L}(\mathcal{U}): \qquad \prod_i \mathcal{L}(U_i) \rightrightarrows \prod_{i<j} \mathcal{L}(U_{ij}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \prod_{i<j<k} \mathcal{L}(U_{ijk}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots,$$

where the coface maps $\partial_h \colon \displaystyle\prod_{i_0 < \cdots < i_{k-1}} \mathcal{L}(U_{i_0 \cdots i_{k-1}}) \to \prod_{i_0 < \cdots < i_k} \mathcal{L}(U_{i_0 \cdots i_k})$ are given by

$$\partial_h(x)_{i_0 \ldots i_k} = x_{i_0 \ldots \widehat{i_h} \ldots i_k \,|U_{i_0 \cdots i_k}}, \qquad \text{for } h = 0, \ldots, k.$$

Thus $N(\mathcal{L}(\mathcal{U}))$ is the Čech complex of $\mathcal{L}$ in the covering $\mathcal{U}$.

V.5 Maurer-Cartan and deformation functors of semicosimplicial Lie algebras

Consider a semicosimplicial Lie algebra

$$\mathfrak{g}^{\Delta}: \qquad \mathfrak{g}_0 \rightrightarrows \mathfrak{g}_1 \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \mathfrak{g}_2 \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots,$$

Denoting by $\alpha_i = \partial_i \colon \mathfrak{g}_0 \to \mathfrak{g}_1$, $\beta_j = \partial_j \colon \mathfrak{g}_1 \to \mathfrak{g}_2$ and $\gamma_k = \partial_k \colon \mathfrak{g}_2 \to \mathfrak{g}_3$ the face morphims we have:

$$\beta_1\alpha_0 = \beta_0\alpha_0, \qquad \beta_2\alpha_0 = \beta_0\alpha_1, \qquad \beta_2\alpha_1 = \beta_1\alpha_1, \tag{V.1}$$

$$\gamma_0\beta_0 = \gamma_1\beta_0, \quad \gamma_0\beta_1 = \gamma_2\beta_0, \quad \gamma_0\beta_2 = \gamma_3\beta_0, \quad \gamma_1\beta_1 = \gamma_2\beta_1, \quad \gamma_1\beta_2 = \gamma_3\beta_1, \quad \gamma_2\beta_2 = \gamma_3\beta_2.$$

Define the Maurer-Cartan functor

$$\mathrm{MC}_{\mathfrak{g}^\Delta} \colon \mathbf{Art} \to \mathbf{Set},$$

$$\mathrm{MC}_{\mathfrak{g}^\Delta}(A) = \{e^x \in \exp(\mathfrak{g}_1 \otimes \mathfrak{m}_A) \mid e^{\beta_0(x)}e^{-\beta_1(x)}e^{\beta_2(x)} = 1\},$$

or equivalently, using the BCH product $\bullet$,

$$\mathrm{MC}_{\mathfrak{g}^\Delta}(A) = \{x \in \mathfrak{g}_1 \otimes \mathfrak{m}_A \mid (\beta_0(x)) \bullet (-\beta_1(x)) \bullet (\beta_2(x)) = 0\}.$$

Lemma V.5.1. *The action*

$$\exp(\mathfrak{g}_0 \otimes \mathfrak{m}_A) \times \exp(\mathfrak{g}_1 \otimes \mathfrak{m}_A) \to \exp(\mathfrak{g}_1 \otimes \mathfrak{m}_A), \qquad (e^a, e^x) \mapsto e^{\alpha_1(a)}e^x e^{-\alpha_0(a)}$$

preserves Maurer-Cartan elements.

Proof. Let $e^x \in \mathrm{MC}_{\mathfrak{g}^\Delta}(A)$, $e^a \in \exp(\mathfrak{g}_0 \otimes \mathfrak{m}_A)$ and denote $e^y = e^{\alpha_1(a)}e^x e^{-\alpha_0(a)}$. Then

$$\begin{aligned} e^{\beta_0(y)} &= e^{\beta_0\alpha_1(a)}e^{\beta_0(x)}e^{-\beta_0\alpha_0(a)}, \\ e^{-\beta_1(y)} &= e^{\beta_1\alpha_0(a)}e^{-\beta_0(x)}e^{-\beta_1\alpha_1(a)}, \\ e^{\beta_2(y)} &= e^{\beta_2\alpha_1(a)}e^{\beta_2(x)}e^{-\beta_2\alpha_0(a)}, \end{aligned}$$

and the proof follows from equations (V.1). □

Moreover, we can define the quotient functor

$$\mathrm{Def}_{\mathfrak{g}^\Delta} \colon \mathbf{Art} \to \mathbf{Set}, \qquad \mathrm{Def}_{\mathfrak{g}^\Delta}(A) = \frac{\mathrm{MC}_{\mathfrak{g}^\Delta}(A)}{\exp(\mathfrak{g}_0 \otimes \mathfrak{m}_A)}$$

Proposition V.5.2. *The projectioon* $\mathrm{MC}_{\mathfrak{g}^\Delta} \to \mathrm{Def}_{\mathfrak{g}^\Delta}$ *is a smooth morphism of deformation functors.*

Proof. Immediate consequence of Proposition IV.2.7. □

Notice that if $\mathfrak{g}_2$ is abelian, then

$$e^{\beta_0(x)}e^{-\beta_1(x)}e^{\beta_2(x)} = e^{\beta_0(x)-\beta_1(x)+\beta_2(x)}$$

and therefore $\mathrm{MC}_{\mathfrak{g}^\Delta}$ is a smooth functor, since

$$\mathrm{MC}_{\mathfrak{g}^\Delta}(A) = Z^1(N(\mathfrak{g}^\Delta)) \otimes \mathfrak{m}_A.$$

Finally **every morphism of semicosimplicial Lie algebras induce a natural transformation of associated MC and Def functors.**

Let's now compute tangent and obstruction space of $\mathrm{MC}_{\mathfrak{g}^\Delta}$ and $\mathrm{Def}_{\mathfrak{g}^\Delta}$.

$$T^1\mathrm{MC}_{\mathfrak{g}^\Delta} = \mathrm{MC}_{\mathfrak{g}^\Delta}(\mathbb{K}[\varepsilon]) = \{x \in \mathfrak{g}_1 \otimes \mathbb{K}\varepsilon \mid e^{\beta_0(x)}e^{-\beta_1(x)}e^{\beta_2(x)} = 1\} =$$
$$\{x \in \mathfrak{g}_1 \otimes \varepsilon \mid \beta_0(x) - \beta_1(x) + \beta_2(x) = 0\} = \ker(\delta) = Z^1(N(\mathfrak{g}^\Delta)).$$

Next
$$\mathrm{Def}_{\mathfrak{g}^\Delta}(\mathbb{K}[\varepsilon]) = \frac{\mathrm{MC}_{\mathfrak{g}^\Delta}(\mathbb{K}[\varepsilon])}{\exp(\mathfrak{g}_0 \otimes \mathbb{K}\varepsilon)} =$$
$$= \frac{Z^1(N(\mathfrak{g}^\Delta))}{\{-\alpha_1(a) + \alpha_0(a) \mid \ a \in \mathfrak{g}_0\}} = H^1(N(\mathfrak{g}^\Delta)).$$

Next goal is to determine a complete obstruction theory for $\mathrm{MC}_{\mathfrak{g}^\Delta}$; by Lemma **??**, this will be also an obstruction theory for $\mathrm{Def}_{\mathfrak{g}^\Delta}$.

Let $0 \to J \to A \to B \to 0$ be a small extension and let $x \in \mathfrak{g}_1 \otimes \mathfrak{m}_A$ be any lifting of an element $y \in \mathrm{MC}_{\mathfrak{g}^\Delta}(B)$, then
$$e^{\beta_0(x)}e^{-\beta_1(x)}e^{\beta_2(x)} = e^r, \qquad \text{where} \quad r \in \mathfrak{g}_2 \otimes J.$$
Since J is annihilated by maximal ideals, the element e^r belongs to the center of the group $\exp(\mathfrak{g}_2 \otimes \mathfrak{m}_A)$ and then we have
$$e^r = e^{\beta_0(x)}e^{-\beta_1(x)}e^{\beta_2(x)} = e^{\beta_2(x)}e^{\beta_0(x)}e^{-\beta_1(x)} = e^{-\beta_1(x)}e^{\beta_2(x)}e^{\beta_0(x)}.$$

Lemma V.5.3. *In the notation above r is a cocycle in $N(\mathfrak{g}^\Delta)$, i.e. $\sum_k(-1)^k\gamma_k(r) = 0$.*

Proof. Notice that
$$\gamma_i(e^r)\gamma_j(e^r) = e^{\gamma_i(r)+\gamma_j(r)} = \gamma_j(e^r)\gamma_i(e^r)$$
for every i, j. It is therefore sufficient to prove that
$$\gamma_0(e^r)\gamma_2(e^r) = \gamma_1(e^r)\gamma_3(e^r).$$
We have
$$\gamma_k(e^r) = e^{\gamma_k\beta_0(x)}e^{-\gamma_k\beta_1(x)}e^{\gamma_k\beta_2(x)}, \qquad k = 0, 1, 2, 3,$$
and then
$$(\gamma_1(e^r))^{-1}\gamma_0(e^r) = e^{-\gamma_1\beta_2(x)}e^{\gamma_1\beta_1(x)}e^{-\gamma_0\beta_1(x)}e^{\gamma_0\beta_2(x)},$$
$$\gamma_2(e^r)(\gamma_3(e^r))^{-1} = e^{\gamma_2\beta_0(x)}e^{-\gamma_2\beta_1(x)}e^{\gamma_3\beta_1(x)}e^{-\gamma_3\beta_0(x)} = e^{-\gamma_3\beta_0(x)}e^{\gamma_2\beta_0(x)}e^{-\gamma_2\beta_1(x)}e^{\gamma_3\beta_1(x)},$$

$$\begin{aligned}(\gamma_1(e^r))^{-1}\gamma_0(e^r)\gamma_2(e^r)(\gamma_3(e^r))^{-1} &= \\ &= e^{-\gamma_1\beta_2(x)}e^{\gamma_1\beta_1(x)}e^{-\gamma_0\beta_1(x)}e^{\gamma_0\beta_2(x)}e^{-\gamma_3\beta_0(x)}e^{\gamma_2\beta_0(x)}e^{-\gamma_2\beta_1(x)}e^{\gamma_3\beta_1(x)} \\ &= e^{-\gamma_1\beta_2(x)}e^{\gamma_1\beta_1(x)}e^{-\gamma_0\beta_1(x)}e^{\gamma_2\beta_0(x)}e^{-\gamma_2\beta_1(x)}e^{\gamma_3\beta_1(x)} \\ &= e^{-\gamma_1\beta_2(x)}e^{\gamma_1\beta_1(x)}e^{-\gamma_2\beta_1(x)}e^{\gamma_3\beta_1(x)} \\ &= e^{-\gamma_1\beta_2(x)}e^{\gamma_3\beta_1(x)} = 1.\end{aligned}$$

□

Therefore, the element r defines a cohomology class $[r] \in H^2(N(\mathfrak{g}^\Delta)) \otimes J$ depending only by the class of y in $\mathfrak{g}_1 \otimes \mathfrak{m}_B$. In fact, any other lifting is equal to $x + u$, with $u \in \mathfrak{g}_1 \otimes J$. Every $e^{\beta_j(u)}$ belongs to the center of $\exp(\mathfrak{g}_2 \otimes \mathfrak{m}_A)$ and so
$$e^{\beta_0(x+u)}e^{-\beta_1(x+u)}e^{\beta_2(x+u)} = e^{\beta_0(x)}e^{-\beta_1(x)}e^{\beta_2(x)}e^{\beta_0(u)-\beta_1(u)+\beta_2(u)} = e^{r+\delta(u)}.$$

The same argument proves that $[r] \in H^2(N(\mathfrak{g}^\Delta)) \otimes J$ is a complete obstruction.

Corollary V.5.4. *Notation as above, if $H^2(N(\mathfrak{g}^\Delta)) = 0$ then $\mathrm{Def}_{\mathfrak{g}^\Delta}$ is smooth.*

Corollary V.5.5. *Let $f\colon \mathfrak{g}^\Delta \to \mathfrak{h}^\Delta$ be a morphism of semicosimplicial Lie algebras. If:*

1. *$f\colon H^1(N(\mathfrak{g}^\Delta)) \to H^1(N(\mathfrak{h}^\Delta))$ is surjective,*
2. *$f\colon H^2(N(\mathfrak{g}^\Delta)) \to H^2(N(\mathfrak{h}^\Delta))$ is injective,*

then the morphism $f\colon \mathrm{Def}_{\mathfrak{g}^\Delta} \to \mathrm{Def}_{\mathfrak{h}^\Delta}$ is smooth.

Proof. Apply standard smoothness criterion. □

V.6 An example: deformations of manifolds

Let $\mathcal{U} = \{U_i\}$ be an affine open cover of a smooth variety X, defined over an algebraically closed field of characteristic 0; denote by Θ_X the tangent sheaf of X. Then, we can define the Čech semicosimplicial Lie algebra $\Theta_X(\mathcal{U})$ as the semicosimplicial Lie algebra

$$\Theta_X(\mathcal{U}): \quad \prod_i \Theta_X(U_i) \rightrightarrows \prod_{i<j} \Theta_X(U_{ij}) \rightrightarrows\!\!\!\rightarrow \prod_{i<j<k} \Theta_X(U_{ijk}) \rightrightarrows\!\!\!\rightrightarrows \cdots,$$

Since every infinitesimal deformation of a smooth affine scheme is trivial [?, Lemma II.1.3], every infinitesimal deformation X_A of X over $\mathrm{Spec}(A)$ is obtained by gluing the trivial deformations $U_i \times \mathrm{Spec}(A)$ along the double intersections U_{ij}, and therefore it is determined by the sequence $\{\theta_{ij}\}_{i<j}$ of automorphisms of sheaves of A-algebras

$$\begin{tikzcd}
 & \mathcal{O}(U_{ij}) & \\
\mathcal{O}(U_{ij}) \otimes A \arrow[ur] \arrow[rr, "\theta_{ij}", "\simeq"'] & & \mathcal{O}(U_{ij}) \otimes A \arrow[ul] \\
 & A \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

satisfying the cocycle condition

$$\theta_{jk}\theta_{ik}^{-1}\theta_{ij} = Id_{\mathcal{O}(U_{ijk})\otimes A}, \qquad \forall\ i < j < k \in I. \tag{V.2}$$

Since we are in characteristic zero, we can take the logarithms and write $\theta_{ij} = e^{d_{ij}}$, where $d_{ij} \in \Theta_X(U_{ij}) \otimes \mathfrak{m}_A$. Therefore, the Equation (V.2) is equivalent to

$$e^{d_{jk}}e^{-d_{ik}}e^{d_{ij}} = 1 \in \exp(\Theta_X(U_{ijk}) \otimes \mathfrak{m}_A), \qquad \forall\ i < j < k \in I.$$

Next, let X'_A be another deformation of X over $\mathrm{Spec}(A)$, defined by the cocycle θ'_{ij}. To give an isomorphism of deformations $X'_A \simeq X_A$ is the same to give, for every i, an automorphism α_i of $\mathcal{O}(U_i) \otimes A$ such that $\theta_{ij} = {\alpha_i}^{-1}{\theta'_{ij}}^{-1}\alpha_j$, for every $i < j$. Taking again logarithms, we can write $\alpha_i = e^{a_i}$, with $a_i \in \Theta_X(U_i) \otimes \mathfrak{m}_A$, and so $e^{-a_i}e^{d'_{ij}}e^{a_j} = e^{d_{ij}}$.

Theorem V.6.1. *Let $\mathcal{U}$ be an affine open cover of a smooth algebraic variety X defined over an algebraically closed field of characteristic 0. Denoting by Def_X the functor of infinitesimal deformations of X, there exist isomorphisms of functors*

$$\mathrm{Def}_X \cong \mathrm{Def}_{\Theta_X(\mathcal{U})}$$

where $\Theta_X(\mathcal{U})$ is the semicosimplicial Lie algebra defined above.

Proof. By definition,

$$\mathrm{MC}_{\Theta_X(\mathcal{U})}(A) = \{\{x_{ij}\} \in \prod_{i<j} \Theta_X(U_{ij}) \otimes \mathfrak{m}_A \mid e^{x_{jk}}e^{-x_{ik}}e^{x_{ij}} = 1\ \forall\ i < j < k\},$$

for each $A \in \mathbf{Art}$. Moreover, given $x = \{x_{ij}\}$ and $y = \{y_{ij}\} \in \prod_{i<j} \Theta_X(U_{ij}) \otimes \mathfrak{m}_A$, we have $x \sim y$ if and only if there exists $a = \{a_i\} \in \prod_i \Theta_X(U_i) \otimes \mathfrak{m}_A$ such that $e^{-a_j}e^{x_{ij}}e^{a_i} = e^{y_{ij}}$ for all $i < j$. □

In particular this proves that if $H^2(\Theta_X) = 0$ then X has unobstructed deformations.

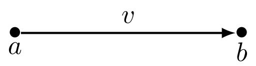
V.7 Rooted trees

Definition V.7.1. A **directed graph** is the data of two sets X_0, X_1 and two maps

$$\partial_0, \partial_1 : X_1 \to X_0.$$

Elements of X_0 will be called vertices, elements of X_1 are called edges. If $\partial_1(v) = b$ we will say that v is an incoming edge of b.

Thus a directed graph is a particular type of semisimplicial set and can be described graphically by taking a point for every vertex and an arrow for every edge:



$$\partial_0(v) = a,\ \partial_1(v) = b.$$

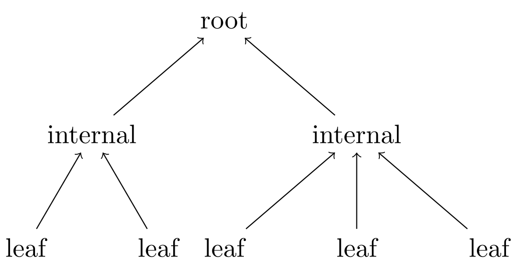
Definition V.7.2. A directed path from of length n from a vertex a to a vertex b is a sequence of edges $(v_1, \dots, v_n)$ such that

$$\partial_0(v_1) = a, \quad \partial_1(v_n) = b, \quad \text{and} \quad \partial_1(v_i) = \partial_0(v_{i+1}) \text{ for } 1 \leq i < n.$$

Definition V.7.3. A directed graph is called a **rooted tree** is there exists a vertex, called **root**, with the property that for every vertex a there exists a unique directed path from a to the root.

A rooted tree is called **planar** if for every vertex a it is given a total ordering in the set of incoming edges of a.

Given two vertices u, v in a rooted tree we shall write $u \to v$ if the vertex v belongs to the directed path from u to the root. A *leaf* is a vertex without incoming edges: equivalently, a vertex u is a leaf if the relation $v \to u$ implies $u = v$. A vertex is called *internal* if it is not a leaf; notice that, if a rooted tree has at least two vertices, then the root is an internal vertex.



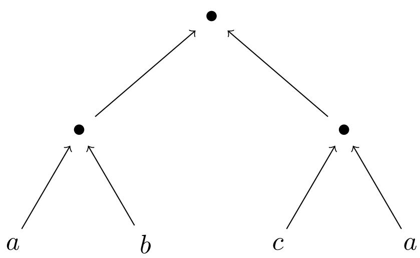
Every finite planar rooted tree can be described graphically in the plane with the root at the top and the leaves at the bottom (i.e., every directed path moves upward), and with the total ordering on incoming edges from the leftmost to the rightmost.

Definition V.7.4. A planar rooted tree Γ is a *binary tree* if every internal vertex has exactly two incoming edges. We use the notation

$$\mathcal{B} = \bigcup_{n>0} \mathcal{B}_n$$

where $\mathcal{B}_n$ is the set of planar binary rooted trees with n leaves.

Let R be a nonassociative algebra and $\Gamma \in \mathcal{B}$ a planar rooted binary tree. Labelling the leaves of Γ with elements of R, we can associate in a natural way the product element in R obtained by performing the product of R at every internal vertex in the order arising from the planar structure of the directed tree. For example:



$$\rightsquigarrow (ab)(ca) \in R$$

More formally, if $L(\Gamma)$ is the set of leaves of a planar rooted tree Γ, for every labelling $f : L(\Gamma) \to R$ we obtain an element $Z_\Gamma(f) \in R$.

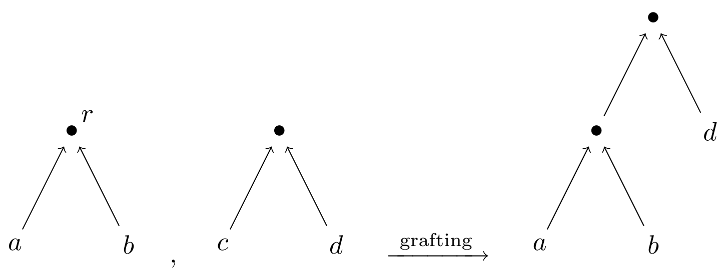


Figure V.1: Grafting the root r in the leaf c

V.8 A tree summation formula for the BCH product

If S is a subset of a nonassociatve algebra R, then the elements $Z_\Gamma(f)$, with $\Gamma \in \mathcal{B}$ and $f\colon L(\Gamma) \to S$, are a set of generators of the subalgebra generated by S. If R is either symmetric or skewsymmetric (e.g. R is a Lie algebra), then we may reduce the set of generators by a suitable choice of the labelling. Keeping in mind our main application (the BCH product), a possible way of doing that is the following, see e.g. [48].

Definition V.8.1. A rightmost branch of a planar rooted tree $\Gamma \in \mathcal{T}$ is a maximal connected subgraph $\Omega \subset \Gamma$, with the property that every edge of Ω is a rightmost edge of Γ. A rightmost branch is called non trivial if it has at least two vertices.

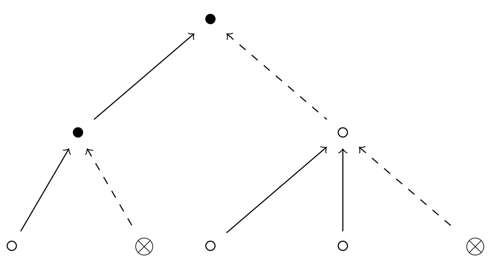
Definition V.8.2. A local rightmost leaf is a leaf lying on a non trivial rightmost branch. Given an internal vertex v, we call $m(v)$ the leaf lying on the rightmost branch containing v. We also denote by $d(v)$ the lenght of the unique directed path from $m(v)$ to v.

Definition V.8.3. A **subroot** is the vertex of a non trivial rightmost branch which is nearest to the root. The set of subroots of a finite planar rooted tree Γ will be denoted by $R(\Gamma)$.

Therefore, we have the natural bijections

$$\{\text{ subroots }\} \cong \{\text{ non trivial rightmost branches }\} \cong \{\text{ local rightmost leaves }\}.$$

Example V.8.4. In the planar rooted tree



the rightmost branches are dashed, subroots are denoted by $\bullet$ and local rightmost leaves by $\otimes$.

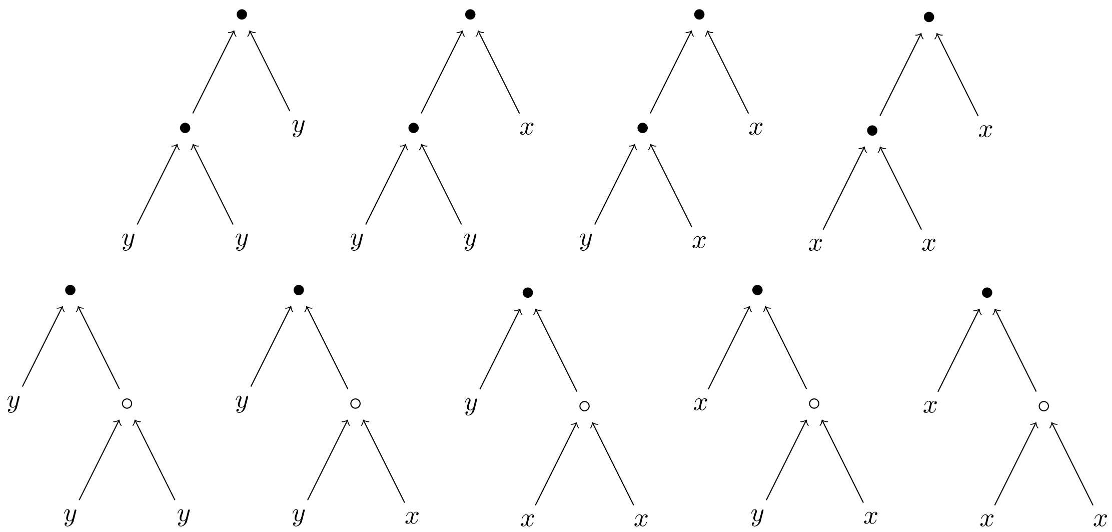
Using the notion of subroot, we can define a partial order $\preceq$ on the set of leaves $L(\Gamma)$.

Definition V.8.5. Given two leaves l_1 and l_2 in a planar rooted tree $\Gamma \in \mathcal{T}$, we say $l_1 \preceq l_2$ if $l_1 = l_2$ or there exists a subroot $v \in R(\Gamma)$ such that $l_2 = m(v)$ and $l_1 \to v$.

Definition V.8.6. For every partially ordered set $(A, \leq)$, we denote

$$\mathcal{B}_n(A) = \{(\Gamma, f) \,|\, \Gamma \in \mathcal{B}_n,\ f : (L(\Gamma), \preceq) \to (A, \leq), f \text{ monotone}\}, \qquad \mathcal{B}(A) = \bigcup_{n>0} \mathcal{B}_n(A).$$

Example V.8.7. The 9 elements of $\mathcal{B}_3(y \leq x)$ are



Definition V.8.8. Given $(\Gamma, f) \in \mathcal{B}(A)$, let us define

$$b_{(\Gamma,f)} := \prod_{v \in R(\Gamma)} \frac{b_{d(v)}}{t(v)},$$

where:

$$\sum_{n \geq 0} b_n t^n = \frac{t}{e^t - 1}, \qquad b_n = \frac{B_n}{n!}$$

and for every subroot $v \in R(\Gamma)$, we have

$$t(v) = \text{ number of leaves } u \in L(\Gamma) \text{ such that } u \to v \text{ and } f(u) = f(m(v)).$$

Theorem V.8.9. *Let L be a nilpotent Lie algebra as above; then, for every positive integer k and every $x_1, \ldots, x_k \in L$, we have*

$$x_k \bullet x_{k-1} \bullet \cdots \bullet x_1 = \sum_{(\Gamma,f) \,\in\, \mathcal{B}\,(x_1 \leq x_2 \leq \cdots \leq x_k)} b_{(\Gamma,f)} Z_\Gamma(f), \tag{V.3}$$

$$x_1 \bullet x_2 \bullet \cdots \bullet x_k = \sum_{n=1}^{+\infty} (-1)^{n-1} \sum_{(\Gamma,f) \,\in\, \mathcal{B}_n\,(x_1 \leq x_2 \leq \cdots \leq x_k)} b_{(\Gamma,f)} Z_\Gamma(f). \tag{V.4}$$

In particular, for $x, y \in L$, we have

$$x \bullet y = \sum_{(\Gamma,f) \in \mathcal{B}(y \leq x)} b_{(\Gamma,f)} Z_\Gamma(f). \tag{V.5}$$

Proof. Let us first prove Formula (V.5). Let $\mathcal{C}'\,(y \leq x) \subset \mathcal{B}\,(y \leq x)$ be the subset of trees having every local rightmost leaf labelled with x and denote by $\mathcal{C}\,(y \leq x) = \mathcal{C}'\,(y \leq x) \cup \mathcal{B}_1(y)$.

Since the bracket is skewsymmetric, we have that $Z_\Gamma(f) = 0$, for every $(\Gamma, f) \notin \mathcal{C}\,(y \leq x)$; therefore,

$$\sum_{(\Gamma,f) \in \mathcal{B}(y \leq x)} b_{(\Gamma,f)} Z_\Gamma(f) = \sum_{(\Gamma,f) \in \mathcal{C}(y \leq x)} b_{(\Gamma,f)} Z_\Gamma(f).$$

Recall that

$$x \bullet y = \sum_{r \geq 0} Z_r,$$

where

$$Z_0 = y, \qquad Z_{r+1} = \frac{1}{r+1} \sum_{m \geq 0} b_m \sum_{i_1 + \cdots + i_m = r} (\operatorname{ad} Z_{i_1})(\operatorname{ad} Z_{i_2}) \cdots (\operatorname{ad} Z_{i_m}) x, \quad \text{for } r \geq 0.$$

For every $r > 0$, let $\mathcal{C}_r \subset \mathcal{C}(y \leq x)$ be the subset of trees with exactly r leaves labelled with x; we prove that, for every $r \geq 0$, we have

$$Z_r = \sum_{(\Gamma,f)\in\mathcal{C}_r} b_{(\Gamma,f)} Z_\Gamma(f). \tag{V.6}$$

This is clear for $r = 0$; for $r = 1$, we have

$$Z_1 = \sum_{m\geq 0} b_m (\operatorname{ad} Z_0)^m x = \sum_{m\geq 0} b_m (\operatorname{ad} y)^m x,$$

whereas $\mathcal{C}_1 = \{\Omega_m\}$, $m \geq 0$, and Ω_m is the unique posetted tree in $\mathcal{C}_1$ with $m+1$ leaves. Therefore, the coefficient $b_{(\Omega_m)}$ is exactly b_m and so

$$Z_1 = \sum_{(\Gamma,f)\in\mathcal{C}_1} b_{(\Gamma,f)} Z_\Gamma(f).$$

Moreover, every element of $\mathcal{C}_{r+1}$ is obtained in a unique way starting from a tree Ω_m and grafting, at each of the m leaves labelled with y, the roots of elements of $\mathcal{C}_{i_1}, \ldots, \mathcal{C}_{i_m}$, with $i_1 + \cdots + i_m = r$. Therefore, the proof of (V.6) follows easily by induction on r.

Next, since $\bullet$ is associative, we have

$$x_1 \bullet x_2 \bullet \cdots \bullet x_k = -((-x_k) \bullet \cdots \bullet (-x_1)),$$

and Formula (V.4) follows immediately from (V.3). Finally, setting $y = x_{k-1} \bullet \cdots \bullet x_1$, we have that every posetted tree of $\mathcal{B}(x_1 \leq x_2 \leq \cdots \leq x_k)$ can be described in a unique way as a posetted tree in $\mathcal{C}(y \leq x_k)$, where at every leaf labelled with y is grafted the root of a posetted tree of $\mathcal{B}(x_1 \leq x_2 \leq \cdots \leq x_{k-1})$. In view of the associativity relation

$$x_k \bullet x_{k-1} \bullet \cdots \bullet x_1 = x_k \bullet y,$$

we obtain that (V.3) is a consequence of $x_k \bullet y = \displaystyle\sum_{(\Gamma,f)\in\mathcal{C}(y\leq x_k)} b_{(\Gamma,f)} Z_\Gamma(f)$.

□

V.9 Exercises

Exercise V.9.1 (Differential operators, [36, 14, 27]). Let A be a unitary commutative $\mathbb{K}$-algebra. Consider A as a subalgebra of $\operatorname{Hom}_{\mathbb{K}}(A, A)$, where every $a \in A$ is identified with the operator

$$a\colon A \to A, \qquad a(b) = ab.$$

1. Prove that
$$\operatorname{Diff}_0(A) := \{f \in \operatorname{Hom}_{\mathbb{K}}(A, A) \mid [f, a] = 0\ \forall a \in A\} = A.$$

2. For $n > 0$ define recursively
$$\operatorname{Diff}_n(A) := \{f \in \operatorname{Hom}_{\mathbb{K}}(A, A) \mid [f, a] \in \operatorname{Diff}_{n-1}(A)\ \forall a \in A\}$$
and prove that for every $f \in \operatorname{Diff}_n(A)$, $g \in \operatorname{Diff}_m(A)$ we have
$$fg \in \operatorname{Diff}_{n+m}(A), \qquad [f, g] \in \operatorname{Diff}_{n+m-1}(A).$$
In particular $\operatorname{Diff}_0(A), \operatorname{Diff}_1(A)$ and $\operatorname{Diff}(A) := \bigcup_n \operatorname{Diff}_n(A)$ are Lie subalgebras of $\operatorname{Hom}_{\mathbb{K}}(A, A)$.

3. (Higher Koszul brackets, [64]) For every $f \in \operatorname{Hom}_{\mathbb{K}}(A, A)$ and every $n > 0$ define
$$\Phi^n_f\colon A^n \to A, \qquad \Phi^n_f(a_1, \ldots, a_n) = (-1)^n [a_1, a_2, \ldots, a_n, f](1).$$
Prove that f is $\mathbb{K}$ multilinear symmetric and $\Phi^n_f = 0$ if and only if $f \in \operatorname{Diff}_{n-1}(A)$.

4. For $A = \mathbb{K}[t]$, prove that every $f \in \mathrm{Diff}_n(A)$ is uniquely determined by $f(1), f(t), \ldots, f(t^n)$ (Hint: compute $\Phi_f^m(t, t, \ldots, t)$ for $m > n$). Deduce that there exist $p_0, \ldots, p_n \in \mathbb{K}[t]$ such that
$$f = p_0 + p_1 \frac{d}{dt} + \cdots + p_n \frac{d^n}{dt^n}.$$

5. For $A = \mathbb{C}[t^2, t^3] \subset \mathbb{C}[t]$ show that the map
$$\mathrm{Diff}_1(A) \otimes \mathrm{Diff}_1(A) \to \mathrm{Diff}_2(A), \qquad f \otimes g \to fg$$
is not surjective.

Exercise V.9.2. Use the associativity of the BCH product for proving the relations
$$b_n(1 + n(-1)^n) = -\sum_{i=1}^{n-1}(-1)^i b_i b_{n-i}, \quad B_n(1 + n(-1)^n) = -\sum_{i=1}^{n-1}(-1)^i \binom{n}{i} B_i B_{n-i}.$$
among Bernoulli numbers. Then prove the same relations in a more elementary way.

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